A COMBINED SINE-GORDON AND MODIFIED KORTEWEG–DE VRIES HIERARCHY AND ITS ALGEBRO-GEOMETRIC SOLUTIONS

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Abstract. We derive a zero-curvature formalism for a combined sine-Gordon (sG) and modified Korteweg–de Vries (mKdV) equation which yields a local sGmKdV hierarchy. In complete analogy to other completely integrable hierarchies of soliton equations, such as the KdV, AKNS, and Toda hierarchies, the sGmKdV hierarchy is recursively constructed by means of a fundamental polynomial formalism involving a spectral parameter. We further illustrate our approach by developing the basic algebro-geometric setting for the sGmKdV hierarchy, including Baker–Akhiezer functions, trace formulas, Dubrovin-type equations, and theta function representations for its algebro-geometric solutions. Although we mainly focus on sG-type equations, our formalism also yields the sinh-Gordon, elliptic sine-Gordon, elliptic sinh-Gordon, and Liouville-type equations combined with the mKdV hierarchy.

1. Introduction

This paper is concerned with two main issues, a systematic derivation of a local hierarchy of nonlinear evolution equations by embedding the sine-Gordon (sG) equation into the modified Korteweg–de Vries (mKdV) hierarchy, resulting in what we call the sGmKdV hierarchy, and a simplified approach to its algebro-geometric solutions.

A careful investigation of the (enormous) literature on the sG equation (in light-cone coordinates)

$$u_{xt} = \sin(u), \tag{1.1}$$

reveals the fact that relatively little effort has been spent on deriving solutions which simultaneously satisfy the whole hierarchy of sine-Gordon equations. More significantly, the generally accepted hierarchy in the sine-Gordon case, as originally derived by Sasaki and Bullough [68], [69] in 1980, in sharp contrast to other hierarchies of soliton equations such as the KdV, AKNS, Toda, and Gelfand–Dickey hierarchies, appears to be nonlocal in u for all but the first element (1.1) in the hierarchy, although attempts at deriving a local sine-Gordon hierarchy (which, however, fell short of providing an explicit formalism) were made by S. J. Al'ber and M. S. Al'ber [3] in 1987. In particular, algebro-geometric (or periodic) solutions and their theta function representations are not derived for higher-order sG equations. The current paper focuses on the close connection between the sG equation and the mKdV hierarchy. We offer an elementary recursive approach to a local

Date: July 2, 1997.

1991 *Mathematics Subject Classification.* Primary 35Q53, 58F07; Secondary 35Q51.

Research supported in part by the US National Science Foundation under Grant No. DMS-9623121 and the Norwegian Research Council.

hierarchy which combines the sG equation and the mKdV hierarchy in a completely integrable manner (and similarly for the sinh-Gordon, elliptic sine-Gordon, elliptic sinh-Gordon, and Liouville-type equations, etc.), in the spirit of previous treatments of the Toda [14], Boussinesq [19], AKNS [46], and KdV [47] hierarchies, respectively. Since, in a sense to be made precise at the end of Section 2, the new hierarchy embeds the sG equation into the mKdV hierarchy, we call it the sGmKdV herarchy.

Our second major aim is to provide a simplified derivation of the algebro-geometric solutions of the sG equation and simultaneously of the sGmKdV hierarchy. Due to the intensive literature devoted to algebro-geometric solutions of the sG equation (and its close relatives), see, for instance [6], [15]–[18], [24]–[27], [29], [30], [33]–[38], [42], [43], [49], [52], [53], [56], [58], [63], [65], [71], [73], [74], this second part of our paper might need some justification. Our reasons for including a detailed treatment of this topic are twofold. First of all, it is natural to test whether the sGmKdV hierarchy permits algebro-geometric solutions in a manner similar to all other known hierarchies of integrable equations mentioned previously. Secondly, we are going to present a fundamental polynomial formalism which considerably simplifies the existing approach to algebro-geometric solutions even in the special case of the original sG equation (1.1).

Before sketching the content of each section, it seems appropriate to point out the enormous popularity of the sG equations and its close allies (the sinh-Gordon, elliptic sine-Gordon, elliptic sinh-Gordon equations) in a great variety of diverse fields. Explicit examples are the study of surfaces with constant negative curvature, or integrable surfaces (see, e.g., [10], [11], [31], [39], [51], [60], [66]), which points to its actual historical origin in the 19th century, elementary particle physics (cf. [20], Sects. 7.1–7.5), quantum optics (see [20], Sect. 7.8), Josephson junctions (cf. [20], Sect. 7.8.1), nonlinear excitations in condensed matter physics (see [12], [13]), vortex structures in fluids and plasmas (cf. [73]), and in connection with the reduction process of Abelian integrals on hyperelliptic curves to elliptic functions (see [4], [5], [6], Sect. 7.9, [7]–[9], [70]–[72]). In particular, it represents one of the celebrated examples of an infinite-dimensional completely integrable Hamiltonian system with associated action-angle variables in the scattering case (cf. [40], Part II, Sects. II.6 and II.7, [64], Sect. I.11). Moreover, the dimensional reductions of the self-dual Yang-Mills equations to the (elliptic) sG equation (see, e.g., [77], [78]) should also be mentioned in this connection.

In Section 2 we describe our zero-curvature formalism for the sGmKdV hierarchy. Following a recursive polynomial approach originally developed by Al'ber in the context of the KdV hierarchy [1], [2] and further developed in [14], [46], and [47] in connection with the Toda, AKNS, and KdV hierarchies, respectively, we derive a hierarchy of local nonlinear evolution equations whose first element is the sG equation (1.1). Section 3 is devoted to a detailed study of the stationary sGmKdV hierarchy. Following a device originally due to Jacobi [50] and applied to the KdV equation by Mumford [61] and McKean [59], we employ the recursive polynomial formalism of Section 2 to describe positive divisors of degree n on a hyperelliptic curve $\mathcal{K}_n$ of genus n associated with the nth equation in the stationary sGmKdV hierarchy. By means of a fundamental meromorphic function ϕ on $\mathcal{K}_n$ we then proceed to derive the theta function representations of the associated Baker–Akhiezer function and all stationary solutions of the sGmKdV hierarchy. In addition, we consider Dubrovin-type equations for auxiliary divisors of degree n

and the corresponding trace formulas for $u(x)$ in terms of these divisors. While mKdV curves (more generally, AKNS curves) are Toda-like curves and hence not branched at infinity, the sG curve is KdV-like with a branch point at infinity. In our combined sGmKdV hierarchy (for $\alpha, \beta \neq 0$) the sG part dominates, resulting again in a KdV-like curve branched at infinity. The corresponding time-dependent formalism is presented in detail in Section 4. Appendix A collects relevant material on hyperelliptic curves and their theta functions. Appendix B sketches a new connection between our polynomial recursion formalism and symmetric functions related to auxiliary divisors due to [32], which provides new insight into the linearization of the sGmKdV flows in Theorems 3.4 and 4.8 and into the construction of integrable hierarchies of soliton equations. Finally, Appendix C proves solvability of the principal recursion relations in Section 2.

2. The sGmKdV hierarchy

In this section we provide the basic construction of a local sGmKdV hierarchy, embedding the sG equation into the mKdV hierarchy in an integrable manner, using a zero-curvature formalism. We follow some ideas of Al'ber [1], [2], who developed a recursive polynomial approach for the KdV hierarchy, and recent extensions of this formalism to the Toda, Boussinesq, AKNS, and KdV hierarchies in [14], [19], [46], and [47], respectively.

Assuming $u \in C^\infty(\mathbb{R})$ (or meromorphic on $\mathbb{C}$) and $z \in \mathbb{C}$, we introduce the 2×2 matrix $U(z,x)$ by

$$U(z,x) = -i \begin{pmatrix} \frac{1}{2}u_x(x) & 1 \\ z & -\frac{1}{2}u_x(x) \end{pmatrix}, \quad x \in \mathbb{R}, \tag{2.1}$$

and for each $n \in \mathbb{N}_0$ the following 2×2 matrix $V_n(z,x)$

$$V_n(z,x) = \begin{pmatrix} -G_{n-1}(z,x) & \frac{1}{z}F_n(z,x) \\ H_n(z,x) & G_{n-1}(z,x) \end{pmatrix}, \quad x \in \mathbb{R}, \tag{2.2}$$

supposing F_n, G_{n-1}, and H_n to be entire with respect to the spectral parameter z and C^∞ (or meromorphic) in x.

Postulating the stationary zero-curvature condition

$$V_{n,x} = [U, V_n] \tag{2.3}$$

($[\,\cdot\,,\,\cdot\,]$ the commutator), equation (2.3) yields the following fundamental relationships between the functions F_n, H_n and G_{n-1},

$$F_{n,x}(z,x) = -iu_x(x)F_n(z,x) - 2izG_{n-1}(z,x), \tag{2.4a}$$

$$H_{n,x}(z,x) = iu_x(x)H_n(z,x) + 2izG_{n-1}(z,x), \tag{2.4b}$$

$$G_{n-1,x}(z,x) = i(H_n(z,x) - F_n(z,x)). \tag{2.4c}$$

From (2.4) one infers that

$$\frac{d}{dx}\Big(zG_{n-1}(z,x)^2 + F_n(z,x)H_n(z,x)\Big) = 0, \tag{2.5}$$

and hence

$$zG_{n-1}(z,x)^2 + F_n(z,x)H_n(z,x) = P_{2n}(z), \tag{2.6}$$

where $P_{2n}(z)$ is x-independent. It turns out that it is more convenient to define $R_{2n+1}(z) = zP_{2n}(z)$ so that (2.6) becomes

$$z^2G_{n-1}(z,x)^2 + zF_n(z,x)H_n(z,x) = R_{2n+1}(z). \tag{2.7}$$

Using equations (2.4) one can derive individual differential equations for F_n and H_n as follows. From (2.4a) and (2.4c) one infers

$$F_{n,xx} = -iu_{xx}F_n + 2z(H_n - F_n) - u_x^2F_n - 2zu_xG_{n-1}. \tag{2.8}$$

Multiplying (2.8) by F_n one can eliminate G_{n-1} to find

$$F_nF_{n,xx} - \frac{1}{2}F_{n,x}^2 + (2z + \frac{1}{2}u_x^2 + iu_{xx})F_n^2 = 2zP_{2n} = 2R_{2n+1}, \tag{2.9}$$

and differentiating with respect to x finally yields

$$F_{n,xxx} + (4z + u_x^2 + 2iu_{xx})F_{n,x} + (u_xu_{xx} + iu_{xxx})F_n = 0. \tag{2.10}$$

A similar analysis for H_n results in

$$H_nH_{n,xx} - \frac{1}{2}H_{n,x}^2 + (2z + \frac{1}{2}u_x^2 - iu_{xx})H_n^2 = 2R_{2n+1} \tag{2.11}$$

and

$$H_{n,xxx} + (4z + u_x^2 - 2iu_{xx})H_{n,x} + (u_xu_{xx} - iu_{xxx})H_n = 0. \tag{2.12}$$

Introducing

$$w_\pm = -\frac{1}{4}(u_x^2 \pm 2iu_{xx}), \tag{2.13}$$

equations (2.10) and (2.12) take the form

$$-\frac{1}{4}F_{n,xxx} + (w_+ - z)F_{n,x} + \frac{1}{2}w_{+,x}F_n = 0, \tag{2.14a}$$

$$-\frac{1}{4}H_{n,xxx} + (w_- - z)H_{n,x} + \frac{1}{2}w_{-,x}H_n = 0, \tag{2.14b}$$

which are identical to the corresponding equations for the KdV hierarchy (see [47]) with potential $V = w_\pm$. Analogous assertions apply to (2.9) and (2.11). Intimate connections between the KdV, mKdV, and sine-Gordon (resp. sinh-Gordon) equations involving Miura-type transformations have been known for a long time, see, for instance, [21], [48] and more recently, [76]. Additional comments in this direction will be made in Remark 2.3. Thus by making the following polynomial ansatz with respect to the spectral parameter z,

$$F_n(z,x) = \sum_{j=0}^{n} f_{n-j}(x)z^j, \quad H_n(z,x) = \sum_{j=0}^{n} h_{n-j}(x)z^j, \tag{2.15}$$

we first deduce by taking $z = 0$ in (2.4a) and (2.4b) that

$$f_{n,x}(x) = -iu_x(x)f_n(x), \quad n \in \mathbb{N}_0, \tag{2.16a}$$

$$h_{n,x}(x) = iu_x(x)h_n(x), \quad n \in \mathbb{N}_0. \tag{2.16b}$$

For $n \in \mathbb{N}$ we conclude by combining (2.14) and (2.15) (cf. [47])

$$f_0(x) = 1, \quad n \in \mathbb{N},$$

$$f_{j,x}(x) = -\frac{1}{4}f_{j-1,xxx}(x) + w_+(x)f_{j-1,x}(x) + \frac{1}{2}w_{+,x}(x)f_{j-1}(x), \tag{2.17a}$$

$$j = 1, \dots, n, \; n \in \mathbb{N},$$

and

$$\begin{aligned}
&h_0(x) = 1, \quad n \in \mathbb{N}, \\
&h_{j,x}(x) = -\frac{1}{4}h_{j-1,xxx}(x) + w_-(x)h_{j-1,x}(x) + \frac{1}{2}w_{-,x}(x)h_{j-1}(x), \\
&\qquad\qquad j = 1, \dots, n,\, n \in \mathbb{N}.
\end{aligned} \tag{2.17b}$$

Furthermore, by taking $z = 0$ in (2.6) we find the constraint

$$f_n(x)h_n(x) = P_{2n}(0). \tag{2.18}$$

Compatibility of f_n and h_n, $n \in \mathbb{N}$ as defined by (2.16) and (2.17) is proved in Lemma C.2.

Explicitly (for n sufficiently large),

$$\begin{aligned}
f_0 &= 1, \\
f_1 &= \tfrac{1}{2}w_+ + c_1 = -\tfrac{1}{8}(u_x^2 + 2iu_{xx}) + c_1, \\
f_2 &= -\tfrac{1}{8}w_{+,xx} + \tfrac{3}{8}w_+^2 + c_1\tfrac{1}{2}w_+ + c_2 \\
&= -\tfrac{1}{32}u_{xx}^2 + \tfrac{1}{16}u_xu_{xxx} + \tfrac{i}{16}u_{xxxx} + \tfrac{3}{128}u_x^4 + \tfrac{3i}{32}u_x^2u_{xx} - \tfrac{c_1}{8}(u_x^2 + 2iu_{xx}) + c_2, \\
f_3 &= \tfrac{1}{32}w_{+,xxxx} - \tfrac{5}{16}w_+w_{+,xx} - \tfrac{5}{32}w_{+,x}^2 + \tfrac{5}{16}w_+^3 + c_1(-\tfrac{1}{8}w_{+,xx} + \tfrac{3}{8}w_+^2) \\
&\quad + c_2\tfrac{1}{2}w_+ + c_3, \text{ etc.},
\end{aligned} \tag{2.19a}$$

and

$$\begin{aligned}
h_0 &= 1, \\
h_1 &= \tfrac{1}{2}w_- + c_1 = -\tfrac{1}{8}(u_x^2 - 2iu_{xx}) + c_1, \\
h_2 &= -\tfrac{1}{8}w_{-,xx} + \tfrac{3}{8}w_-^2 + c_1\tfrac{1}{2}w_- + c_2 \\
&= -\tfrac{1}{32}u_{xx}^2 + \tfrac{1}{16}u_xu_{xxx} - \tfrac{i}{16}u_{xxxx} + \tfrac{3}{128}u_x^4 - \tfrac{3i}{32}u_x^2u_{xx} - \tfrac{c_1}{8}(u_x^2 - 2iu_{xx}) + c_2, \\
h_3 &= \tfrac{1}{32}w_{-,xxxx} - \tfrac{5}{16}w_-w_{+,xx} - \tfrac{5}{32}w_{-,x}^2 + \tfrac{5}{16}w_-^3 + c_1(-\tfrac{1}{8}w_{-,xx} + \tfrac{3}{8}w_-^2) \\
&\quad + c_2\tfrac{1}{2}w_- + c_3, \text{ etc.}
\end{aligned} \tag{2.19b}$$

Here $\{c_\ell\}_{\ell\in\mathbb{N}}$ denote integration constants, which for compatibility reasons (see Lemma C.1) have to be chosen identical in (2.19a) and (2.19b).

Next, making the ansatz

$$G_{-1}(z,x) = 0, \quad G_{n-1}(z,x) = \sum_{j=0}^{n-1} g_{n-1-j}(x)z^j, \quad n \in \mathbb{N}, \tag{2.20}$$

equation (2.4a) yields

$$g_j(x) = \frac{i}{2}(f_{j,x}(x) + iu_x(x)f_j(x)), \quad j = 0, \dots, n-1,\, n \in \mathbb{N}, \tag{2.21}$$

and $f_{n,x} + iu_xf_n = 0$, implying

$$f_n(x) = \alpha e^{-iu(x)}, \quad \alpha \in \mathbb{C}, \quad n \in \mathbb{N}_0. \tag{2.22}$$

Recording the first few coefficients in (2.20) then yields (for n sufficiently large)

$$\begin{aligned}
g_0 &= -\tfrac{1}{2}u_x, \\
g_1 &= \tfrac{1}{16}u_x^3 + \tfrac{1}{8}u_{xxx} - \tfrac{c_1}{2}u_x, \\
g_2 &= -\tfrac{3}{256}u_x^5 - \tfrac{1}{32}u_{xxxxx} - \tfrac{i}{32}u_{xx}u_{xxx} - \tfrac{5}{64}u_x^2u_{xxx} - \tfrac{5}{64}u_xu_{xx}^2
\end{aligned} \tag{2.23}$$

$$+\, c_1(\tfrac{1}{16}u_x^3 + \tfrac{1}{8}u_{xxx}) - \tfrac{c_2}{2}u_x, \text{ etc.}$$

Equation (2.4b) yields

$$g_j(x) = \frac{i}{2}(-h_{j,x}(x) + iu_x(x)h_j(x)), \quad j = 0, \dots, n-1,\, n \in \mathbb{N}, \tag{2.24}$$

and $h_{n,x} - iu_x h_n = 0$, implying

$$h_n(x) = \beta e^{iu(x)}, \quad \beta \in \mathbb{C}, \quad n \in \mathbb{N}_0. \tag{2.25}$$

Finally, equation (2.4c) implies

$$g_{j,x}(x) = i(h_{j+1}(x) - f_{j+1}(x)), \quad j = 0, \dots, n-1,\, n \in \mathbb{N}. \tag{2.26}$$

Compatibility of (2.21), (2.22), and (2.24)–(2.26) is proven in Lemmas C.1 and C.2.

For notational convenience we define

$$g_{-1}(x) = 0. \tag{2.27}$$

Then

$$g_{n-1,x}(x) = i(\beta e^{iu(x)} - \alpha e^{-iu(x)}), \quad n \in \mathbb{N}_0, \tag{2.28}$$

defines the nth stationary sGmKdV equation, subject to the constraint (cf. (2.18))

$$\alpha\beta = P_{2n}(0). \tag{2.29}$$

We record the first few equations explicitly,

$$\begin{aligned} (\beta e^{iu} - \alpha e^{-iu}) &= 0, \\ -\tfrac{i}{2}u_{xx} + (\beta e^{iu} - \alpha e^{-iu}) &= 0, \\ \tfrac{i}{16}(u_x^3 + 2u_{xxx})_x - c_1\tfrac{i}{2}u_{xx} + (\beta e^{iu} - \alpha e^{-iu}) &= 0, \text{ etc.} \end{aligned} \tag{2.30}$$

In particular, for $\alpha = \beta \neq 0$, the first equation in (2.30) yields the stationary sine-Gordon equation (in light-cone coordinates), that is,

$$\sin(u) = 0. \tag{2.31}$$

In the special case $\alpha = \beta = 0$ one obtains $f_n = h_n = 0$, and hence the $(n-1)$th stationary KdV equation is satisfied for the potential $w_\pm$. Introducing

$$v = \frac{u_x}{2i}, \tag{2.32}$$

we see that $w_\pm$ and v are related by the Miura-transformation

$$w_\pm = v^2 \pm v_x, \tag{2.33}$$

and we may conclude that $g_{n-1,x} = 0$ equals the $(n-1)$th stationary mKdV equation with solution v for $n \in \mathbb{N}$.

For later purposes, it will be convenient to record the homogeneous case defined by the vanishing of all integration constants c_ℓ. Denote

$$\begin{aligned} &\hat{f}_0 = f_0 = 1, && \hat{g}_0 = g_0 = -\tfrac{1}{2}u_x, && \hat{h}_0 = h_0 = 1, \quad n \in \mathbb{N}, \\ &\hat{f}_j = f_j\big|_{c_1=\cdots=c_j=0}, && \hat{g}_j = g_j\big|_{c_1=\cdots=c_j=0}, && \hat{h}_j = h_j\big|_{c_1=\cdots=c_j=0}, \\ &&&&& j = 1, \dots, n-1,\, n \geq 2, \end{aligned} \tag{2.34}$$

$$\hat{f}_n = f_n = \alpha e^{-iu}, \qquad \hat{h}_n = h_n = \beta e^{iu}, \qquad n \in \mathbb{N}_0. \tag{2.35}$$

The connection between homogenous and nonhomogeneous quantities then reads,

$$f_j = \sum_{\ell=0}^{j} c_\ell \hat{f}_{j-\ell}, \quad g_j = \sum_{\ell=0}^{j} c_\ell \hat{g}_{j-\ell}, \quad h_j = \sum_{\ell=0}^{j} c_\ell \hat{h}_{j-\ell}, \qquad c_0 = 1, \quad j = 0, \dots, n, \, n \in \mathbb{N}_0. \tag{2.36}$$

Introducing

$$\widehat{F}_j(z,x) = \sum_{k=0}^{j} \hat{f}_{j-k}(x) z^k, \quad \widehat{H}_j(z,x) = \sum_{k=0}^{j} \hat{h}_{j-k}(x) z^k,$$
$$\widehat{G}_{j-1}(z,x) = \sum_{k=0}^{j-1} \hat{g}_{j-1-k}(x) z^k, \quad j = 0, \dots, n, \, n \in \mathbb{N}_0 \tag{2.37}$$

then results in

$$F_n = \sum_{j=0}^{n} c_{n-j} \widehat{F}_j, \quad G_{n-1} = \sum_{j=0}^{n-1} c_{n-1-j} \widehat{G}_j, \quad H_n = \sum_{j=0}^{n} c_{n-j} \widehat{H}_j, \quad n \in \mathbb{N}_0, \tag{2.38}$$

with $\hat{f}_n = f_n = \alpha e^{-iu}$ and $\hat{h}_n = h_n = \beta e^{iu}$ as in (2.35).

Finally, we turn to the time-dependent sGmKdV hierarchy. Introducing a deformation parameter $t_n \in \mathbb{R}$ into u (i.e., $u(x) \to u(x, t_n)$), the corresponding zero-curvature relation reads

$$U_{t_n} - V_{n,x} + [U, V_n] = 0, \quad n \in \mathbb{N}_0, \tag{2.39}$$

which results in the following set of equations

$$u_{xt_n}(x,t_n) = -2iG_{n-1,x}(x,t_n) - 2(H_n(x,t_n) - F_n(x,t_n)), \tag{2.40a}$$
$$F_{n,x}(x,t_n) = -iu_x(x,t_n)F_n(x,t_n) - 2izG_{n-1}(x,t_n), \tag{2.40b}$$
$$H_{n,x}(x,t_n) = iu_x(x,t_n)H_n(x,t_n) + 2izG_{n-1}(x,t_n). \tag{2.40c}$$

We follow a more elaborate notation inspired by Hirota's τ-function approach and indicate the individual nth sGmKdV flow by a separate time variable $t_n \in \mathbb{R}$. (The latter notation suggests consideration of all sGmKdV flows simultaneously by introducing $\underline{t} = (t_0, t_1, t_2, \dots)$.)

Inserting the polynomial expressions for F_n, H_n, and G_{n-1} into (2.40b) and (2.40c), respectively, first yields

$$f_{n,x}(x,t_n) = -iu_x(x,t_n) f_n(x,t_n), \quad h_{n,x}(x,t_n) = iu_x(x,t_n) h_n(x,t_n), \quad n \in \mathbb{N}_0. \tag{2.41}$$

In the general case we find

$$f_0(x,t_n) = 1, \quad n \in \mathbb{N}, \tag{2.42a}$$
$$f_{j,x}(x,t_n) = -\frac{1}{4} f_{j-1,xxx}(x,t_n) + w_+(x,t_n) f_{j-1,x}(x,t_n) + \frac{1}{2} w_{+,x}(x,t_n) f_{j-1}(x,t_n), \qquad j = 1, \dots, n-1, \, n \geq 2,$$

and

$$h_0(x,t_n) = 1, \quad n \in \mathbb{N}, \tag{2.42b}$$
$$h_{j,x}(x,t_n) = -\frac{1}{4} h_{j-1,xxx}(x,t_n) + w_-(x,t_n) h_{j-1,x}(x,t_n) + \frac{1}{2} w_{-,x}(x,t_n) h_{j-1}(x,t_n), \qquad j = 1, \dots, n-1, \, n \geq 2,$$

and (recall our convention (2.27))

$$u_{xt_n}(x,t_n) = -2ig_{n-1,x}(x,t_n) - 2(h_n(x,t_n) - f_n(x,t_n)), \quad n \in \mathbb{N}_0, \tag{2.43}$$

in addition to equations (2.21), (2.22), (2.24), (2.25), and (2.26) (the latter equation for $j = 0, \dots, n-2$ and only for $n \geq 2$). Varying $n \in \mathbb{N}_0$ then defines the time-dependent sGmKdV hierarchy by

$$\begin{aligned} \mathrm{sGmKdV}_n(u(x,t_n)) &= u_{xt_n}(x,t_n) + 2ig_{n-1,x}(x,t_n) + 2(\beta e^{iu(x,t_n)} - \alpha e^{-iu(x,t_n)}) \\ &= 0, \qquad n \in \mathbb{N}_0. \end{aligned} \tag{2.44}$$

Explicitly, the first few equations read

$$\begin{aligned} &\mathrm{sGmKdV}_0(u) = u_{xt_0} + 2(\beta e^{iu} - \alpha e^{-iu}) = 0, \\ &\mathrm{sGmKdV}_1(u) = u_{xt_1} - iu_{xx} + 2(\beta e^{iu} - \alpha e^{-iu}) = 0, \\ &\mathrm{sGmKdV}_2(u) = u_{xt_2} + \tfrac{i}{8}(u_x^3 + 2u_{xxx})_x - c_1 iu_{xx} + 2(\beta e^{iu} - \alpha e^{-iu}) = 0, \text{ etc.} \end{aligned} \tag{2.45}$$

In contrast to the stationary case, the constraint (2.29) does not apply in the t_n-dependent context (2.44) (since the left-hand side of (2.40a) is nonvanishing). Moreover, the compatibility requirements for f_n and h_n in (2.16) and (2.17), which were of great significance in the stationary case, are absent in the present time-dependent situation.

Remark 2.1. Here $\mathrm{sGmKdV}_0(u) = \mathrm{sG}(u) = 0$ is *the* sine-Gordon equation in light-cone coordinates choosing $\alpha = \beta = i/4$,

$$u_{xt_0} = \sin(u). \tag{2.46}$$

We note that $u_{xt_0} = 2(\alpha e^{-iu} - \beta e^{iu})$ reduces to $v_{xt_0} = \sin(v)$, where $v(x,t_0) = u(x/a,t_0) + (2i)^{-1}\ln(\beta/\alpha)$, $a = -2i(\alpha\beta)^{1/2}$, $\alpha, \beta \in \mathbb{C} \setminus \{0\}$. Writing $v = iu$ and setting $\alpha = \beta = 1/4$, one finds

$$v_{xt_0} = \sinh(v), \tag{2.47}$$

that is, the sinh-Gordon equation. Similarly, introducing $\xi = x + t_0$, $\eta = x - t_0$ produces the (hyperbolic) sine-Gordon and (hyperbolic) sinh-Gordon equations in laboratory coordinates

$$u_{\xi\xi} - u_{\eta\eta} = \sin(u), \quad v_{\xi\xi} - v_{\eta\eta} = \sinh(v) \tag{2.48}$$

and $\xi = x + t_0$, $\tau = i(x - t_0)$ produces the elliptic sine-Gordon and elliptic sinh-Gordon equations

$$u_{\xi\xi} + u_{\tau\tau} = \sin(u), \quad v_{\xi\xi} + v_{\tau\tau} = \sinh(v). \tag{2.49}$$

Similarly, the case $\alpha = 0$ yields

$$u_{xt_0} = -2\beta e^{iu} \tag{2.50}$$

and a similar equation in the case $\beta = 0$. Hence writing $v = iu$ and changing coordinates $(x,t_0) \to (\xi,\tau)$ yields the Liouville hierarchy for v (cf., e.g., [75]) starting with

$$v_{\xi\xi} + v_{\tau\tau} = 2i\beta e^{-v}. \tag{2.51}$$

In particular, the results in Sections 3 and 4 extend to these hierarchies and hence we omit further distinctions and focus on the sGmKdV hierarchy in light-cone coordinates for the rest of this paper.

Finally, in the case $\alpha = \beta = 0$, we define

$$v(x, t_n) = \frac{u_x(x, it_n)}{2i}, \tag{2.52}$$

and the sGmKdV hierarchy reduces to

$$v_{t_n}(x, t_n) - ig_{n-1,x}(x, t_n) = 0, \quad n \in \mathbb{N}, \tag{2.53}$$

which equals the $(n-1)$th modified KdV equation with solution v.

Remark 2.2. In addition to the nonlocal sine-Gordon hierarchies constructed by Sasaki and Bullough [68], [69], we are aware of a few other attempts introducing a nonlocal sG hierarchy. For instance, Newell [62], Sect. 5k introduces a nonlocal sG hierarchy using an extension of the AKNS hierarchy, Tracy and Widom [76] discuss the sG hierarchy in close connection with the mKdV hierarchy, and Gu [48] derives a generalized mKdV-sG hierarchy. On the other hand, the existence of an infinite sequence of conservation laws for (2.46) polynomial in u and its x-derivatives (see, e.g., [67]) suggests the existence of a local hierarchy involving the sG equation. Moreover, the fact that the usual zero-curvature representation for (2.46) is gauge equivalent to that of the nonlinear Schrödinger (nS) equation as discussed in [40], Part II, Sect. II.7 (see also [10], Sect. 5 and [66]), and given the fact that the nS hierarchy is well-known to be local, also hints to the existence of a local hierarchy involving the sG equation even though no further details seem to have been worked out. Finally, the strongest indication for the existence of such a local hierarchy is provided by the so called $\underline{\mu}$-representation of the algebro-geometric sG solutions (cf. Remark 4.7) as discussed in [3], [32], [34], [42], [73], [74]. The only work we are aware of that clearly aimed at a recursive construction of a local hierarchy involving the sG equation is due to Al'ber and Al'ber [3]. Their approach, however, seems different from ours and does not start from a zero-curvature representation. While a set of recurrence relations is claimed in equations (4.4) and (4.5) of [3], no explicit formulas are offered and the precise form of the higher-order equations is not detailed.

Remark 2.3. The relationship between the KdV and mKdV and the sGmKdV hierarchy (2.44), alluded to in (2.13), (2.14), the paragraph following (2.14), (2.32), (2.33), (2.52), and (2.53), can be made precise as follows. The equation

$$\Psi_x = U\Psi, \quad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{2.54}$$

is equivalent to

$$\psi_{1,xx} = (w_+ - z)\psi_1, \quad \psi_{2,xx} = (w_- - z)\psi_2, \tag{2.55}$$

with

$$w_\pm = v^2 \pm v_x, \quad v = u/(2i). \tag{2.56}$$

Equation (2.56) represents the familiar Miura transformation between solutions $w_\pm$ of the KdV hierarchy and v of the mKdV hierarchy. Since $\mathrm{sGmKdV}_n(u) = 0$ reduces to $\mathrm{mKdV}_{n-1}(v) = 0$, $n \in \mathbb{N}$ for $\alpha = \beta = 0$ as discussed in Remark 2.1, $\mathrm{sGmKdV}_n(u) = 0$ for general $\alpha, \beta \in \mathbb{C}$ represents a completely integrable linear combination of the $\mathrm{sG}(u)$ equation and the $\mathrm{mKdV}_{n-1}(v)$ equation. This corresponds to our choice (2.2) of $zV_n(z,x)$ being a polynomial with respect to z. The usually considered nonlocal sG hierarchy then corresponds to a choice of $zV_n(z,x)$ being rational in z with a nontrivial principal part.

3. The stationary sGmKdV formalism

This section is devoted to a detailed study of the stationary sGmKdV hierarchy and its algebro-geometric solutions. Our principal tool will be a combination of the polynomial recursion formalism introduced in Section 2 and a meromorphic function ϕ on a hyperelliptic curve $\mathcal{K}_n$ defined in terms of $R_{2n+1}(z)$. Moreover, we discuss in detail the associated stationary Baker–Akhiezer vector $\Psi(z,x,x_0)$ and associated auxiliary positive divisors of degree n on $\mathcal{K}_n$.

Throughout this section we assume (2.3) (resp. (2.4)), (2.15), (2.17), (2.19), and freely employ the formalism developed in (2.1)–(2.38).

Returning to (2.7) we infer from (2.15) that $R_{2n+1}(z) = zP_{2n}(z)$ is a monomial in z of degree $2n+1$ of the form

$$R_{2n+1}(z) = \prod_{m=0}^{2n} (z - E_m), \quad E_0 = 0,\, E_1, \dots, E_{2n} \in \mathbb{C}. \tag{3.1}$$

Computing

$$\begin{aligned} \det(wI_2 - iV_n(z,x)) &= w^2 - \det(V_n(z,x)) \\ &= w^2 + G_{n-1}(z,x)^2 + \frac{1}{z} F_n(z,x) H_n(z,x) \\ &= w^2 + \frac{1}{z^2} R_{2n+1}(z) \end{aligned} \tag{3.2}$$

(with I_2 the identity matrix in $\mathbb{C}^2$), we are led to introduce the (possibly singular) hyperelliptic curve $\mathcal{K}_n$ of arithmetic genus n defined by

$$\mathcal{K}_n \colon \mathcal{F}_n(z,y) = y^2 - R_{2n+1}(z) = 0. \tag{3.3}$$

We compactify $\mathcal{K}_n$ by adding the point P_∞, still denoting its projective closure by $\mathcal{K}_n$. Hence $\mathcal{K}_n$ becomes a two-sheeted Riemann surface of arithmetic genus n. Finite points P on $\mathcal{K}_n$ are denoted by $P = (z,y)$, where $y(P)$ denotes the meromorphic function on $\mathcal{K}_n$ satisfying $\mathcal{F}_n(z,y) = 0$. The complex structure on $\mathcal{K}_n$ is then defined in a standard manner. Furthermore, we introduce the involution

$$* \colon \mathcal{K}_n \to \mathcal{K}_n, \quad P = (z,y) \mapsto P^* = (z,-y). \tag{3.4}$$

For further properties and notation concerning hyperelliptic curves we refer to Appendix A.

In order to avoid numerous case distinctions because of the trivial case $n = 0$, we shall assume $n \in \mathbb{N}$ for the remainder of this section. Moreover, to simplify our presentation in the following, we will subsequently focus on sG-type equations and hence assume

$$\alpha,\, \beta \in \mathbb{C} \setminus \{0\}. \tag{3.5}$$

By (2.29) this is equivalent to

$$P_{2n}(0) = \prod_{m=1}^{2n} E_m = \alpha\beta \neq 0. \tag{3.6}$$

Hence, from this point on, we suppose

$$E_0 = 0, \quad E_m \in \mathbb{C} \setminus \{0\}, \quad m = 0, \dots, 2n. \tag{3.7}$$

In the following we need the roots of the polynomials F_n and H_n, and hence introduce

$$F_n(z,x) = \prod_{j=1}^{n}(z-\mu_j(x)), \quad H_n(z,x) = \prod_{j=1}^{n}(z-\nu_j(x)). \tag{3.8}$$

Next, define

$$\hat{\mu}_j(x) = (\mu_j(x), -\mu_j(x)G_{n-1}(\mu_j(x),x)) \in \mathcal{K}_n, \quad j=1,\dots,n, \tag{3.9a}$$

$$\hat{\nu}_j(x) = (\nu_j(x), \nu_j(x)G_{n-1}(\nu_j(x),x)) \in \mathcal{K}_n, \quad j=1,\dots,n \tag{3.9b}$$

and

$$P_0 = (0,0). \tag{3.10}$$

Define the fundamental meromorphic function ϕ on $\mathcal{K}_n$ by

$$\phi(P,x) = \frac{y - zG_{n-1}(z,x)}{F_n(z,x)} = \frac{zH_n(z,x)}{y + zG_{n-1}(z,x)}, \quad x \in \mathbb{R},\ P=(z,y) \in \mathcal{K}_n, \tag{3.11}$$

with divisor $(\phi(P,x))$ (cf. the notation for divisors introduced in (A.30) and (A.31)) given by

$$(\phi(P,x)) = \mathcal{D}_{P_0\underline{\hat{\nu}}(x)} - \mathcal{D}_{P_\infty\underline{\hat{\mu}}(x)}, \tag{3.12}$$

where we abbreviated

$$\underline{\hat{\mu}}(x) = (\hat{\mu}_1(x),\dots,\hat{\mu}_n(x)), \quad \underline{\hat{\nu}}(x) = (\hat{\nu}_1(x),\dots,\hat{\nu}_n(x)). \tag{3.13}$$

The stationary Baker–Akhiezer function

$$\Psi(P,x,x_0) = \begin{pmatrix} \psi_1(P,x,x_0) \\ \psi_2(P,x,x_0) \end{pmatrix}, \tag{3.14}$$

is defined on $\mathcal{K}_n \setminus \{P_\infty\}$ by

$$\psi_1(P,x,x_0) = \exp\left(-\frac{i}{2}(u(x)-u(x_0)) + i\int_{x_0}^{x} dx'\,\phi(P,x')\right), \tag{3.15}$$

$$\psi_2(P,x,x_0) = -\psi_1(P,x,x_0)\phi(P,x), \tag{3.16}$$

$$(x,x_0) \in \mathbb{R}^2,\ P \in \mathcal{K}_n \setminus \{P_\infty\}.$$

We summarize the fundamental properties of ϕ and Ψ in the following lemma.

Lemma 3.1. *Assume* (3.7), $P=(z,y) \in \mathcal{K}_n \setminus \{P_\infty\}$, *and let* $(z,x,x_0) \in \mathbb{C}\times\mathbb{R}^2$. *Then* ϕ *satisfies*

$$\phi(P,x)^2 - i\phi_x(P,x) = z + u_x(x)\phi(P,x), \tag{3.17}$$

$$\phi(P,x) + \phi(P^*,x) = -2z\frac{G_{n-1}(z,x)}{F_n(z,x)}, \tag{3.18}$$

$$\phi(P,x)\phi(P^*,x) = -z\frac{H_n(z,x)}{F_n(z,x)}, \tag{3.19}$$

$$\phi(P,x) - \phi(P^*,x) = 2\frac{y(P)}{F_n(z,x)}, \tag{3.20}$$

while Ψ *is meromorphic on* $\mathcal{K}_n \setminus \{P_\infty\}$ *and fulfills*

$$\Psi_x(P,x,x_0) = U(z,x)\Psi(P,x,x_0), \tag{3.21}$$

$$-\frac{y(P)}{z}\Psi(P,x,x_0) = V_n(z,x)\Psi(P,x,x_0), \tag{3.22}$$

$$\psi_1(P,x,x_0)=\left(\frac{F_n(z,x)}{F_n(z,x_0)}\right)^{1/2}\exp\left(iy(P)\int_{x_0}^{x}\frac{dx'}{F_n(z,x')}\right), \tag{3.23}$$

$$\psi_1(P,x,x_0)\psi_1(P^*,x,x_0)=\frac{F_n(z,x)}{F_n(z,x_0)}, \tag{3.24}$$

$$\psi_2(P,x,x_0)\psi_2(P^*,x,x_0)=-z\frac{H_n(z,x)}{F_n(z,x_0)}, \tag{3.25}$$

$$\psi_1(P,x,x_0)\psi_2(P^*,x,x_0)+\psi_1(P^*,x,x_0)\psi_2(P,x,x_0)=2z\frac{G_{n-1}(z,x)}{F_n(z,x_0)}, \tag{3.26}$$

$$\psi_1(P,x,x_0)\psi_2(P^*,x,x_0)-\psi_1(P^*,x,x_0)\psi_2(P,x,x_0)=2\frac{y(P)}{F_n(z,x_0)}. \tag{3.27}$$

Moreover, u satisfies the following trace relations

$$u(x)=i\ln\left((-1)^n\alpha^{-1}\prod_{j=1}^{n}\mu_j(x)\right), \tag{3.28a}$$

$$u(x)=-i\ln\left((-1)^n\beta^{-1}\prod_{j=1}^{n}\nu_j(x)\right), \tag{3.28b}$$

where (cf. (3.6)*)*

$$\alpha\beta=\prod_{m=1}^{2n}E_m\neq 0. \tag{3.29}$$

Proof. Equation (3.17) follows using the definition (3.11) of ϕ as well as relations (2.4). The other relations, (3.18)–(3.20), are easy consequences of $y(P^*)=-y(P)$ in addition to (2.4).

By (3.15) and (3.16), Ψ is meromorphic away from the poles $\hat\mu_j(x')$ of $\phi(\,\cdot\,,x')$. By (2.4a), (3.11), and (3.9a),

$$i\phi(P,x')\underset{P\to\hat\mu_j(x')}{=}-\frac{d}{dx'}\ln(F_n(z,x'))\big|_{z=\mu_j(x')}+O(1) \tag{3.30}$$

and hence ψ_1 is meromorphic on $\mathcal{K}_n\setminus\{P_\infty\}$ by (3.15). Since ϕ is meromorphic on $\mathcal{K}_n$, ψ_2 is meromorphic on $\mathcal{K}_n\setminus\{P_\infty\}$. The remaining properties of Ψ can be verified by using the definition (3.14)–(3.16) as well as the relations (3.17)–(3.20). In particular, equation (3.23) follows by inserting the definition of ϕ, (3.11), into (3.15), using (2.4a).

Finally, the trace relation (3.28a) follows by considering the constant term in F_n in (2.15) combined with (2.22) and (3.8). Equation (3.28b) can be deduced in a similar way studying the polynomial H_n. ☐

Next, we derive differential equations, (i.e., Dubrovin-type equations) for μ_j and ν_j. For this purpose, and for the theta function representations to follow in Theorem 3.4 we need to assume that $\mathcal{K}_n$ is nonsingular. Summing up all our conditions on $\{E_m\}_{m=0,\dots,2n}\subset\mathbb{C}$ we thus assume

$$E_0=0,\quad E_m\neq E_n \text{ for } m\neq n,\ m,n=0,1,\dots,2n \tag{3.31}$$

for the rest of this section.

Lemma 3.2. *Assume* (3.31) *and suppose that the zeros* $\{\mu_j(x)\}_{j=1,\dots,n}$ *of* $F_n(\,\cdot\,,x)$ *remain distinct for* $x\in\Omega$, *where* $\Omega\subseteq\mathbb{R}$ *is an open interval. Then* $\{\mu_j(x)\}_{j=1,\dots,n}$ *satisfies the following first-order system of differential equations*

$$\mu_{j,x}(x)=-2i\frac{y(\hat\mu_j(x))}{\prod_{\substack{\ell=1\\ \ell\neq j}}^n(\mu_j(x)-\mu_\ell(x))},\quad j=1,\dots,n,\, x\in\Omega, \tag{3.32}$$

with initial conditions

$$\hat\mu_j(x_0)\in\mathcal{K}_n,\quad j=1,\dots,n \tag{3.33}$$

for some fixed $x_0\in\Omega$. *The initial value problem* (3.32), (3.33) *has a unique solution* $\{\hat\mu_j(x)\}_{j=1,\dots,n}$ *such that*

$$\hat\mu_j(x)\in C^\infty(\Omega,\mathcal{K}_n),\quad j=1,\dots,n. \tag{3.34}$$

For the zeros $\{\nu_j(x)\}_{j=1,\dots,n}$ *of* $H_n(\,\cdot\,,x)$ *identical statements hold with* μ *replaced by* ν. *In particular,* $\{\nu_j(x)\}_{j=1,\dots,n}$ *satisfies*

$$\nu_{j,x}(x)=-2i\frac{y(\hat\nu_j(x))}{\prod_{\substack{\ell=1\\ \ell\neq j}}^n(\nu_j(x)-\nu_\ell(x))},\quad j=1,\dots,n,\, x\in\Omega. \tag{3.35}$$

Proof. We only prove equation (3.32) since the proof of (3.35) follows in an identical manner. Inserting $z=\mu_j$ into equation (2.4a), one concludes from (3.9a)

$$F_{n,x}(\mu_j)=-\mu_{j,x}\prod_{\substack{\ell=1\\ \ell\neq j}}^n(\mu_j-\mu_\ell)=-2i\mu_jG_{n-1}(\mu_j)=2iy(\hat\mu_j), \tag{3.36}$$

proving (3.32). The smoothness assertion (3.34) is clear as long as $\hat\mu_j(x)$ stays away from the branch points $(E_m,0)$. In case $\hat\mu_j(x)$ hits such a branch point, one can use the local chart around $(E_m,0)$ (with local coordinate $\zeta=\sigma(z-E_m)^{1/2}$, $\sigma=\pm1$) to verify (3.34). ☐

Remark 3.3. If $\alpha=0$ (as in Liouville-type equations (2.51)), the fact that $f_n(x)=(-1)^n\prod_{j=1}^n\mu_j(x)=0$ forces (at least) one $\hat\mu_{j_0}(x)$ to coincide with $P_0=(0,0)$ and hence to be x-independent. Similarly, if $\alpha=\beta=0$, then $\hat\mu_{j_1}(x)=\hat\nu_{j_2}(x)=P_0$ for some $j_1,j_2\in\{1,\dots,n\}$.

We end this section by deriving representations of the functions ϕ, Ψ as well as u, in terms of the Riemann theta function θ of $\mathcal{K}_n$. We will be relatively brief in our exposition and refer to [6], Chs. 2–4, [22]–[24], [26]–[28], [45], [54], [55], [57], and [64], Ch. II for the general algebro-geometric approach to integrable soliton equations. We freely use the notation established in Appendix A.

We choose an arbitrary but fixed base point Q_0 on $\mathcal{K}_n\setminus\{P_0,P_\infty\}$. Let $\omega^{(3)}_{P_\infty,P_0}$ be a normal differential of the third kind (cf. (A.20) and (A.21)) with simple poles at P_∞ and P_0 with residues 1 and -1, respectively. Thus

$$\omega^{(3)}_{P_\infty,P_0}\underset{\zeta\to0}{=}(\zeta^{-1}+O(1))d\zeta \text{ as } P\to P_\infty, \tag{3.37a}$$

$$\omega^{(3)}_{P_\infty,P_0}\underset{\zeta\to0}{=}(-\zeta^{-1}+O(1))d\zeta \text{ as } P\to P_0, \tag{3.37b}$$

where

$$\zeta=\sigma/z^{1/2} \text{ for } P \text{ near } P_\infty,\quad \zeta=\sigma z^{1/2} \text{ for } P \text{ near } P_0,\quad \sigma=\pm1, \tag{3.38}$$
$$z^{1/2}=|z^{1/2}|e^{\frac{i}{2}\arg(z)},\quad 0\le\arg(z)<2\pi.$$

Hence we find

$$\exp\Big(-\int_{Q_0}^{P}\omega^{(3)}_{P_\infty,P_0}\Big)\underset{\zeta\to 0}{=}e_{-1}\zeta^{-1}+e_0+O(\zeta)\text{ as }P\to P_\infty, \tag{3.39a}$$

$$\exp\Big(-\int_{Q_0}^{P}\omega^{(3)}_{P_\infty,P_0}\Big)\underset{\zeta\to 0}{=}d_1\zeta+O(\zeta^2)\text{ as }P\to P_0, \tag{3.39b}$$

for appropriate coefficients e_{-1}, e_0, and d_1 depending on the base point Q_0. In the remainder of this paper we choose $Q_0\in\mathcal{K}_n\setminus\{P_0,P_\infty\}$ such that

$$e_{-1}=1 \tag{3.40}$$

in (3.39a). Let $\omega^{(2)}_{P_\infty,0}$ be the normalized differential of the second kind holomorphic on $\mathcal{K}_n\setminus\{P_\infty\}$ such that (cf. (A.16)–(A.18))

$$\omega^{(2)}_{P_\infty,0}\underset{\zeta\to 0}{=}(\zeta^{-2}+O(1))d\zeta\text{ as }P\to P_\infty \tag{3.41}$$

and denote the vector of b-periods of $\omega^{(2)}_{P_\infty,0}/(2\pi i)$ by $\underline{U}^{(2)}_0$, that is,

$$\underline{U}^{(2)}_0=(U^{(2)}_{0,1},\dots,U^{(2)}_{0,n}),\quad U^{(0)}_{0,j}=\frac{1}{2\pi i}\int_{b_j}\omega^{(2)}_{P_\infty,0}=-2c_j(n),\quad j=1,\dots,n, \tag{3.42}$$

applying (A.18).

Theorem 3.4. *Assume* (3.31), (3.40), $P\in\mathcal{K}_n\setminus\{P_\infty\}$, *and* $x,x_0\in\Omega$, *where* $\Omega\subseteq\mathbb{R}$ *is an open interval. Suppose that* $\mathcal{D}_{\underline{\hat\mu}(x)}$, *or equivalently,* $\mathcal{D}_{\underline{\hat\nu}(x)}$ *is nonspecial for* $x\in\Omega$. *Then* $\phi(P,x)$ *admits the representation*

$$\begin{aligned}\phi(P,x)=&\frac{\theta(\Xi_{Q_0}-\underline{A}_{Q_0}(P_\infty)+\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat\mu}(x)}))}{\theta(\Xi_{Q_0}-\underline{A}_{Q_0}(P_\infty)+\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat\mu}(x)})+\underline{\Delta})}\times\\&\times\frac{\theta(\Xi_{Q_0}-\underline{A}_{Q_0}(P)+\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat\mu}(x)})+\underline{\Delta})}{\theta(\Xi_{Q_0}-\underline{A}_{Q_0}(P)+\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat\mu}(x)}))}\exp\Big(-\int_{Q_0}^{P}\omega^{(3)}_{P_\infty,P_0}\Big),\end{aligned} \tag{3.43}$$

with $\underline{\Delta}$ *a halfperiod defined as*

$$\underline{\Delta}=\underline{A}_{P_0}(P_\infty),\quad 2\underline{\Delta}=0\pmod{L_n}. \tag{3.44}$$

The components $\psi_j(P,x,x_0)$, $j=1,2$ *of the Baker–Akhiezer function* $\Psi(P,x,x_0)$ *read,*

$$\begin{aligned}\psi_1(P,x,x_0)=&\frac{\theta(\Xi_{Q_0}-\underline{A}_{Q_0}(P)+\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat\mu}(x)}))\theta(\Xi_{Q_0}-\underline{A}_{Q_0}(P_\infty)+\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat\mu}(x_0)}))}{\theta(\Xi_{Q_0}-\underline{A}_{Q_0}(P_\infty)+\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat\mu}(x)}))\theta(\Xi_{Q_0}-\underline{A}_{Q_0}(P)+\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat\mu}(x_0)}))}\times\\&\times\exp\Big(-i(x-x_0)\int_{Q_0}^{P}\omega^{(2)}_{P_\infty,0}\Big)\end{aligned} \tag{3.45}$$

and

$$\begin{aligned}&\psi_2(P,x,x_0)\\&=-\frac{\theta(\Xi_{Q_0}-\underline{A}_{Q_0}(P)+\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat\mu}(x)})+\underline{\Delta})}{\theta(\Xi_{Q_0}-\underline{A}_{Q_0}(P_\infty)+\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat\mu}(x)})+\underline{\Delta})}\frac{\theta(\Xi_{Q_0}-\underline{A}_{Q_0}(P_\infty)+\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat\mu}(x_0)}))}{\theta(\Xi_{Q_0}-\underline{A}_{Q_0}(P)+\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat\mu}(x_0)}))}\times\\&\quad\times\exp\Big(-i(x-x_0)\int_{Q_0}^{P}\omega^{(2)}_{P_\infty,0}-\int_{Q_0}^{P}\omega^{(3)}_{P_\infty,P_0}\Big).\end{aligned} \tag{3.46}$$

The Abel map linearizes the auxiliary divisors in the sense that

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\underline{\mu}}(x)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\underline{\mu}}(x_0)}) - 2i\underline{c}(n)(x - x_0), \tag{3.47a}$$

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\underline{\nu}}(x)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\underline{\nu}}(x_0)}) - 2i\underline{c}(n)(x - x_0), \tag{3.47b}$$

where $\underline{c}(n)$ *is defined in* (A.7). *The stationary solution* $u(x)$ *of* $\mathrm{sGmKdV}_n(u) = 0$ *finally reads*

$$\begin{aligned} u(x) &= u(x_0) \\ &\quad + 2i \ln\bigg(\frac{\theta(P_\infty, \hat{\underline{\mu}}(x), \underline{\Delta})\theta(P_\infty, \hat{\underline{\mu}}(x_0))}{\theta(P_\infty, \hat{\underline{\mu}}(x_0), \underline{\Delta})\theta(P_\infty, \hat{\underline{\mu}}(x))} \exp\big(-ie_0(x - x_0)\big)\bigg), \end{aligned} \tag{3.48}$$

abbreviating

$$\begin{aligned} \theta(P, \underline{Q}) &= \theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{Q}})), \\ \theta(P, \underline{Q}, \underline{\Delta}) &= \theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{Q}}) + \underline{\Delta}), \\ &\qquad\qquad \underline{Q} = (Q_1, \dots, Q_n) \in \sigma^n \mathcal{K}_n. \end{aligned} \tag{3.49}$$

Proof. Assume temporarily that

$$\mu_j(x) \neq \mu_{j'}(x),\ \nu_m(x) \neq \nu_{m'}(x) \text{ for } j \neq j',\ m \neq m', \text{ and } x \in \widetilde{\Omega} \subseteq \Omega, \tag{3.50}$$

where $\widetilde{\Omega}$ is an open interval. Denote the right-hand side of (3.43) by Φ. By the Riemann-Roch theorem (A.36), to prove that $\phi = \Phi$ it suffices to show that ϕ and Φ have the same poles and zeros as well as the same value at one point on $\mathcal{K}_n$. From the definition (3.11) of ϕ we conclude that it has simple zeros at $\hat{\underline{\nu}}(x)$ and P_0 and simple poles at $\hat{\underline{\mu}}(x)$ and P_∞. Note the linear equivalence $\mathcal{D}_{P_\infty \hat{\underline{\mu}}(x)} \sim \mathcal{D}_{P_0 \hat{\underline{\nu}}(x)}$, that is,

$$\underline{A}_{Q_0}(P_\infty) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\underline{\mu}}(x)}) = \underline{A}_{Q_0}(P_0) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\underline{\nu}}(x)}) \tag{3.51}$$

and hence

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\underline{\nu}}(x)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\underline{\mu}}(x)}) + \underline{\Delta}, \tag{3.52}$$

with

$$\underline{\Delta} = \underline{A}_{P_0}(P_\infty). \tag{3.53}$$

Since P_0 and P_∞ are branch points of $\mathcal{K}_n$, the right-hand side of (3.53) is a half-period, proving (3.44). By Theorem A.3 one concludes that Φ and ϕ share the same zeros and poles. Thus Φ/ϕ equals a constant. From (3.11) one infers that

$$\phi(P, x) \underset{\zeta \to 0}{=} \zeta^{-1} + O(1) \text{ as } P \to P_\infty, \tag{3.54}$$

and using the leading term in the expansion (3.39a) with $e_{-1} = 1$ according to (3.40) then proves $\phi = \Phi$ subject to (3.50).

Next we turn to the proof of (3.45). One infers from (3.24) that ψ_1 has first-order zeros at $\hat{\underline{\mu}}(x)$, and first-order poles at $\hat{\underline{\mu}}(x_0)$. Thus, by Theorem A.3, ψ_1 contains a factor $\theta(P, \hat{\underline{\mu}}(x))/\theta(P, \hat{\underline{\mu}}(x_0))$. A high-energy expansion (cf. (B.6)) then reveals that

$$\int_{x_0}^{x} dx' \, \frac{iy(P)}{F_n(z, x')} \underset{\zeta \to 0}{=} i(x - x_0)\zeta^{-1} + O(1) \text{ as } P \to P_\infty, \tag{3.55}$$

and upon inserting (3.11) into (3.15) one concludes that ψ_1 has an essential singularity at P_∞ precisely of the type (3.55). Since

$$\int_{P_0}^{P} \omega^{(2)}_{P_\infty,0} \underset{\zeta\to 0}{=} -\zeta^{-1} + O(1) \text{ as } P \to P_\infty, \tag{3.56}$$

the Riemann-Roch-type uniqueness result, Lemma A.5, applied to ψ_1 then yields (3.45) subject to (3.50). The corresponding expression (3.46) for ψ_2 is then obvious from (3.16), (3.43), and (3.45).

Next we prove the linearization property (3.47a) using Lemma 3.2 (still assuming (3.50)). Equation (3.47b) then follows from (3.47a) and (3.51). From

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x)}) = \bigg(\sum_{j=1}^{n} \int_{Q_0}^{\hat{\mu}_j(x)} \underline{\omega}\bigg) \pmod{L_n} \tag{3.57}$$

and the Dubrovin equations (3.32) one infers

$$\begin{aligned}\frac{d}{dx}\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x)}) &= \sum_{j=1}^{n} \mu_j'(x) \sum_{k=1}^{n} \underline{c}(k) \frac{\mu_j(x)^{k-1}}{y(\hat{\mu}_j(x))} \\ &= -2i \sum_{j,k=1}^{n} \underline{c}(k) \frac{\mu_j(x)^{k-1}}{\prod_{\substack{\ell=1\\ \ell\neq j}}^{n}(\mu_j(x)-\mu_\ell(x))}, \quad x \in \widetilde{\Omega}.\end{aligned} \tag{3.58}$$

Lagrange's interpolation formula (cf. Theorem B.1)

$$\sum_{j=1}^{n} \mu_j^{k-1} \prod_{\substack{\ell=1\\ \ell\neq j}}^{n} (\mu_j - \mu_\ell)^{-1} = \delta_{k,n}, \quad k = 1, \dots, n \tag{3.59}$$

then yields

$$\frac{d}{dx}\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x)}) = -2i\underline{c}(n) = i\underline{U}_0, \quad x \in \widetilde{\Omega} \tag{3.60}$$

and hence (3.47a) subject to (3.50). Moreover, applying (A.18) establishes the relation between the left-hand side of (3.60) and the vector $\underline{U}_0$ of b-periods of $\omega^{(2)}_{P_\infty,0}/(2\pi i)$ introduced in (3.42).

To prove formula (3.48) for $u(x)$ we employ a high-energy expansion for $\phi(P,x)$ and the Riccati equation (3.17). First we conclude from (A.10)

$$\underline{\omega} \underset{\zeta\to 0}{=} (-2\underline{c}(n) + O(\zeta^2))d\zeta \text{ as } P \to P_\infty, \tag{3.61}$$

and hence

$$\begin{aligned}\underline{A}_{Q_0}(P) = \int_{Q_0}^{P} \underline{\omega} \pmod{L_n} &\underset{\zeta\to 0}{=} \underline{A}_{Q_0}(P_\infty) - 2\underline{c}(n)\zeta + O(\zeta^3) \\ &= \underline{A}_{Q_0}(P_\infty) + \underline{U}_0^{(2)}\zeta + O(\zeta^2),\end{aligned} \tag{3.62}$$

using (A.18) and (3.42). Expanding the subsequent ratios of Riemann theta functions in (3.43) one finds

$$\frac{\theta(P,\underline{\hat{\mu}}(x))}{\theta(P_\infty,\underline{\hat{\mu}}(x))}$$

$$\underset{\zeta\to 0}{=} 1-\frac{\sum_{j=1}^n U_{0,j}^{(2)}\frac{\partial}{\partial w_j}\theta(\underline{\Xi}_{Q_0}-\underline{A}_{Q_0}(P_\infty)+\underline{w}+\underline{\alpha}_{Q_0}(\underline{\hat{\mu}}(x)))|_{\underline{w}=0}}{\theta(\underline{\Xi}_{Q_0}-\underline{A}_{Q_0}(P_\infty)+\underline{\alpha}_{Q_0}(\underline{\hat{\mu}}(x)))}\zeta+O(\zeta^3), \tag{3.63}$$

and the same formula for the theta function ratio involving the additional half-period $\underline{\Delta}$. Here $\sum_{j=1}^n U_{0,j}^{(2)}\partial/\partial w_j$ denotes the directional derivative in $\underline{U}_0^{(2)}$-direction. Given (3.42) we may write

$$\sum_{j=1}^n iU_{0,j}^{(2)}\frac{\partial}{\partial w_j}\theta(\underline{\Xi}_{Q_0}-\underline{A}_{Q_0}(P_\infty)+\underline{w}+\underline{\alpha}_{Q_0}(\underline{\hat{\mu}}(x_0))+i\underline{U}_0^{(2)}(x-x_0))|_{\underline{w}=0}$$
$$=\frac{d}{dx}\theta(\underline{\Xi}_{Q_0}-\underline{A}_{Q_0}(P_\infty)+\underline{\alpha}_{Q_0}(\underline{\hat{\mu}}(x_0))+i\underline{U}_0^{(2)}(x-x_0)), \tag{3.64}$$

and hence obtain from (3.63),

$$\frac{\theta(P,\underline{\hat{\mu}}(x))}{\theta(P_\infty,\underline{\hat{\mu}}(x))}\underset{\zeta\to 0}{=}1+i\frac{d}{dx}\ln(\theta(P_\infty,\underline{\hat{\mu}}(x)))\zeta+O(\zeta^3) \text{ as } P\to P_\infty, \tag{3.65}$$

and the identical formula for the theta function ratio involving $\underline{\Delta}$. Together with (3.39a) this shows that

$$\phi(P,x)\underset{\zeta\to 0}{=}\zeta^{-1}+i\frac{d}{dx}\ln\left(\frac{\theta(P_\infty,\underline{\hat{\mu}}(x),\underline{\Delta})}{\theta(P_\infty,\underline{\hat{\mu}}(x))}\right)+e_0+O(\zeta) \text{ as } P\to P_\infty. \tag{3.66}$$

Expanding the Riccati equation (3.17) for P near P_∞ then yields

$$u_x=2i\frac{d}{dx}\ln\left(\frac{\theta(P_\infty,\underline{\hat{\mu}}(x),\underline{\Delta})}{\theta(P_\infty,\underline{\hat{\mu}}(x))}\right)+2e_0, \quad x\in\widetilde{\Omega} \tag{3.67}$$

and hence (3.48), assuming (3.50). The extension of all these results from $x\in\widetilde{\Omega}$ to $x\in\Omega$ then simply follows from the continuity of $\underline{\alpha}_{P_0}$ and the hypothesis of $\mathcal{D}_{\underline{\hat{\mu}}(x)}$ being nonspecial for $x\in\Omega$.

To verify that $u(x)$ indeed satisfies nth stationary sGmKdV equation it suffices to note

$$(V_{n,x}-[U,V_n])\Psi=\frac{d}{dx}(V_n\Psi+\frac{y}{z}\Psi)=0, \tag{3.68}$$

applying (3.21) and (3.22) repeatedly. Since $P\in\mathcal{K}_n\setminus\{P_\infty\}$ is arbitrary, one concludes $V_{n,x}-[U,V_n]=0$ which completes the proof. ☐

While this approach to the algebro-geometric solutions of the sGmKdV hierarchy resembles that of the AKNS hierarchy (which includes the mKdV hierarchy) in many ways (cf. [46]), there are, however, some characteristic differences. In particular, the branch point $P_0=(0,0)$ is an unusual necessity in the sG context and P_∞ (as in the KdV context) is a branch point as opposed to the AKNS case. This shows that sGmKdV curves are actually special KdV curves (with 0 a branch point).

4. The time-dependent sGmKdV formalism

In our final section we indicate how to employ the polynomial formalism of Sections 2 and 3 to derive the algebro-geometric solutions of the sGmKdV hierarchy.

The basic problem in the analysis of algebro-geometric solutions of completely integrable soliton equations is to solve the time-dependent rth sGmKdV equation with initial data a stationary solution of the nth equation in the hierarchy. More precisely, consider a solution $u^{(0)}(x)$ of the nth stationary sGmKdV equation, that is,

$$g_{n-1,x} - i(\beta e^{iu^{(0)}} - \alpha e^{-iu^{(0)}}) = 0, \quad n \in \mathbb{N}, \tag{4.1}$$

subject to the constraint

$$\alpha\beta = P_{2n}(0). \tag{4.2}$$

Let $r \in \mathbb{N}_0$ be fixed (and independent of n). We want to find a solution u of $\mathrm{sGmKdV}_r(u) = 0$ with $u(x, t_{0,r}) = u^{(0)}(x)$. To emphasize that the integration constants that enter in the definitions of the sGmKdV hierarchy may be different for the stationary initial data and the time-dependent equation, we indicate this by adding a tilde on all the time-dependent quantities. Hence we shall employ the notation $\widetilde V_r$, $\widetilde F_r$, $\widetilde G_{r-1}$, $\widetilde H_r$, $\tilde f_\ell$, $\tilde g_\ell$, $\tilde h_\ell$, $\tilde c_\ell$, $\tilde\alpha$, $\tilde\beta$ in order to distinguish them from V_n, F_n, G_{n-1}, H_n, f_ℓ, g_ℓ, h_ℓ, c_ℓ, α, β, in the following.

Thus we are seeking a solution u of

$$\mathrm{sGmKdV}_r(u) = u_{xt_r} + 2i\tilde g_{r-1,x} - i(\tilde\beta e^{iu} - \tilde\alpha e^{-iu}) = 0, \quad u(x, t_{0,r}) = u^{(0)}(x). \tag{4.3}$$

Actually, since we need to rely on the isospectral property of the sGmKdV flows we go a step further and assume (4.1) not only at $t_r = t_{0,r}$ but for all $t_r \in \mathbb{R}$. Together with (4.3) we are thus led to start with

$$U_{t_r} - \widetilde V_{r,x} + [U, \widetilde V_r] = 0, \quad V_{n,x} = [U, V_n], \quad (x, t_r) \in \mathbb{R}^2. \tag{4.4}$$

Explicitly, this yields for each entry of V_n and $\widetilde V_r$ (cf. (2.4), (2.40))

$$F_{n,x} = -iu_x F_n - 2izG_{n-1}, \tag{4.5a}$$

$$H_{n,x} = iu_x H_n + 2izG_{n-1}, \tag{4.5b}$$

$$G_{n-1,x} = i(H_n - F_n). \tag{4.5c}$$

and

$$u_{xt_r} = -2i\widetilde G_{r-1,x} - 2(\widetilde H_r - \widetilde F_r), \tag{4.6a}$$

$$\widetilde F_{r,x} = -iu_x \widetilde F_r - 2iz\widetilde G_{r-1}, \tag{4.6b}$$

$$\widetilde H_{r,x} = iu_x \widetilde H_r + 2iz\widetilde G_{r-1}. \tag{4.6c}$$

At this point we proceed assuming there is a solution of (4.3), or rather, equations (4.5) and (4.6). Once explicit expressions for algebro-geometric solutions are obtained, it is a straightforward manner to verify directly that they satisfy (4.4) (resp., (4.5) and (4.6)).

It will turn out later (cf. Remark 4.3) that $\alpha, \beta, \tilde\alpha, \tilde\beta$ in (4.1) and (4.3) are not independent of each other but constrained by

$$f_n \tilde h_r = \tilde f_r h_n \text{ or } \alpha\tilde\beta = \tilde\alpha\beta. \tag{4.7}$$

Throughout this section we assume (4.4) (resp. (4.5) and (4.6)) and in accordance with Section 2 we suppose that F_n, G_{n-1}, H_n, $\widetilde F_r$, $\widetilde G_{r-1}$, and $\widetilde H_r$ have the polynomial structure (2.15), (3.8), with $t_r \in \mathbb{R}$ the additional deformation parameter in u. We also recall our conventions in (2.20) and (2.27) and set

$$\widetilde G_{-1}(x, t_0) = 0, \quad \tilde g_{-1}(x, t_0) = 0 \text{ for } r = 0. \tag{4.8}$$

Hence (2.3)–(2.38) apply to F_n, G_{n-1}, H_n, f_j, g_j, and h_j and (2.15), (2.19)–(2.25), (2.26) for $j = 0, \dots, n-2$, (2.34)–(2.44), with $n \to r$, $c_\ell \to \tilde{c}_\ell$, $\alpha \to \tilde{\alpha}$, $\beta \to \tilde{\beta}$ apply to $\widetilde{F}_r$, $\widetilde{G}_{r-1}$, $\widetilde{H}_r$, $\tilde{f}_j$, $\tilde{g}_j$, and $\tilde{h}_j$. In particular, the fundamental identity (2.7) holds,

$$z^2 G_{n-1}(z,x,t_r)^2 + zF_n(z,x,t_r)H_n(z,x,t_r) = R_{2n+1}(z), \tag{4.9}$$

with

$$F_n(z,x,t_r) = \prod_{j=1}^n (z - \mu_j(x,t_r)), \quad H_n(z,x,t_r) = \prod_{j=1}^n (z - \nu_j(x,t_r)), \tag{4.10}$$

assuming (3.7) for the remainder of this section. Hence the hyperelliptic curve $\mathcal{K}_n$ is still given by

$$\mathcal{K}_n \colon \mathcal{F}_n(z,y) = y^2 - R_{2n+1}(z) = 0. \tag{4.11}$$

In analogy to equations (3.9) and (3.11) we define

$$\hat{\mu}_j(x,t_r) = (\mu_j(x,t_r), -\mu_j(x,t_r)G_{n-1}(\mu_j(x,t_r),x,t_r)) \in \mathcal{K}_n, \quad j = 1, \dots, n, \tag{4.12a}$$

$$\hat{\nu}_j(x,t_r) = (\nu_j(x,t_r), \nu_j(x,t_r)G_{n-1}(\nu_j(x,t_r),x,t_r)) \in \mathcal{K}_n, \quad j = 1, \dots, n. \tag{4.12b}$$

and

$$\phi(P,x,t_r) = \frac{y(P) - zG_{n-1}(z,x,t_r)}{F_n(z,x,t_r)} = \frac{zH_n(z,x,t_r)}{y(P) + zG_{n-1}(z,x,t_r)}, \tag{4.13}$$
$$(x,t_r) \in \mathbb{R}^2,\ P = (z,y) \in \mathcal{K}_n \setminus \{P_\infty\}$$

Hence the divisor $(\phi(P,x,t_r))$ of $\phi(P,x,t_r)$ reads

$$(\phi(P,x,t_r)) = \mathcal{D}_{P_0 \underline{\hat{\nu}}(x,t_r)} - \mathcal{D}_{P_\infty \underline{\hat{\mu}}(x,t_r)}, \tag{4.14}$$

with

$$\underline{\hat{\mu}}(x,t_r) = (\hat{\mu}_1(x,t_r), \dots, \hat{\mu}_n(x,t_r)), \quad \underline{\hat{\nu}}(x,t_r) = (\hat{\nu}_1(x,t_r), \dots, \hat{\nu}_n(x,t_r)). \tag{4.15}$$

The time-dependent Baker–Akhiezer function

$$\Psi(P,x,x_0,t_r,t_{0,r}) = \begin{pmatrix} \psi_1(P,x,x_0,t_r,t_{0,r}) \\ \psi_2(P,x,x_0,t_r,t_{0,r}) \end{pmatrix} \tag{4.16}$$

is defined by

$$\psi_1(P,x,x_0,t_r,t_{0,r}) = \exp\bigg(-\frac{i}{2}(u(x,t_r) - u(x_0,t_r)) \tag{4.17}$$
$$+ i \int_{x_0}^x dx'\, \phi(P,x',t_r) - \int_{t_{0,r}}^{t_r} ds\, \big(\frac{1}{z}\widetilde{F}_r(z,x_0,s)\phi(P,x_0,s) + \widetilde{G}_{r-1}(z,x_0,s)\big)\bigg),$$

$$\psi_2(P,x,x_0,t_r,t_{0,r}) = -\psi_1(P,x,x_0,t_r,t_{0,r})\phi(P,x,t_r), \tag{4.18}$$
$$P \in \mathcal{K}_n \setminus \{P_\infty\},\ (x,x_0,t_r,t_{0,r}) \in \mathbb{R}^4.$$

The properties of ϕ can now be summarized as follows.

Lemma 4.1. *Assume* (3.7), $P = (z,y) \in \mathcal{K}_n \setminus \{P_\infty\}$, *and let* $(z,x,x_0,t_r,t_{0,r}) \in \mathbb{C} \times \mathbb{R}^4$. *Then the function* ϕ *satisfies*

$$\phi(P,x,t_r)^2 - i\phi_x(P,x,t_r) = z + u_x(x,t_r)\phi(P,x,t_r), \tag{4.19}$$

$$\phi_{t_r}(P,x,t_r) = \widetilde{F}_r(z,x,t_r) - \widetilde{H}_r(z,x,t_r) + \frac{i}{z}(\phi(P,x,t_r)\widetilde{F}_r(z,x,t_r))_x, \tag{4.20}$$

$$\phi(P,x,t_r)+\phi(P^*,x,t_r)=-2z\frac{G_{n-1}(z,x,t_r)}{F_n(z,x,t_r)}, \tag{4.21}$$

$$\phi(P,x,t_r)\phi(P^*,x,t_r)=-z\frac{H_n(z,x,t_r)}{F_n(z,x,t_r)}, \tag{4.22}$$

$$\phi(P,x,t_r)-\phi(P^*,x,t_r)=2\frac{y(P)}{F_n(z,x,t_r)}. \tag{4.23}$$

Proof. Equation (4.19) follows as in Lemma 3.1. To prove (4.20) we first observe that

$$\big(\partial_x+i(2\phi-u_x)\big)\big(\phi_{t_r}+(\widetilde H_r-\widetilde F_r)-\frac{i}{z}(\phi\widetilde F_r)_x\big)=0 \tag{4.24}$$

using (4.19) and relations (4.6) repeatedly. Thus,

$$\phi_{t_r}+(\widetilde H_r-\widetilde F_r)-\frac{i}{z}(\phi\widetilde F_r)_x=C\exp\bigg(-i\int^x dx'\,(2\phi-u_{x'})\bigg), \tag{4.25}$$

where the left-hand side is meromorphic at P_∞, while the right-hand side is meromorphic at P_∞ only if $C=0$, which proves (4.20). Equations (4.21)–(4.23) are proved as in Lemma 3.1. □

Next we prove that relations (4.5) and (4.6) determine the time-development of F_n, G_n, and H_n.

Lemma 4.2. *Suppose* (3.7) *and let* $(z,x,x_0,t_r,t_{0,r})\in\mathbb{C}\times\mathbb{R}^4$. *Then*

$$F_{n,t_r}=2(G_{n-1}\widetilde F_r-\widetilde G_{r-1}F_n), \tag{4.26a}$$

$$zG_{n-1,t_r}=(F_n\widetilde H_r-\widetilde F_rH_n), \tag{4.26b}$$

$$H_{n,t_r}=2(\widetilde G_{r-1}H_n-G_{n-1}\widetilde H_r). \tag{4.26c}$$

Equations (4.26) *are equivalent to*[1]

$$V_{n,t_r}=[\widetilde V_r,V_n]. \tag{4.27}$$

Proof. We prove (4.26a) by using (4.23) which shows that

$$(\phi(P)-\phi(P^*))_{t_r}=-2y(P)\frac{F_{n,t_r}}{F_n^2}. \tag{4.28}$$

However, the left-hand side of (4.28) also equals

$$\phi(P)_{t_r}-\phi(P^*)_{t_r}=\frac{4y(P)}{F_n^2}(\widetilde G_{r-1}F_n-\widetilde F_rG_{n-1}), \tag{4.29}$$

using (4.20), (4.23), (4.5a), and (4.6a). Combining (4.28) and (4.29) proves (4.26a). Similarly, to prove (4.26b), we use (4.21) to write

$$(\phi(P)+\phi(P^*))_{t_r}=-\frac{2z}{F_n^2}\big(G_{n-1,t_r}F_n-G_{n-1}F_{n,t_r}\big). \tag{4.30}$$

Here the left-hand side can be expressed as

$$\phi(P)_{t_r}+\phi(P^*)_{t_r}=-2\widetilde H_r+2\frac{H_n}{F_n}\widetilde F_r+4z(\frac{G_{n-1}}{F_n})^2\widetilde F_r-4z\frac{G_{n-1}}{F_n}\widetilde G_{r-1}, \tag{4.31}$$

[1] Equation (4.27) is almost equivalent with the consistency requirement that $V_{n,xt_r}=V_{n,t_rx}$. Using (4.4) we find $[\tilde V_r,V_n]_x=V_{n,t_rx}$. This shows that $[\tilde V_r,V_n]$ equals V_{n,t_r} up to a 2×2 matrix depending on z and t_r, but not on x. (4.27) shows that this matrix is identically zero.

using (4.5a) and (4.6a). Combining (4.30) and (4.31), using (4.26a), proves (4.26b). Finally, (4.26c) follows by differentiating (4.9), that is, $(zG_{n-1})^2+zF_nH_n = R_{2n+1}$, with respect to t_r, and using (4.26a) and (4.26b). $\square$

Remark 4.3. Taking $z=0$ in (4.26b), one infers the compatibility relation

$$f_n(x)\tilde{h}_r(x) = \tilde{f}_r(x)h_n(x), \tag{4.32}$$

or equivalently, using $f_n = \alpha e^{-iu}$, $\tilde{f}_r = \tilde{\alpha}e^{-iu}$, $h_n = \beta e^{iu}$, and $\tilde{h}_r = \tilde{\beta}e^{iu}$, one obtains the constraint

$$\alpha\tilde{\beta} = \tilde{\alpha}\beta. \tag{4.33}$$

Lemmas 4.1 and 4.2 permit us to characterize Ψ.

Lemma 4.4. *Assume* (3.7), $P=(z,y)\in\mathcal{K}_n\setminus\{P_\infty\}$, *and let* $(x,x_0,t_r,t_{0,r})\in\mathbb{R}^4$. *Then the Baker–Akhiezer function* Ψ *is meromorphic on* $\mathcal{K}_n\setminus\{P_\infty\}$ *and satisfies*

$$\Psi_x(P,x,x_0,t_r,t_{0,r}) = U(z,x,t_r)\Psi(P,x,x_0,t_r,t_{0,r}), \tag{4.34}$$

$$-\frac{y(P)}{z}\Psi(P,x,x_0,t_r,t_{0,r}) = V_n(z,x,t_r)\Psi(P,x,x_0,t_r,t_{0,r}), \tag{4.35}$$

$$\Psi_{t_r}(P,x,x_0,t_r,t_{0,r}) = \widetilde{V}_r(z,x,t_r)\Psi(P,x,x_0,t_r,t_{0,r}), \tag{4.36}$$

$$\begin{aligned}&\psi_1(P,x,x_0,t_r,t_{0,r}) \\ &= \left(\frac{F_n(z,x,t_r)}{F_n(z,x_0,t_{0,r})}\right)^{1/2}\exp\left(iy(P)\int_{x_0}^{x}\frac{dx'}{F_n(z,x',t_r)} - \frac{y(P)}{z}\int_{t_{0,r}}^{t_r}ds\,\frac{\widetilde{F}_r(z,x_0,s)}{F_n(z,x_0,s)}\right),\end{aligned} \tag{4.37}$$

$$\psi_1(P,x,x_0,t_r,t_{0,r})\psi_1(P^*,x,x_0,t_r,t_{0,r}) = \frac{F_n(z,x,t_r)}{F_n(z,x_0,t_{0,r})}, \tag{4.38}$$

$$\psi_2(P,x,x_0,t_r,t_{0,r})\psi_2(P^*,x,x_0,t_r,t_{0,r}) = -z\frac{H_n(z,x,t_r)}{F_n(z,x_0,t_{0,r})}, \tag{4.39}$$

$$\begin{aligned}&\psi_1(P,x,x_0,t_r,t_{0,r})\psi_2(P^*,x,x_0,t_r,t_{0,r}) \\ &\quad + \psi_1(P^*,x,x_0,t_r,t_{0,r})\psi_2(P,x,x_0,t_r,t_{0,r}) = 2z\frac{G_{n-1}(z,x,t_r)}{F_n(z,x_0,t_{0,r})},\end{aligned} \tag{4.40}$$

$$\begin{aligned}&\psi_1(P,x,x_0,t_r,t_{0,r})\psi_2(P^*,x,x_0,t_r,t_{0,r}) \\ &\quad - \psi_1(P^*,x,x_0,t_r,t_{0,r})\psi_2(P,x,x_0,t_r,t_{0,r}) = 2\frac{y(P)}{F_n(z,x_0,t_{0,r})}.\end{aligned} \tag{4.41}$$

Proof. Relations (4.34) and (4.35) follow as in Lemma 3.1, while the time evolution (4.36) is a consequence of the definition of Ψ in (4.17), (4.18) as well as (4.20), rewriting

$$\phi_{t_r} = \big(\frac{i}{z}\phi\widetilde{F}_r + i\widetilde{G}_{r-1} + \frac{1}{2}u_{t_r}\big)_x, \tag{4.42}$$

using (4.6a) and

$$\phi_{t_r} = -\widetilde{H}_r + \frac{1}{z}\widetilde{F}_r\phi^2 + 2\phi\widetilde{G}_{r-1}, \tag{4.43}$$

using (4.19) and (4.6b). To prove (4.37) we recall the definition (4.17), that is,

$$\psi_1(P,x,x_0,t_r,t_{0,r}) = \exp\bigg(-\frac{i}{2}(u(x,t_r)-u(x_0,t_r)) \tag{4.44}$$

$$
\begin{aligned}
&+ i\int_{x_0}^{x} dx'\,\phi(P,x',t_r) - \int_{t_{0,r}}^{t_r} ds\,\big(\frac{1}{z}\widetilde F_r(z,x_0,s)\phi(P,x_0,s) + \widetilde G_{r-1}(z,x_0,s)\big)\bigg),\\
&= \exp\bigg(iy(P)\int_{x_0}^{x} dx'\,\frac{1}{F_n(z,x',t_r)} - \frac{y(P)}{z}\int_{t_{0,r}}^{t_r} ds\,\frac{\widetilde F_r(z,x_0,s)}{F_n(z,x_0,s)}\\
&\quad - \frac{i}{2}(u(x,t_r)-u(x_0,t_r)) - iz\int_{x_0}^{x} dx'\,\frac{G_{n-1}(z,x',t_r)}{F_n(z,x',t_r)}\\
&\quad + \int_{t_{0,r}}^{t_r} ds\,\big(\frac{\widetilde F_r(z,x_0,s)}{F_n(z,x_0,s)}G_{n-1}(z,x_0,s) - \widetilde G_{r-1}(z,x_0,s)\big)\bigg),
\end{aligned}
$$

using (4.13). By first invoking (4.5a) and subsequently (4.26a), relation (4.37) follows. Evaluating (4.37) at the points P and P^* and multiplying the resulting expressions yields (4.38). The remaining statements are direct consequences of (4.21)–(4.23) and (4.37). ☐

We also note the following trace formulas, the t_r-dependent analog of (3.28).

Lemma 4.5. *Suppose* (3.7). *Then $u(x,t_r)$ satisfies*

$$u(x,t_r) = i\ln\bigg((-1)^n\alpha^{-1}\prod_{j=1}^{n}\mu_j(x,t_r)\bigg), \tag{4.45a}$$

$$u(x,t_r) = -i\ln\bigg((-1)^n\beta^{-1}\prod_{j=1}^{n}\nu_j(x,t_r)\bigg), \tag{4.45b}$$

where $\alpha\beta = \prod_{m=1}^{2n}E_m \neq 0$.

Proof. The proof is identical to that of relations (3.28). ☐

Next we turn to the time evolution of the quantities μ_j and ν_j and, as in Section 3, we will assume that $\mathcal{K}_n$ is nonsingular for the rest of this section.

Lemma 4.6. *Assume* (3.31) *and suppose the zeros $\{\mu_j(x,t_r)\}_{j=1,\dots,n}$ of $F_n(\,\cdot\,,x,t_r)$ remain distinct for $(x,t_r)\in\Omega$, where $\Omega\subseteq\mathbb{R}^2$ is open and connected. Then $\{\mu_j(x,t_r)\}_{j=1,\dots,n}$ satisfies the following system of differential equations*

$$\mu_{j,x}(x,t_r) = -2i\frac{y(\hat\mu_j(x,t_r))}{\prod_{\substack{\ell=1\\ \ell\neq j}}^{n}(\mu_j(x,t_r)-\mu_\ell(x,t_r))}, \tag{4.46}$$

$$\mu_{j,t_r}(x,t_r) = 2\frac{\widetilde F_r(\mu_j(x,t_r),x,t_r)}{\mu_j(x,t_r)}\frac{y(\hat\mu_j(x,t_r))}{\prod_{\substack{\ell=1\\ \ell\neq j}}^{n}(\mu_j(x,t_r)-\mu_\ell(x,t_r))}, \tag{4.47}$$

$$j=1,\dots,n,\ (x,t_r)\in\Omega.$$

with initial conditions

$$\hat\mu_j(x_0,t_{0,r})\in\mathcal{K}_n,\quad j=1,\dots,n, \tag{4.48}$$

for some fixed $(x_0,t_{0,r})\in\Omega$. The initial value problem (4.46)–(4.48) *has a unique solution $\{\hat\mu_j(x,t_r)\}_{j=1,\dots,n}$ such that*

$$\hat\mu_j(x,t_r)\in C^\infty(\Omega,\mathcal{K}_n),\quad j=1,\dots,n. \tag{4.49}$$

For the zeros $\{\nu_j(x,t_r)\}_{j=1,\dots,n}$ of $H_n(\,\cdot\,,x,t_r)$ identical statements hold with μ replaced by ν. In particular, $\{\hat\nu_j(x,t_r)\}_{j=1,\dots,n}$ satisfies

$$\nu_{j,x}(x,t_r) = -2i\frac{y(\hat\nu_j(x,t_r))}{\prod_{\substack{\ell=1\\ \ell\neq j}}^n(\nu_j(x,t_r)-\nu_\ell(x,t_r))}, \tag{4.50}$$

$$\nu_{j,t_r}(x,t_r) = 2\frac{\widetilde H_r(\nu_j(x,t_r),x,t_r)}{\nu_j(x,t_r)}\frac{y(\hat\nu_j(x,t_r))}{\prod_{\substack{\ell=1\\ \ell\neq j}}^n(\nu_j(x,t_r)-\nu_\ell(x,t_r))}, \tag{4.51}$$
$$j=1,\dots,n,\ (x,t_r)\in\Omega.$$

Proof. It suffices to prove (4.47) since the argument for (4.51) is analogous and that for (4.46) and (4.50) has been given in the proof of Lemma 3.2. Inserting $z=\mu_j(x,t_r)$ into (4.26a), observing (4.12a), yields

$$F_{n,t_r}(\mu_j) = -\mu_{j,t_r}\prod_{\substack{\ell=1\\ \ell\neq j}}^n(\mu_j-\mu_\ell) = 2\frac{\widetilde F_r(\mu_j)}{\mu_j}\mu_jG_{n-1}(\mu_j) = -2\frac{\widetilde F_r(\mu_j)}{\mu_j}y(\hat\mu_j). \tag{4.52}$$

The rest is identical to the proof of Lemma 3.2. □

Remark 4.7. Consider the case $n\in\mathbb{N}$, and $r=0$, and $\alpha=\beta=i/4$. Then, as explicitly proven in [3], [34], [42], [73], [74],

$$u(x,t_0) = i\ln\left((-1)^n\alpha^{-1}\prod_{j=1}^n\mu_j(x,t_0)\right) \tag{4.53}$$

satisfies

$$u_{xt_0} = \sin(u) \tag{4.54}$$

whenever $\{\hat\mu_j(x,t_0)\}_{j=1,\dots,n}$ satisfies (4.46) and (4.47) for $r=0$. As discussed in detail in [32], the trace relation (4.45a) for $u(x,t_r)$ subject to the Dubrovin-type equations (4.46), (4.47) yields algebro-geometric solutions of (higher-order) sGmKdV equations. In fact, a systematic study of $\widetilde F_r(\mu_j)$ in terms of certain symmetric functions of $\mu_1,\dots,\mu_n$ (cf. Appendix B) yields an alternative approach to the sGmKdV hierarchy.

Next we turn to the principal result of this section, the representation of ϕ, Ψ, and u in terms of the Riemann theta function associated with $\mathcal{K}_n$. First we need to introduce some notation. Let $\omega^{(2)}_{P_\infty,2q}$ be a normalized differential of second kind with unique pole at P_∞ and principal part $\zeta^{-2q-2}d\zeta$ near P_∞. Define

$$\widetilde\Omega^{(2)}_{P_\infty,2r-2} = \begin{cases}\sum_{q=0}^{r-1}(2q+1)\tilde c_{r-1-q}\,\omega^{(2)}_{P_\infty,2q} & \text{for } r\in\mathbb{N},\\ 0 & \text{for } r=0.\end{cases} \tag{4.55}$$

Then one infers

$$\int_{Q_0}^P\widetilde\Omega^{(2)}_{P_\infty,2r-2} \underset{\zeta\to0}{=} \begin{cases}-\sum_{q=0}^{r-1}\tilde c_{r-1-q}\zeta^{-2q-1}+O(1) & \text{for } r\in\mathbb{N}\\ 0 & \text{for } r=0\end{cases} \quad \text{as } P\to P_\infty. \tag{4.56}$$

The corresponding vector of b-periods of $\widetilde{\Omega}^{(2)}_{P_\infty,2r-2}/(2\pi i)$ is then denoted by $\underline{\widetilde{U}}^{(2)}_{2r-2}$,

$$\underline{\widetilde{U}}^{(2)}_{2r-2} = (\widetilde{U}^{(2)}_{2r-2,1}, \dots, \widetilde{U}^{(2)}_{2r-2,n}), \quad \widetilde{U}^{(2)}_{2r-2,j} = \frac{1}{2\pi i}\int_{b_j} \widetilde{\Omega}^{(2)}_{P_\infty,2r-2}, \quad j=1,\dots,n. \tag{4.57}$$

One computes from (A.18),

$$\widetilde{U}^{(2)}_{2r-2,j} = -2\sum_{q=0}^{r-1} \tilde{c}_{r-1-q} \sum_{k=1}^{n} c_j(k) c_{k-n+q}(\underline{E}), \quad j=1,\dots,n, \tag{4.58}$$

with $c_k(\underline{E})$ defined in (B.8). Moreover, let $\omega^{(2)}_{\hat{P},0}$ be a normalized differential of the second kind, holomorphic on $\mathcal{K}_n \setminus \{\widehat{P}\}$, such that

$$\omega^{(2)}_{\hat{P},0} = (\zeta^{-2} + O(1))d\zeta \text{ as } P \to \widehat{P}. \tag{4.59}$$

In the special case $\widehat{P} = P_0$, the vector of b-periods of $(\tilde{\alpha}/\alpha)Q^{1/2}\omega^{(2)}_{P_0,0}/(2\pi i)$ is denoted by $\underline{W}^{(2)}_0$ and one obtains (cf. (A.19)),

$$\underline{W}^{(2)}_0 = (W^{(2)}_{0,1}, \dots, W^{(2)}_{0,n}), \quad W^{(2)}_{0,j} = \frac{1}{2\pi i}\frac{\tilde{\alpha}}{\alpha}Q^{1/2}\int_{b_j} \omega^{(2)}_{P_0,0} = 2\frac{\tilde{\alpha}}{\alpha}c_j(1), \quad j=1,\dots,n. \tag{4.60}$$

Theorem 4.8. *Assume* (3.31), (3.40), $P \in \mathcal{K}_n \setminus \{P_\infty\}$ *and* $(x,t_n), (x_0,t_{0,n}) \in \Omega$, *where* $\Omega \subseteq \mathbb{R}^2$ *is open and connected. Suppose that* $\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}$, *or equivalently,* $\mathcal{D}_{\underline{\hat{\nu}}(x,t_r)}$ *is nonspecial for* $(x,t_r) \in \Omega$. *Then* $\phi(P,x,t_r)$ *admits the representation*

$$\begin{aligned}\phi(P,x,t_r) =& \frac{\theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_\infty) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}))}{\theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_\infty) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}) + \underline{\Delta})} \times \\ &\times \frac{\theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}) + \underline{\Delta})}{\theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}))} \exp\bigg(-\int_{Q_0}^{P} \omega^{(3)}_{P_\infty,P_0}\bigg),\end{aligned} \tag{4.61}$$

with the halfperiod $\underline{\Delta}$ *defined in* (3.44). *The components* $\psi_j(P,x,x_0,t_r,t_{0,r})$, $j = 1,2$ *of the Baker–Akhiezer vector are given by*

$$\begin{aligned}\psi_1(P,x,x_0,t_r,t_{0,r}) = & \frac{\theta(P,\underline{\hat{\mu}}(x,t_r))\theta(P_\infty,\underline{\hat{\mu}}(x_0,t_{0,r}))}{\theta(P_\infty,\underline{\hat{\mu}}(x,t_r))\theta(P,\underline{\hat{\mu}}(x_0,t_{0,r}))} \exp\bigg(-i(x-x_0)\int_{Q_0}^{P}\omega^{(2)}_{P_\infty,0} \\ & + (t_r - t_{0,r})(\tilde{\alpha}/\alpha)Q^{1/2}\int_{Q_0}^{P}\omega^{(2)}_{P_0,0} + (t_r - t_{0,r})\int_{Q_0}^{P}\widetilde{\Omega}^{(2)}_{P_\infty,2r-2}\bigg)\end{aligned} \tag{4.62}$$

and

$$\begin{aligned}\psi_2(P,x,x_0,t_r,t_{0,r}) = & -\frac{\theta(P,\underline{\hat{\mu}}(x,t_r)),\underline{\Delta})\theta(P_\infty,\underline{\hat{\mu}}(x_0,t_{0,r}))}{\theta(P_\infty,\underline{\hat{\mu}}(x,t_r)),\underline{\Delta})\theta(P,\underline{\hat{\mu}}(x_0,t_{0,r}))} \exp\bigg(-\int_{Q_0}^{P}\omega^{(3)}_{P_\infty,P_0} \\ & - i(x-x_0)\int_{Q_0}^{P}\omega^{(2)}_{P_\infty,0} + (t_r - t_{0,r})(\tilde{\alpha}/\alpha)Q^{1/2}\int_{Q_0}^{P}\omega^{(2)}_{P_0,0} \\ & + (t_r - t_{0,r})\int_{Q_0}^{P}\widetilde{\Omega}^{(2)}_{P_\infty,2r-2}\bigg).\end{aligned} \tag{4.63}$$

The Abel map linearizes the auxiliary divisors in the sense that

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x_0,t_{0,r})}) - 2i\underline{c}(n)(x-x_0) + \underline{d}_r(t_r - t_{0,r}), \tag{4.64a}$$

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(x,t_r)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(x_0,t_{0,r})}) - 2i\underline{c}(n)(x-x_0) + \underline{d}_r(t_r - t_{0,r}), \tag{4.64b}$$

where $\underline{c}(n)$ *is defined in* (A.7) *and* $\underline{d}_r$ *is given by*

$$\underline{d}_r = \begin{cases} -2(\tilde{\alpha}/\alpha)\underline{c}(1) + 2\sum_{q=0}^{r-1}\tilde{c}_{r-1-q}\sum_{k=1}^{n}\underline{c}(k)\,c_{k-n+q}(\underline{E}) & \text{for } r\in\mathbb{N}, \\ -2(\tilde{\alpha}/\alpha)\underline{c}(1) & \text{for } r=0, \end{cases} \tag{4.65}$$

with $c_k(\underline{E})$ *defined in* (B.8) *and the convention* $c_{-k}(\underline{E})=0$ *for* $k\in\mathbb{N}$.

The solution $u(x,t_r)$ *of equations* (4.1) *and* (4.3) *reads*

$$\begin{aligned} u(x,t_r) &= u(x_0,t_r) \\ &\quad + 2i\ln\bigg(\frac{\theta(P_\infty,\underline{\hat{\mu}}(x,t_r),\underline{\Delta})\theta(P_\infty,\underline{\hat{\mu}}(x_0,t_{0,r}))}{\theta(P_\infty,\underline{\hat{\mu}}(x_0,t_{0,r}),\underline{\Delta})\theta(P_\infty,\underline{\hat{\mu}}(x,t_r))}\exp\big(-ie_0(x-x_0)\big)\bigg), \end{aligned} \tag{4.66}$$

where we had to resort to the abbreviations (3.49).

Proof. The proofs of (4.61) and (4.66) carry over without changes from the stationary situation described in Theorem 3.4 since they are based on the Riccati-type equation (4.19) in both cases, with essentially the same expression for ϕ. Moreover, that the constructed $u(x,t_r)$ indeed satisfies $\mathrm{sGmKdV}_r(u)=0$ and the nth stationary sGmKdV equation readily follows from

$$(U_{t_r} - \widetilde{V}_{r,x} + [U,\widetilde{V}_r])\Psi = \Psi_{xt_r} - \Psi_{t_rx} = 0 \tag{4.67}$$

and

$$(V_{n,x} - [U,V_n])\Psi = 0 \tag{4.68}$$

applying (4.34)–(4.36) repeatedly. Hence we turn to the Baker–Akhiezer function Ψ whose first component ψ_1 is given by (4.37),

$$\begin{aligned} &\psi_1(P,x,x_0,t_r,t_{0,r}) \\ &= \bigg(\frac{F_n(z,x,t_r)}{F_n(z,x_0,t_{0,r})}\bigg)^{1/2}\exp\bigg(iy(P)\int_{x_0}^{x}\frac{dx'}{F_n(z,x',t_r)} - \frac{y(P)}{z}\int_{t_{0,r}}^{t_r}ds\,\frac{\widetilde{F}_r(z,x_0,s)}{F_n(z,x_0,s)}\bigg). \end{aligned} \tag{4.69}$$

The time-dependent term in the exponential has two potential singularities, one at P_∞, and the other at P_0. We first study this term as $P\to P_\infty$ (using the local coordinate $\zeta=\sigma/z^{1/2}\to 0$, $\sigma=\pm 1$, cf. (3.38)). First assume that $\widetilde{F}_r$ is homogeneous, that is, $\widetilde{F}_r=\widehat{F}_r$. Then, using the high-energy expansion (B.6) at $x=x_0$, equation (2.37) at $j=r$, as well as (2.35) (with n replaced by r, α by $\tilde{\alpha}$, and β by $\tilde{\beta}$),

$$\begin{aligned} \frac{y(P)\widehat{F}_r(z,x_0,t_r)}{zF_n(z,x_0,t_r)} &= \zeta^2\bigg(\sum_{q=0}^{r}\hat{f}_{r-q}(x_0,t_r)z^q\bigg)\frac{y(P)}{F_n(z,x_0,t_r)} \\ &= \zeta^{-2r+1} + (\tilde{f}_r(x_0,t_r) - \hat{f}_r(x_0,t_r))\zeta + O(\zeta^3) \text{ as } P\to P_\infty. \end{aligned} \tag{4.70}$$

Here $\hat{f}_r$ is computed from the KdV recursion (2.17a), whereas $\tilde{f}_r=\tilde{\alpha}e^{-iu}$. Similarly,

$$\frac{y(P)\widehat{F}_q(z,x_0,t_r)}{zF_n(z,x_0,t_r)} \underset{\zeta\to 0}{=} \zeta^{-2q+1} + O(\zeta^3) \text{ as } P\to P_\infty, \quad q=0,\dots,r-1. \tag{4.71}$$

Hence one concludes

$$\frac{y(P)\widetilde{F}_r(z,x_0,t_r)}{zF_n(z,x_0,t_r)} = \sum_{q=0}^{r}\tilde{c}_{r-q}\frac{y(P)\widehat{F}_q(z,x_0,t_r)}{zF_n(z,x_0,t_r)}$$

$$\underset{\zeta\to 0}{=} \begin{cases} \sum_{q=1}^{r} \tilde{c}_{r-q}\zeta^{-2q+1} + O(\zeta) & \text{for } r\in\mathbb{N}, \\ \zeta + O(\zeta^3) & \text{for } r=0 \end{cases} \tag{4.72}$$

and thus

$$\begin{aligned} &\frac{y(P)}{z}\int_{t_{0,r}}^{t_r} ds\, \frac{\widetilde{F}_r(z,x_0,s)}{F_n(z,x_0,s)} \\ &\underset{\zeta\to 0}{=} \begin{cases} (t_r - t_{0,r})\big(\sum_{q=0}^{r-1} \tilde{c}_{r-1-q}\zeta^{-2q-1}\big) + O(\zeta) & \text{for } r\in\mathbb{N}, \\ (t_r-t_{0,r})\zeta + O(\zeta^3) & \text{for } r=0, \end{cases} \quad \text{as } P\to P_\infty. \end{aligned} \tag{4.73}$$

Secondly, we study the behavior near P_0 (using the local coordinate $\zeta = \sigma z^{1/2}\to 0$, $\sigma=\pm 1$, cf. (3.38)). One finds

$$\frac{y(P)\widetilde{F}_r(z,x_0,t_r)}{zF_n(z,x_0,t_r)} \underset{\zeta\to 0}{=} \frac{\tilde{\alpha}}{\alpha} Q^{1/2}\zeta^{-1} + O(\zeta) \text{ as } P\to P_0 \tag{4.74}$$

and hence

$$\frac{y(P)}{z}\int_{t_{0,r}}^{t_r} ds\, \frac{\widetilde{F}_r(z,x_0,s)}{F_n(z,x_0,s)} \underset{\zeta\to 0}{=} \frac{\tilde{\alpha}}{\alpha}Q^{1/2}(t_r-t_{0,r})\zeta^{-1} + O(\zeta) \text{ as } P\to P_0. \tag{4.75}$$

Here the sign of $Q^{1/2}$ is determined by the compatibility of the charts near P_∞ and P_0. Together with our choice of signs in (A.13) one obtains

$$Q^{1/2} = (-1)^n \prod_{m=1}^{2n} |E_m^{1/2}|. \tag{4.76}$$

Equations (4.69), (4.73), and (4.75) then yield (4.62) as in Theorem 3.4.

Next we indicate the proof of (4.64a) using Lemma 4.6 and following the corresponding argument in the proof of Theorem 3.4. We temporarily assume,

$$\mu_j(x,t_r) \neq \mu_{j'}(x,t_r) \text{ for } j\neq j' \text{ and } (x,t_r)\in\widetilde{\Omega}\subseteq\Omega, \tag{4.77}$$

where $\widetilde{\Omega}$ is open and connected. Equations (4.47), (B.11), Lagrange's interpolation theorem, Theorem B.1, and (A.18) then yield for $r\in\mathbb{N}$,

$$\begin{aligned} &\frac{\partial}{\partial t_r}\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}) = \sum_{j=1}^n \mu_{j,t_r}(x,t_r)\sum_{k=1}^n \underline{c}(k)\frac{\mu_j(x,t_r)^{k-1}}{y(\hat{\mu}_j(x,t_r))} \\ &= 2\sum_{j,k=1}^n \underline{c}(k)\frac{\mu_j(x,t_r)^{k-1}}{\prod_{\substack{\ell=1\\ \ell\neq j}}(\mu_j(x,t_r)-\mu_\ell(x,t_r))}\, \frac{\widetilde{F}_r(\mu_j(x,t_r),x,t_r)}{\mu_j(x,t_r)} \\ &= 2\sum_{j,k=1}^n \underline{c}(k)\frac{\mu_j(x,t_r)^{k-1}}{\prod_{\substack{\ell=1\\ \ell\neq j}}(\mu_j(x,t_r)-\mu_\ell(x,t_r))}\Bigg(\sum_{q=0}^{r-1}\sum_{p=(q-n)\vee 0}^{q} \tilde{c}_{r-1-q}\, c_p(\underline{E})\Phi_{q-p}^{(j)}(\underline{\mu}) \\ &\qquad\qquad - \frac{\tilde{\alpha}}{\alpha}\Phi_{n-1}^{(j)}(\underline{\mu})\Bigg) \\ &= 2\sum_{k=1}^n \underline{c}(k)\Bigg(\sum_{q=0}^{r-1}\sum_{p=(q-n)\vee 0}^{q}\tilde{c}_{r-1-q}\, c_p(\underline{E})\sum_{j=1}^n \frac{\mu_j(x,t_r)^{k-1}}{\prod_{\substack{\ell=1\\ \ell\neq j}}(\mu_j(x,t_r)-\mu_\ell(x,t_r))}\Phi_{q-p}^{(j)}(\underline{\mu}) \\ &\qquad\qquad - \frac{\tilde{\alpha}}{\alpha}\sum_{j=1}^n \frac{\mu_j(x,t_r)^{k-1}}{\prod_{\substack{\ell=1\\ \ell\neq j}}(\mu_j(x,t_r)-\mu_\ell(x,t_r))}\Phi_{n-1}^{(j)}(\underline{\mu})\Bigg) \end{aligned}$$

$$= 2\sum_{k=1}^{n} \underline{c}(k)\bigg(\sum_{q=0}^{r-1} \sum_{p=(q-n)\vee 0}^{q} \tilde{c}_{r-1-q}\, c_p(\underline{E})\delta_{k,n-(q-p)} - \frac{\tilde{\alpha}}{\alpha}\delta_{k,1}\bigg)$$

$$= \underline{d}_r = -(\underline{\widetilde{U}}^{(2)}_{2r-2} + \underline{W}^{(2)}_0) \tag{4.78}$$

and hence (4.64a) subject to (4.77). The extension from $(x,t_r) \in \widetilde{\Omega}$ to $(x,t_r) \in \Omega$ then follows by continuity of $\underline{\alpha}_{Q_0}$ as in Theorem 3.4. (4.64b) then follows from (4.64a) and the linear equivalence of $\mathcal{D}_{P_\infty \underline{\hat{\mu}}(x,t_r)}$ and $\mathcal{D}_{P_0 \underline{\hat{\nu}}(x,t_r)}$, that is,

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(x,t_r)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}) + \underline{\Delta}. \tag{4.79}$$

□

Remark 4.9. Equations (4.62)–(4.66) have been derived for the sG equation (and its close allies), that is, in the case $r = 1$, by a variety of authors, we refer, for instance, to [6], Sects. 4.2, 4.3, [17], [18], [34], [53], [57], and [73]. Moreover, considerable efforts have been devoted to the construction of real-valued solutions by imposing certain symmetry conditions on the (nonzero) branch points $(E_m, 0)$, $m = 1, \dots, 2n$, see [6], Sect. 4.3.3, [10], [11], [18], [25], [29], [30], [33], [34], [36], [39], [42], [43], [63], [66], [70], [71], [74], and the references therein.

Remark 4.10. The linearization property (4.64a) (and (3.47a)) can also be obtained via an alternative procedure which we briefly sketch. Following a standard argument (see, e.g., [64], p. 139–144) one introduces the meromorphic differential

$$\Omega_1(x,x_0,t_r,t_{0,r}) = \frac{\partial}{\partial z}\ln(\psi_1(\,\cdot\,,x,x_0,t_r,t_{0,r}))dz \tag{4.80}$$

and hence infers from the representation (4.62)

$$\begin{aligned}\Omega_1(x,x_0,t_r,t_{0,r}) = & -i(x-x_0)\omega^{(2)}_{P_\infty,0} + (t_r - t_{0,r})\widetilde{\Omega}^{(2)}_{P_\infty,2r-2} + (t_r - t_{0,r})\frac{\tilde{\alpha}}{\alpha}Q^{1/2}\omega^{(2)}_{P_0,0} \\ & - \sum_{j=1}^{n} \omega^{(3)}_{\hat{\mu}_j(x_0,t_{0,r}),\hat{\mu}_j(x,t_r)} + \omega.\end{aligned} \tag{4.81}$$

Here ω denotes a holomorphic differential on $\mathcal{K}_n$, that is,

$$\omega = \sum_{j=1}^{n} c_j \omega_j \tag{4.82}$$

for some $c_j \in \mathbb{C}$, $j = 1, \dots, n$. Since $\psi_1(\,\cdot\,,x,x_0,t_r,t_{0,r})$ is single-valued on $\mathcal{K}_n$, all a and b-periods of Ω_1 are integer multiples of $2\pi i$ and hence

$$2\pi i m_k = \int_{a_k} \Omega_1(x,x_0,t_r,t_{0,r}) = \int_{a_k} \omega = c_k, \quad j = 1, \dots, n \tag{4.83}$$

for some $m_k \in \mathbb{Z}$ identifies c_k as integer multiples of $2\pi i$. Similarly, for some $n_k \in \mathbb{Z}$,

$$\begin{aligned}2\pi i n_k = & \int_{b_k} \Omega_1(x,x_0,t_r,t_{0,r}) \\ = & -i(x-x_0)\int_{b_k} \omega^{(2)}_{P_\infty,0} + (t_r - t_{0,r})\frac{\tilde{\alpha}}{\alpha}Q^{1/2}\int_{b_k} \omega^{(2)}_{P_0,0} \\ & + (t_r - t_{0,r})\int_{b_k} \widetilde{\Omega}^{(2)}_{P_\infty,2r-2} - \sum_{j=1}^{n}\int_{b_k} \omega^{(3)}_{\hat{\mu}_j(x_0,t_{0,r}),\hat{\mu}_j(x,t_r)} + 2\pi i \sum_{j=1}^{n} m_j \int_{b_k} \omega_j\end{aligned}$$

$$= 2\pi U^{(2)}_{0,k}(x-x_0) + 2\pi(\widetilde{U}^{(2)}_{2r-2,k} + W^{(2)}_{0,k})(t_r - t_{0,r})$$
$$- 2\pi i \sum_{j=1}^{n} \big(\underline{A}_{\hat\mu_j(x,t_r)}(\hat\mu_j(x_0,t_{0,r}))\big)_k + 2\pi i \sum_{j=1}^{n} m_j \tau_{j,k}, \quad k=1,\dots,n, \quad (4.84)$$

using (3.42), (4.57), (4.60), and (A.21). By symmetry of τ (cf. (A.23)), (4.84) is equivalent to

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat\mu}(x,t_r)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat\mu}(x_0,t_{0,r})}) + i\underline{U}^{(2)}_0(x-x_0) - (\underline{\widetilde{U}}^{(2)}_{2r-2} + \underline{W}^{(2)}_0)(t_r - t_{0,r}). \quad (4.85)$$

Finally, we remark on an interesting property of the time-dependent sGmKdV hierarchy in connection with its algebro-geometric solutions. In fact, equations (4.1) and (4.3) can be rewritten so that $u(x,t_r)$ satisfies a differential equation with a pure first-order time derivative.

Remark 4.11. The solution u of equations (4.1) and (4.3) satisfies

$$u_{t_r} = 2i\alpha^{-1}(\tilde\alpha g_{n-1} - \alpha\tilde g_{r-1}), \quad u(x,t_{0,r}) = u^{(0)}(x). \quad (4.86)$$

Indeed, (2.22) yields

$$f_n = \alpha e^{-iu}, \quad \tilde f_r = \tilde\alpha e^{-iu}, \quad (4.87)$$

which inserted into the constant term (i.e., the coefficient of z^0) in (4.26a) results in

$$-i\alpha u_{t_r} e^{-iu} = 2(g_{n-1}\tilde f_r - \tilde g_{r-1} f_n). \quad (4.88)$$

Remark 4.11 can be illustrated as follows.
(i) Consider $n=1$.
(ia) $r=0$. Then (4.86) becomes

$$u_{t_0} + i\frac{\tilde\alpha}{\alpha}u_x = 0, \quad (4.89)$$

with solution

$$u(x,t_0) = u^{(0)}(x - i\frac{\tilde\alpha}{\alpha}t_0). \quad (4.90)$$

(ib) $r=1$. Then (4.86) reads

$$u_{t_1} + i(\frac{\tilde\alpha}{\alpha} - 1)u_x = 0, \quad (4.91)$$

with solution

$$u(x,t_1) = u^{(0)}(x - i(\frac{\tilde\alpha}{\alpha} - 1)t_1). \quad (4.92)$$

(ii) Consider $n=2$.
(iia) $r=0$. Then (4.86) becomes

$$u_{t_0} = i\frac{\tilde\alpha}{8\alpha}(u_x^3 + 2u_{xxx}) - i\frac{\tilde\alpha}{\alpha}c_1 u_x, \quad u(x,t_{0,0}) = u^{(0)}(x). \quad (4.93)$$

(iib) $r=1$. Then (4.86) reads

$$u_{t_1} = i\frac{\tilde\alpha}{8\alpha}(u_x^3 + 2u_{xxx}) + i(1 - c_1\frac{\tilde\alpha}{8\alpha})u_x, \quad u(x,t_{0,1}) = u^{(0)}(x). \quad (4.94)$$

(iic) $r = 2$. Then (4.86) becomes

$$u_{t_2} = -\frac{i}{8}(\frac{\tilde\alpha}{\alpha} - 1)(u_x^3 + 2u_{xxx}) + i(\tilde c_1 - \frac{\tilde\alpha}{\alpha}c_1)u_x, \quad u(x, t_{0,2}) = u^{(0)}(x). \tag{4.95}$$

Appendix A. Hyperelliptic curves and their theta functions

We give a brief summary of some of the fundamental properties and notations needed from the theory of hyperelliptic curves. More details can be found in some of the standard textbooks [41] and [61], as well as monographs dedicated to integrable systems such as [6], Ch. 2, [45], App. A–C, and [64].

Fix $n \in \mathbb{N}$. The hyperelliptic curve $\mathcal{K}_n$ of genus n is defined by

$$\begin{aligned} &\mathcal{K}_n : \mathcal{F}_n(z, y) = y^2 - R_{2n+1}(z) = 0, \quad R_{2n+1}(z) = \prod_{m=0}^{2n} (z - E_m), \\ &\quad E_0 = 0, \quad E_1, \dots, E_{2n} \in \mathbb{C}, \quad E_m \neq E_n \text{ for } m \neq n,\, m, n = 0, \dots, 2n. \end{aligned} \tag{A.1}$$

The curve (A.1) is compactified by adding one point P_∞ at infinity. One introduces an appropriate set of n nonintersecting cuts $\mathcal{C}_j$ joining $E_{m(j)}$ and $E_{m'(j)}$ and $\mathcal{C}_\infty$ joining E_{2n} and ∞. Denote

$$\mathcal{C} = \bigcup_{j \in \{1,\dots,n\} \cup \{\infty\}} \mathcal{C}_j, \quad \mathcal{C}_j \cap \mathcal{C}_k = \emptyset, \quad j \neq k. \tag{A.2}$$

Define the cut plane

$$\Pi = \mathbb{C} \setminus \mathcal{C}, \tag{A.3}$$

and introduce the holomorphic function

$$R_{2n+1}(\,\cdot\,)^{1/2} \colon \Pi \to \mathbb{C}, \quad z \mapsto \left(\prod_{m=0}^{2n} (z - E_m) \right)^{1/2} \tag{A.4}$$

on Π with an appropriate choice of the square root branch in (A.4). Define

$$\mathcal{M}_n = \{(z, \sigma R_{2n+1}(z)^{1/2}) \mid z \in \mathbb{C},\ \sigma \in \{\pm 1\}\} \cup \{P_\infty\} \tag{A.5}$$

by extending $R_{2n+1}(\,\cdot\,)^{1/2}$ to $\mathcal{C}$. The hyperelliptic curve $\mathcal{K}_n$ is then the set $\mathcal{M}_n$ with its natural complex structure obtained upon gluing the two sheets of $\mathcal{M}_n$ crosswise along the cuts. Finite points P on $\mathcal{K}_n$ are denoted by $P = (z, y)$, where $y(P)$ denotes the meromorphic function on $\mathcal{K}_n$ satisfying $\mathcal{F}_n(z, y) = y^2 - R_{2n+1}(z) = 0$; $\mathcal{K}_n$ has genus n.

One verifies that dz/y is a holomorphic differential on $\mathcal{K}_n$ with zeros of order $2(n-1)$ at P_∞ and hence

$$\eta_j = \frac{z^{j-1} dz}{y}, \quad j = 1, \dots, n \tag{A.6}$$

form a basis for the space of holomorphic differentials on $\mathcal{K}_n$. Introducing the invertible matrix C in $\mathbb{C}^n$,

$$\begin{aligned} C &= (C_{j,k})_{j,k=1,\dots,n}, \quad C_{j,k} = \int_{a_k} \eta_j, \\ \underline{c}(k) &= (c_1(k), \dots, c_n(k)), \quad c_j(k) = C^{-1}_{j,k}, \end{aligned} \tag{A.7}$$

the normalized differentials ω_j for $j = 1, \dots, n$,

$$\omega_j = \sum_{\ell=1}^{n} c_j(\ell)\eta_\ell, \quad \int_{a_k} \omega_j = \delta_{j,k}, \quad j,k = 1,\dots,n \tag{A.8}$$

form a canonical basis for the space of holomorphic differentials on $\mathcal{K}_n$. Here $\{a_j, b_j\}_{j=1,\dots,n}$ is a homology basis for $\mathcal{K}_n$. Near P_∞ one infers

$$\begin{aligned}
\underline{\omega} = (\omega_1, \dots, \omega_n) &= -2\bigg(\sum_{j=1}^{n} \frac{\underline{c}(j)\zeta^{2(n-j)}}{\big(\prod_{m=0}^{2n}(1 - E_m\zeta^2)\big)^{1/2}}\bigg) d\zeta \\
&\underset{\zeta\to 0}{=} -2\bigg(\sum_{q=0}^{\infty}\sum_{k=1}^{n} \underline{c}(k) c_{k-n+q}(\underline{E})\zeta^{2q}\bigg) d\zeta \qquad (A.9)\\
&\underset{\zeta\to 0}{=} -2\bigg(\underline{c}(n) + \big(\frac{1}{2}\underline{c}(n)\sum_{m=0}^{2n} E_m + \underline{c}(n-1)\big)\zeta^2 + O(\zeta^4)\bigg) d\zeta, \qquad (A.10)\\
&\qquad\qquad \zeta = \sigma/z^{1/2}, \quad \sigma = \pm 1,
\end{aligned}$$

with $c_k(\underline{E})$ defined in (B.8) and

$$y(P) \underset{\zeta\to 0}{=} \bigg(1 - \frac{1}{2}\bigg(\sum_{m=0}^{2n} E_m\bigg)\zeta^2 + O(\zeta^4)\bigg)\zeta^{-2n-1} \text{ as } P \to P_\infty. \tag{A.11}$$

Similarly, near $P_0 = (0,0)$ one computes

$$\underline{\omega} \underset{\zeta\to 0}{=} 2\bigg(\frac{\underline{c}(1)}{Q^{1/2}} + O(\zeta^2)\bigg) d\zeta, \quad Q^{1/2} = (-1)^n \prod_{m=1}^{2n} |E_m^{1/2}|, \tag{A.12}$$
$$\zeta = \sigma z^{1/2}, \quad \sigma = \pm 1,$$

using

$$y(P) \underset{\zeta\to 0}{=} Q^{1/2}\zeta + O(\zeta^3) \text{ as } P \to P_0. \tag{A.13}$$

Associated with the homology basis $\{a_j, b_j\}_{j=1,\dots,n}$ we also recall the canonical dissection of $\mathcal{K}_n$ along its cycles yielding the simply connected interior $\widehat{\mathcal{K}}_n$ of the fundamental polygon $\partial\widehat{\mathcal{K}}_n$ given by

$$\partial\widehat{\mathcal{K}}_n = a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_n^{-1} b_n^{-1}. \tag{A.14}$$

Let $\mathcal{M}(\mathcal{K}_n)$ and $\mathcal{M}^1(\mathcal{K}_n)$ denote the set of meromorphic functions (0-forms) and meromorphic differentials (1-forms) on $\mathcal{K}_n$. The residue of a meromorphic differential $\nu \in \mathcal{M}^1(\mathcal{K}_n)$ at a point $Q \in \mathcal{K}_n$ is defined by

$$\operatorname{res}_Q(\nu) = \frac{1}{2\pi i}\int_{\gamma_Q} \nu, \tag{A.15}$$

where γ_Q is a counterclockwise oriented smooth simple closed contour encircling Q but no other pole of ν. Holomorphic differentials are also called Abelian differentials of the first kind (dfk). Abelian differentials of the second kind (dsk) $\omega^{(2)} \in \mathcal{M}^1(\mathcal{K}_n)$ are characterized by the property that all their residues vanish. They are normalized, for instance, by demanding that all their a-periods vanish, that is,

$$\int_{a_j} \omega^{(2)} = 0, \quad j = 1, \dots, n. \tag{A.16}$$

If $\omega^{(2)}_{P_1,n}$ is a dsk on $\mathcal{K}_n$ whose only pole is $P_1 \in \widehat{\mathcal{K}}_n$ with principal part $\zeta^{-n-2}\,d\zeta$, $n \in \mathbb{N}_0$ near P_1 and $\omega_j = (\sum_{m=0}^{\infty} d_{j,m}(P_1)\zeta^m)\,d\zeta$ near P_1, then

$$\int_{b_j} \omega^{(2)}_{P_1,n} = \frac{2\pi i}{n+1} d_{j,n}(P_1). \tag{A.17}$$

In particular,

$$\frac{1}{2\pi i}\int_{b_j} \omega^{(2)}_{P_\infty,0} = -2c_j(n),$$

$$\frac{2q+1}{2\pi i}\int_{b_j} \omega^{(2)}_{P_\infty,2q} = -2\sum_{k=1}^{n} c_j(k)c_{k-n+q}(\underline{E}), \quad j = 1,\dots,n,\ q \in \mathbb{N}_0, \tag{A.18}$$

$$\frac{1}{2\pi i}\int_{b_j} \omega^{(2)}_{P_0,0} = 2\frac{c_j(1)}{Q^{1/2}}, \quad j = 1,\dots,n, \tag{A.19}$$

using (A.9) and (A.12).

Any meromorphic differential $\omega^{(3)}$ on $\mathcal{K}_n$ not of the first or second kind is said to be of the third kind (dtk). A dtk $\omega^{(3)} \in \mathcal{M}^1(\mathcal{K}_n)$ is usually normalized by the vanishing of its a-periods, that is,

$$\int_{a_j} \omega^{(3)} = 0, \quad j = 1,\dots,n. \tag{A.20}$$

A normal dtk $\omega^{(3)}_{P_1,P_2}$ associated with two points P_1, $P_2 \in \widehat{\mathcal{K}}_n$, $P_1 \neq P_2$ by definition has simple poles at P_j with residues $(-1)^{j+1}$, $j = 1,2$ and vanishing a-periods. If $\omega^{(3)}_{P,Q}$ is a normal dtk associated with P, $Q \in \widehat{\mathcal{K}}_n$, holomorphic on $\mathcal{K}_n \setminus \{P,Q\}$, then

$$\int_{b_j} \omega^{(3)}_{P,Q} = 2\pi i \int_Q^P \omega_j, \quad j = 1,\dots,n, \tag{A.21}$$

where the path from Q to P lies in $\widehat{\mathcal{K}}_n$ (i.e., does not touch any of the cycles a_j, b_j).

We shall always assume (without loss of generality) that all poles of dsk's and dtk's on $\mathcal{K}_n$ lie on $\widehat{\mathcal{K}}_n$ (i.e., not on $\partial\widehat{\mathcal{K}}_n$).

Define the matrix $\tau = (\tau_{j,\ell})_{j,\ell=1,\dots,n}$ by

$$\tau_{j,\ell} = \int_{b_j} \omega_\ell, \quad j,\ell = 1,\dots,n. \tag{A.22}$$

Then

$$\operatorname{Im}(\tau) > 0, \quad \text{and} \quad \tau_{j,\ell} = \tau_{\ell,j}, \quad j,\ell = 1,\dots,n. \tag{A.23}$$

Associated with τ one introduces the period lattice

$$L_n = \{\underline{z} \in \mathbb{C}^n \mid \underline{z} = \underline{m} + \tau\underline{n},\ \underline{m},\underline{n} \in \mathbb{Z}^n\} \tag{A.24}$$

and the Riemann theta function associated with $\mathcal{K}_n$ and the given homology basis $\{a_j, b_j\}_{j=1,\dots,n}$,

$$\theta(\underline{z}) = \sum_{\underline{n}\in\mathbb{Z}^n} \exp\big(2\pi i(\underline{n},\underline{z}) + \pi i(\underline{n},\tau\underline{n})\big), \quad \underline{z} \in \mathbb{C}^n, \tag{A.25}$$

where $(\underline{u},\underline{v}) = \sum_{j=1}^n \overline{u}_j v_j$ denotes the scalar product in $\mathbb{C}^n$. It has the fundamental properties

$$\theta(z_1,\dots,z_{j-1},-z_j,z_{j+1},\dots,z_n) = \theta(\underline{z}), \tag{A.26}$$

$$\theta(\underline{z}+\underline{m}+\tau\underline{n}) = \exp\big(-2\pi i(\underline{n},\underline{z})-\pi i(\underline{n},\tau\underline{n})\big)\theta(\underline{z}), \quad \underline{m},\underline{n}\in\mathbb{Z}^n. \tag{A.27}$$

Next, fix a base point $Q_0\in\mathcal{K}_n\setminus\{P_0,P_\infty\}$, denote by $J(\mathcal{K}_n)=\mathbb{C}^n/L_n$ the Jacobi variety of $\mathcal{K}_n$, and define the Abel map $\underline{A}_{Q_0}$ by

$$\underline{A}_{Q_0}\colon \mathcal{K}_n\to J(\mathcal{K}_n), \quad \underline{A}_{Q_0}(P)=\Big(\int_{Q_0}^P\omega_1,\dots,\int_{Q_0}^P\omega_n\Big) \pmod{L_n},\ P\in\mathcal{K}_n. \tag{A.28}$$

Similarly, we introduce

$$\underline{\alpha}_{Q_0}\colon \operatorname{Div}(\mathcal{K}_n)\to J(\mathcal{K}_n), \quad \mathcal{D}\mapsto\underline{\alpha}_{Q_0}(\mathcal{D})=\sum_{P\in\mathcal{K}_n}\mathcal{D}(P)\underline{A}_{Q_0}(P), \tag{A.29}$$

where $\operatorname{Div}(\mathcal{K}_n)$ denotes the set of divisors on $\mathcal{K}_n$. Here $\mathcal{D}\colon\mathcal{K}_n\to\mathbb{Z}$ is called a divisor on $\mathcal{K}_n$ if $\mathcal{D}(P)\neq 0$ for only finitely many $P\in\mathcal{K}_n$.

In connection with divisors on $\mathcal{K}_n$ we shall employ the following (additive) notation,

$$\mathcal{D}_{Q_0\underline{Q}}=\mathcal{D}_{Q_0}+\mathcal{D}_{\underline{Q}}, \quad \mathcal{D}_{\underline{Q}}=\mathcal{D}_{Q_1}+\cdots+\mathcal{D}_{Q_n}, \\ \underline{Q}=(Q_1,\dots,Q_n)\in\sigma^n\mathcal{K}_n, \quad Q_0\in\mathcal{K}_n, \tag{A.30}$$

where for any $Q\in\mathcal{K}_n$,

$$\mathcal{D}_Q\colon\mathcal{K}_n\to\mathbb{N}_0, \quad P\mapsto\mathcal{D}_Q(P)=\begin{cases}1 & \text{for } P=Q,\\ 0 & \text{for } P\in\mathcal{K}_n\setminus\{Q\},\end{cases} \tag{A.31}$$

and $\sigma^n\mathcal{K}_n$ denotes the nth symmetric product of $\mathcal{K}_n$. In particular, $\sigma^m\mathcal{K}_n$ can be identified with the set of nonegative divisors $0\le\mathcal{D}\in\operatorname{Div}(\mathcal{K}_n)$ of degree m.

For $f\in\mathcal{M}(\mathcal{K}_n)\setminus\{0\}$, $\omega\in\mathcal{M}^1(\mathcal{K}_n)\setminus\{0\}$ the divisors of f and ω are denoted by (f) and (ω), respectively. Two divisors $\mathcal{D}$, $\mathcal{E}\in\operatorname{Div}(\mathcal{K}_n)$ are called equivalent, denoted by $\mathcal{D}\sim\mathcal{E}$, if and only if $\mathcal{D}-\mathcal{E}=(f)$ for some $f\in\mathcal{M}(\mathcal{K}_n)\setminus\{0\}$. The divisor class $[\mathcal{D}]$ of $\mathcal{D}$ is then given by $[\mathcal{D}]=\{\mathcal{E}\in\operatorname{Div}(\mathcal{K}_n)\mid\mathcal{E}\sim\mathcal{D}\}$. We recall that

$$\deg((f))=0,\ \deg((\omega))=2(n-1),\ f\in\mathcal{M}(\mathcal{K}_n)\setminus\{0\},\ \omega\in\mathcal{M}^1(\mathcal{K}_n)\setminus\{0\}, \tag{A.32}$$

where the degree $\deg(\mathcal{D})$ of $\mathcal{D}$ is given by $\deg(\mathcal{D})=\sum_{P\in\mathcal{K}_n}\mathcal{D}(P)$. It is custom to call (f) (respectively, (ω)) a principal (respectively, canonical) divisor.

Introducing the complex linear spaces

$$\mathcal{L}(\mathcal{D})=\{f\in\mathcal{M}(\mathcal{K}_n)\mid f=0 \text{ or } (f)\ge\mathcal{D}\},\ r(\mathcal{D})=\dim_{\mathbb{C}}\mathcal{L}(\mathcal{D}), \tag{A.33}$$

$$\mathcal{L}^1(\mathcal{D})=\{\omega\in\mathcal{M}^1(\mathcal{K}_n)\mid \omega=0 \text{ or } (\omega)\ge\mathcal{D}\},\ i(\mathcal{D})=\dim_{\mathbb{C}}\mathcal{L}^1(\mathcal{D}), \tag{A.34}$$

($i(\mathcal{D})$ the index of speciality of $\mathcal{D}$) one infers that $\deg(\mathcal{D})$, $r(\mathcal{D})$, and $i(\mathcal{D})$ only depend on the divisor class $[\mathcal{D}]$ of $\mathcal{D}$. Moreover, we recall the following fundamental facts.

Theorem A.1. *Let $\mathcal{D}\in\operatorname{Div}(\mathcal{K}_n)$, $\omega\in\mathcal{M}^1(\mathcal{K}_n)\setminus\{0\}$. Then*

$$i(\mathcal{D})=r(\mathcal{D}-(\omega)), \quad n\in\mathbb{N}_0. \tag{A.35}$$

The Riemann-Roch theorem reads

$$r(-\mathcal{D})=\deg(\mathcal{D})+i(\mathcal{D})-n+1, \quad n\in\mathbb{N}_0. \tag{A.36}$$

By Abel's theorem, $\mathcal{D}\in\operatorname{Div}(\mathcal{K}_n)$, $n\in\mathbb{N}$ is principal if and only if

$$\deg(\mathcal{D})=0 \text{ and } \underline{\alpha}_{Q_0}(\mathcal{D})=\underline{0}. \tag{A.37}$$

Finally, assume $n \in \mathbb{N}$. Then $\underline{\alpha}_{Q_0} : \operatorname{Div}(\mathcal{K}_n) \to J(\mathcal{K}_n)$ is surjective (Jacobi's inversion theorem).

Lemma A.2. *Let $\mathcal{D}_{\underline{Q}} \in \sigma^n \mathcal{K}_n$, $\underline{Q} = (Q_1, \dots, Q_n)$. Then*

$$1 \leq i(\mathcal{D}_{\underline{Q}}) = s (\leq n/2) \tag{A.38}$$

if and only if there are s pairs of the type $(P, P^*) \in \{Q_1, \dots, Q_n\}$ (this includes, of course, branch points for which $P = P^*$).

Denote by $\underline{\Xi}_{Q_0} = (\Xi_{Q_{0,1}}, \dots, \Xi_{Q_{0,n}})$ the vector of Riemann constants,

$$\Xi_{Q_{0,j}} = \frac{1}{2}(1 + \tau_{j,j}) - \sum_{\substack{\ell=1 \\ \ell \neq j}}^{n} \int_{a_\ell} \omega_\ell(P) \int_{Q_0}^{P} \omega_j, \quad j = 1, \dots, n. \tag{A.39}$$

Theorem A.3. *Let $\underline{Q} = (Q_1, \dots, Q_n) \in \sigma^n \mathcal{K}_n$ and assume $\mathcal{D}_{\underline{Q}}$ to be nonspecial, that is, $i(\mathcal{D}_{\underline{Q}}) = 0$. Then*

$$\theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \alpha_{Q_0}(\mathcal{D}_{\underline{Q}})) = 0 \text{ if and only if } P \in \{Q_1, \dots, Q_n\}. \tag{A.40}$$

Remark A.4. While $\theta(\underline{z})$ is well-defined (in fact, entire) for $\underline{z} \in \mathbb{C}^n$, it is not well-defined on $J(\mathcal{K}_n) = \mathbb{C}^n / L_n$ because of (A.27). Nevertheless, θ is a "multiplicative function" on $J(\mathcal{K}_n)$ since the multipliers in (A.27) cannot vanish. In particular, if $\underline{z}_1 = \underline{z}_2 \pmod{L_n}$, then $\theta(\underline{z}_1) = 0$ if and only if $\theta(\underline{z}_2) = 0$. Hence it is meaningful to state that θ vanishes at points of $J(\mathcal{K}_n)$. Since the Abel map $\underline{A}_{Q_0}$ maps $\mathcal{K}_n$ into $J(\mathcal{K}_n)$, the function $\theta(\underline{A}_{Q_0}(P) - \underline{\xi})$ for $\underline{\xi} \in \mathbb{C}^n$, becomes a multiplicative function on $\mathcal{K}_n$. Again it makes sense to say that $\theta(\underline{A}_{Q_0}(\,\cdot\,) - \underline{\xi})$ vanishes at points of $\mathcal{K}_n$.

Finally, we formulate the following auxiliary result (cf., e.g., Lemma 3.4 in [46]).

Lemma A.5. *Let $\psi(\,\cdot\,, x)$, $x \in \mathcal{U}$, $\mathcal{U} \subseteq \mathbb{R}$ open, be meromorphic on $\mathcal{K}_n \setminus \{P_\infty\}$ with an essential singularity at P_∞ such that $\tilde{\psi}(\,\cdot\,, x)$ defined by*

$$\tilde{\psi}(\,\cdot\,, x) = \psi(\,\cdot\,, x) \exp\bigg(- i(x - x_0) \int_{Q_0}^{P} \Omega_0^{(2)} \bigg) \tag{A.41}$$

is multi-valued meromorphic on $\mathcal{K}_n$ and its divisor satisfies

$$(\tilde{\psi}(\,\cdot\,, x)) \geq -\mathcal{D}_{\underline{\hat{\mu}}(x)}. \tag{A.42}$$

Define a divisor $\mathcal{D}_0(x)$ by

$$(\tilde{\psi}(\,\cdot\,, x)) = \mathcal{D}_0(x) - \mathcal{D}_{\underline{\hat{\mu}}(x)}. \tag{A.43}$$

Then

$$\mathcal{D}_0(x) \in \sigma^n \mathcal{K}_n, \ \mathcal{D}_0(x) \geq 0, \ \deg(\mathcal{D}_0(x)) = n. \tag{A.44}$$

Moreover, if $\mathcal{D}_0(x)$ is nonspecial for all $x \in \mathcal{U}$, that is, if $i(\mathcal{D}_0(x)) = 0$, then $\psi(\,\cdot\,, x)$ is unique up to a constant multiple (which may depend on x).

Appendix B. Lagrange's interpolation formula and symmetric functions

In this appendix we quote a few useful facts from [32], which facilitate the proofs of the linearization properties of (3.47) and (4.64).

Let $n \in \mathbb{N}$ be fixed, introduce

$$\mathcal{S}_k = \{\underline{\ell} = (\ell_1, \dots, \ell_k) \in \mathbb{N}^k \mid \ell_1 < \cdots < \ell_k \leq n\}, \quad k \leq n, \tag{B.1a}$$

$$\mathcal{I}_k^{(j)} = \{\underline{\ell} = (\ell_1, \dots, \ell_k) \in \mathcal{S}_k \mid \ell_m \neq j\}, \quad k \leq n-1, \tag{B.1b}$$

and define

$$\Psi_0(\underline{\mu}) = 1, \quad \Psi_k(\underline{\mu}) = (-1)^k \sum_{\underline{\ell} \in \mathcal{S}_k} \mu_{\ell_1} \cdots \mu_{\ell_k}, \quad k \leq n, \tag{B.2a}$$

$$\Phi_0^{(j)}(\underline{\mu}) = 1, \quad \Phi_k^{(j)}(\underline{\mu}) = (-1)^k \sum_{\underline{\ell} \in \mathcal{I}_k^{(j)}} \mu_{\ell_1} \cdots \mu_{\ell_k}, \quad k \leq n-1, \tag{B.2b}$$

$$\Phi_n^{(j)}(\underline{\mu}) = 0,$$

where $\underline{\mu} = (\mu_1, \dots, \mu_n) \in \mathbb{C}^n$. We recall our notation

$$F_n(z) = \prod_{j=1}^n (z - \mu_j) \tag{B.3}$$

and note

$$F_n'(\mu_j) = \prod_{\substack{\ell=1 \\ \ell \neq j}}^n (\mu_j - \mu_\ell) \tag{B.4}$$

(the (x, t_r)-dependence of μ_j being immaterial here and hence suppressed in the following).

Theorem B.1 (Lagrange's interpolation formula)**.** *Assume that $\mu_1, \dots, \mu_n$ are n distinct complex numbers. Then*

$$\sum_{j=1}^n \frac{\mu_j^{m-1}}{F_n'(\mu_j)} \Phi_k^{(j)}(\underline{\mu}) = \delta_{m,n-k} - \Psi_{k+1}(\underline{\mu}) \delta_{m,n+1}, \qquad m = 1, \dots, n+1, \quad k = 0, \dots, n-1. \tag{B.5}$$

For a proof of Theorem B.1 see, for instance, [32], Appendix A.

We will also need the following high-energy expansion[2] ([44], [47])

$$\frac{F_n(z, x, t_r)}{y(P)} \underset{\zeta \to 0}{=} \zeta \sum_{j=0}^\infty \hat{f}_j(x, t_r) \zeta^{2j}, \quad \zeta = \sigma / z^{1/2}, \quad \sigma = \pm 1. \tag{B.6}$$

Furthermore, we have[3] (cf. [32], Lemma 4.3)

$$\hat{f}_j = \sum_{k=0}^{j \wedge n} c_{j-k}(\underline{E}) \Psi_k(\underline{\mu}), \tag{B.7}$$

[2] Observe that the right-hand side gives the homogeneous quantities $\hat{f}_j$ even with a non-homogeneous left-hand side F_n.

[3] $n \wedge m = \min\{n, m\}$.

where $\underline{E} = (E_0, \dots, E_{2n})$ and[4]

$$c_0(\underline{E}) = 1,$$
$$c_k(\underline{E}) = \sum_{\substack{j_0,\dots,j_{2n}=0 \\ j_0+\dots+j_{2n}=k}}^{k} \frac{(2j_0-1)!!\cdots(2j_{2n}-1)!!}{2^k j_0!\cdots j_{2n}!} E_0^{j_0}\cdots E_{2n}^{j_{2n}}, \quad k \in \mathbb{N}. \tag{B.8}$$

In addition we recall[5] (see [32], Lemma 4.4)

$$\widehat{F}_q(\mu_j) = \sum_{p=(q-n)\vee 0}^{q} c_p(\underline{E})\Phi_{q-p}^{(j)}(\underline{\mu}), \quad q = 0,\dots,r-1. \tag{B.9}$$

We note that relations (B.7) and (B.9) coincide with those in the KdV case since the recursion relations are identical (but with the KdV solution V replaced by $-(u_x^2 + 2iu_{xx})/4$, see (2.14a)). Observe the identity

$$\widetilde{F}_r(z) = z\widetilde{F}_{r-1}(z) + \tilde{f}_r \tag{B.10}$$

(using the same set of constants c_ℓ, $\ell = 1,\dots,r-1$ in $\widetilde{F}_{r-1}$ and $\widetilde{F}_r$), which implies

$$\frac{\widetilde{F}_r(\mu_j)}{\mu_j} = \widetilde{F}_{r-1}(\mu_j) + \frac{\tilde{f}_r}{\mu_j} = \sum_{q=0}^{r-1}\tilde{c}_{r-1-q}\sum_{p=(q-n)\vee 0}^{q} c_p(\underline{E})\Phi_{q-p}^{(j)}(\underline{\mu}) - \frac{\tilde{\alpha}}{\alpha}\Phi_{n-1}^{(j)}(\underline{\mu}), \tag{B.11}$$

using $\tilde{f}_r = \tilde{\alpha}e^{-iu}$ and the trace relation (4.45a).

Appendix C. Solvability of the recursion relation

In this appendix we prove solvability of the principal recursion relations in Section 2 and show that the system of equations (2.21), (2.22), and (2.24)–(2.26) is compatible.

We first prove that equations (2.21), (2.24), and (2.26) are consistent for all $j \in \mathbb{N}_0$, that is, without reference to a fixed $n \in \mathbb{N}_0$. To this end we define $\{f_j\}_{j\in\mathbb{N}_0}$ by

$$f_0^\pm = 1, \quad f_{j,x}^\pm = -\frac{1}{4}f_{j-1,xxx}^\pm + w_\pm f_{j-1,x}^\pm + \frac{1}{2}w_{\pm,x}f_{j-1}^\pm$$
$$= -\frac{1}{4}(\partial_x \mp iu_x)\partial_x(\partial_x \pm iu_x)f_{j-1}^\pm, \quad j \in \mathbb{N}, \tag{C.1}$$

with integration constants $c_j^\pm \in \mathbb{C}$. When choosing the integration constants equal, that is, $c_j^+ = c_j^- = c_j$ for $j \in \mathbb{N}$, we conclude that

$$f_j = f_j^+, \quad h_j = f_j^-, \quad j \in \mathbb{N}_0, \tag{C.2}$$

temporarily ignoring (2.22) and (2.25). Observe that $f_j^\pm$ are defined by the same recursion used to construct the quantity f_j for the KdV hierarchy with the solution V replaced by $w_\pm = -(u_x^2 \pm 2iu_{xx})/4$. It is known that $f_{j,x}^\pm$ equal a total derivative, and hence that $f_j^\pm$ are differential polynomials in $w_\pm$. Furthermore, let

$$g_{-1}^\pm = 0, \quad g_j^\pm = \frac{i}{2}(\pm f_{j,x}^\pm + iu_x f_j^\pm) = \pm\frac{i}{2}(\partial_x \pm iu_x)f_j^\pm, \quad j \in \mathbb{N}_0. \tag{C.3}$$

[4] $(2n-1)!! = 1\cdot 3\cdots(2n-1)$, and $(-1)!! = 1$.
[5] $n \vee m = \max\{n, m\}$.

Combining (C.1) and (C.3) yields

$$f^{\pm}_{j,x} = \pm\frac{i}{2}(\partial_x \mp iu_x)g^{\pm}_{j-1,x}, \quad j \in \mathbb{N}_0. \tag{C.4}$$

Write

$$f^{\pm}_{j} = \pm\frac{i}{2}g^{\pm}_{j-1,x} + \rho^{\pm}_j + c^{\pm}_j, \tag{C.5}$$

where $\rho^{\pm}_j$ is a differential polynomial in $w_{\pm}$ with

$$\rho^{\pm}_{j,x} = f^{\pm}_{j,x} \mp \frac{i}{2}g^{\pm}_{j-1,xx} = \frac{1}{2}u_x g^{\pm}_{j-1,x}, \tag{C.6}$$

using (C.4).

Lemma C.1. *Let $f^{\pm}_j$ and $g^{\pm}_j$ be defined by* (C.1) *and* (C.3)*, respectively. Assume that*

$$c^{+}_j = c^{-}_j = c_j, \quad j \in \mathbb{N}. \tag{C.7}$$

Then

$$g^{+}_j = g^{-}_j = g_j, \quad j \in \mathbb{N}_0, \tag{C.8}$$

where g_j is defined as in (2.21) *or* (2.24)*. Moreover,*

$$g_{j,x} = i(f^{-}_{j+1} - f^{+}_{j+1}), \quad j \in \mathbb{N}_0. \tag{C.9}$$

Proof. Equation (C.8) is certainly true for $j = 0$. Proceeding inductively upon j we assume that $g^{+}_{j-1} = g^{-}_{j-1} = g_{j-1}$ holds. Then

$$f^{\pm}_{j,x} = \pm\frac{i}{2}(\partial_x \mp iu_x)g_{j-1,x}. \tag{C.10}$$

Using this, (C.4), (C.5), and (C.6) we conclude that $\rho^{+}_j = \rho^{-}_j = \rho_j$ and hence

$$f^{\pm}_{j} = \pm\frac{i}{2}g_{j-1,xx} + \rho_j + c^{\pm}_j, \tag{C.11}$$

with $\rho_{j,x} = u_x g_{j-1,x}/2$. Consequently,

$$\begin{aligned} g^{+}_j - g^{-}_j &= \frac{i}{2}(\partial_x + iu_x)f^{+}_j + \frac{i}{2}(\partial_x - iu_x)f^{-}_j \\ &= \frac{i}{2}\big((\partial_x + iu_x)(\frac{i}{2}g_{j-1,x} + \rho_j + c^{+}_j) + (\partial_x - iu_x)(-\frac{i}{2}g_{j-1,x} + \rho_j + c^{-}_j)\big) \\ &= \frac{1}{2}u_x(c^{+}_j - c^{-}_j) - \frac{i}{2}u_x g_{j-1,x} + i\rho_{j,x} = 0 \end{aligned} \tag{C.12}$$

using (C.7). This proves (C.8).

By (C.3)–(C.8) one infers

$$\begin{aligned} 0 = g^{+}_{j+1} - g^{-}_{j+1} &= \frac{i}{2}(f^{+}_{j+1,x} + f^{-}_{j+1,x}) - \frac{1}{2}u_x(f^{+}_{j+1} - f^{-}_{j+1}) \\ &= \frac{i}{2}\big(\frac{i}{2}(\partial_x - iu_x)g_{j,x} - \frac{i}{2}(\partial_x + iu_x)g_{j,x}\big) - \frac{1}{2}u_x(f^{+}_{j+1} - f^{-}_{j+1}) \\ &= \frac{i}{2}u_x(g_{j,x} - i(f^{-}_{j+1} - f^{+}_{j+1})) \end{aligned} \tag{C.13}$$

and hence (C.9) as long as $u_x \not\equiv 0$. The latter case can only occur for $n = 0$ but then (C.9) can easily be verified directly. □

At this point we proved that equations (2.21), (2.24) and (2.26) are compatible. To satisfy the additional equations (2.22), (2.25) and hence (2.28), we proceed as follows. Fix $n \in \mathbb{N}$ and let $\alpha, \beta \in \mathbb{C}$. In accordance with the notation used in this appendix we write $\alpha = \alpha^+$ and $\beta = \alpha^-$. We define the nth stationary sGmKdV equation by the relation (2.28), (2.29), that is,

$$g_{n-1,x} = i(\alpha^- e^{iu} - \alpha^+ e^{-iu}), \quad n \in \mathbb{N}_0, \tag{C.14}$$

subject to the constraint

$$f_n^+(x) f_n^-(x) = \alpha^+ \alpha^- = P_{2n}(0). \tag{C.15}$$

Lemma C.2. *Let $f_j^\pm$ and g_j be defined according to* (C.1)*,* (C.7)*,* (C.3)*,* (C.8) *and assume* (C.14) *and* (C.15)*. Then*

$$f_n^\pm = \alpha^\pm e^{\mp iu}, \quad n \in \mathbb{N}. \tag{C.16}$$

Proof. Using (C.10) for $j = n$ one finds

$$\begin{aligned} f_{n,x}^\pm &= \pm\frac{i}{2}(\partial_x \mp iu_x) g_{n-1,x} = \pm\frac{i}{2} g_{n-1,xx} + \frac{1}{2} u_x g_{n-1,x} \\ &= \mp\frac{1}{2}\partial_x(\alpha^- e^{iu} - \alpha^+ e^{-iu}) + \frac{1}{2} u_x g_{n-1,x} \\ &= \mp\frac{i}{2}(\alpha^- e^{iu} + \alpha^+ e^{-iu}) u_x + \frac{1}{2} u_x g_{n-1,x} \\ &= \mp\frac{i}{2}(\alpha^- e^{iu} - \alpha^+ e^{-iu}) u_x \mp i\alpha^\pm e^{\mp iu} u_x + \frac{1}{2} u_x g_{n-1,x} \\ &= -\frac{1}{2} u_x g_{j-1,x} \mp i\alpha^\pm e^{\mp iu} u_x + \frac{1}{2} u_x g_{n-1,x} = \mp i\alpha^\pm e^{\mp iu} u_x \\ &= (\alpha^\pm e^{\mp iu})_x, \end{aligned} \tag{C.17}$$

using relation (C.14) twice. Thus

$$f_n^\pm(x) = \alpha^\pm e^{\mp iu(x)} + \gamma^\pm \tag{C.18}$$

for some constants $\gamma^\pm \in \mathbb{C}$. By (C.15),

$$P_{2n}(0) = f_n^+(x) f_n^-(x) = \alpha^+\alpha^- = \alpha^+\alpha^- + \gamma^+\gamma^- + \alpha^+\gamma^- e^{-iu(x)} + \alpha^-\gamma^+ e^{iu(x)} \tag{C.19}$$

then yields $\gamma^+ = \gamma^- = 0$ and hence (C.16). □

The case $n = 0$ yields the trivial situation $u_x \equiv 0$, $f_0^+ f_0^- = P_0(0)$, $f_0^+ = f_0^-$ by (2.4), (2.6), (2.27), and (2.28).

Acknowledgments. H.H. is indebted to the Department of Mathematics at the University of Missouri, Columbia for the great hospitality extended to him during his sabbatical 1996–97 when this work was done. Financial support by the Norwegian Research Council is gratefully acknowledged.

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