The New Identity for the Scattering Matrix of Exactly Solvable Models[1]

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Abstract

We discovered a simple quadratic equation, which relates scattering phases of particles on Fermi surface. We consider one dimensional Bose gas and XXZ Heisenberg spin chain.

[1]Dedicated to J. Zittartz on the occasion of his 60th birthday

1 Introduction

To define everything precisely we consider a specific model. Let us concentrate our attention on Bose gas with delta interaction (quantum Nonlinear Schrödinger equation).

The Hamiltonian of the model is

$$H = \int dx \left(\partial_x \Psi^\dagger(x) \partial_x \Psi(x) + c \Psi^\dagger(x) \Psi^\dagger(x) \Psi(x) \Psi(x) - h \Psi^\dagger(x) \Psi(x) \right). \quad (1.1)$$

Here $0 < c < \infty$, $h > 0$ are the coupling constant and the chemical potential respectively. The canonical Bose fields $\Psi(x,t)$, $\Psi^\dagger(x,t)$, $(x, t \in \mathbf{R})$ obey the standard commutation relations

$$[\Psi(x,t), \Psi^\dagger(y,t)] = \delta(x-y). \quad (1.2)$$

and act in the Fock space with the vacuum vector $|0\rangle$, which is characterized by the relation:

$$\Psi(x,t)|0\rangle = 0. \quad (1.3)$$

Alternatively the model can be formulated on the language of many-body quantum mechanics system, containing N identical particles. In this case the Hamiltonian of the Bose gas can be represented as

$$\mathcal{H}_N = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2c \sum_{N \geq j > k \geq 1} \delta(x_j - x_k) - hN. \quad (1.4)$$

For nonzero value of the coupling constant the Pauli principal is valid (chapter VII of [1]).

The model was solved by Bethe Ansatz [2]. The ground state is a Fermi sphere. In order to describe it precisely it is convenient to introduce spectral parameter λ (similar to rapidity). The derivative of the momentum of the particle with respect to the spectral parameter is

$$\frac{\partial k(\lambda)}{\partial \lambda} = 2\pi \rho(\lambda), \quad (1.5)$$

where the function $\rho(\lambda)$ is defined by an integral equation

$$\rho(\lambda) - \frac{1}{2\pi} \int_{-q}^{q} K(\lambda, \mu) \rho(\mu) \, d\mu = \frac{1}{2\pi}. \quad (1.6)$$

Here q is the value of the spectral parameter on the Fermi surface, and

$$K(\lambda, \mu) = \frac{2c}{c^2 + (\lambda - \mu)^2}. \tag{1.7}$$

One can prove that the integral operator $\hat{I} - \frac{1}{2\pi}\hat{K}$ is not degenerated, and hence, the equation (1.6) has unique solution ([3], chapter I of [1]). The density of the gas is given by

$$D = \int_{-q}^{q} \rho(\lambda)\, d\lambda. \tag{1.8}$$

There is one particle in the model. It is defined at $\lambda \geq q$ or $\lambda \leq -q$. The energy of the particle $\varepsilon(\lambda)$ is

$$\varepsilon(\lambda) - \frac{1}{2\pi}\int_{-q}^{q} K(\lambda, \mu)\varepsilon(\mu)\, d\mu = \lambda^2 - h. \tag{1.9}$$

It vanishes on the Fermi surface $\varepsilon(\pm q) = 0$. The momentum is

$$k(\lambda) = \lambda + \int_{-q}^{q} \theta(\lambda - \mu)\rho(\mu)\, d\mu. \tag{1.10}$$

Here

$$\theta(\lambda) = i \ln\left(\frac{ic + \lambda}{ic - \lambda}\right). \tag{1.11}$$

One can calculate a scattering matrix of particle with spectral parameter λ on another particle with spectral parameter μ (chapter I of [1]). There is no multi-particle production or reflection. Transition coefficient is

$$\exp\{2\pi i F(\lambda|\mu)\}. \tag{1.12}$$

The phase $F(\lambda|\mu)$ is defined by an integral equation

$$F(\lambda|\mu) - \frac{1}{2\pi}\int_{-q}^{q} K(\lambda, \nu)F(\nu|\mu)\, d\nu = \frac{1}{2\pi}\theta(\lambda - \mu). \tag{1.13}$$

The most important are scattering phases of particles on the Fermi edges $F(q|q)$ and $F(q|-q)$. In this paper we prove the identity

$$\det\begin{pmatrix} 1 - F(q|q) & F(q|-q) \\ -F(-q|q) & 1 + F(-q|-q) \end{pmatrix} = 1. \tag{1.14}$$

This is the main result of the paper. Another way to rewrite this identity is

$$\Big(1 - F(q|q)\Big)^2 - F^2(q|-q) = 1. \tag{1.15}$$

Here we have used the property $F(-\lambda|-\mu) = -F(\lambda|\mu)$, which follows immediately from the antisymmetry of $\theta(\lambda-\mu) = -\theta(\mu-\lambda)$.

This identity also permit us to relate "fractional" charge to the phase shift on the Fermi surface. Fractional charge $\mathcal{Z}$ appears in formulæ for finite size corrections (chapter I of [1]). This value is necessary for conformal description of the model (chapter XVIII of [1]) and it is equal to

$$\mathcal{Z} = 2\pi\rho(q). \tag{1.16}$$

Using the equations (1.6), (1.13) for $\rho(\lambda)$ and $F(\lambda|\mu)$, one can find the relationship between the fractional charge and the scattering phase on the Fermi surface

$$\mathcal{Z} = 1 + F(q|-q) - F(q|q). \tag{1.17}$$

Indeed, it follows from (1.6) that

$$[2\pi\rho(\lambda)-1] - \frac{1}{2\pi}\int_{-q}^{q} K(\lambda,\mu)[2\pi\rho(\mu)-1]\,d\mu = \frac{1}{2\pi}[\theta(\lambda+q)-\theta(\lambda-q)]. \tag{1.18}$$

Comparing this equation with (1.13) we find

$$2\pi\rho(\lambda) = 1 + F(\lambda|-q) - F(\lambda|q), \tag{1.19}$$

what, in turns, implies (1.17). The identity (1.15) allows us to find new relation

$$\mathcal{Z}^{-1} = 1 - F(q|-q) - F(q|q). \tag{1.20}$$

2 The proof of the main identity

In this section we give the proof of the identity (1.14). In order to do this, one should calculate the derivatives of the function $F(\lambda|\mu)$ with respect to λ, μ and q. Using the basic equation (1.13), we have

$$\frac{\partial F(\lambda|\mu)}{\partial\lambda} - \frac{1}{2\pi}\int_{-q}^{q} K(\lambda,\nu)\frac{\partial F(\nu|\mu)}{\partial\nu}\,d\nu = \frac{1}{2\pi}K(\lambda,\mu)$$

$$-\frac{1}{2\pi}K(\lambda,q)F(q|\mu) + \frac{1}{2\pi}K(\lambda,-q)F(-q|\mu), \tag{2.1}$$

$$\frac{\partial F(\lambda|\mu)}{\partial \mu} - \frac{1}{2\pi}\int_{-q}^{q} K(\lambda,\nu)\frac{\partial F(\nu|\mu)}{\partial \mu}\,d\nu = -\frac{1}{2\pi}K(\lambda,\mu), \tag{2.2}$$

$$\frac{\partial F(\lambda|\mu)}{\partial q} - \frac{1}{2\pi}\int_{-q}^{q} K(\lambda,\nu)\frac{\partial F(\nu|\mu)}{\partial q}\,d\nu$$
$$= \frac{1}{2\pi}K(\lambda,q)F(q|\mu) + \frac{1}{2\pi}K(\lambda,-q)F(-q|\mu). \tag{2.3}$$

Here we have used that $\frac{\partial}{\partial\lambda}\theta(\lambda-\mu) = K(\lambda,\mu)$.

As we have mentioned already, the resolvent of the operator $\hat{I} - \frac{1}{2\pi}\hat{K}$ exists and it is equal to

$$\hat{R} = \left(\hat{I} - \frac{1}{2\pi}\hat{K}\right)^{-1} \cdot \frac{1}{2\pi}\hat{K}. \tag{2.4}$$

The derivatives of the function $F(\lambda|\mu)$ can be expressed in terms of the resolvent

$$\begin{aligned}
\frac{\partial F(\lambda|\mu)}{\partial \lambda} &= R(\lambda,\mu) - R(\lambda,q)F(q|\mu) + R(\lambda,-q)F(-q|\mu)\\
\frac{\partial F(\lambda|\mu)}{\partial \mu} &= -R(\lambda,\mu)\\
\frac{\partial F(\lambda|\mu)}{\partial q} &= R(\lambda,q)F(q|\mu) + R(\lambda,-q)F(-q|\mu)
\end{aligned} \tag{2.5}$$

Using these equations one can find the complete derivatives with respect to q of functions $F(q|q)$, $F(q|-q)$ etc.:

$$\begin{aligned}
\frac{d}{dq}F(q|\pm q) &= \left(\frac{\partial}{\partial\lambda} \pm \frac{\partial}{\partial\mu} + \frac{\partial}{\partial q}\right) F(\lambda|\mu)\Big|_{\substack{\lambda=q\\ \mu=\pm q}},\\
\frac{d}{dq}F(-q|\pm q) &= \left(-\frac{\partial}{\partial\lambda} \pm \frac{\partial}{\partial\mu} + \frac{\partial}{\partial q}\right) F(\lambda|\mu)\Big|_{\substack{\lambda=-q\\ \mu=\pm q}}.
\end{aligned} \tag{2.6}$$

Substituting here equations (2.5) we find

$$
\begin{aligned}
&\frac{d}{dq}F(q|-q) = 2R(q,-q)\Big(1 + F(-q|-q)\Big),\\
&\frac{d}{dq}F(-q|q) = -2R(-q,q)\Big(1 - F(q|q)\Big),\\
&\frac{d}{dq}F(q|q) = 2R(q,-q)F(-q|q),\\
&\frac{d}{dq}F(-q|-q) = 2R(-q,q)F(q|-q).
\end{aligned}
\tag{2.7}
$$

Now it is sufficient to take the derivative with respect to q of the l.h.s. of the equation (1.14):

$$
\begin{aligned}
&\frac{d}{dq}\det\begin{pmatrix} 1 - F(q|q) & F(q|-q) \\ -F(-q|q) & 1 + F(-q|-q) \end{pmatrix}\\
&= -\frac{dF(q|q)}{dq}\Big(1 + F(-q|-q)\Big) + \Big(1 - F(q|q)\Big)\frac{dF(-q|-q)}{dq}\\
&\quad + \frac{dF(q|-q)}{dq}F(-q|q) + F(q|-q)\frac{dF(-q|q)}{dq}\\
&= -2R(q,-q)F(-q|q)\Big(1 + F(-q|-q)\Big)\\
&\quad +2R(-q,q)F(q|-q)\Big(1 - F(q|q)\Big)\\
&\quad +2R(q,-q)F(-q|q)\Big(1 + F(-q|-q)\Big)\\
&\quad -2R(-q,q)F(q|-q)\Big(1 - F(q|q)\Big) = 0.
\end{aligned}
\tag{2.8}
$$

On the other hand, it is clear that for $q=0$ we have

$$\det \begin{pmatrix} 1-F(q|q) & F(q|-q) \\ -F(-q|q) & 1+F(-q|-q) \end{pmatrix} \Bigg|_{q=0} = 1. \tag{2.9}$$

Thus, the identity (1.14) is proved.

We would like to emphasize the we did not use the explicit expressions for the kernel $K(\lambda,\mu)$ and the function $\theta(\lambda-\mu)$. In fact, we have used only three properties:

a) the existence of the resolvent of the operator $\hat{I}-\frac{1}{2\pi}\hat{K}$;

b) the kernel $K(\lambda,\mu)$ and the function $\theta(\lambda-\mu)$ depend on the difference;

c) the derivative of $\theta(\lambda-\mu)$ is equal to the kernel $K(\lambda,\mu)$.

In order to reduce (1.14) to the identity (1.15) one should use also the antisymmetry property $\theta(\lambda-\mu)=-\theta(\mu-\lambda)$.

Thus, the quadratic identity for the scattering phase is valid for a wide class of completely integrable models, but not only for the one-dimensional Bose gas. In particular, it is valid for scattering phases of elementary particles (spin waves) of XXZ Heisenberg spin chain in a magnetic field ([4], see also chapter II of [1]).

Acknowledgments

This work was supported by the National Science Foundation (NSF) Grant No. PHY-9321165, and the Russian Foundation of Basic Research Grant No. 96-01-00344.

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