A Pair of Optimal Inequalities Related to the Error Function

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November 4, 1997

The function

$$\begin{aligned} V(x) &\equiv \sqrt{\pi} e^{x^2}[1 - \text{erf}(x)] \qquad (1) \\ &= \int_0^\infty \frac{e^{-u}}{\sqrt{x^2+u}} du = 2e^{x^2} \int_x^\infty e^{-t^2} dt \end{aligned}$$

arises in many contexts, from probability to mathematical physics, and satisifies the differential equation

$$V'(x) = 2xV(x) - 2 \quad \text{with} \quad V(0) = \sqrt{\pi}. \qquad (2)$$

In this note, we restrict attention to $x \geq 0$, in which case $0 < V(x) < 1/x$ and $V(x)$ is decreasing. For $x \geq 0$ we show that

$$g_\pi(x) \leq V(x) < g_4(x) \qquad (3)$$

*supported by National Science Foundation Grant DMS-97-06981

where

$$g_k(x) = \frac{k}{(k-1)x + \sqrt{x^2+k}}. \tag{4}$$

We also show that these inequalities are optimal for functions of the form (4) with equality only at $g_\pi(0) = V(0) = \sqrt{\pi}$. The bounds in (3) are considerably sharper than the classical inequalities of Komatsu [1]. The upper bound, which implies the convexity of $1/V(x)$, was established independently by Wirth [4] for $x \geq 0$ and by Szarek and Werner [3] for $x > -\frac{1}{\sqrt{2}}$. The weaker lower bound $g_3(x) < V(x)$ was used in [2] to show that the function $[1/V(x) - x]^2 / V(x)$ is decreasing for $x \geq 0$. It is easy to see that the family of functions $g_k(x)$ is increasing in k and that $0 < g_k(x) < 1/x$.

In order to prove that the upper bound is optimal, we first observe that

$$\begin{aligned} & g_k'(x) > 2[x g_k(x) - 1] \\ \iff \quad & (k-2)x^2 + k(k-3) < (k-2)x\sqrt{x^2+k} \\ \iff \quad & x^2(k-2)(k-4) + k(k-3)^2 < 0 \end{aligned} \tag{5}$$

when $k > 3$. We now restrict attention to $3 \leq k \leq 4$ and let $h_k(x) = g_k(x) - V(x)$. For $k = 4$, the expression (5) implies $g_4'(x) < 2[x g_k(x) - 1]$ so that $h_4'(x) < 2x h_4(x)$ for all $x \geq 0$; whereas for $k < 4$ this holds only for $x < a_k = \sqrt{\frac{k(k-3)^2}{(k-2)(4-k)}}$. Since both $V(x)$ and $g_k(x)$ are positive and bounded above by $1/x$, their difference also satisfies $|h_k(x)| < 1/x \to 0$.

For $k = 4$, if $h_4(x) \leq 0$ for some $x > 0$, then $h_4'(x) < 2x h_4(x)$ is negative and thus h_4 is negative and strictly decreasing from a certain x on, which contradicts $\lim_{x\to\infty} h_4(x) = 0$. Thus $h_4(x) > 0$ so that $g_4(x) > V(x)$, for all x. Now suppose that for some $k < 4$, g_k is an upper bound, i.e. $h_k(x) \geq 0$ for all $x \geq 0$. In particular, $h_k(x) \geq 0$ for all $x > a_k$. For $k < 4$, we find however that $h_k'(x) > 2x h_k(x)$ holds for $x > a_k$. Thus we get $h_k(x) \geq 0$ and strictly increasing for all $x > a_k$ which contradicts $\lim_{x\to\infty} h_k(x) = 0$. Thus the upper bound can *not* hold when $x > a_k$ and $k < 4$. The lower bound also fails for $k > \pi$ since then $h_k(0) = g_k(0) - V(0) = \sqrt{k} - \sqrt{\pi} > 0$.

To establish the improved lower bound $g_\pi \leq V(x)$ we note that the argument above implies that $h_k(x)$ is negative for $x > a_k$ and $3 < k \leq \pi$. However for $k < \pi$ we have $h_k(0) < 0$ so that $h_k(x)$ is also negative for very small x. If $h_k(x)$ is ever non-negative, we can let b denote the first place $h_k(x)$ touches or crosses the x-axis, i.e., $h_k(b) = 0$ and $h_k(x) < 0$ for $x < b$. Then h_k must be increasing on some interval of the form (x_0, b). However, by the remarks above, $h_k(b) = 0$ implies $b \leq a_k$ so that $h_k'(x) < 2xh_k(x) < 0$ on (x_0, b). Since this contradicts h_k increasing on (x_0, b), we must have $h_k(x) < 0$ for all $x \geq 0$ if $k < \pi$.

Thus we have proved the lower bound $g_k(x) < V(x)$ on $[0, \infty)$ for $k < \pi$. Since g_k is continuous and increasing in k, it follows that $g_\pi(x) \leq V(x)$. To show that this inequality is strict except at $x = 0$, note that the right derivative of h_k at 0 satisfies $h_k'(0) = 2 - k$ so that $h_\pi'(0) < 0$ and $h_\pi(x)$ is negative at least on some small interval $(0, x_1)$. Then we can repeat the argument above to show that $h_\pi(x) < 0$ if $x > 0$.

Acknowledgment We are grateful for the hospitality of the School of Mathematics at Georgia Tech where this work was done.

References

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