The Complex Ginzburg-Landau Equation with Weak Initial Data

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Abstract

In this paper we investigate the existence and regularity of the solutions to the complex Ginzburg-Landau equation, $\partial_t u = Au + (a + i\nu)\Delta u - (b + i\mu)|u|^{2\sigma} u$, on the phase space $L_{r,p}(\mathbb{R}^n)$ of weighted L^p functions in infinite domain $\mathbb{R}^n$ of arbitrary spatial dimensions. The unique local strong solutions are established for subcritical σ, i.e., $r > \frac{n}{p} - \frac{1}{\sigma}$. Especially, the classical Lebesgue phase space L^p and the Hilbert phase space H^r are included.

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1 Introduction

The complex Ginzburg-Landau (CGL for short) equation is derived as a model equation to provide a quantitative description of fluid dynamical systems near the onset of the instabilities [6]. It also has important applications to chemical reactions [17], as well as the phenomenon of spatial pattern formation [6] and superconductivity. A general form of the CGL equation is

$$\partial_t u = Au + (a + i\nu)\Delta u - (b + i\mu)|u|^{2\sigma}u, \quad (x,t) \in \mathbb{R}^n \times (0,\infty) \qquad (1.1)$$

where u is a complex-valued function of a space variable $x \in \mathbb{R}^n$ and of a time variable $t \in (0,\infty)$, and $\sigma > 0, A \geq 0, a > 0, b > 0, \nu, \mu$ are real parameters.

The CGL equation, derived from the Navier-Stokes equations via multiple scaling methods in convection [20], allows for the quantitative investigation of some aspects of turbulent behavior that is currently not accessible in the Navier-Stokes equations. For example, the nonlinear Schrödinger equation (NLS for short) as the inviscid limit of the CGL equation is shown to have blow-up solution in finite time. But whether the solutions of the Euler equations, the inviscid limit of the Navier-Stokes equations ([4], [5]), can become singular in finite time remain outstandingly open.

In this paper the CGL equation is considered in an infinite spatial domain, i.e., $\mathbb{R}^n$. It is advantageous to study the spatially extended systems in many physical applications such as the pattern formation in which the size of the structure is much smaller than the size of the domain. The first mathematical question to resolve is of course the existence and regularity of the solutions to the initial value problem (IVP for short) of (1.1) with prescribed initial data

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^n \qquad (1.2)$$

In order to tell properly the time evolution of the dynamical systems, we needs to decide a phase space describing the instantaneous state of the system. Very often physics is done in Lebesgue spaces L^p or Hilbert spaces H^r, but these spaces usually involve decay at infinity and such spaces exclude some interesting solutions. The weighted Lebesgue spaces $L_{r,p}$ (defined below), especially when $r < 0$, seem more adequate. Actually the weighted Lebesgue space has been chosen as the phase space in studying the dynamics of many other nonlinear partial differential equations, for example, the

Navier-Stokes equations ([14],[21]), Burgers' equation [7] and the nonlinear heat (NLH for short) equation ([16], [23]).

For $s \in \mathbb{R}$, $1 \leq q < \infty$, $L_{s,q} \equiv L_{s,q}(\mathbb{R}^n)$ denotes the weighted Lebesgue space consisting of all functions f such that

$$(1-\Delta)^{\frac{s}{2}} f \in L^q(\mathbb{R}^n), \quad \text{i.e.,} \quad \|f\|_{s,q} \equiv \left(\int_{\mathbb{R}^n} \left[(1+|x|^2)^{\frac{s}{2}} |f(x)| \right]^q dx \right)^{\frac{1}{q}} < \infty$$

where L^q is the classical Lebesgue space over $\mathbb{R}^n$ and $\|f\|_{s,q}$ denotes the norm in $L_{s,q}$. It is easy to see from this definition that for $s = 0$ the space $L_{0,q}$ is simply the Lebesgue space L^q and for $q = 2$ the space $L_{s,2}$ reduces to the Hilbert space H^s. Some other properties concerning $L_{s,q}$ can be found in [1]. Especially, we will need the imbedding theorem

$$L_{s_1,q_1} \subset L_{s_2,q_2}, \quad \text{if} \quad s_1 - \frac{n}{q_1} = s_2 - \frac{n}{q_2}, \quad 1 < q_1 \leq q_2 < \infty,$$

(see e.g. [1], p.153).

The nonlinearity in (1.1) is of order $2\sigma + 1$. The value of σ plays an important role in whether a solution can exhibit blow up phenomena. At the level of L^p, σ is said to be critical (respectively subcritical, supercritical) if $\sigma n = p$ (respectively $\sigma n < p$, $\sigma n > p$). At the level of weighted Lebesgue space $L_{r,p}$, σ is said to be critical (respectively subcritical, supercritical) if $\sigma(n - pr) = p$ (respectively $\sigma(n - pr) < p$, $\sigma(n - pr) > p$). These terms are defined to be consistent with those for the NLS equation. For the NLS equation, the terms *critical* and *supercritical* indicate physically the collapse of self-focusing beams and mathematically the blowup of solutions.

Even though there is a large literature on the CGL equation, the exploration of existence and regularity of solutions to the CGL equation with given weak initial data is recent. Doering, Gibbon and Levermore [8] establish the existence and uniqueness of a local (in time) strong solution to the CGL equation on periodic domain $\mathbb{T}^n$. Their assumption is that either the initial data is in the Lebesgue space L^p and σ is subcritical or the data is bounded and σ is an integer. Levermore and Oliver [19] obtain a unique local strong solution for distributional data in $H^r(\mathbb{T}^n)$ in the subcritical case, i.e., $r > \frac{n}{2} - \frac{1}{\sigma}$ and this result directly corresponds to a theorem for the nonlinear Schrödinger equation of Cazenave and Weissler [2].

The purpose of this paper is to answer the question of existence and regularity of solutions to the CGL equation if the given initial data u_0 is in the weighted Lebesgue space $L_{r,p}(\mathbb{R}^n)$. Two important special cases, $u_0 \in L^p$ and $u_0 \in H^r$, are automatically included.

This paper is organized as follows. In Section 2 we establish estimates for the solution operator K to the linear equation and an integral operator G (defined in (2.5)) over spaces of weighted continuous functions in time (see Definition 2.2). In Section 3 we prove the existence and uniqueness of a local strong solution to the CGL equation in $\mathbb{R}^n$ given initial data in $L_{r,p}$ and σ subcritical, i.e., $r > \frac{n}{p} - \frac{1}{\sigma}$. The proof is based on the contraction mapping arguments with the help of the estimates in Section 2.

2 Preliminary estimates

We now consider the linear equation

$$\partial_t u = Au + (a+i\nu)\Delta u, \qquad (x,t) \in \mathbb{R}^n \times (0,\infty) \tag{2.1}$$

and its solution operator $K(t) = e^{At+((a+i\nu)\Delta t}$. $K(t)$ is represented by the convolution in x, i.e.,

$$K(t)f = k_t * f \equiv \frac{1}{(4\pi(a+i\nu)t)^{n/2}} e^{At-\frac{|x|^2}{4(a+i\nu)t}} * f \tag{2.2}$$

and the kernel k_t satisfies

$$|k_t(x)| \le \frac{1}{(4\pi|a+i\nu|t)^{n/2}} e^{At-\frac{a|x|^2}{4(a^2+\nu^2)t}}$$

Furthermore, we have

Lemma 2.1 *Let $A \ge 0$, $a > 0$ and ν be real parameters in (2.1).*

(a) *For $t > 0$, $\|k_t\|_{L^1} = \left(1 + \left[\frac{\nu}{a}\right]^2\right)^{\frac{n}{4}} e^{At}$ and therefore*

$$\|k_t * f\|_{L^p} \le \left(1 + \left[\frac{\nu}{a}\right]^2\right)^{\frac{n}{4}} e^{At}\|f\|_{L^p}, \qquad 1 \le p < \infty \tag{2.3}$$

(b) *For $s \ge 0$ and $q \in [1,\infty)$,*

$$\|k_t\|_{s,q} = \|(1-\Delta)^{\frac{s}{2}} k_t\|_{L^q} \le C_k e^{At} t^{\frac{1}{2}\left[-s-n\left(1-\frac{1}{q}\right)\right]} \tag{2.4}$$

for some constant C_k depending on a, ν, s, q.

Proof (a) The value of $\|k_t\|_{L^1}$ is obtained by a simple calculation. The inequality (2.3) is a consequence of Young's inequality.

(b) If $\frac{s}{2}$ is an integer, (2.4) is obtained by a direct calculation. The general case can be dealt with by interpolation.

We shall solve the IVP (1.1), (1.2) in the space of weighted continuous functions in time, which we now introduce. Kato and his collaborators define this type of spaces in solving the IVP for the Navier-Stokes equations with various initial data ([11],[12],[13],[14],[15], [9]). The class of weighted continuous functions in time are also appropriate in dealing with other types of evolution equations such as the NLH equation ([9], [22],[23]).

Definition 2.2 *For $T > 0$ and $\alpha \geq 0$, we define the space*

$$C_{\alpha,s,q} \equiv \{f \in C((0,T); L_{s,q}), \quad \|f\|_{\alpha,s,q} < \infty\}$$

where the norm $\|f\|_{\alpha,s,q}$ is given by

$$\|f\|_{\alpha,s,q} = \sup\{t^\alpha \|f(\cdot,t)\|_{s,q} : \quad t \in (0,T)\}$$

$\dot{C}_{\alpha,s,q}$ is defined to be a subspace of $C_{\alpha,s,q}$

$$\dot{C}_{\alpha,s,q} = \{f \in C_{\alpha,s,q}((0,T); L_{s,q}) : \quad \lim_{t\to 0^+} t^\alpha \|f(\cdot,t)\|_{s,q} = 0\}$$

$\dot{C}_{\alpha,s,q}$ is a closed subspace of $C_{\alpha,s,q}$. $f \in C_{\alpha,s,q}$ (resp. $f \in \dot{C}_{\alpha,s,q}$) implies

$$\|f(\cdot,t)\|_{s,q} = O(t^{-\alpha}), \quad (\text{resp.} \quad \|f(\cdot,t)\|_{s,q} = o(t^{-\alpha})), \quad \text{as} \quad t \to 0^+$$

If $T < \infty$, $C_{\alpha,s,q} \subset \dot{C}_{\beta,s,q}$ when $\alpha < \beta$. The case $\alpha = 0$ is special. Clearly, $\dot{C}_{0,s,q} \subset BC([0,T); L_{s,q}) \subset C_{0,s,q}$, where $BC([0,T); L_{s,q})$ denotes the space of $L_{s,q}-$valued bounded continuous functions on $[0,T)$. For simplicity, we'll write $\bar{C}_{s,q}$ for $BC([0,T); L_{s,q})$.

To solve the IVP (1.1), (1.2), we need to establish estimates for the operators K and G (defined in (2.5)) over the type of spaces we just defined.

Proposition 2.3 *Let $A \geq 0, a > 0$ and ν be real parameters.*

(i) *For $q \in [1,\infty), s \in \mathbb{R}$,*

$$K(t)f \to f \quad in \quad L_{s,q} \quad as\ t \to 0^+$$

and K maps continuously from $L_{s,q}$ to $\bar{C}_{s,q} \subset C_{0,s,q}$.

(ii) *If q_1, q_2, s_1, s_2 and α_2 satisfy*

$$q_1 \leq q_2, \qquad s_1 \leq s_2$$

$$\alpha_2 = \frac{1}{2}\left[s_2 - s_1 + \frac{n}{q_1} - \frac{n}{q_2}\right]$$

then K maps continuously from L_{s_1,q_1} to $\dot{C}_{\alpha_2,s_2,q_2}$ (When $\alpha_2 = 0$, $\dot{C}_{\alpha_2,s_2,q_2}$ should be replaced by $\bar{C}_{s_2,q_2}$).

Proof. (i) follows from the semi-group properties of K on L^q.

We now show (ii). Without loss of generality, we assume $s_1 = 0$. Let $f \in L^{q_1}$ and $s_0 = s_2 - s_1 = s_2 \geq 0$. By Young's inequality,

$$\|K(t)f\|_{s_0,q_2} = \|(1-\Delta)^{\frac{s_0}{2}} k_t * f\|_{L^{q_2}} \leq \|(1-\Delta)^{\frac{s_0}{2}} k_t\|_{L^{q_3}} \|f\|_{L^{q_1}}$$

where q_3 satisfies $1 - \frac{1}{q_3} = \frac{1}{q_1} - \frac{1}{q_2}$. Using Lemma 2.1,

$$\|K(t)f\|_{s_0,q_2} \leq C_k e^{At} t^{\frac{1}{2}\left(-s_0 - \frac{n}{q_1} + \frac{n}{q_2}\right)} \|f\|_{L^{q_1}}$$

which implies that K maps continuously from L_{s_1,q_1} to C_{α_2,s_2,q_2}. Furthermore, for $\alpha_2 > 0$, K maps into $\dot{C}_{\alpha_2,s_2,q_2}$, not just C_{α_2,s_2,q_2}. This is because K maps smooth functions into $C_{\alpha_2,s_2,q_2} \cap C_{0,s_2,q_2} \subset \dot{C}_{\alpha_2,s_2,q_2}$.

For $\alpha_2 = 0$, $\dot{C}$ is replaced by $\bar{C}$. Another easy proof can also be given in this case. Since now

$$s_1 - \frac{n}{q_1} = s_2 - \frac{n}{q_2}, \qquad q_1 \leq q_2$$

we can use the imbedding $L_{s_1,q_1} \subset L_{s_2,q_2}$ and (i) to show that K maps L_{s_1,q_1} to $\bar{C}_{s_2,q_2}$.

We now give estimate for G:

$$G(g)(t) = \int_0^t K(t-\tau) g(\tau) d\tau \tag{2.5}$$

Proposition 2.4 *If $q_1, q_2, s_1, s_2, \alpha_1$ and α_2 satisfy*

$$q_1 \leq q_2 \tag{2.6}$$

$$0 \leq s_2 - s_1 < 2 - \left[\frac{n}{q_1} - \frac{n}{q_2}\right] \tag{2.7}$$

$$\alpha_1 < 1, \quad \alpha_2 = \alpha_1 - 1 + \frac{1}{2}\left[s_2 - s_1 + \frac{n}{q_1} - \frac{n}{q_2}\right] \tag{2.8}$$

Then G maps continuously from $\dot{C}_{\alpha_1,s_1,q_1}$ to $\dot{C}_{\alpha_2,s_2,q_2}$.

Proof. Let $g \in \dot{C}_{\alpha_1,s_1,q_1}$ and $s_0 = s_2 - s_1$.

$$\|G(g)(t)\|_{s_2,q_2} = \int_0^t \|(1-\Delta)^{\frac{s_0}{2}} k_t * \left((1-\Delta)^{\frac{s_1}{2}} g(\tau)\right) \|_{L^{q_2}} d\tau$$

Using Young's inequality with $1 - \frac{1}{q_3} = \frac{1}{q_1} - \frac{1}{q_2}$ and Lemma 2.1,

$$\|G(g)(t)\|_{s_2,q_2} \leq \int_0^t \|(1-\Delta)^{\frac{s_0}{2}} k_t\|_{L^{q_3}} \|g(\tau)\|_{s_1,q_1} d\tau$$

$$\leq C_k e^{At} \|g\|_{\alpha_1,s_1,q_1} \int_0^t (t-\tau)^{\frac{1}{2}\left(-s_0 - \frac{n}{q_1} + \frac{1}{q_2}\right)} \tau^{-\alpha_1} d\tau$$

$$= C_k e^{At} \|g\|_{\alpha_1,s_1,q_1} t^{-\alpha_1 + 1 - \frac{1}{2}\left(-s_0 - \frac{n}{q_1} + \frac{n}{q_2}\right)} \int_0^1 (1-\rho)^{\frac{1}{2}\left(-s_0 - \frac{n}{q_1} + \frac{n}{q_2}\right)} \rho^{-\alpha_1} d\rho$$

Taking $\alpha_2 = \alpha_1 - 1 + \frac{1}{2}\left[s_2 - s_1 + \frac{n}{q_1} - \frac{n}{q_2}\right]$,

$$\|G(g)\|_{\alpha_2,s_2,q_2} \leq C_k e^{At} \|g\|_{\alpha_1,s_1,q_1} B\left(1 - \frac{1}{2}\left[s_2 - s_1 + \frac{n}{q_1} - \frac{n}{q_2}\right], 1 - \alpha_1\right)$$

We conclude the proof by noticing that the Beta function B is finite if

$$0 \leq s_2 - s_1 < 2 - \left[\frac{n}{q_1} - \frac{n}{q_2}\right], \quad \alpha_1 < 1$$

3 Well-posedness

We first state the main result.

Theorem 3.1 *Let p and r satisfy*

$$1 < p < \infty, \qquad \frac{1}{\sigma(2\sigma+1)} \leq \frac{n}{p} < \frac{1}{\sigma}, \tag{3.1}$$

$$r_0 < r < -2\sigma r_0, \qquad r_0 \equiv \frac{n}{p} - \frac{1}{\sigma} \tag{3.2}$$

Then for any $\delta > 0$, there is a $T = T(\delta) > 0$ such that for every $u_0 \in L_{r,p}$ with $\|u_0\|_{r,p} < \delta$ there is a unique solution u of the IVP (1.1), (1.2) on $(0, T)$ satisfying

$$u \in Z_T \equiv \bar{C}_{r,p} \cap \left(\cap_{p \leq q < \infty} \cap_{\lambda > r} \dot{C}_{\frac{\lambda - r}{2} + \frac{1}{2}\left(\frac{n}{p} - \frac{n}{q}\right), \lambda, q} \right) \tag{3.3}$$

Moreover, for any $T' \in (0, T)$, there is a neighborhood V of u_0 in $L_{r,p}$ such that the solution map

$$\mathfrak{P}: \qquad V \longmapsto Z_{T'}, \quad u_0 \longmapsto u$$

is Lipschitz.

We make several remarks about this theorem.

Remark 3.2 *Since we are mainly concerned about the case $r < 0$, the constraint $r < -2\sigma r_0$ for $r_0 < 0$ is virtually no condition.*

Remark 3.3 *By taking $p = 2$, Theorem 3.1 reduces to H^r theory for the CGL on $\mathbb{R}^n$, parallel to Theorem 4 in [19] where the IVP (1.1), (1.2) with data in $H^r(\mathbb{T}^n)$ is considered.*

Remark 3.4 *We observe that the time interval of existence depends only on $\|u_0\|_{r,p}$, the norm of the initial data. Therefore it is sufficient to obtain global control of $L_{r,p}$ norm in order to show global existence.*

Theorem 3.1 is proved by contraction mapping arguments. To this end, we write Equation 1.1 in the integral form

$$u = Ku_0 + G(N(u)), \qquad N(u) = -(b+i\mu)|u|^{2\sigma}u \tag{3.4}$$

where K is the solution operator (2.2) to the linear equation (2.1) and G is defined in (2.5). Then we show there is a fixed point for the integral equation (3.4) in an appropriate space. As mentioned earlier, the space of weighted continuous functions in time is suitable.

Proof of Theorem 3.1. For simplicity of notations, we may include the factor e^{At} in the constants $C_1, C_2, C_3, \cdots$ that appear below. This practice causes no side effects because of the insignificant role played by e^{At} here.

Let the space X_T and the norm for $u \in X_T$ given by

$$X_T = \bar{C}_{r,p} \cap \dot{C}_{-\frac{r}{2},0,p}, \qquad \|u\|_{X_T} = \|u\|_{0,r,p} + \|u\|_{-\frac{r}{2},0,p}$$

Let $X_{T,R} \subset X_T$ be the closed ball of radius R centered at 0 and $X_{T,R}$ is a complete metric space. Consider the mapping $\mathcal{A}$ on $X_{T,R}$ with T, R yet to be determined:

$$\mathcal{A}(u) = Ku_0 + G(N(u)), \qquad N(u) = -(b+i\mu)|u|^{2\sigma}u$$

Applying (i) of Proposition 2.3 and (ii) with

$$q_1 = p = q_2, \quad s_1 = r, \quad s_2 = 0, \quad \alpha_2 = -\frac{r}{2}$$

we obtain for two constants C_1 and C_2

$$\|Ku_0\|_{0,r,p} \le C_1\|u_0\|_{r,p}, \qquad \|Ku_0\|_{-\frac{r}{2},0,p} \le C_2\|u_0\|_{r,p}$$

That is, $\|Ku_0\|_{X_T} \le C_3\|u_0\|_{r,p}$ with $C_3 = C_1 + C_2$.

We now estimate $G(N(u))$. Choose p_1 and r_1 such that

$$p_1 = \frac{p}{2\sigma+1} < p, \qquad r_1 - \frac{n}{p_1} = r - \frac{n}{p}$$

and use the imbedding $L_{r_1,p_1} \subset L_{r,p}$ (see e.g. [1] , p.153) to obtain

$$\|G(N(u))\|_{0,r,p} \le C_4\|G(N(u))\|_{0,r_1,p_1}$$

We can now estimate $\|G(N(u))\|_{0,r_1,p_1}$ by applying Proposition 2.4 with

$$q_1 = q_2 = \frac{p}{2\sigma+1}$$

$$s_1 = 0, \qquad s_2 = r_1 \geq 0$$

$$\alpha_1 = -\frac{r}{2} - \sigma r_0, \qquad \alpha_2 = 0$$

because the conditions (2.6), (2.7) and (2.8) are now satisfied under the constraints (3.1), (3.2)

$$r_1 = r - \frac{n}{p} + \frac{n}{p_1} > (2\sigma+1)\left(\frac{n}{p} - \frac{1}{\sigma(2\sigma+1)}\right) \geq 0$$

$$0 < s_2 - s_1 = r_1 = r + 2\sigma\frac{n}{p} < -2\sigma r_0 + 2\sigma\frac{n}{p} = 2$$

$$\alpha_1 = -\frac{r}{2} - \sigma r_0 < -\frac{r_0}{2}(2\sigma+1) = -\left(\frac{n}{p} - \frac{1}{\sigma(2\sigma+1)}\right)\left(\sigma + \frac{1}{2}\right) + 1 < 1$$

We obtain

$$\|G(N(u))\|_{0,r_1,p_1} \leq C_5\|N(u)\|_{-\frac{r}{2}-\sigma r_0,0,\frac{p}{2\sigma+1}} \leq C_5 T^{\sigma(r-r_0)}\|N(u)\|_{-\frac{(2\sigma+1)r}{2},0,\frac{p}{2\sigma+1}}$$

Here we've used $r > r_0$ to pick up the factor $T^{\sigma(r-r_0)}$. Thus,

$$\|G(N(u))\|_{0,r,p} \leq C_6 T^{\sigma(r-r_0)}\|u\|^{2\sigma+1}_{-\frac{r}{2},0,p} \leq C_6 T^{\sigma(r-r_0)}\|u\|^{2\sigma+1}_{X_T} \tag{3.5}$$

where $C_6 = C_4 C_5\sqrt{b^2+\mu^2}$.

To estimate $\|G(N(u))\|_{-\frac{r}{2},0,p}$, we apply Proposition 2.4 with

$$q_1 = \frac{p}{2\sigma+1}, \qquad q_2 = p,$$

$$s_1 = 0, \qquad s_2 = 0,$$

$$\alpha_1 = -\frac{r}{2} - \sigma r_0, \quad \alpha_2 = -\frac{r}{2}$$

and it is easy to check that the conditions (2.6), (2.7) and (2.8) are satisfied because of the constraints (3.1) , (3.2) and $r_0 < 0$. We obtain for some constant C_7

$$\|G(N(u))\|_{-\frac{r}{2},0,p} \leq C_7\|N(u)\|_{-\frac{r}{2}-\sigma r_0,0,\frac{p}{2\sigma+1}}$$

$$\leq C_8 T^{\sigma(r-r_0)} \|u\|^{2\sigma+1}_{-\frac{r}{2},0,p} \leq C_8 T^{\sigma(r-r_0)} \|u\|^{2\sigma+1}_{X_T} \tag{3.6}$$

where $C_8 = C_7\sqrt{b^2+\mu^2}$.

Now we show that $\mathcal{A}$ is a contraction. Using the inequality

$$||u|^{2\sigma}u - |\tilde{u}|^{2\sigma}\tilde{u}| \leq C_9 \left(|u|^{2\sigma} + |\tilde{u}|^{2\sigma}\right)|u-\tilde{u}|$$

for $\sigma > 0$ and some constant C_9, we have for any $u, \tilde{u} \in X_T$

$$\|\mathcal{A}(u) - \mathcal{A}(\tilde{u})\|_{X_T} = \|G(N(u)) - G(N(\tilde{u}))\|_{X_T}$$
$$\leq C_{10}\|G((|u|^{2\sigma} + |\tilde{u}|^{2\sigma})|u-\tilde{u}|)\|_{X_T}$$

where $C_{10} = \sqrt{b^2+\mu^2}C_9$. Estimating G in X_T similarly as in (3.5) and (3.6),

$$\|\mathcal{A}(u) - \mathcal{A}(\tilde{u})\|_{X_T} \leq C_{11}T^{\sigma(r-r_0)}\|(|u|^{2\sigma} + |\tilde{u}|^{2\sigma})|u-\tilde{u}|\|_{-(2\sigma+1)\frac{r}{2},0,\frac{p}{2\sigma+1}}$$

$$\leq C_{11}T^{\sigma(r-r_0)}\left(\|u\|^{2\sigma}_{-\frac{r}{2},0,p} + \|\tilde{u}\|^{2\sigma}_{-\frac{r}{2},0,p}\right)\|u-\tilde{u}\|_{-\frac{r}{2},0,p}$$
$$= C_{11}T^{\sigma(r-r_0)}\left(\|u\|^{2\sigma}_{X_T} + \|\tilde{u}\|^{2\sigma}_{X_T}\right)\|u-\tilde{u}\|_{X_T}$$

Summing up, we've found that for some constants C_H, C_I, C_J

$$\|\mathcal{A}(u)\|_{X_T} \leq C_H\|u_0\|_{r,p} + C_I T^{\sigma(r-r_0)}\|u\|^{2\sigma+1}_{X_T}$$

$$\|\mathcal{A}(u) - \mathcal{A}(\tilde{u})\|_{X_T} \leq C_J T^{\sigma(r-r_0)}\left(\|u\|^{2\sigma}_{X_T} + \|\tilde{u}\|^{2\sigma}_{X_T}\right)\|u-\tilde{u}\|_{X_T}$$

So the conditions that $\mathcal{A}$ map $X_{T,R}$ into itself and be a contraction are

$$C_H\delta + C_I T^{\sigma(r-r_0)}R^{2\sigma+1} \leq R, \qquad 2C_J T^{\sigma(r-r_0)}R^{2\sigma} < 1 \tag{3.7}$$

For $\delta > 0$, we choose $R > 0$ and $T = T(\delta) > 0$ such that (3.7) is satisfied. Hence it follows from the contraction mapping principle that there is a fixed point $u \in X_{T,R}$ for $\mathcal{A}$, i.e., $u = \mathcal{A}(u)$. That is, u solves (3.4).

We now show that u is actually in Z_T (defined in (3.3)). First we notice that

$$u = \mathcal{A}(u) = Ku_0 + G(N(u))$$

For $p \le q < \infty$, $\lambda > r$, $Ku_0 \in \dot{C}_{\frac{\lambda-r}{2}+\frac{1}{2}\left(\frac{n}{p}-\frac{n}{q}\right),\lambda,q}$ is an easy consequence of Proposition 2.3. Thus $Ku_0 \in Z_T$. To show $G(N(u)) \in Z_T$, we follow similar procedures of establishing (3.5) and obtain for any $p \le q < \infty$ and $\lambda > r$

$$\|G(N(u))\|_{\frac{\lambda-r}{2}+\frac{1}{2}\left(\frac{n}{p}-\frac{n}{q}\right),\lambda,q}$$

$$\le C_{12}T^{\sigma(r-r_0)}\|N(u)\|_{-\frac{(2\sigma+1)r}{2},0,\frac{p}{2\sigma+1}} \le C_{13}T^{\sigma(r-r_0)}\|u\|_{X_T}^{2\sigma+1}$$

Therefore $u \in Z_T$.

The proof of the uniqueness and the local Lipschitz continuity is routine and thus omitted.

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