On error estimates for the Trotter-Kato product formula

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Abstract

We study the error bound in the operator norm topology for the Trotter exponential product formula as well as for its generalization à la Kato. In the frame of the abstract setting we give a simple proof of error estimates which improve some of recent results in this direction.

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1 Introduction and Setup of the Problem

For simplicity let A and B be two non-negative self-adjoint operators on the separable Hilbert space $\mathcal{H}$. Without loss of generality we can assume that

$$A \geq I \qquad \text{and} \qquad B \geq I. \tag{1.1}$$

In this general setting Kato [6] has shown that the Trotter product formula [1, 2, 10] always converges in the strong operator sense to a semigroup which is generated by the form sum $H := A \dot{+} B$, i.e.

$$s - \lim_{n\to\infty} \left(e^{-tA/n}e^{-tB/n}\right)^n = e^{-tH}. \tag{1.2}$$

Since the Sophus Lie observation for finite matrices we know (see e.g. [8]) that for bounded self-adjoint operators A and B there is a constant $C = C(t, \|A\|, \|B\|)$ such that

$$\left\| \left(e^{-tA/n}e^{-tB/n}\right)^n - e^{-tH} \right\| \leq \frac{C}{n}, \qquad n = 1, 2, \ldots. \tag{1.3}$$

Hence, instead of (1.2) one has convergence in operator norm. However, the boundedness of A and B is very strong and far from practical needs. Therefore, few natural problems which arise are:

(a) to indicate an optimal topology γ in which the Trotter product formula (1.2) converges,

(b) to give a corresponding error estimate à la Lie,

(c) to generalize (1.2) à la Kato [6] in the corresponding topology γ to a class of real-valued, Borel measurable functions $f(\cdot)$ and $g(\cdot)$:

$$\gamma - \lim_{n\to\infty} \left(f(tA/n)g(tB/n)\right)^n = e^{-tH}. \tag{1.4}$$

In fact we have performed this program for the case of Gibbs semigroups [7], i.e., for the trace-norm topology $\gamma = \|\cdot\|_1$. The aim of the present letter is to show that one can perform it in operator-norm topology $\gamma = \|\cdot\|$.

Up to our knowledge it was Rogava [9] who first has announced a result concerning (a) and (b) for unbounded operators A and B and $\gamma = \|\cdot\|$. Under assumption that the self-adjoint operator H coincides with the algebraic sum of A and B, i.e. the operator H is given by

$$Hf = Af + Bf, \qquad f \in \mathrm{dom}(H) = \mathrm{dom}(A) \subseteq \mathrm{dom}(B), \tag{1.5}$$

in [9] it is claimed that there is a constant C such that

$$\left\| \left(e^{-tA/n}e^{-tB/n}\right)^n - e^{-tH} \right\| \leq \frac{C\log(n)}{\sqrt{n}}, \qquad n = 2, 3, \ldots, \tag{1.6}$$

uniformly in $t \in [0,T]$, $0 < T < \infty$.

For a symmetrized version of the Trotter product formula a stronger estimate has been proven by Ichinose and Tamura [3]. Under the assumption

$$\mathrm{dom}(A^\alpha) \subseteq \mathrm{dom}(B), \qquad 0 \le \alpha < 1, \tag{1.7}$$

they prove that there is a constant C such that

$$\left\| F(t/n)^n - e^{-tH} \right\| \le C n^{-2/(3+\alpha)}, \qquad n = 1, 2, \ldots, \tag{1.8}$$

uniformly in $t \in [0,T]$, $0 < T < \infty$, where

$$F(\tau) := e^{-\tau A/2} e^{-\tau B} e^{-\tau A/2}, \qquad \tau > 0. \tag{1.9}$$

In the present paper we give further improvement and generalization of these results following our programm (a) - (c) for $\gamma = \|\cdot\|$ in (a). Under weaker assumptions than in [3, 9] we show that the error bounds (1.6) and (1.8) can be refined and generalized to the Trotter-Kato formula (1.4). We relax (1.7) and assume that

$$\mathrm{dom}(A) \subseteq \mathrm{dom}(B) \tag{1.10}$$

and

$$\|Bf\| \le a\|Af\|, \qquad f \in \mathrm{dom}(A), \qquad a > 0. \tag{1.11}$$

Further, instead of (1.6) or (1.8) we consider general Trotter-Kato product formula (1.4) and its symmetrized version. To this end let us introduce Borel functions $f(\cdot)$ and $g(\cdot)$ defined on $[0, +\infty)$ such that

$$0 \le f(x) \le 1, \qquad f(0) = 1, \qquad f'(0) = -1 \tag{1.12}$$

$$0 \le g(x) \le 1, \qquad g(0) = 1, \qquad g'(0) = -1. \tag{1.13}$$

We assume that

$$C_0 := \sup_{x>0} \frac{x\sqrt{f(x)}}{1 - f(x)} < +\infty \tag{1.14}$$

$$C_1 := \sup_{x>0} \frac{1 - f(x)}{x} < +\infty \tag{1.15}$$

$$C_2 := \sup_{x>0} \left| \left(f(x) - \frac{1}{1+x} \right) \frac{1}{x^2} \right| < +\infty \tag{1.16}$$

$$S_1 := \sup_{x>0} \frac{1 - g(x)}{x} < +\infty \tag{1.17}$$

$$S_2 := \sup_{x>0} \left| \left(g(x) - \frac{1}{1+x} \right) \frac{1}{x^2} \right| < +\infty \tag{1.18}$$

as well as

$$0 < C_0 S_1 a < 1. \tag{1.19}$$

We note that by (1.12) and (1.13) one always has $C_0 \geq 1$ and $S_1 \geq 1$. Therefore the relation (1.19) implies $0 < a < (C_0 S_1)^{-1}$ in (1.11). In our main theorem (see Section 2) we show that there are constants $C > 0$ and $C' > 0$ such that

$$\left\| \left(\sqrt{f(tA/n)} g(tB/n) \sqrt{f(tA/n)} \right)^n - e^{-tH} \right\| \leq C \, \frac{\log(n)}{n}, \qquad n = 3, 4, \ldots, \tag{1.20}$$

and

$$\left\| (f(tA/n) g(tB/n))^n - e^{-tH} \right\| \leq C' \, \frac{\log(n)}{n}, \qquad n = 3, 4, \ldots, \tag{1.21}$$

uniformly in $t \geq 0$, cf. (1.6) and (1.8). Since (1.19) always implies $0 < a < 1$, we conclude that under conditions (1.10) and (1.11) the self-adjoint operator H is given by (1.5). We would like to note that $f(x) = \exp(-x)$ and $g(x) = \exp(-x)$ belong to the class of Borel functions described by (1.12) - (1.18) with $C_0 = C_1 = C_2 = S_1 = S_2 = 1$, which in particular yields that instead of (1.19) we have $0 < a < 1$. Hence, the relation (1.21) recovers standard Trotter product formula (1.2) for operator norm convergence with better error bound than in (1.6). Similar remark is valid with respect to (1.8) and (1.20).

Finally, we note that the class of Borel functions (1.12) - (1.18) contains for example $f(x) = (1 + x/2)^{-2}$ with $C_0 = 2$ but not $f(x) = (1 + x)^{-1}$ because in this case $C_0 = +\infty$. On the other hand $g(x) = (1 + x)^{-1}$ belongs to the class (1.12) - (1.18).

2 Main Theorem

The goal of this section is to proved the announced results (1.20) and (1.21). The proof relies on the following series of lemmas.

Lemma 2.1 *If X is a self-adjoint operator such that $0 \leq X \leq I$, then*

$$\|X^k (I - X)\| \leq \frac{1}{1 + k}, \qquad k = 0, 1, \ldots. \tag{2.1}$$

Proof: This estimate is a consequence of the estimate

$$|x^k (1 - x)| \leq \frac{1}{1 + k}, \qquad k = 0, 1, \ldots, \quad 0 \leq x \leq 1, \tag{2.2}$$

and the spectral theorem for self-adjoint operators. □

Lemma 2.2 *If X is a self-adjoint operator such that $X \geq 0$, then*

$$\left\| X e^{-qX} \right\| \leq \frac{1}{qe}, \qquad q \geq 0. \tag{2.3}$$

Proof: Since

$$\left| x e^{-qx} \right| \leq \frac{1}{qe}, \qquad q \geq 0, \qquad 0 \leq x < +\infty \tag{2.4}$$

we obtain (2.3) by the spectral theorem for self-adjoint operators. □

For brevity we set below

$$T(\tau) := \sqrt{f(\tau A)}\, g(\tau B) \sqrt{f(\tau A)}, \qquad \tau \geq 0, \tag{2.5}$$

and

$$U(\tau) := e^{-\tau H}, \qquad \tau \geq 0, \tag{2.6}$$

Lemma 2.3 *Let A be a self-adjoint operator which satisfies (1.1). If the Borel function f obeys (1.12) and (1.14), then the operator $I - f(\tau A)$ is boundedly invertible and*

$$\left\| (I - f(\tau A))^{-1} h \right\| \leq \|h\| + \frac{C_0}{\tau} \left\| A^{-1} h \right\|, \qquad h \in \mathcal{H}. \tag{2.7}$$

for each $\tau > 0$.

Proof: From (1.14) we obtain

$$0 \leq f(x) \leq \left(1 + \frac{x^2}{2C_0^2} + \frac{x}{C_0} \sqrt{1 + \frac{x^2}{4C_0^2}} \right)^{-1}, \qquad x \geq 0, \tag{2.8}$$

which yields

$$0 \leq f(x) \leq \left(1 + \frac{x}{C_0} \right)^{-1}, \qquad x \geq 0. \tag{2.9}$$

Since $A \geq I$ (1.1), this gives

$$0 \leq f(\tau A) \leq \left(1 + \frac{\tau}{C_0} \right)^{-1} I \leq \frac{C_0}{C_0 + \tau} I, \qquad \tau \geq 0, \tag{2.10}$$

which leads to

$$I - f(\tau A) \geq \frac{\tau}{C_0 + \tau} I, \qquad \tau > 0. \tag{2.11}$$

Therefore, the operator $I - f(\tau A)$ is boundedly invertible. Since

$$(I - f(\tau A))^{-1} = I + f(\tau A)(I - f(\tau A))^{-1} \tag{2.12}$$

and

$$\left\|\tau A f(\tau A)(I-f(\tau A))^{-1}\right\| \leq \left\|\tau A\sqrt{f(\tau A)}(I-f(\tau A))^{-1}\right\|, \tag{2.13}$$

we obtain from (1.14) that

$$\left\|\tau A f(\tau A)(I-f(\tau A))^{-1}\right\| \leq C_0. \tag{2.14}$$

Therefore,

$$\begin{aligned}
&\left\|(I-f(\tau A))^{-1}h\right\| \\
\leq \; & \|h\| + \left\|f(\tau A)(I-f(\tau A))^{-1}h\right\| \leq \|h\| + \frac{C_0}{\tau}\|A^{-1}h\|, \qquad h \in \mathcal{H},
\end{aligned} \tag{2.15}$$

which proves (2.7). □

Lemma 2.4 *Let A and B be self-adjoint operators such that the conditions (1.1), (1.10) and (1.11) are satisfied. If Borel functions f and g satisfy (1.12), (1.13), (1.14) and (1.17) and if condition (1.19) is fulfilled, then operator $I-T(\tau)$ is boundedly invertible for each $\tau > 0$ and*

$$\left\|(I-T(\tau))^{-1}h\right\| \leq \frac{1}{1-C_0S_1a}\left(\|h\| + \frac{C_0}{\tau}\|A^{-1}h\|\right), \qquad h \in \mathcal{H}, \tag{2.16}$$

Proof: We use representation

$$I - T(\tau) = I - f(\tau A) + \sqrt{f(\tau A)}(I-g(\tau B))\sqrt{f(\tau A)}. \tag{2.17}$$

By Lemma 2.3 the operator $I - f(\tau A)$ is boundedly invertible provided $\tau > 0$. Hence, one has representation

$$I - T(\tau) = (I-f(\tau A))\left\{I + \sqrt{f(\tau A)}(I-f(\tau A))^{-1}(I-g(\tau B))\sqrt{f(\tau A)}\right\}. \tag{2.18}$$

Since

$$\begin{aligned}
&\sqrt{f(\tau A)}(I-f(\tau A))^{-1}(I-g(\tau B))\sqrt{f(\tau A)} \\
= \; & \tau A\sqrt{f(\tau A)}(I-f(\tau A))^{-1}A^{-1}B\frac{1}{\tau}(I-g(\tau B))B^{-1}\sqrt{f(\tau A)},
\end{aligned} \tag{2.19}$$

one gets estimate

$$\begin{aligned}
&\left\|\sqrt{f(\tau A)}(I-f(\tau A))^{-1}(I-g(\tau B))\sqrt{f(\tau A)}\right\| \\
\leq \; & \left\|\tau A\sqrt{f(\tau A)}(I-f(\tau A))^{-1}\right\| \|BA^{-1}\|\frac{1}{\tau}\left\|(I-g(\tau B))B^{-1}\right\| \left\|\sqrt{f(\tau A)}\right\|.
\end{aligned} \tag{2.20}$$

Using (1.11), (1.14), (1.17), (1.19) and (2.10) we obtain

$$\left\|\sqrt{f(\tau A)}(I-f(\tau A))^{-1}(I-g(\tau B))\sqrt{f(\tau A)}\right\| \leq C_0S_1a < 1 \tag{2.21}$$

Hence, operator $I+\sqrt{f(\tau A)}(I-f(\tau A))^{-1}(I-g(\tau B))\sqrt{f(\tau A)}$ is boundedly invertible and obeys the estimate

$$\left\|\left(I+\sqrt{f(\tau A)}(I-f(\tau A))^{-1}(I-g(\tau B))\sqrt{f(\tau A)}\right)^{-1}\right\| \le \frac{1}{1-C_0S_1a}. \tag{2.22}$$

By (2.7), (2.18) and (2.22) we finally get

$$\begin{aligned} &\|(I-T(\tau))^{-1}h\| \qquad (2.23)\\ \le\ & \frac{1}{1-C_0S_1a}\left\|(I-f(\tau A))^{-1}h\right\| \le \frac{1}{1-C_0S_1a}\left\{\|h\|+\frac{C_0}{\tau}\|A^{-1}h\|\right\}, \qquad h\in\mathcal{H}\end{aligned}$$

which proves (2.16) □

Lemma 2.5 *Let A and B be self-adjoint operators which satisfy (1.1), (1.10) and (1.11). If Borel functions f and g obey (1.12), (1.13), (1.15) and (1.17), then there is a constant $L_1>0$ such that*

$$\frac{1}{\tau}\left\|A^{-1}\left((T(\tau)-U(\tau)\right)\right\| \le L_1 \tag{2.24}$$

for $\tau>0$.

Proof: By (2.5) and (2.6) one gets

$$\begin{aligned} &T(\tau)-U(\tau) \qquad (2.25)\\ =\ & -\sqrt{f(\tau A)}(I-g(\tau B))\sqrt{f(\tau A)}-(I-f(\tau A))+(I-U(\tau))\end{aligned}$$

which yields

$$\begin{aligned} &A^{-1}\left(T(\tau)-U(\tau)\right) \qquad (2.26)\\ =\ & -\sqrt{f(\tau A)}A^{-1}B(I-g(\tau B))B^{-1}\sqrt{f(\tau A)}-(I-f(\tau A))A^{-1}+A^{-1}H(I-U(\tau))H^{-1}.\end{aligned}$$

Hence, by (2.10) we obtain

$$\begin{aligned} &\frac{1}{\tau}\left\|A^{-1}\left(T(\tau)-U(\tau)\right)\right\| \qquad (2.27)\\ \le\ & \|BA^{-1}\|\frac{1}{\tau}\|(I-g(\tau B))B^{-1}\|+\frac{1}{\tau}\|(I-f(\tau A))A^{-1}\|+\|HA^{-1}\|\frac{1}{\tau}\|(I-U(\tau))H^{-1}\|.\end{aligned}$$

Therefore, by (1.15) and (1.17) we find

$$\frac{1}{\tau}\left\|A^{-1}\left(T(\tau)-U(\tau)\right)\right\| \le \|BA^{-1}\|S_1+C_1+\|HA^{-1}\|D_1, \tag{2.28}$$

where

$$D_1 := \sup_{x>0}\left(1-e^{-x}\right)\frac{1}{x}. \tag{2.29}$$

Since $\|HA^{-1}\|\le\|I+BA^{-1}\|\le 1+\|BA^{-1}\|\le 1+a$, we obtain

$$\frac{1}{\tau}\left\|A^{-1}\left(T(\tau)-U(\tau)\right)\right\| \le C_1+D_1+(S_1+D_1)a. \tag{2.30}$$

Setting $L_1 := C_1+D_1+(S_1+D_1)a$ we prove (2.24). □

Lemma 2.6 *Let A and B be self-adjoint operators which satisfy (1.1), (1.10) and (1.11). If Borel functions f and g obey (1.12) - (1.18), then there is a constant $L_2 > 0$ such that*

$$\frac{1}{\tau^2}\left\|A^{-1}(T(\tau) - U(\tau))A^{-1}\right\| \leq L_2 \tag{2.31}$$

for $\tau > 0$.

Proof: From (2.25) we obtain representation

$$\begin{aligned} &T(\tau) - U(\tau) \\ &= \left(I - \sqrt{f(\tau A)}\right)(I - g(\tau B))\sqrt{f(\tau A)} + (I - g(\tau B))\left(I - \sqrt{f(\tau A)}\right) \\ &\quad + \left(f(\tau A) - (I + \tau A)^{-1}\right) + \left(g(\tau B) - (I + \tau B)^{-1}\right) + \left((I + \tau H)^{-1} - U(\tau)\right) \\ &\quad + \tau H(I + \tau H)^{-1} - \tau A(I + \tau A)^{-1} - \tau B(I + \tau B)^{-1}. \end{aligned} \tag{2.32}$$

Since

$$\begin{aligned} &A^{-1}\left\{\tau H(I + \tau H)^{-1} - \tau A(I + \tau A)^{-1} - \tau B(I + \tau B)^{-1}\right\}A^{-1} \\ &= \tau^2\left\{(I + \tau A)^{-1} + A^{-1}B(I + \tau B)^{-1}BA^{-1} - A^{-1}H(I + \tau H)^{-1}HA^{-1}\right\} \end{aligned} \tag{2.33}$$

we get

$$\begin{aligned} &A^{-1}\{T(\tau) - U(\tau)\}A^{-1} \\ &= \left(I - \sqrt{f(\tau A)}\right)A^{-1}(I - g(\tau B))B^{-1}BA^{-1}\sqrt{f(\tau A)} \\ &\quad + A^{-1}B(I - g(\tau B))B^{-1}\left(I - \sqrt{f(\tau A)}\right)A^{-1} + \left(f(\tau A) - (I + \tau A)^{-1}\right)A^{-2} \\ &\quad + A^{-1}B\left(g(\tau B) - (I + \tau B)^{-1}\right)B^{-2}BA^{-1} \\ &\quad + A^{-1}H\left((I + \tau H)^{-1} - U(\tau)\right)H^{-2}HA^{-1} + \tau^2(I + \tau A)^{-1} \\ &\quad + \tau^2 A^{-1}B(I + \tau B)^{-1}BA^{-1} - \tau^2 A^{-1}H(I + \tau H)^{-1}HA^{-1}. \end{aligned} \tag{2.34}$$

Therefore, we have the estimate

$$\begin{aligned} &\frac{1}{\tau^2}\left\|A^{-1}\{T(\tau) - U(\tau)\}A^{-1}\right\| \\ &\leq \frac{1}{\tau}\left\|\left(I - \sqrt{f(\tau A)}\right)A^{-1}\right\|\frac{1}{\tau}\|(I - g(\tau B))B^{-1}\|\|BA^{-1}\| \\ &\quad + \|BA^{-1}\|\frac{1}{\tau}\|(I - g(\tau B))B^{-1}\|\frac{1}{\tau}\left\|\left(I - \sqrt{f(\tau A)}\right)A^{-1}\right\| \\ &\quad + \frac{1}{\tau^2}\left\|\left(f(\tau A) - (I + \tau A)^{-1}\right)A^{-2}\right\| + \frac{1}{\tau^2}\left\|\left(g(\tau B) - A^{-1}B(I + \tau B)^{-1}\right)B^{-2}\right\|\|BA^{-1}\|^2 \\ &\quad + \left\|\left((I + \tau H)^{-1} - U(\tau)\right)H^{-2}\right\|\|HA^{-1}\|^2 + 1 + \|BA^{-1}\|^2 + \|HA^{-1}\|^2. \end{aligned} \tag{2.35}$$

By

$$\frac{1-\sqrt{f(x)}}{x} \le \frac{1-f(x)}{x} \le C_1 \tag{2.36}$$

one obtains

$$\frac{1}{\tau}\|(I-\sqrt{f(\tau A)}A^{-1}\| \le C_1. \tag{2.37}$$

Further, setting

$$D_2 := \sup_{x>0}\left|\left(\frac{1}{1+x}-e^{-x}\right)\frac{1}{x^2}\right| < +\infty, \tag{2.38}$$

one gets estimate

$$\frac{1}{\tau^2}\left\|((I+\tau H)^{-1}-U(\tau))H^{-2}\right\| \le D_2. \tag{2.39}$$

Using (2.36) and (2.39) we find from (2.35) and (1.16) - (1.18) that

$$\begin{aligned} &\frac{1}{\tau^2}\left\|A^{-1}\left\{T(\tau)-U(\tau)\right\}A^{-1}\right\| \\ &\le\ 2C_1S_1\|BA^{-1}\| + C_2 + S_2\|BA^{-1}\|^2 + D_2\|HA^{-1}\|^2 + 1 + \|BA^{-1}\|^2 + \|HA^{-1}\|^{-2}. \end{aligned} \tag{2.40}$$

Since $\|BA^{-1}\| \le a$ and $\|HA^{-1}\| \le \|I+BA^{-1}\| \le 1+a$, we obtain from (2.40) the estimate

$$\frac{1}{\tau^2}\left\|A^{-1}\left\{T(\tau)-U(\tau)\right\}A^{-1}\right\| \le 2 + C_2 + D_2(2+2D_2+2C_1S_1)a + (2+S_2+D_2)a^2. \tag{2.41}$$

Setting $L_2 = 2 + C_2 + D_2(2+2D_2+2C_1S_1)a + (2+S_2+D_2)a^2$ we prove (2.31). □

Theorem 2.7 *Let A and B be self-adjoint operators which satisfy (1.1), (1.10) and (1.11). If Borel functions f and g obey (1.12) - (1.18) and conditon (1.19) is valid, then there is a constant $L>0$ such that estimate*

$$\left\|\left(\sqrt{f(tA/n)}g(tB/n)\sqrt{f(tA/n)}\right)^n - e^{-tH}\right\| \le \frac{L}{(1-C_0S_1a)(1-a)}\,\frac{\log(n)}{n}, \qquad n = 3,4,\ldots, \tag{2.42}$$

holds uniformly in $t \ge 0$.

Proof: Let $\tau = t/n$. We start with representation

$$\begin{aligned} &T(\tau)^n - U(\tau)^n \\ &=\ T(\tau)^{n-1}(T(\tau)-U(\tau)) + \sum_{m=1}^{n-1} T(\tau)^{n-m-1}(T(\tau)-U(\tau))U(\tau)^m. \end{aligned} \tag{2.43}$$

Using Lemma 2.4 we find

$$\begin{aligned} T(\tau)^n - U(\tau)^n &= T(\tau)^{n-1}(T(\tau)-U(\tau)) \\ &+ \sum_{m=1}^{n-1} T(\tau)^{n-m-1}(I-T(\tau))(I-T(\tau))^{-1}(T(\tau)-U(\tau))U(\tau)^m. \end{aligned} \tag{2.44}$$

This leads to estimate

$$\begin{aligned}\|T(\tau)^n - U(\tau)^n\| &\le \|T(\tau)^{n-1}(T(\tau) - U(\tau))\| \\ &+ \sum_{m=1}^{n-1} \|T(\tau)^{n-m-1}(I - T(\tau))\|\|(I - T(\tau))^{-1}(T(\tau) - U(\tau))U(\tau)^m\|.\end{aligned} \tag{2.45}$$

By (2.16) we obtain

$$\begin{aligned}&\|(I - T(\tau))^{-1}(T(\tau) - U(\tau))U(\tau)^m\| \\ \le\ & \frac{1}{1 - C_0 S_1 a}\left\{\|(T(\tau) - U(\tau))U(\tau)^m\| + \frac{C_0}{\tau}\left\|A^{-1}(T(\tau) - U(\tau))U(\tau)^m\right\|\right\}.\end{aligned} \tag{2.46}$$

Since

$$(T(\tau) - U(\tau))U(\tau)^m = (T(\tau) - U(\tau))A^{-1}AH^{-1}HU(\tau)^m, \tag{2.47}$$

one gets

$$\|(T(\tau) - U(\tau))U(\tau)^m\| \le \left\|(T(\tau) - U(\tau))A^{-1}\right\| \|AH^{-1}\|\|HU(\tau)^m\|. \tag{2.48}$$

Applying now Lemma 2.2 we obtain

$$\|(T(\tau) - U(\tau))U(\tau)^m\| \le \frac{1}{e\tau}\left\|A^{-1}(T(\tau) - U(\tau))\right\| \|AH^{-1}\|\frac{1}{m}. \tag{2.49}$$

By $AH^{-1} = (I + BA^{-1})^{-1}$ one gets

$$\|AH^{-1}\| \le \frac{1}{1 - a}. \tag{2.50}$$

Hence

$$\|(T(\tau) - U(\tau))U(\tau)^m\| \le \frac{1}{e(1 - a)\tau}\left\|A^{-1}(T(\tau) - U(\tau))\right\|\frac{1}{m}. \tag{2.51}$$

Similarly we get

$$\|A^{-1}(T(\tau) - U(\tau))U(\tau)^m\| \le \frac{1}{e(1 - a)\tau}\left\|A^{-1}(T(\tau) - U(\tau))A^{-1}\right\|\frac{1}{m}. \tag{2.52}$$

Inserting (2.51) and (2.52) into (2.46) we find

$$\begin{aligned}&\|(I - T(\tau))^{-1}(T(\tau) - U(\tau))U(\tau)^m\| \\ \le\ & \frac{1}{e(1 - C_0 S_1 a)(1 - a)}\left\{\frac{1}{\tau}\|(A^{-1}(T(\tau) - U(\tau))\| + \frac{C_0}{\tau^2}\left\|A^{-1}(T(\tau) - U(\tau))A^{-1}\right\|\right\}\frac{1}{m}.\end{aligned} \tag{2.53}$$

Applying to (2.53) Lemma 2.5 and Lemma 2.6 we find

$$\begin{aligned}&\|(I - T(\tau))^{-1}(T(\tau) - U(\tau))U(\tau)^m\| \\ \le\ & \frac{L_1 + C_0 L_2}{e(1 - C_0 S_1 a)(1 - a)}\frac{1}{m} = \frac{L_3}{(1 - C_0 S_1 a)(1 - a)}\frac{1}{m},\end{aligned} \tag{2.54}$$

where we put $L_3 := (L_1 + C_0 L_2)e^{-1}$. Further, by Lemma 2.1 we have

$$\|T(\tau)^{n-m-1}(I - T(\tau)\| \le \frac{1}{n-m}, \qquad n = 2, 3, \dots, \qquad m = 0, 1, 2, \dots, n-1. \tag{2.55}$$

Inserting (2.54) and (2.55) into (2.45) we get

$$\|T(\tau)^n - U(\tau)^n\| \le \|T(\tau)^{n-1}(T(\tau) - U(\tau))\| + \frac{L_3}{(1 - C_0 S_1 a)(1-a)} \sum_{m=1}^{n-1} \frac{1}{(n-m)m}. \tag{2.56}$$

Since

$$\begin{aligned} &\sum_{m=1}^{n-1} \frac{1}{(n-m)m} \\ &= \frac{2}{n} \sum_{m=1}^{n-1} \frac{1}{m} \le \frac{2}{n}(1 + \log(n-1)) \le 4\frac{\log(n)}{n}, \qquad n = 2, 3, \dots, \end{aligned} \tag{2.57}$$

one has

$$\|T(\tau)^n - U(\tau)^n\| \le \|T(\tau)^{n-1}(T(\tau) - U(\tau))\| + \frac{4L_3}{(1 - C_0 S_1 a)(1-a)} \frac{\log(n)}{n}. \tag{2.58}$$

It remains to estimate $\|T(\tau)^{n-1}(T(\tau) - U(\tau))\|$. Again by invertibility of $I - T(\tau)$ (see Lemma 2.4) we find

$$\|T(\tau)^{n-1}(T(\tau) - U(\tau))\| \le \|T(\tau)^{n-1}(I - T(\tau)\| \|(I - T(\tau))^{-1}(T(\tau) - U(\tau))\|. \tag{2.59}$$

By (2.16) we get

$$\begin{aligned} &\|(I - T(\tau))^{-1}(T(\tau) - U(\tau))\| \\ &\le \frac{1}{1 - C_0 S_1 a}\left\{\|T(\tau) - U(\tau)\| + \frac{C_0}{\tau}\left\|A^{-1}(T(\tau) - U(\tau))\right\|\right\} \\ &\le \frac{1}{1 - C_0 S_1 a}\left\{2 + \frac{C_0}{\tau}\left\|A^{-1}(T(\tau) - U(\tau))\right\|\right\} \end{aligned} \tag{2.60}$$

Applying to (2.60) Lemma 2.5 we find

$$\|(I - T(\tau))^{-1}(T(\tau) - U(\tau))\| \le \frac{2 + C_0 L_1}{1 - C_0 S_1 a}. \tag{2.61}$$

Inserting (2.61) into (2.59) one gets

$$\|T(\tau)^{n-1}(T(\tau) - U(\tau))\| \le \frac{2 + C_0 L_1}{1 - C_0 S_1 a}\|T(\tau)^{n-1}(I - T(\tau)\|. \tag{2.62}$$

Then applying Lemma 2.1 to the right-hand side of (2.62) we obtain

$$\|T(\tau)^{n-1}(T(\tau) - U(\tau))\| \le \frac{2 + C_0 L_1}{1 - C_0 S_1 a}\frac{1}{n}. \tag{2.63}$$

Inserting (2.63) into (2.58) we find

$$\|T(\tau)^n - U(\tau)^n\| \le \frac{(2 + C_0 L_1)(1-a) + 4L_3}{(1 - C_0 S_1 a)(1-a)} \frac{\log(n)}{n}, \qquad n = 3, 4, \dots. \tag{2.64}$$

Setting $L := (2 + C_0 L_1)(1-a) + 4L_3$ we get the estimate (2.42). □

Corollary 2.8 *Let A and B be self-adjoint operators which satisfy (1.1), (1.10) and (1.11). If Borel functions f and g obey (1.12) - (1.18) and conditon (1.19) is valid, then there is a constant $L' > 0$ such that estimate*

$$\left\|(f(tA/n)g(tB/n))^n - e^{-tH}\right\| \leq \frac{L'}{(1-C_0S_1a)^2(1-a)}\frac{\log(n)}{n}, \qquad n = 3,4,\ldots, \tag{2.65}$$

holds uniformly in $t \geq 0$.

Proof: Let $\tau = t/n$. We use identity

$$(f(\tau A)g(\tau B))^n - T(\tau)^n = \sum_{m=0}^{n-1} (f(\tau A)g(\tau B))^{n-m-1}(f(\tau A)g(\tau B) - T(\tau))T(\tau)^m. \tag{2.66}$$

Since (cf. (2.5))

$$f(\tau A)g(\tau B) - T(\tau) = \sqrt{f(\tau A)}\left(\sqrt{f(\tau A)}g(\tau B) - g(\tau B)\sqrt{f(\tau A)}\right) \tag{2.67}$$

one gets

$$(f(\tau A)g(\tau B))^n - T(\tau)^n = \sqrt{f(\tau A)}\sum_{m=0}^{n-1} T(\tau)^{n-m-1}\left(\sqrt{f(\tau A)}g(\tau B) - g(\tau B)\sqrt{f(\tau A)}\right)T(\tau)^m. \tag{2.68}$$

By Lemma 2.4 one has $I = (I - T(\tau))(I - T(\tau))^{-1} = (I - T(\tau))^{-1}(I - T(\tau))$. So, the relation (2.68) can be represented identically as

$$\begin{aligned}
&(f(\tau A)g(\tau B))^n - T(\tau)^n \qquad (2.69)\\
&= \sqrt{f(\tau A)}\sum_{m=0}^{n-1} T(\tau)^{n-m-1}(I - T(\tau))(I - T(\tau))^{-1}\\
&\quad \times \left(\sqrt{f(\tau A)}g(\tau B) - g(\tau B)\sqrt{f(\tau A)}\right)(I - T(\tau))^{-1}(I - T(\tau))T(\tau)^m.
\end{aligned}$$

Therefore, we get the estimate

$$\begin{aligned}
\|(f(\tau A)g(\tau B))^n - T(\tau)^n\| &\leq \sum_{m=0}^{n-1}\|T(\tau)^{n-m-1}(I - T(\tau))\| \qquad (2.70)\\
&\leq \times\left\|(I - T(\tau))^{-1}\left(\sqrt{f(\tau A)}g(\tau B) - g(\tau B)\sqrt{f(\tau A)}\right)(I - T(\tau))^{-1}\right\|\|(I - T(\tau))T(\tau)^m\|.
\end{aligned}$$

Again by Lemma 2.4 and (2.16) we find

$$\begin{aligned}
&\left\|(I - T(\tau))^{-1}\left(\sqrt{f(\tau A)}g(\tau B) - g(\tau B)\sqrt{f(\tau A)}\right)(I - T(\tau))^{-1}\right\| \qquad (2.71)\\
&\leq \frac{1}{(1-C_0S_1a)^2}\left\{\left\|\sqrt{f(\tau A)}g(\tau B) - g(\tau B)\sqrt{f(\tau A)}\right\|\right.\\
&\quad + \frac{2C_0}{\tau}\left\|A^{-1}\left(\sqrt{f(\tau A)}g(\tau B) - g(\tau B)\sqrt{f(\tau A)}\right)\right\|\\
&\quad \left. + \frac{C_0^2}{\tau^2}\left\|A^{-1}\left(\sqrt{f(\tau A)}g(\tau B) - g(\tau B)\sqrt{f(\tau A)}\right)A^{-1}\right\|\right\}
\end{aligned}$$

Since

$$\sqrt{f(\tau A)}g(\tau B)-g(\tau B)\sqrt{f(\tau A)}=(I-g(\tau B))\sqrt{f(\tau A)}-\sqrt{f(\tau A)}(I-g(\tau B)),\tag{2.72}$$

one has

$$\begin{aligned}&A^{-1}\left(\sqrt{f(\tau A)}g(\tau B)-g(\tau B)\sqrt{f(\tau A)}\right)\\ =\ &A^{-1}B(I-g(\tau B))B^{-1}\sqrt{f(\tau A)}-\sqrt{f(\tau A)}A^{-1}B(I-g(\tau B))B^{-1}.\end{aligned}\tag{2.73}$$

Hence, by (2.10) we get the estimate

$$\begin{aligned}&\frac{1}{\tau}\left\|A^{-1}\left(\sqrt{f(\tau A)}g(\tau B)-g(\tau B)\sqrt{f(\tau A)}\right)\right\|\\ \leq\ &\|BA^{-1}\|\frac{1}{\tau}\|(I-g(\tau B))\|+\frac{1}{\tau}\|BA^{-1}\|\|(I-g(\tau B))B^{-1}\|,\end{aligned}\tag{2.74}$$

which by (1.11) and (1.17) yields

$$\frac{1}{\tau}\left\|A^{-1}\left(\sqrt{f(\tau A)}g(\tau B)-g(\tau B)\sqrt{f(\tau A)}\right)\right\|\leq 2S_1a\tag{2.75}$$

¿From (2.72) we obtain the representation

$$\begin{aligned}&\sqrt{f(\tau A)}g(\tau B)-g(\tau B)\sqrt{f(\tau A)}\\ &=\left(I-\sqrt{f(\tau A)}\right)(I-g(\tau B))-(I-g(\tau B))\left(I-\sqrt{f(\tau A)}\right)\end{aligned}\tag{2.76}$$

which leads to

$$\begin{aligned}&A^{-1}\left\{\sqrt{f(\tau A)}g(\tau B)-g(\tau B)\sqrt{f(\tau A)}\right\}A^{-1}\\ =\ &\left(I-\sqrt{f(\tau A)}\right)A^{-1}(I-g(\tau B))B^{-1}BA^{-1}-A^{-1}B(I-g(\tau B))B^{-1}\left(I-\sqrt{f(\tau A)}\right)A^{-1}.\end{aligned}\tag{2.77}$$

Hence by (1.11), (1.17) and (2.37) we obtain

$$\begin{aligned}&\frac{1}{\tau^2}\left\|A^{-1}\left\{\sqrt{f(\tau A)}g(\tau B)-g(\tau B)\sqrt{f(\tau A)}\right\}A^{-1}\right\|\\ \leq\ &2\|BA^{-1}\|\frac{1}{\tau}\|(I-\sqrt{f(\tau A)})A^{-1}\|\frac{1}{\tau}\|(I-g(\tau B))B^{-1}\|\leq 2S_1C_1a.\end{aligned}\tag{2.78}$$

Inserting (2.75) and (2.78) into (2.71) one has

$$\|(I-T(\tau))^{-1}\left(\sqrt{f(\tau A)}g(\tau B)-g(\tau B)\sqrt{f(\tau A)}\right)(I-T(\tau))^{-1}\|\leq\frac{L_4}{(1-C_0S_1)^2},\tag{2.79}$$

where $L_4:=2+(4C_0S_1+2C_0^2C_1S_1)a$. Taking into account (2.70) we get

$$\begin{aligned}&\|\left(f(\tau A)g(\tau B)\right)^n-T(\tau)^n\|\\ \leq\ &\frac{L_4}{(1-C_0S_1a)^2}\sum_{m=0}^{n-1}\|T(\tau)^{n-m-1}(I-T(\tau))\|\|(I-T(\tau))T(\tau)^m\|.\end{aligned}\tag{2.80}$$

Now applying Lemma 2.1 we obtain

$$\| (f(\tau A)g(\tau B))^n - T(\tau)^n\| \leq \frac{L_4}{(1-C_0S_1a)^2} \sum_{m=0}^{n-1} \frac{1}{(n-m)(m+1)} \tag{2.81}$$

Since

$$\sum_{m=0}^{n-1} \frac{1}{(n-m)(m+1)} = \frac{2}{n+1} \sum_{m=1}^{n} \frac{1}{m} \leq 4\frac{\log(n)}{n}, \qquad n = 3, 4, \ldots, \tag{2.82}$$

from (2.80) we obtain

$$\| (f(\tau A)g(\tau B))^n - T(\tau)^n\| \leq \frac{4L_4}{(1-C_0S_1a)^2} \frac{\log(n)}{n}, \qquad n = 3, 4, \ldots. \tag{2.83}$$

By the estimate

$$\| (f(\tau A)g(\tau B))^n - U(\tau)^n\| \leq \| (f(\tau A)g(\tau B))^n - T(\tau)^n\| + \|T(\tau)^n - U(\tau)^n\| \tag{2.84}$$

and (2.42) we get

$$\| (f(\tau A)g(\tau B))^n - U(\tau)^n\| \leq \frac{L'}{(1-C_0S_1a)^2(1-a)} \frac{\log(n)}{n}, \qquad n = 3, 4, \ldots, \tag{2.85}$$

where $L' := L(1-C_0S_1a) + 4L_4(1-a)$ which proves (2.65). □

3 Concluding Remarks

It would be instructive to compare our results (Theorem 2.7 and Corollary 2.8) with an important recent paper by Ichinose and Tamura [3] which corresponds to symmetrized Trotter formula (1.8) or (2.42) for $f(x) = g(x) = \exp(-x)$. In Section 3 we propose a line of reasoning which being straightforward gives a simple proof of stronger estimates for less restrictive conditions and in more abstract settings:

(i) We relax the condition (1.7) of [3] up to (1.10) and (1.11) and we obtain a better error bound for $f(x) = g(x) = \exp(-x)$ both for symmetric (1.20) and for non-symmetric (1.21) versions of the Trotter product formula. We recall that for $f(x) = g(x) = \exp(-x)$ inqualilty (1.19) reduces to $0 < a < 1$. The conditions (1.10) and (1.11) are obviously weaker than (1.7). In fact, replacing the pair $\{A, B\}$ by $\{A + \eta I, B\}$, $\eta > 0$, we get from (1.7) that $\mathrm{dom}(A + \eta I) = \mathrm{dom}(A) \subseteq \mathrm{dom}(A^\alpha) \subseteq \mathrm{dom}(B)$ and

$$\|B(A+\eta I)^{-1}f\| \leq \|BA^{-\alpha}\|\|A^\alpha(A+\eta I)^{-1}f\| \leq \eta^{-(1-\alpha)}\|BA^{-\alpha}\|\|f\| \qquad f \in \mathcal{H}.$$

Hence, for the pair $\{A + \eta I, B\}$ the conditon (1.11) is satisfied for arbitrary small $a > 0$ if η gets sufficiently large. Note that the shift of A by ηI changes only constant C in the error estimate (1.20). Moreover, in contrast to [3] our estimates (1.20) and (1.21) are uniform in $t \geq 0$.

(ii) We get error bound for symmetric generalization of the Trotter formula à la Kato (2.42), where Borel functions $f(\cdot)$ and $g(\cdot)$ satisfy (1.12) - (1.18). We show that only coefficient C (1.20) is dependent on the choice of functions $f(\cdot)$ and $g(\cdot)$, see (2.42) and (2.64).

(iii) We show that for non-symmetric Trotter-Kato formula the error estimate has the same asymptotics in n (see (1.21) or (2.65)) as the symmetric one modulus coefficient.

(iv) We have to note that in both cases (ii) and (iii) the relative constant a does depend on the choice of the Borel functions $f(\cdot)$ and $g(\cdot)$, see (1.19).

(v) Note that if B is Kato-small with zero-lower bound, then our condition (1.19) becomes superfluous. In particular, this case is ensured by inclusion relation (1.7).

Comparing the present paper with [7] we would like to note that in both of them we have operator-norm convergence (in fact in [7] this convergence is in the trace-norm topology), while assumptions in these papers are far to be comparable. Finally, we would like to note that in another recent paper [4] Ichinose and Tamura give an interesting analysis of the trace-norm convergence of the symmetrized exponential Trotter formula for Gibbs semigroups generated by Schrödinger operators.

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