Remarks on decay of correlations and Witten Laplacians
– Brascamp-Lieb inequalities and semiclassical limit –

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Abstract

As it appears in recent articles by Helffer or Sjöstrand and Naddaf-Spencer, the analysis, in the context of the statistical mechanics, of measures of the type $\exp -\Phi(x)\ dx$ is connected with the analysis of suitable Witten Laplacians on 1-forms. For illustrating this point of view, we present here remarks about the Brascamp-Lieb inequalities and its extensions and prove the decay of the correlation in some cases when Φ is weakly non convex.

1 Introduction

Our aim[1] is to analyze Laplace integrals associated to a measure whose density with respect to the Lebesgue measure takes the form $\exp -\frac{1}{h}\Phi$, up to a multiplicative normalization constant, in the case when the potential Φ is weakly convex or weakly non convex. We analyze as a starting point the

[1] A first version of these remarks was diffused in June 1996.

Brascamp-Lieb inequality or the Poincaré inequality in connection with the lowest eigenvalue of a suitable Witten Laplacian on one-forms. The role of this Laplacian which appears implicitly in [Sj2], [Sj3], [HeSj2] and [He3] was emphasized in [Sj6] with new applications. This was then used in [NaSp] and [He5] in connection with the Brascamp-Lieb inequality. We have in mind applications to a potential of the form

$$\Phi(x) = \sum_{j=1}^{m} v(x_j) + \frac{\mathcal{J}}{2} \sum_{j=1}^{m} |x_j - x_{j+1}|^2 \tag{1.1}$$

with the convention that $x_{m+1} = x_1$ and where the one particle phase v takes the form $v(x) = \lambda x^4 + \nu x^2$. The parameters λ and ν (possibly depending on h) satisfy for the moment

$$\lambda > 0 \; , \tag{1.2}$$

and

$$\nu \geq 0 \; , \tag{1.3}$$

but the case $\nu < 0$ will also be analyzed under a condition of the type $\nu \geq -\epsilon(\lambda)$.

More generally we will be interested in the similar problem attached to a d-dimensional (periodic) lattice Λ (identified (modulo translation) with a subset of $\mathbb{Z}^d$)

$$\Phi(x) = \sum_{\ell \in \Lambda} v(x_\ell) + \frac{\mathcal{J}}{2} \sum_{j \sim k} |x_j - x_k|^2 \; , \tag{1.4}$$

where $j \sim k$ means that j and k are nearest neighbors in Λ considered as living on a torus.
With this technique we can also consider examples like

$$\Phi(x) = \sum_{\ell \in \Lambda} v_\ell(x_\ell) + \Phi_i(x) \; , \tag{1.5}$$

where $\Phi_i(x)$ is a convex interaction potential with uniformly bounded second derivatives and $t \mapsto v_\ell(t)$ is a family of potentials whose dependence with respect to ℓ is controlled uniformly.
Our main problem will be to analyze the properties of the measure

$$d\mu := \exp -\frac{\Phi(x)}{h} dx \Big/ \left(\int \exp -\frac{\Phi(x)}{h} dx \right) \tag{1.6}$$

and more precisely the covariance associating to (f, g)

$$\text{Cov}\,(f, g) = \langle (f - \langle f\rangle)(g - \langle g\rangle)\rangle \tag{1.7}$$

where $\langle \cdot \rangle$ denotes the mean value with respect to the measure $d\mu$.
As usual we denote the variance by

$$\text{var}\; g := \text{Cov}\,(g, g)\;. \tag{1.8}$$

We shall sometimes use the notation var ${}^{(m)}g$ and Cov ${}^{(m)}(f, g)$ if we want to refer to the family of phases $\Phi = \Phi^\Lambda$ with $\Lambda = \{1, \cdots, m\}$ and follow the uniformity with respect to m.

2 About Brascamp-Lieb

We follow partially [Sj6] or [He5]. H.J.Brascamp and E.Lieb [BraLi] have proved the following celebrated inequality which plays an important role in different contexts in the study of the Schrödinger equation.

Theorem 2.1 *:*
Let $F(x) = \exp(-\Phi(x)), x \in I\!R^m$, with Φ in C^2 and strictly convex. We assume that Φ has a minimum and consequently F decays exponentially in all directions. Let $g \in C^1(I\!R^m)$, and let us assume that var $g < \infty$. *Then*

$$\text{var}\; g \leq \langle\; \nabla g \cdot ((\text{Hess}\,\Phi\,)^{-1}\nabla g)\;\rangle \tag{2.1}$$

where ∇g is the gradient of g.

Remark 2.2 *:*
In the semi-classical case, we obtain (for the normalized measure associated with $\exp -\frac{\Phi}{h}dx$*),*

$$\text{var}\; g \leq h\,\langle\; \nabla g \cdot \Big((\text{Hess}\,\Phi\,)^{-1}\nabla g\Big)\;\rangle \tag{2.2}$$

We first recall a proof of this theorem inspired by [HeSj2], [He3], [Sj6] and [He5] which was only given for Φ uniformly strictly convex and with bounded second derivatives. We will remain rather sketchy and refer to the complete study given by J. Johnsen in [Jo] for a general justification. Note nevertheless that the proof is easier to verify in the case of our main example (1.1). For g in C^1, such that ∇g is bounded, we have seen that there exists f such that $\langle f \rangle = 0$ and

$$g - \langle g \rangle = \nabla\Phi \cdot \nabla f - \Delta f =: A_0 f \ . \tag{2.3}$$

The operator A_0 is selfadjoint[2] on $L^2(I\!R^m; \exp -\Phi)$.
If we take the formula giving the variance, we get

$$\begin{aligned} \text{var}\ (g) &\stackrel{\text{def}}{=} \textstyle\int (g - \langle g \rangle)^2 \exp -\Phi \ dx / \int \exp -\Phi \ dx \\ &= \textstyle\int (g - \langle g \rangle)(\nabla f \cdot \nabla\Phi - \text{div}\, \nabla f) \exp -\Phi \ dx / \int \exp -\Phi \ dx \\ &= \textstyle(\int \nabla f \cdot \nabla g \ \exp -\Phi \ dx) / \int \exp -\Phi \ dx \end{aligned} \tag{2.4}$$

As in [HeSj2], we get by differentiation of (2.3) and with $v := \nabla f$,

$$\nabla g = A_1 v \ , \tag{2.5}$$

where A_1 is the operator

$$v \mapsto A_1 v = (\nabla\Phi \cdot \nabla - \Delta) v + \text{Hess}\, \Phi \cdot v \ ,$$

which can be interpreted as a Witten Laplacian. This operator is positive and actually strictly positive (as first observed in [Sj6]) under rather weak assumptions[3] but **without any assumption of strict convexity**. The proof given in [Sj6] has been extended in [Jo] where technical conditions previously imposed by J. Sjöstrand are eliminated. In this paper, we shall actually get the strict positivity by explicit lower bounds.
We can then rewrite (2.4) as

$$\text{var}\ (g) = \left(\int A_1^{-1} \nabla g \cdot \nabla g \ \exp -\Phi \ dx \right) / \left(\int \exp -\Phi \ dx \right) \tag{2.6}$$

In the case when Φ is uniformly strictly convex, we observe the following inequality between selfadjoint operators

$$A_1 \geq \text{Hess}\, \Phi \geq \sigma > 0 \ , \tag{2.7}$$

[2]We do not discuss in detail the problem of the essential selfadjointness of A_0 and later of A_1. These problems are interesting in themselves. Let us just observe that we consider here the Friedrichs extension and that, for our particular examples, the essential selfadjointness can be easily analyzed because the operators are globally quasielliptic (see Helffer-Robert [HeRo], Helffer [He1]). A more general study is developed in Johnsen [Jo].

[3]The main assumption is the existence of $\delta > 0$ such that $x \cdot \nabla\Phi(x) \geq \delta < x >^{1+\delta} - \frac{1}{\delta}$.

and, using abstract analysis (extending the result mentioned in [Ru]), we obtain

$$A_1^{-1} \leq (\mathrm{Hess}\,\Phi)^{-1}\ . \tag{2.8}$$

The Brascamp-Lieb inequality is then an immediate consequence of (2.6).
In the general case, let us denote by ρ_1 the lowest eigenvalue of A_1 which satisfies (as accounted for above)

$$\rho_1 > 0\ . \tag{2.9}$$

We now deduce from (2.6) the upper bound

$$\mathrm{var}\,(g) \leq \rho_1^{-1} \left(\int \nabla g \cdot \nabla g\ \exp -\Phi\ dx\right) / \left(\int \exp -\Phi\ dx\right)\ . \tag{2.10}$$

This is of course a stronger result than the following consequence of the Brascamp-Lieb inequality in the convex case :

$$\mathrm{var}\,(g) \leq \left(\inf_x \lambda_{min}(\mathrm{Hess}\,\Phi(x))\right)^{-1} \left(\int \nabla g \cdot \nabla g\ \exp -\Phi\ dx\right) / \left(\int \exp -\Phi\ dx\right)\ . \tag{2.11}$$

We have indeed in this case

$$\rho_1 \geq \inf_x \lambda_{min}(\mathrm{Hess}\,\Phi(x)) \tag{2.12}$$

as a consequence of (2.7). But what is the most interesting here is that the proof of (2.10) is independent of the convexity!
Coming back to the semi-classical situation, J. Sjöstrand ([Sj6]) observed that, after conjugation by the operator of multiplication by $\exp -\frac{\phi}{2h}$, the operator $h^2 A_1$ (defined by starting from the potential Φ/h) becomes the following more standard Witten Laplacian W_1,

$$W_1 := \left[\sum_j \left(-h\partial/\partial x_j + \frac{1}{2}\partial\Phi/\partial x_j\right)\left(h\partial/\partial x_j + \frac{1}{2}\partial\Phi/\partial x_j\right)\right] \otimes I + h\ \mathrm{Hess}\,\Phi\ , \tag{2.13}$$

defined on the L^2 $1-$forms with respect to the standard Lebesgue measure on $I\!R^m$, with $m = |\Lambda|$. Let us recall also that W_1 is related to the Witten Laplacian W_0 on the 0-forms by

$$W_0 := \left[\sum_j \left(-h\partial/\partial x_j + \frac{1}{2}\partial\Phi/\partial x_j\right)\left(h\partial/\partial x_j + \frac{1}{2}\partial\Phi/\partial x_j\right)\right]\ , \tag{2.14}$$

through the identity

$$W_1 = W_0 \otimes I + h \,\mathrm{Hess}\,\Phi \,. \tag{2.15}$$

The basic philosophy that we want to develop is that in many cases occuring in statistical mechanics, the results obtained in the uniformly strictly convex situation by use of the strictly positive constant

$$\rho_0 := \inf_x \lambda_{min}(\mathrm{Hess}\,\Phi(x)) \,, \tag{2.16}$$

will also be true in non-convex situations with ρ_0 replaced by ρ_1. This will be particularly important in the case when one can find strictly positive lower bounds of ρ_0 or ρ_1 which are suitably controlled with respect to Λ or the parameter h.

3 Lower bound for the spectrum of the Witten Laplacian in the semi-classical case

The aim of this section is to show that the approach developed in the preceding section is performant. As a typical example, we prove the following

Theorem 3.1 *:*
Let $m \in I\!N$ and $\Phi^{(m)} = \Phi$ the phase on $I\!R^m$

$$\Phi(x) = \sum_{j=1}^{m} \lambda_j x_j^4 + \sum_j \nu_j x_j^2 + \frac{\mathcal{J}}{2} \sum_j |x_j - x_{j+1}|^2 \,, \tag{3.1}$$

where the λ_j satisfy

$$0 < \underline{\lambda} \leq \lambda_j \,. \tag{3.2}$$

and ν_j satisfies for some j and m independent sufficiently small $\epsilon_0 > 0$

$$\nu_j \geq -\epsilon_0 h \,. \tag{3.3}$$

Then there exists $c > 0$ and h_0 such that, for all m and all h such that $0 < h < h_0$,

$$\langle W_1 u \mid u \rangle_{L^2} \geq c\, h^2 ||u||^2 \,, \tag{3.4}$$

for all u in $C_0^\infty(I\!R^m \,;\, I\!R^m)$.

This proposition gives the following version of a " uniform " Poincaré inequality

Corollary 3.2 *:*
There exists $C > 0$ and h_0 such that, for all m and all h, such that $0 < h < h_0$,

$$\mathrm{var}\; g \leq C\, ||\nabla g||^2_{L^2(\exp -\Phi/h)} \;/\; \left(\int \exp -\Phi/h \; dx\right)\;, \tag{3.5}$$

for any "temperate" C^∞ function g.

In particular this corollary can be applied with

$$g = \frac{1}{|\Lambda|}\,\left(\sum_{j\in\Lambda} x_j\right)$$

and we deduce for this case that

$$\mathrm{var}\; g \leq \frac{C}{|\Lambda|}\;. \tag{3.6}$$

In particular this tends to 0 as $|\Lambda| \to +\infty$ and this can be interpreted as a sign of no phase transition.

Remark 3.3 *:*

- *This result contains non convex examples when ν_j is negative.*
- *Our results can easily be extended to the case $d > 1$.*
- *In the case of a one dimensional lattice ($d = 1$), these problems can also be analyzed through the technique of the transfer matrix, that is by the analysis of a spectral problem attached to the operator*

$$K_v = \exp -\frac{v}{2h}\;\cdot\;\exp h\frac{d^2}{dx^2}\;\cdot\;\exp -\frac{v}{2h}\;.$$

- *This problem can also be analyzed through Sokal's approach [Sok].*

Proof of Theorem 3.1:
Letting $X_j = h\partial_j + \frac{1}{2}\partial_j\Phi$, we start from

$$\langle W_1 u \mid u\rangle_{L^2} = \sum_{j,k} ||X_k u_j||^2 + h\sum_{j,k}\int \frac{\partial^2\Phi}{\partial x_j\,\partial x_k} u_j\; u_k\; dx\;. \tag{3.7}$$

We first "omit" the terms $||X_k\; u_j||^2$ with $k \neq j$

$$\begin{aligned}\langle W_1 u \mid u\rangle_{L^2} & \geq \sum_j \left(||X_j u_j||^2 + h\int \frac{\partial^2\Phi}{\partial x_j\,\partial x_j} u_j^2 dx\right)\\ & +h\sum_{j\neq k}\int \frac{\partial^2\Phi}{\partial x_j\,\partial x_k} u_j\; u_k\; dx\;.\end{aligned} \tag{3.8}$$

We then analyze for fixed j the term

$$\langle w_j u_j \mid u_j\rangle := ||X_j u_j||^2 + h\int \frac{\partial^2\Phi}{\partial x_j\,\partial x_j} u_j^2\; dx\;.$$

Easy computations give

$$\langle w_j u_j \mid u_j\rangle = ||h\partial_{x_j} u_j||^2 + \frac{1}{4}||(\frac{\partial\Phi}{\partial x_j})\; u_j||^2 + \frac{h}{2}\int \frac{\partial^2\Phi}{\partial x_j\,\partial x_j} u_j^2\; dx\;. \tag{3.9}$$

Of course we have the lower bound

$$||h\partial_{x_j} u_j||^2 + \frac{1}{4}||(\frac{\partial\Phi}{\partial x_j})\; u_j||^2 \geq \frac{h}{2}\int \frac{\partial^2\Phi}{\partial x_j\,\partial x_j} u_j^2 dx\;, \tag{3.10}$$

using the standard commutator argument but this is of no interest because this does not give any new inequality. In order to go further, we introduce a possibly h-dependent ϵ with $0 < \epsilon \leq 1$ and get first, using (3.9) and (3.10),

$$\begin{aligned}\langle w_j u_j \mid u_j\rangle & = (1-\epsilon)\langle w_j u_j \mid u_j\rangle + \epsilon\langle w_j u_j \mid u_j\rangle\\ & \geq \epsilon||h\partial_{x_j} u_j||^2 + (2-\epsilon)\frac{h}{2}\int \frac{\partial^2\Phi}{\partial x_j\,\partial x_j} u_j^2\; dx\;.\end{aligned} \tag{3.11}$$

Let us treat the case $\nu_j = 0$. We introduce the following decomposition of the phase Φ

$$\Phi(x) = \sum_{j=1}^m \lambda_j x_j^4 + \frac{J}{2}\sum_{j\sim k}|x_j - x_k|^2 =: \Phi_d + \Phi_i\;, \tag{3.12}$$

where Φ_d is the sum of the single spin potentials v_j defined by $v_j(t) = \lambda_j t^4$,

$$\Phi_d(x) = \sum_{j=1}^{m} v_j(x_j)\ .$$

We rewrite (3.11) in the form

$$\begin{array}{l} \langle w_j u_j \mid u_j \rangle \\ \quad \geq \epsilon h \left(\|h^{\frac{1}{2}} \partial_{x_j} u_j\|^2 + \frac{2-\epsilon}{2\epsilon} \int \frac{\partial^2 \Phi_d}{\partial x_j\, \partial x_j} u_j^2 dx \right) + (1 - \frac{\epsilon}{2}) h \int \frac{\partial^2 \Phi_i}{\partial x_j\, \partial x_j} u_j^2 dx. \end{array} \tag{3.13}$$

We observe now the property that

$$\|h^{\frac{1}{2}} \partial_{x_j} u_j\|^2 + \frac{2-\epsilon}{2\epsilon} \int \frac{\partial^2 \Phi_d}{\partial x_j^2} u_j^2 dx - \frac{1}{2} \int \frac{\partial^2 \Phi_i}{\partial x_j^2} u_j^2 dx$$

is the quadratic form attached to a new "harmonic oscillator" (in the x_j variable) and a new commutator argument, using for the first time the structure of Φ_d, gives

$$\|h^{\frac{1}{2}} \partial_{x_j} u_j\|^2 + \frac{2-\epsilon}{2\epsilon} \int \frac{\partial^2 \Phi_d}{\partial x_j\, \partial x_j} u_j^2 dx \geq h^{\frac{1}{2}} (12 \underline{\lambda} \frac{2-\epsilon}{2\epsilon})^{\frac{1}{2}} \int u_j^2\ dx\ . \tag{3.14}$$

We realize that $\frac{h}{\epsilon}$ has to be chosen sufficiently large in order to control $-\frac{1}{2} \int \frac{\partial^2 \Phi_i}{\partial x_j^2} u_j^2 dx$.
We consequently look for $0 < \epsilon < 1$ in the form $\epsilon = \frac{h}{C_1}$ with C_1 to be determined large enough and get

$$\langle w_j u_j \mid u_j \rangle \geq \frac{1}{C_1} h^2 \left((6 \underline{\lambda} C_1)^{\frac{1}{2}} - J \right) \|u_j\|^2 + h \int \frac{\partial^2 \Phi_i}{\partial x_j\, \partial x_j} u_j^2 dx\ . \tag{3.15}$$

The constant C_1 is now chosen in order to get

$$(6 \underline{\lambda} C_1)^{\frac{1}{2}} - J > 0\ . \tag{3.16}$$

Returning to (3.8), we obtain the existence of C for which

$$\begin{array}{ll} \langle W_1 u \mid u \rangle_{L^2} & \geq \frac{h^2}{C} \sum_j \|u_j\|^2 \\ & + h \sum_{j,k} \int \frac{\partial^2 \Phi_i}{\partial x_j\, \partial x_k} u_j\ u_k\ dx\ . \end{array} \tag{3.17}$$

But Φ_i is convex and we have finally the existence of C such that

$$\langle W_1 u \mid u \rangle_{L^2} \geq \frac{h^2}{C} ||u||^2 \ , \tag{3.18}$$

as announced in the theorem.
The case when $\nu_j \neq 0$ is then easily obtained by a variant of the argument leading to (3.14).

Remark 3.4 :
As observed by V. Bach, T. Jecko and J. Sjöstrand in recent discussions (see also [BaJeSj]), the omission in the proof, of the positive terms $\sum_{j\neq k} ||X_k u_j||^2$, when going from (3.7) to (3.8) will surely limit the class of interactions in consideration. We hope to come back to this point elsewhere [He10].

Remark 3.5 :
The condition " h small enough " appears when we assume that $\epsilon < 1$ in our estimates.

Remark 3.6 :
We have only used in the proof the property that the interaction phase Φ_i is convex and that $|\partial^2 \Phi_i / \partial x_j^2|$ is uniformly bounded on $I\!R^m$.

Remark 3.7 :
The paper by Sokal [Sok] treats similar models but an important assumption in the argument seems, when the interaction potential is given by

$$\Phi_i(x) = \sum_{jk} J_{jk} x_j \, x_k \ ,$$

the condition that $J_{jk} \leq 0$ for $j \neq k$. Our assumption is simply a "weak" convexity assumption. This convexity of the interaction appears also in a recent contribution by A&A. Antoniouk [AA].
The other point that we have to explore is when $x_j \in I\!R^n$ $(n > 1)$. The use of the GHS and zero-field Lebowitz inequalities is only possible for $n \leq 4$. Our approach apparently does not meet such a restriction but this will probably give weaker results.

4 A proof of the correlation decay without the Maximum Principle

Inspired by a recent paper by J. Sjöstrand [Sj6], we use only an L^2 theory and avoid the use of the Maximum Principle which was playing an important role in [Sj5] or [HeSj2]. This was also used in a somewhat different context by A. Naddaf and T. Spencer [NaSp] (Theorem B).
This will permit us to weaken the assumption of convexity. We consider only the case when the lattice is of dimension 1 but this is only for simplification and we could also analyze correlations attached to periodic lattices in $\mathbb{Z}^d$ $(d > 1)$.
The starting point (we take for simplification $h = 1$ and assume, after renormalization, $\int \exp -\Phi \, dx = 1$) is the formula for the correlation

$$\mathrm{Cov}\,(f,g) = \left(\int \left[(A_1^{-1})\nabla f\right] \cdot \nabla g \;\exp -\Phi \; dx\right) \tag{4.1}$$

We have in mind to take $f = x_i$, $g = x_j$ with $|i - j|$ large but much smaller than the size of the lattice $|\Lambda|$. We recall that we first consider the thermodynamic limit $|\Lambda| \to +\infty$ and then the behavior $|i - j|$ large. We consider for simplicity $\Lambda := \mathbb{Z}/m\,\mathbb{Z}$ that we identify with $\{1, \cdots, m\}$.
The idea[4], which was already present in [Sj1], [HeSj1], [HeSj2] and also in the more recent [NaSp] or [BaJeSj], is to introduce weighted spaces $\ell^2_\rho(\mathbb{Z}/m\,\mathbb{Z})$, for suitable strictly positive weights satisfying

$$\exp -\kappa \leq \rho(\ell)/\rho(\ell+1) \leq \exp \kappa \;, \tag{4.2}$$

with κ to be determined later.
For a given j satisfying

$$1 \leq j \leq \frac{m}{2}\;,$$

we are mainly thinking of weights of the form

$$\rho_j(\ell) = \exp\left(\kappa \sup[0, (\inf(\ell - 1, 2j - \ell)]\right)$$

or

$$\rho_j(\ell) = \exp - \left(\kappa \sup[0, (\inf(\ell - 1, 2j - \ell)]\right)\;.$$

[4] This is actually not different in spirit from the much older techniques by Combes-Thomas introduced for the study of the decay of eigenfunctions [CT].

Let us now associate with a given weight ρ the $m \times m$ diagonal matrix M defined by

$$M_{k\ell} = \delta_{k\ell}\ \rho(\ell)\ . \tag{4.3}$$

For arbitrary slowly increasing functions f, g, we can rewrite (4.1) in the form

$$\mathrm{Cov}\,(f,g) = \left(\int \left((M^{-1}A_1^{-1}M)M^{-1}\nabla f\right)\cdot (M\nabla g)\exp -\Phi\ dx\right) \tag{4.4}$$

and we deduce the estimate

$$|\mathrm{Cov}\,(f,g)| \leq ||M^{-1}\,A_1^{-1}\,M||\cdot ||M^{-1}\nabla f||\cdot ||M\nabla g||\ . \tag{4.5}$$

We now take $f(x) = x_1$, $g(x) = x_j$ and $j < \frac{m}{2}$ and choose ρ_j as above so that (4.2) is satisfied. We immediately observe that for this choice

$$||M^{-1}\nabla f|| = 1\ ,\ ||M\nabla g|| = \exp -\kappa(j-1)\ . \tag{4.6}$$

Everything is then reduced to the control of $M^{-1}\,A_1^{-1}\,M$ in weighted L^2-norms. We have only here to analyze the effect of the " distorsion " by M. This will be done by a simple perturbation argument, once we have characterized the domain of the selfadjoint operator A_1 and verified that the domain is conserved in the distorsion. This is easily done in the case of our example (see [Jo] for more general situations). We observe that for this example (cf (3.1))

$$\begin{aligned} &||\mathrm{Hess}\,\Phi(x) - M^{-1}\mathrm{Hess}\,\Phi(x)M||_{\mathcal{L}(\ell^2)} = \\ &||\mathrm{Hess}\,\Phi_i(x) - M^{-1}\mathrm{Hess}\,\Phi_i(x)M||_{\mathcal{L}(\ell^2)}\ . \end{aligned} \tag{4.7}$$

In this example, observing that the coefficients of

$$\delta_M(\mathrm{Hess}\,\Phi_i) := \mathrm{Hess}\,\Phi_i(x) - M^{-1}\mathrm{Hess}\,\Phi_i(x)M$$

vanish if $k \not\sim \ell$, it is immediate to get, uniformly with respect to m, that

$$||\delta_M(\mathrm{Hess}\,\Phi_i)||_{\mathcal{L}(\ell^2)} \leq \mathcal{J}\sup_{\ell\sim k}|(1-\frac{\rho(\ell)}{\rho(k)})| = \mathcal{O}(\kappa)\ . \tag{4.8}$$

We now estimate the operator $M^{-1}\,A_1^{-1}\,M$. An immediate computation gives

$$M^{-1}A_1^{-1}M = A_1^{-1}[I + \delta_M(\mathrm{Hess}\,\Phi)A_1^{-1}]^{-1}\ , \tag{4.9}$$

where $\delta_M(\mathrm{Hess}\,\Phi) = \delta_M(\mathrm{Hess}\,\Phi_i)$ is now considered as an operator (of order 0) on the L^2 1-forms. But the norm of this operator is $\mathcal{O}(\kappa)$ according to (4.7) and (4.8). We finally obtain the existence of C such that, if $0 < \kappa < \frac{1}{C}\rho_1$, then

$$||M^{-1}A_1^{-1}M|| \leq \frac{1}{\rho_1}[1 - C\frac{\kappa}{\rho_1}]^{-1} \tag{4.10}$$

We are done and this gives more generally, each time that some lattice-independent lower bound of ρ_1 (the bottom of the spectrum of A_1) is available, a general scheme to get the decay for the correlation (without use of the Maximum Principle).

Returning to the semiclassical situation, and following the proof with respect to h, we obtain the following

Theorem 4.1 *:*
Under the same assumptions as in Theorem 3.1, there exists D and h_0, such that the correlation pair function $\mathrm{Cov}^{(m)}(x_1, x_j)$*, for any pair (j,m) s.t.*

$$1 \leq j \leq \frac{m}{2}\;,$$

and any h s.t. $0 < h \leq h_0$, satisfies

$$|\mathrm{Cov}^{(m)}(x_1, x_j)| \leq D \exp -\frac{j\,h}{D}\;. \tag{4.11}$$

Remark 4.2 *:*
All the assumptions we have met are strongly related to the assumptions $(H8)$ and $(H9)$ given in [Sj6].
An important role is played in [Sj6] by the mean value of $\mathrm{Hess}\,\Phi$*. This is probably another way to measure the effect of the quartic term.*
This could be interesting to compare also with the arguments by A. Sokal [Sok].

Remark 4.3 *:*
This theorem is not related to the property that the lattice is one dimensional. In the case when $d = 1$, the theorem is probably not optimal (See the next section). In the strictly convex case $\nu_j \geq \underline{\nu}$, we get the better result that, for some strictly positive D, and for $j \ll m$,

$$|\mathrm{Cov}^{(m)}(x_1, x_j)| \leq D \exp -\frac{j}{D}\;. \tag{4.12}$$

5 Comparison with the transfer matrix approach

In the case when $d = 1$, $v_\ell = v$ and $\Phi_i(x) = \frac{1}{4}\sum_{j\sim k}|x_j - x_k|^2$, we know from [He5] that all the interesting quantities are related to the spectral properties of the operator $K_v = \exp -\frac{v}{2h} \; \cdot \; \exp h\Delta \; \cdot \; \exp -\frac{v}{2h}$ and particularly to the estimate of the quantity μ_2/μ_1 where $\mu_1 > \mu_2$ are the two largest eigenvalues of this compact operator, which is called in statistical mechnics " transfer operator ". Let us analyze the case when

$$\Phi(x) = \lambda \sum_{j=1}^{m} x_j^4 + \nu \sum_{j=1}^{m} x_j^2 + \frac{1}{4}\sum_{j=1}^{m} \; |x_j - x_{j+1}|^2 \,. \tag{5.1}$$

Here λ is strictly positive and ν may be of indefinite sign and h-dependent. We are interested in the correlation pair function that is

$$\mathrm{Cov}^{\,(\infty)}(x_1, x_j) =: \lim_{m\to+\infty} \left(\int x_1 x_j \exp -\frac{\Phi(x)}{h}\, dx\right) / \left(\int \exp -\frac{\Phi(x)}{h}\, dx\right) \,.$$

It is easy to prove that this correlation pair function behaves like $(\mu_2/\mu_1)^j$. This operator K_v takes the form

$$c(h)\exp -\frac{\lambda x^4 + \nu x^2}{2h} \; \cdot \; \exp h\frac{d^2}{dx^2} \; \cdot \; \exp -\frac{\lambda x^4 + \nu x^2}{2h} \,. \tag{5.2}$$

We shall analyze the "splitting" μ_2/μ_1 between the two first largest eigenvalues of the transfer operator. We will be rather sketchy and leave the details to the reader. We recall (cf for example [He7]) that in the case $\lambda = 0$, $\nu > 0$, a dilation $x = h^{\frac{1}{2}}y$ reduces the problem to an h-independent "Kac" harmonic oscillator for which the splitting is explicitely computable. In the case when $\lambda > 0$, we use another dilation and introduce the change $x = h^{\frac{1}{3}}y$ which leads to the new (unitarily equivalent) operator

$$c(h)\exp -\frac{1}{2}\, h^{\frac{1}{3}}\left(\lambda y^4 + \nu\, h^{-\frac{2}{3}} y^2\right) \; \cdot \; \exp h^{\frac{1}{3}}\frac{d^2}{dy^2} \; \cdot \; \exp -\frac{1}{2}\, h^{\frac{1}{3}}\left(\lambda y^4 + \nu\, h^{-\frac{2}{3}} y^2\right) \,. \tag{5.3}$$

If ν satisfies for some constant C the condition

$$\nu h^{-\frac{2}{3}} \geq C \,, \tag{5.4}$$

then the possible non-convexity due to the presence of ν has no effect. One finds an estimate of the splitting in the form

$$\frac{\mu_2}{\mu_1} \sim \exp -Dh^{\frac{1}{3}} \tag{5.5}$$

where D is a smooth function of λ and $\tilde{\nu} = \nu h^{-\frac{2}{3}}$.
The comparison between the Kac operator and the Schrödinger operator can be done by the Trotter-Kato formula (See for example [He4]). Modulo an error of $o(h^{\frac{1}{3}})$ (cf in the quartic case [IT] or [DS]), this comparison leads to the study of the operator $\exp -h^{\frac{1}{3}}S$ where

$$S := -\frac{d^2}{dy^2} + \lambda y^4 + \nu\, h^{-\frac{2}{3}} y^2\ .$$

The study of the Schrödinger operator S is relatively standard and was analyzed for example in [He2].
This gives eventually the following decay for the correlation

Proposition 5.1 *:*
Under the condition that ν satisfies (5.4), there exists D, C_1 and h_0 such that the correlation decays, for $h < h_0$, like

$$|\mathrm{Cov}^{(\infty)}(x_1, x_j)| \leq C_1 \exp -Djh^{\frac{1}{3}}\ .$$

In particular this is a sign of no phase transition.

6 Conclusion

In this paper, we have tried to show how simple the analysis through the Witten Laplacian can be for analyzing the decay of the correlations and other properties of Laplace Integrals. This was efficient in particular for considering weakly non-convex situations. We hope to come back in other publications [He8], [He9], [He10] and [He11] to the application to other problems like the uniqueness of the limiting measure or the logarithmic Sobolev inequality and to the case of strongly non-convex single spin phases but with small interaction [He10] and [He11].

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