RELAXATION PATTERNS FOR COMPETING METASTABLE STATES: A NUCLEATION AND GROWTH MODEL

F.Manzo[(1)], E.Olivieri[(2)]

(1) Dipartimento di Matematica - II Università di Roma - Tor Vergata
Via della Ricerca Scientifica - 00133 ROMA - Italy
E-mail: MANZO@MAT.UTOVRM.IT

(2) Dipartimento di Matematica - II Università di Roma - Tor Vergata
Via della Ricerca Scientifica - 00133 ROMA - Italy
E-mail: OLIVIERI@MAT.UTOVRM.IT

Abstract We study, at infinite volume and very low temperature, the relaxation mechanisms towards stable equilibrium in presence of two competing metastable states. Following Dehghanpour and Schonmann we introduce a simplified nucleation-growth irreversible model as an approximation for the stochastic Blume-Capel model, a ferromagnetic lattice system with spins taking three possible values: $-1, 0, 1$. Starting from the less stable state $-\underline{1}$ (all minuses) we look at a local observable. We find that, when crossing a special line in the space of the parameters, there is a change in the mechanism of transition towards the stable state $+\underline{1}$: we pass from a situation:
1) Where the intermediate phase $\underline{0}$ is really observable before the final transition with a permanence in $\underline{0}$ typically much longer than the first hitting time to $\underline{0}$;
to the situation:
2) Where $\underline{0}$ is not observable since the typical permanence in $\underline{0}$ is much shorter than the first hitting time to $\underline{0}$ and, moreover, large growing 0-droplets are almost full of $+1$ in their interior so that there are only relatively thin layers of zeroes between $+1$ and -1.

Key words: metastability, nucleation, Blume-Capel model

1991 Mathematics Subject Classification. Primary 60K35, 82A05

Section 1. Introduction.

This note concerns the study, from the point of view of mathematical physics, of some problems arising in the context of metastability and nucleation.

Metastability is a relevant phenomenon for thermodynamic systems close to a first order phase transition (like e.g. a vapour near its condensation point). Suppose that initially the system is in a pure thermodynamic phase and switch some thermodynamic parameter (like the specific volume) to a value corresponding to the coexistence of two or more phases. In some particular experimental situations the system, instead of undergoing a phase transition, persists in a pure phase which is called *metastable*, in an apparent (wrong) equilibrium until an external perturbation or a spontaneous fluctuation leads to the formation and growth of suitable (in size and shape) nuclei of the stable phase inside the metastable phase. It is widely accepted that metastability cannot be included in the Gibbsian approach to the equilibrium statistical mechanics but, rather, it is a genuine dynamical phenomenon. Since a general theory of non-equilibrium statistical mechanical phenomena is still lacking, a particularly important role is played by microscopic models like Glauber dynamics (also called stochastic or kinetic Ising models) .

We refer to [PL2] and [OS3] for a general discussion of metastability from a rigorous point of view. Here we only say that the heuristic description based on the competition between bulk and surface free energies leads to the introduction of the "critical nucleus": droplets of the stable phase inside the metastable phase which are smaller than the critical nucleus tend to shrink whereas larger droplets tend to indefinitely grow.

In [PL1] for the first time the dynamical and statical aspects of metastability were coupled in a rigorous way. The point of view of [PL1] can be called the one of "evolution of ensembles"; metastability is mainly characterized, there, as a slow evolution of the averages over the process towards the stable equilibrium values. The study of the Glauber dynamics in the framework of this point of view was carried out in [CCO].

In [CGOV] a new "pathwise approach" to metastability has been introduced where the analysis of the behaviour of typical trajectories has a crucial importance. The approach of [CGOV] turns out to be the starting point of many recent developments on the rigorous theory of metastability.

In [MOS1], [MOS2], in the framework of the study of relaxation to equilibrium of infinite volume stochastic Ising models at low temperature and small magnetic field, some aspects of the decay from the metastable situation were analysed in terms of the formation of a suitable critical droplet.

A complete analysis of metastability for the kinetic Ising model in large but finite volumes, with periodic boundary conditions, small but fixed magnetic field, in the limit of vanishing temperature was carried out in [NSch1], [NSch2] [Sch1]. The three-dimensional Ising model was studied in [N], [BC]. Different boundary conditions were studied in [CiL]. Different models, consisting in single spin flip markovian dynamics reversible w.r.t. the Gibbs measure corresponding to different hamiltonians, in the same asymptotic regime, were studied in [KO1], [KO2], [CiO], [NO]. In particular, as we will see, the paper in [CiO] refers to Blume–Capel model and is at the basis of the present work.

The above described asymptotic regime can be called "Freidlin-Wentzell regime", in analogy to the case of small random perturbations of dynamical systems (see [FW]). It is mainly relevant for the description of the local aspects of nucleation. As it has been pointed out in [Sch2], [Sch3], [SchSh], a much more interesting asymptotic regime, from a physical point of view, is the one corresponding, for the stochastic Ising model, to infinite volume, low but fixed temperature in the limit of vanishing magnetic field.

It follows from the analysis developed by Schonmann in [Sch2] and [Sch3] that the prescriptions on metastable behaviour introduced in [CGOV] and in particular the asymptotic unpredictability of the transition are only suited for describing nucleation in finite volumes. Indeed they apply to a regime of "single nucleation". In large enough or infinite volume we have the appearance and coalescence of many nuclei. The mechanism of transition is definitely different w.r.t. the one taking place in finite volume. In infinite volume the transition from the metastable to the stable situation, say, near the origin, is driven by the formation and subsequent long growth of supercritical droplets sufficiently far away from the origin rather than a nucleation close to it without a significant growth. In order for this more efficient mechanism to take place the system must have enough room at its disposal.

In conclusion, we can say that finite and infinite volume metastable behaviors involve physically distinct phenomena, and also for the models whose finite volume metastability is known, the specific infinite volume features deserve investigation.

An interesting case is the one of a kinetic Ising system in infinite volume, small but fixed (say positive) magnetic field in the limit of vanishing temperature. It has been studied in [DSch1], [DSch2]. Let, for simplicity, the dimension of the lattice be $d = 2$ and suppose to start from all minuses. The following heuristics is at the basis of the determination of the relaxation time t_{rel}, namely the typical time needed by the spin at the origin to become plus (see [DSch1], [DSch2]): At very low temperature droplets of pluses with squared shape play an overwhelming role. The growth of a squared droplet is ruled by two distinct mechanisms:

1) adding a unit square protuberance (with a plus spin) from the exterior to a face at the rate $e^{-\beta\gamma}$ (β being the inverse temperature and γ being the energy formation of this

protuberance, namely the energy needed to flip a minus spin with one nearest neighbor plus);
2) filling up, at rate 1, of the exterior layer to which the protuberance belongs, starting from the sites nearest neighbor to the protuberance; the rate of this process is one since the energy needed to flip a minus spin with two positive nearest neighbors is non-positive.

Suppose that Γ is the energy formation of a critical droplet so that $e^{-\beta\Gamma}$ is the probability per unit volume and per unit time of the formation of this critical droplet. Let v be the speed of growth of the supercritical droplets. As we will see it is possible to show that $v = e^{-\beta\frac{\gamma}{2}}$.

The relaxation time t_{rel} can be deduced by the following argument: consider the space-time cone with height t_{rel} and basis given by the square centered at the origin of side vt_{rel} and time coordinate 0. The order of magnitude of t_{rel} can be deduced by:

$$e^{-\beta\Gamma}(vt_{rel})^2 t_{rel} \cong 1. \tag{1.1}$$

One gets:

$$t_{rel} \cong e^{\beta\frac{(\Gamma+\gamma)}{3}} \tag{1.2}$$

Notice that a similar argument applied to the displacement of a (one-dimensional) side yelds to the relation $v \cong e^{-\beta\frac{\gamma}{2}}$. Indeed now the dimension is $d = 1$ and the speed of growth of a protuberance (to complete a new layer) is of order one and, finally, Γ must be replaced by the energy formation of a unit square protuberance, namely γ. The factor $\frac{1}{3}$ in the exponent in the r.h.s. of (1.2) which for general dimension d is $1/(d+1)$, drastically reduces the lifetime w.r.t. the finite volume situation. See [DSch] for more details.

The present paper concerns the study of metastability and nucleation in the framework of a simplified dynamical Blume-Capel model. Blume-Capel model describes a spin system on $\mathbb{Z}^d$ where the single spin variable can take three possible values: $-1, 0, +1$. It was originally introduced to study the $He^3 - He^4$ phase transition.

One can think of it as a system of particles with spin. The value $\sigma_x = 0$ of the spin at the lattice site x will correspond to absence of particles (a "vacancy"), whereas the values $\sigma_x = +1, -1$ will correspond to the presence, at x, of a particle with spin $+1, -1$, respectively.

The formal hamiltonian is given by:

$$H(\sigma) = J \sum_{<x,y>} (\sigma_x - \sigma_y)^2 - \lambda \sum_x \sigma_x^2 - h \sum_x \sigma_x \ , \tag{1.3}$$

where λ and h are two real parameters, having the meaning of the chemical potential and the external magnetic field, respectively; J is a real positive constant (ferromagnetic interaction) and $< x, y >$ denotes a generic pair of nearest neighbour sites in $\mathbb{Z}^2$.

Let $-\underline{1}$, $\underline{0}$ and $+\underline{1}$ denote the configurations with all spins equal to $-1, 0, +1$, respectively. It is easy to see that for small enough λ and h these three configurations are local minima for the energy

In [CiO] metastability and nucleation were studied for a two-dimensional Blume-Capel model in the above mentioned Freidlin-Wentzell asymptotic regime: system enclosed in a finite torus Λ, with given λ, h, in the limit of zero temperature. The interesting region of parameters that was analysed in [CiO] is

$$0 < |\lambda| < h \tag{1.4}$$

with sufficiently small h .

In the whole region (1.4) we have (in any finite volume) :

$$H(-\underline{1}) > H(\underline{0}) > H(+\underline{1})$$

In other words $-\underline{1}$ and $\underline{0}$ are metastable whereas $+\underline{1}$ is stable. The main question that can be risen is, when starting from the highest metastable configuration $-\underline{1}$, whether or not the system reaches the intermediate metastable configuration $\underline{0}$ before relaxing to the stable configuration $+\underline{1}$.

In finite volume the question is perfectly well posed . In [CiO] an answer was found in the Freidlin-Wentzell asymptotic regime. It has been shown, in [CiO], that there is a change in the mechanism of transition from $-\underline{1}$ to $+\underline{1}$ when crossing the line $h = 2\lambda$. On the right side of this line ($h < 2\lambda$) the transition is *direct* i.e. the system is driven to $+\underline{1}$ via the formation of a special critical droplet: a squared "picture frame" with suitably large size, made by a square of pluses encircled by a unit layer of zeroes in a sea of minuses. It is easy to see that a direct interface between minus and plus is unstable in our region of parameters. This explains the persistence of a thin layer of zeroes between the pluses and the minuses.

In the other region ($h > 2\lambda$) the transition is *indirect* in the sense that the system first reaches the $\underline{0}$ configuration via the formation of a supercritical squared droplet of zeroes in a sea of minuses. Subsequently, via the formation of a supercritical squared droplet of pluses in a sea of zeroes, the system is driven to the final stable state $+\underline{1}$. In this region of parameters $h > 2\lambda$ the two transitions are "Ising-like" whereas in the previous region $h < 2\lambda$ the mechsnism of transition and in particular the associated interface dynamics are much more complicated.

As we argued before, it is natural to pose the problem of the behavior of the kinetic two-dimensional Blume-Capel model in infinite volume and, in particular, in the Dehghanpour-Schonmann regime ($\beta \to \infty$ for small but fixed λ and h in infinite volume).

In particular one asks himself whether the sort of "dynamical phase transition" that has been detected in finite volume persists and in which form in infinite volume. One easily realizes that it is reasonable to expect a change in the mechanism of transition over the line $\lambda = 0$. Indeed passing from $\lambda < 0$ to $\lambda > 0$ the local energy barrier between $-\underline{1}$ and $\underline{0}$ becomes higher than the local energy barrier between $\underline{0}$ and $+\underline{1}$. Moreover the speed of growth of a droplet of pluses in a sea of zeroes becomes larger than the speed of growth of a droplet of zeroes in a sea of minuses. In other words one expects that, starting from $-\underline{1}$ and looking at an observable localized close to the origin, for $\lambda < 0$ one first sees a large droplet of zeroes coming from a large distance and after a much larger time one observes the arrival of a large droplet of pluses; on the contrary, for $\lambda > 0$ the time of the first arrival of the zeroes near the origin is much longer than the time interval needed for the subsequent arrival of the pluses. One expects that, when a large droplet Δ of zeroes in a sea of minuses is formed, quite soon many nuclei of pluses inside Δ are formed; moreover, due to the difference in speed, these pluses are likely to grow much faster than the zeroes grow inside the sea of minuses so that we expect that they rapidly almost reach the boundary of Δ, leaving only a thin layer of zeroes.

The complete proof of these conjectures appears quite complicated; in the present paper, following Dehghanpour and Schonmann, we introduce a simplified model which is nothing but an adaptation of the Ising-like simplified model introduced in [DSch1] to our Blume-Capel case. Only nucleation and growth of two species of particles (representing the zeroes and the pluses, respectively) will be allowed. Our dynamics is irreversible since the spins can olny grow: a zero cannot turn into a minus and a plus, once created, cannot be destroyed. To mimic the behavior of Blume-Capel model we only allow nucleation of pluses in sites which are surrounded by all nearest neighbors zero so that direct minus-plus interfaces are never permitted. As in the Dehghanpour-Schonmann model we take for simplicity the size of the critical droplets as 1.

In the framework of our simplified irreversible model, using mainly results of references [DS], [KS], we are able to prove the above conjectures in a quite weak form, as it will appear clear from the precise statements contained in the theorems.

We indeed prove (Theorems 1, 2 and 3) a change in the asymptotic behavior of the ratio between the time of first appearance of a zero at the origin, denoted by τ_0, and the time interval, denoted by τ_{0+}, between τ_0 and the first appearance of a + at the origin. When $\tau_{0+} << \tau_0$, we also give information on the shape of large droplets: we show, in Theorem 4, that large droplets of zeroes tend to be invaded by pluses in their interior so that, asymptotically, they become completely full of pluses with only a relatively thin layer of minuses between the internal pluses and the sea of minuses. Theorem 5 deals with the "pursuit" of the zeroes made by the internal pluses as seen from the origin at time τ_0.

The rest of the paper is organized in the following way: in Section 2 we will give definitions and precise results. In Section 3 we will prove Theorems 1, 2 and 3; in Section 4 we will prove Theorems 4 and 5. Section 5 contains a discussion of the results.

Section 2. Definitions and results.

In what follows we use $\lfloor a \rfloor$ to denote the integer part of the real number a (i.e. the largest integer smaller than or equal to a) and by $\lceil a \rceil$ the smallest integer larger than or equal to a.

We construct our model using "exponential clocks". To each site $x \in \mathbb{Z}^2$, we associate six independent exponential random variables $\zeta_o^\Gamma(x)$, $\zeta_o^\gamma(x)$, $\zeta_o^1(x)$, $\zeta_1^\Gamma(x)$, $\zeta_1^\gamma(x)$, and $\zeta_1^1(x)$, of rates $e^{-\beta\Gamma_o}$, $e^{-\beta\gamma_o} - e^{-\beta\Gamma_o}$, $1 - e^{-\beta\gamma_o}$, $e^{-\beta\Gamma_1}$, $e^{-\beta\gamma_1} - e^{-\beta\Gamma_1}$, and $1 - e^{-\beta\gamma_1}$ respectively. The parameters Γ_i, γ_i have the meaning of activation energies of critical droplets and protuberances respectively.

We will always assume

Hypothesis H_0:

$$\begin{array}{lllll} \Gamma_0 & > & \gamma_0 & > & 0 \\ \Gamma_1 & > & \gamma_1 & > & 0 \end{array} \tag{2.1}$$

The meaning of (2.1) is that the formation rate of a protuberance is, exponentially in β, larger than that of a new critical droplet.

The choice of parameters corresponding to Blume–Capel model is made in the following way (see [CiO]):

Let $\delta' := \frac{2J}{h-\lambda} - \lfloor \frac{2J}{h-\lambda} \rfloor$ and $\delta'' := \frac{2J}{h+\lambda} - \lfloor \frac{2J}{h+\lambda} \rfloor$. We have:

$$\Gamma_0^{BC} := \frac{4J^2}{h-\lambda} + 2J - h + \lambda + h(\delta' - \delta'^2),$$

$$\Gamma_1^{BC} := \frac{4J^2}{h+\lambda} + 2J - h - \lambda + h(\delta'' - \delta''^2),$$

$$\gamma_0^{BC} := 2J - (h - \lambda), \tag{2.2}$$

$$\gamma_1^{BC} := 2J - (h + \lambda).$$

Note that $\Gamma_0^{BC} \geq 5\gamma_0^{BC}$, $\Gamma_1^{BC} \geq 5\gamma_1^{BC}$.

Supposing to start from the $-\underline{1}$ configuration, we recursively define the times of first appearance of 0 and + spin at the site x as follows: let $V^{(i)}(x), i = 1, 2$, be the first time when the minus site x has at least i 0-neighbors (all other neighbours being -1) and let $U^{(i)}(x), i = 1, 2$, be the first time when the zero site x has at least i $(+1)$-neighbors and no (-1)-neighbors. Then the transition time to 0 of a minus site x is simply defined to be

$$V(x) = \min\{\zeta_o^\Gamma(x), V^{(1)}(x) + \zeta_o^\gamma(x), V^{(2)}(x) + \zeta_o^1(x)\}; \tag{2.3}$$

the transition time of a zero site to $+1$ is:

$$U(x) = \max_{y:|x-y|\leq 1}(V(y)) + \min\{\zeta_1^\Gamma(x), U^{(1)}(x) + \zeta_1^\gamma(x), U^{(2)}(x) + \zeta_1^1(x)\}. \tag{2.4}$$

We call *acceptable* a configuration $\eta \in \{-1, 0, +1\}^{\mathbb{Z}^2}$ with no pairs of nearest neighbours $-1, +1$. It is easily seen that, starting from $-\underline{1}$, with the above updating rule we only reach acceptable configurations. Using (2.3), (2.4), we can define the times of first appearance of zero and $+1$ starting from any acceptable configuration as well.

Note that, by definition, the growth of a droplet of 0 does not depend on the possible $+1$ inside the droplet.

We denote by σ_t^η the process at time t starting from the acceptable configuration η. Sometimes we omit the initial condition η from notation if $\eta = -\underline{1}$. Given the acceptable configuration $\eta \neq -\underline{1}$, we denote by $\tilde{\sigma}_t^\eta$ the process, starting from η, in which nucleation of isolated zeroes is not allowed, namely the transition time to zero of a negative site x is:

$$\tilde{V}(x) = \min\{V^{(1)}(x) + \zeta_o^\Gamma(x), V^{(1)}(x) + \zeta_o^\gamma(x), V^{(2)}(x) + \zeta_o^1(x)\} \tag{2.5}$$

Note that we introduce the time ζ_o^Γ in (2.5) in order to have the right transition rates and that this is not related with nucleation, i.e. with the appearance of an isolated zero which is forbidden.

Finally, given a configuration η free of minuses, $\eta \in \{0, 1\}^{\mathbb{Z}^2}$, we denote by $\bar{\sigma}_t^\eta$ the Ising-like process on $\{0, 1\}^{\mathbb{Z}^2}$, starting from η, in which nucleation of isolated pluses is not allowed, namely the transition time to $+1$ of a zero site x is:

$$\bar{U}(x) = \min\{U^{(1)}(x) + \zeta_1^\Gamma(x), U^{(1)}(x) + \zeta_1^\gamma(x), U^{(2)}(x) + \zeta_1^1(x)\}. \tag{2.6}$$

Beyond $-\underline{1}$, we will consider two classes of auxiliary (acceptable) initial conditions η_0^y, η_1^y $(y \in \mathbb{Z}^2)$ defined, respectively, as:

$$\eta_0^y(x) = -1 \ \forall\, x \in \mathbb{Z}^2 \backslash \{y\},\ \ \eta_0(y) = 0, \tag{2.7}$$

$$\eta_1^y(x) = 0 \ \forall\, x \in \mathbb{Z}^2 \backslash \{y\},\ \ \eta_1^y(y) = 1. \tag{2.8}$$

The configurations are naturally partially ordered as follows:

$$\eta \le \eta' \text{ if } \eta(x) \le \eta'(x) \text{ for all } x \in \mathbb{Z}^2.$$

Note that the processes σ_t^η, $\sigma_t^{\eta'}$ are defined on the same probability space (of exponential clocks), so by attractiveness, if $\eta \le \eta'$ are any two acceptable configurations, we have the following inequality for arbitrary $t \ge 0$:

$$\sigma_t^\eta \le \sigma_t^{\eta'}. \tag{2.9}$$

Moreover:

$$\tilde{\sigma}_t^\eta \le \sigma_t^\eta \tag{2.10}$$

$$\bar{\sigma}_t^\eta \le \sigma_t^\eta, \tag{2.11}$$

where in the last inequality η is a configuration in $\{0,1\}^{\mathbb{Z}^2}$.

Inequalities analogous to (2.9) also hold for the processes $\tilde{\sigma}_t^\eta$ and $\bar{\sigma}_t^\eta$.

For $x \in \mathbb{Z}^2$ we set $||x||_\infty := \max_{j=1,2} |x_j|$. Given an integer i and $x \in \mathbb{Z}^2$, let

$$Q_i(x) = \{y \in \mathbb{Z}^2 : ||x-y||_\infty \le i\}$$

be the squared box of side–length $2i+1$ centered at x and $Q_i := Q_i(0)$.

For the process with initial condition $-\underline{1}$, let

$$\tau_0 := V(0)$$

$$\tau_1 := U(0)$$

$$\tau_{0+} := \tau_1 - \tau_0.$$

We also set

$$k_0 := \max\left\{\gamma_0, \frac{\Gamma_0 + \gamma_0}{3}\right\}. \tag{2.12}$$

$$k_1 := \max\left\{\gamma_1, \frac{\Gamma_1 + \gamma_1}{3}\right\}. \tag{2.13}$$

As it has been shown by [DSch1] these are the logarithms, divided by β, of the typical occupation times of the origin by the $0, +1$ spins starting from $-\underline{1}$, $\underline{0}$, respectively.

We introduce, at this point, the quantity:

$$k^+ := \max\left\{ k_1, \frac{\Gamma_1 + \gamma_0}{3}, \gamma_0 \right\} \tag{2.14}$$

We will show (see Proposition 1 below) that $e^{\beta k^+}$ is an upper bound for the typical value of τ_{0+} in the limit of large β.

In the following Theorem DS, we give a reformulation, in the specific context of our model, of the main result obtained by Dehghanpour and Schonmann for their Ising-like nucleation and growth model. This theorem concerns the asymptotic behaviour, for large β, of the first appearance of a zero (plus) at the origin starting from all minuses (zeroes).

Theorem DS

Let hypotesis H_0 be satisfied.
then

$$\forall\, k < k_0 \quad \lim_{\beta\to\infty} \mathbb{P}\left(\sigma^{-\underline{1}}_{e^{\beta k}}(0) = -1\right) = 1 \tag{2.15}$$

$$\forall\, k > k_0 \quad \lim_{\beta\to\infty} \mathbb{P}\left(\sigma^{-\underline{1}}_{e^{\beta k}}(0) > -1\right) = 1 \tag{2.16}$$

$$\forall\, k < k_1 \quad \lim_{\beta\to\infty} \mathbb{P}\left(\sigma^{\underline{0}}_{e^{\beta k}}(0) = 0\right) = 1 \tag{2.17}$$

$$\forall\, k > k_1 \quad \lim_{\beta\to\infty} \mathbb{P}\left(\sigma^{\underline{0}}_{e^{\beta k}}(0) = 1\right) = 1 \tag{2.18}$$

Remark 1:

Notice that equation (2.17) *in Theorem DS and inequality* (2.9) *immediately imply that*

$$\forall\, k < k_1 : \qquad \lim_{\beta\to\infty} \mathbb{P}\left(\sigma^{-\underline{1}}_{e^{\beta k}}(0) = 1\right) = 0 \tag{2.19}$$

We are now ready to state our theorems.

Theorem 1

Let hypothesis H_0 be satisfied. If

$$k_0 < k_1,$$

then $\exists\, C > 0$ *such that*

$$\lim_{\beta\to\infty} \mathbb{P}\left(\frac{\tau_{0+}}{\tau_0} \geq e^{\beta C}\right) = 1 \tag{2.20}$$

Theorems 2, 3, 4 and 5 refer to the other region $k_0 > k_1$ possibly with some further restrictions.

Theorem 2

Let hypothesis H_0 be satisfied. If

$$k_0 > k_1$$

then $\forall\, k > k_0$

$$\lim_{\beta\to\infty} \mathbb{P}\left(\sigma^{-1}_{e^{\beta k}}(0) = 1\right) = 1 \tag{2.21}$$

Theorem 2, together with Theorem 1 and Remark 1, asserts that there is a sort of "dynamical phase transition" for $k_0 = k_1$. Indeed in the region $k_0 < k_1$ the spin at the origin, after a relatively small interval of time in which it takes the value -1, before τ_1 is most of the time zero; more precisely on a time scale $t = e^{\beta k}$, by equation (2.15) $\forall\, k < k_0$

$$\lim_{\beta\to\infty} \mathbb{P}\left(\sigma_{e^{\beta k}}(0) = -1\right) = 1; \tag{2.22}$$

by equation (2.16) and equation (2.17) $\forall\, k \in]k_0, k_1[$

$$\lim_{\beta\to\infty} \mathbb{P}\left(\sigma_{e^{\beta k}}(0) = 0\right) = 1 \tag{2.23}$$

and finally by equation (2.18) $\forall\, k > k_1$

$$\lim_{\beta\to\infty} \mathbb{P}\left(\sigma_{e^{\beta k}}(0) = 1\right) = 1; \tag{2.24}$$

On the contrary, in the region $k_0 > k_1$, the zero phase is "not observable" in the sense that, as a consequence of (2.15), (2.21), given any $k > 0$, on a time-scale $e^{\beta k}$, with high probability for large β, we do not see the value zero for the spin at the origin:

$\forall\, k \neq k_0$

$$\lim_{\beta\to\infty} \mathbb{P}\left(\sigma_{e^{\beta k}}(0) = 0\right) = 0; \tag{2.25}$$

Theorem 3

Let hypothesis H_0 be satisfied. If

$$\begin{cases} k_0 & > k_1 \\ \Gamma_0 & > \Gamma_1 \\ \Gamma_0 & \geq 2\gamma_0 \end{cases}, \tag{2.26}$$

then $\exists\, C' > 0$ such that

$$\lim_{\beta\to\infty} \mathbb{P}\left(\frac{\tau_{0+}}{\tau_0} \leq e^{-\beta C'}\right) = 1 \tag{2.27}$$

Theorem 1 and Theorem 3 imply (at least when $\Gamma_0 > \Gamma_1,\ \ \Gamma_0 > 2\gamma_0$) a stronger statement about the above described dynamical phase transition. Indeed they say that when going from $k_0 < k_1$ to $k_0 > k_1$, not only τ_{0+} passes from a value typically larger than τ_0 to a value typically smaller than τ_0, but that this happens *exponentially in* β.

Notice that if $h > -\lambda > 0$, $\Gamma_0^{BC}, \Gamma_1^{BC}, \gamma_0^{BC}$ and γ_1^{BC} (see (2.2)) fulfil the hypoteses of Theorem 1 whereas if $h > \lambda > 0$ they fulfil hypotheses of Theorems 2 and 3.

In the following Theorem 4 we will consider the process $\tilde{\sigma}^{\eta_0^0}$ (see (2.7)). Since nucleation of zeroes is not allowed, there is, in $\tilde{\sigma}^{\eta_0^0}$, only one droplet of zeroes (possibly containing pluses in its interior).

Theorem 4

Let $\Gamma_0 > \Gamma_1$, $\gamma_0 > \gamma_1$, $\Gamma_0 \geq 2\gamma_0$, $\Gamma_1 \geq 2\gamma_1$.

Let $R^0_{ext}(t)$ be the smallest rectangle containing the unique droplet of zeroes in $\tilde{\sigma}^{\eta_0^0}$ at time t. Consider the process at time $t = e^{\beta k}$ where $k > k_1 + \frac{\gamma_0}{2}$.

Then there is a rectangle $R^1_{int}(t)$ inside $R^0_{ext}(t)$ with $\tilde{\sigma}_t^{\eta_0^0}(x) = 1\ \forall\, x \in R^1_{int}(t)$ and such that $\forall\, r < \min\{\frac{\gamma_0}{2}, k - k_1 - \frac{\gamma_0}{2}\}$,

$$\lim_{\beta\to\infty} \mathbb{P}\left(\frac{\mathrm{diam}(R^0_{ext}(t))}{\mathrm{diam}(R^1_{int}(t))} \leq 1 + C'' e^{-\beta r}\right) \;=\; 1 \tag{2.28}$$

where C'' is a suitable positive constant.

Again the Hypotheses of Theorem 4 are fulfilled by Blume–Capel parameters in the region $h > \lambda > 0$.

The following Theorem 5 asserts that at time τ_0 the pluses are close to the origin more than $e^{\beta k^+}$. This is by far an upper bound (since we can argue, using heuristic arguments, that coalescence between large droplets should not change too much the situation w.r.t. the case of a single droplet) but still shows that the first droplet that reaches the origin contains a wide droplet of pluses in its interior.

Theorem 5

Let $\Gamma_0 > \Gamma_1 + \gamma_0$, $\gamma_0 > \gamma_1$, $\Gamma_0 \geq 2\gamma_0$, $\Gamma_1 \geq 2\gamma_1$, *and* $\forall\, \varepsilon > 0$ *let* $\check{L} := e^{\beta(k^+ + \varepsilon)}$*; then*

$$\mathbb{P}\left(\exists\, x \in Q_{\check{L}} : \sigma_{\tau_0}(x) = 1\right) \to 1 \text{ as } \beta \to \infty \tag{2.29}$$

The hypoteses of Theorem 5 are fulfilled by Blume–Capel parameters in the region $h > \lambda > 0$ in the vicinities of the "triple point" $h = \lambda = 0$. In this case, which is the most interesting one, we have that k^+ is strictly smaller than k_0. In this case Theorem 5 says that, for sufficiently small h and λ, when the first zero hits the origin the pluses are at a much smaller distance than the typical size of a zero droplet.

Section 3. Proof of Theorems 1, 2 and 3.

Proof of theorem 1.

This is a corollary of Theorem DS:
By (2.9), we get

$$\tau_1 \geq \min\{t : \sigma_t^{\underline{0}}(0) = 1\}. \tag{3.1}$$

By Theorem DS (see (2.16), (2.17)), we have, $\forall\, \varepsilon > 0$,

$$\mathbb{P}\left(\tau_0 < e^{\beta(k_0+\varepsilon)}\right) \to 1$$

while

$$\mathbb{P}\left(\tau_1 > e^{\beta(k_1-\varepsilon)}\right) \to 1$$

Let us choose $\varepsilon < \frac{k_1 - k_0}{2}$. We have:

$$\mathbb{P}\left(\frac{\tau_{0+}}{\tau_0} > \frac{e^{\beta(k_1-\varepsilon)} - e^{\beta(k_0+\varepsilon)}}{e^{\beta(k_0+\varepsilon)}}\right) \to 1 \text{ as } \beta \to \infty \tag{3.2}$$

This concludes the proof.

■

We will sometimes make use of super-exponential estimates in β in the following way: we tile a square having exponential sidelength $e^{\beta M}$ with smaller squares with exponential sidelength $e^{\beta m}$. If $p < e^{-e^{\beta C}}$ is the probability that a given event takes place in a tile, then, if the events taking place in different tiles are positively correlated, by virtue of (2.9) the probability of having the event in the whole square is less than $e^{\beta(M-m)}p \to 0$.

To prove Theorem 2, we first take a square Q_{L_0} of side $L_0 := e^{\beta(\kappa_0 - \frac{\gamma_0}{2})}$, and show that within a time $\frac{1}{2}e^{\beta k}$ a zero appears in the interior of Q_{L_0} and subsequently it grows up to completely fill it, before the time $e^{\beta k}$, with a probability that goes to 1 super-exponentially fast in β. Therefore we can tile any square centered at the origin with sides exponentially long in β with the translates of Q_{L_0} and show that during the same interval of time it becomes free of minuses (full of zeroes and possibly pluses).

Then we are reduced to an Ising-like $0, +1$ system and we can apply similar methods to conclude the proof.

Given $L \in \mathbb{N}$, let

$$\tau_b^0(L) := \min\{t : \exists\, x \in Q_L \text{ such that } \sigma_t(x) = 0\} \tag{3.3}$$

be the time needed to see the appearance of a zero in Q_L, and

$$\tau_g^0(L) := \max_{y \in Q_L} \min\{t : \tilde{\sigma}_t^{\eta_0^y}(x) = 0\ \forall\, x \in Q_L\} \tag{3.4}$$

be the maximum over y in Q_L of the time needed by the 0 droplet emerging from y to invade the whole Q_L for the process $\tilde{\sigma}_t^{\eta_0^y}$.

Inequalities (2.9), (2.10) imply that at time $\tau_b^0(L) + \tau_g^0(L)$, Q_L is full of zeroes.

The key estimate of $\tau_g^0(L)$ is contained in the following Lemma:

Lemma 1

$\forall\, M \geq \frac{\gamma_0}{2}$ let $k(M) := M + \frac{\gamma_0}{2}$ and $2L + 1 = \lceil e^{\beta M} \rceil$.
Then $\exists\, C_1$ such that $\forall\, k > k(M)$ and sufficiently large β:

$$\mathbb{P}(\tau_g^o(L) > e^{\beta k}) \leq e^{-C_1 e^{\beta(k-k(M))}} \tag{3.5}$$

Proof:

We set

$$s := e^{\beta k} \tag{3.6}$$

and

$$D := \lceil e^{\beta \frac{\gamma_0}{2}} \rceil.$$

In order to estimate τ_g^o, we use a construction analogous to the one introduced in [DSch1] in the proof of Theorem DS: we fill the square one shell at a time, using a suitable number of sites becoming zero at rate $e^{-\beta\gamma_0}$ and then filling the other sites of the shell at rate 1. The main differences with the work of Dehghanpour and Schonmann are that we consider the filling of squares larger than D and that we obtain estimates decayng super-exponentially fast in β.

Let $\partial Q_i := Q_i \backslash Q_{i-1}$ and denote by $\mathcal{C}_i$ the set of four sites at the corners of Q_i. In order to estimate the time of growth of Q_i we split ∂Q_i into suitable intervals A_i^h (horizontal or vertical).

An efficient way to fill a shell ∂Q_i larger than D is to divide it into intervals A_i^h of length of order D and wait until in each of them a first site is occupied so that all other sites can be subsequently occupied at rate 1. Let $n_i := 4\lceil \frac{2i-1}{D} \rceil$. Starting from the upper left corner, we split the upper side of ∂Q_i into $n_i/4$ intervals A_i^h: if $2i-1 > D$ we take the first $n_i/4 - 1$ intervals of length D and the last one of length larger or equal to D but less than $2D+1$; we continue clockwise by sequentially partitioning, with the same criterion, the other sides without the corners that we have previously considered. With this construction, if $2i-1 \le D$, then $n_i = 4$ and A_i^h can be seen as the four sides of Q_i (possibly without some of the corners in $\mathcal{C}_i$).

Let us set $\hat{A}_i^h := A_i^h \backslash \mathcal{C}_i$.

Let Ψ_L be the subset of $\mathbb{N}^L$ given by:

$$\Psi_L := \{\underline{h} = (h_1, \dots, h_L) \ : \ h_1 \le n_1, ..., h_L \le n_L\}$$

Let

$$\check{\tau}_i^h := \min_{x \in \hat{A}_i^h} \{\zeta_0^\gamma(x)\} \tag{3.7}$$

$$\hat{\tau}_i^h := (\check{\tau}_i^h + \sum_{y \in A_i^h} \zeta_0^1(y)) \tag{3.8}$$

using inequality (2.9), we estimate from above $\tau_g^o(L)$ as

$$\hat{\tau}_g^o := \max_{\underline{h} \in \Psi_L} \left\{ \sum_{i=1}^{L} \hat{\tau}_i^{h_i} \right\} \tag{3.9}$$

Note that in (3.7) we use $\hat{A}_i^h$ instead A_i^h because we need that every site that we consider has at least one zero neighbour. In what follows we will refer to the "exponential

Chebychev's inequality" for a random variable ξ as

$$\mathbb{P}(\xi > s) \le e^{-\theta s}\mathbb{E}(e^{\theta\xi})\ \forall\,\theta > 0, \quad s \in \mathbb{R}$$

We split the proof into two cases:

1) case $M = \frac{\gamma_0}{2}$ (so that $L = D$ and $k(M) = \gamma_0$).
Consider $A_i^{h_i}$, $h_i = 1, .., 4$. We have:

$$\mathbb{P}\left(\sum_{i=1}^{D} \max_{h_i \le 4}\left\{\check{\tau}_i^{h_i} + \sum_{y \in A_i^{h_i}} \zeta_0^1(y)\right\} > s\right) \le$$

$$\le 4\mathbb{P}\left(\sum_{i=1}^{D} \min_{j \le 2i+1}\{\zeta_0^{\gamma}(x_j)\} > \frac{s}{2}\right) + \mathbb{P}\left(\sum_{j=1}^{(2D+1)^2} \zeta_0^1(x_j) > \frac{s}{2}\right) \tag{3.10}$$

where in the r.h.s. of (3.10) $\{\zeta_0^1(x_j)\}_j$ as well as $\{\zeta_0^1(x_j)\}_j$ are families of i.i.d. variables. We estimate both terms of (3.10) as large deviations. Using the fact that the minimum among N exponential variables of rate ω is an exponential variable with rate ωN and using exponential Chebychev's inequality with $\theta = e^{-\beta\gamma_0}$, we obtain

$$\text{r.h.s. of (3.10)} \ \le 4e^{-\theta\frac{s}{2}} \prod_{i=1}^{D}\left(\frac{1}{1 - \frac{\theta}{(2i+1)e^{-\beta\gamma_0}}}\right) + e^{-\theta\frac{s}{2}}\left(\frac{1}{1-\theta}\right)^{(2D+1)^2} \le$$

$$\le e^{-c_1 e^{-\beta\gamma_0} s}\left(c_2 e^{-\sum_{i=1}^{D}\log(1-\frac{1}{(2i+1)})} + c_3 e^{c_4\beta}\right) \le$$

$$\le c_5 e^{-c_1 e^{-\beta\gamma_0} s}\left(e^{c_6\sum_{i=1}^{D}\frac{1}{2i+1}} + e^{c_4\beta}\right) \le$$

$$\le c_5 e^{-c_7 e^{\beta(k-k(M))}}$$

where in the last inequality we used (3.6) and $\sum_1^D 1/(2i+1) \sim \text{const}\ \beta$.

2) case $M > \frac{\gamma_0}{2}$ (so that $k(M) > \gamma_0$).
Recalling (3.9) we have:

$$\mathbb{P}\left(\tau_g^o(L) > s\right) \le \mathbb{P}\left(\max_{\underline{h}\in\Psi_D}\left\{\sum_{i=1}^{D}\hat{\tau}_i^{h_i}\right\} > \frac{s}{2}\right) + \mathbb{P}\left(\max_{\underline{h}\in\Psi_L}\left\{\sum_{i=D+1}^{L}\hat{\tau}_i^{h_i}\right\} > \frac{s}{2}\right) \tag{3.11}$$

The first term in the r.h.s. of (3.11) can be bounded as in case 1). To estimate the second one, we use exponential Chebychev's inequality with $\theta = \frac{1}{2}e^{-\beta\frac{\gamma_0}{2}}$. We have, for sufficiently large β:

$$\mathbb{P}\left(\sum_{i=D+1}^{L} \hat{\tau}_i^{h_i} > e^{\beta k}\right) = \mathbb{P}\left(\sum_{i=D+1}^{L} (\check{\tau}_i^{h_i} + \sum_{y\in A_i^{h_i}} \zeta_0^1(y)) > e^{\beta k}\right) \leq$$

$$\leq e^{-c_8 e^{\beta(k-\frac{\gamma_0}{2})}} \prod_{i=D+1}^{L} \left\{ \mathbb{E}\left(e^{\frac{1}{2}e^{-\beta\frac{\gamma_0}{2}}\check{\tau}_i^{h_i}}\right)\left[\mathbb{E}\left(e^{\frac{1}{2}e^{-\beta\frac{\gamma_0}{2}}\zeta_0^1}\right)\right]^{|A_i^{h_i}|}\right\} =$$

$$= e^{-c_8 e^{\beta(k-\frac{\gamma_0}{2})}} \prod_{i=D+1}^{L} \left\{ \frac{e^{-\beta\gamma_0}|A_i^{h_i}|}{e^{-\beta\gamma_0}|A_i^{h_i}| - \frac{1}{2}e^{-\beta\frac{\gamma_0}{2}}} \left(\frac{1}{1-\frac{1}{2}e^{-\beta\frac{\gamma_0}{2}}}\right)^{2D+1}\right\} \leq \tag{3.12}$$

$$\leq e^{-c_9 e^{\beta(k-\frac{\gamma_0}{2})}}.$$

where, to get the last inequality we have bounded the generic term of the product in (3.12) by a constant independent of β and used that $k - \frac{\gamma_0}{2} > M$.

So we get

$$\mathbb{P}\left(\max_{\underline{h}\in\Psi_L}\left(\sum_{i=D+1}^{L} \hat{\tau}_i^{h_i}\right) > \frac{s}{2}\right) \leq \left(4\prod_{i=1}^{L}\lceil\frac{2i-1}{D}\rceil\right) e^{-c_9 e^{\beta(k-\frac{\gamma_0}{2})}} \leq$$

$$\leq 4e^{-c_9 e^{\beta(k-\frac{\gamma_0}{2})}} e^{e^{\beta M}(\beta M-\beta\frac{\gamma_0}{2}+c_{10})} \leq 4e^{-c_{11}e^{\beta(k-\frac{\gamma_0}{2})}} \leq 4e^{-c_{11}e^{\beta(k-k(M))}}$$

The Lemma is then proved.

■

Proof of Theorem 2

Let us take $M = k_0 - \frac{\gamma_0}{2}$ so that $k(M) = k_0$. We have, on one hand, for $t = \frac{1}{2}e^{\beta k}$ and $k > k_0$:

$$\mathbb{P}\left(\{\sigma_t(x) = -1 \;\forall\, x \in Q_{L_0}\}\right) \leq \left(\mathbb{P}\left(\zeta_o^{\Gamma}(0) \leq t\right)\right)^{{L_0}^2} = e^{-te^{-\beta\Gamma_0}{L_0}^2}$$

$$= e^{-\frac{1}{2}(e^{\beta k}e^{-\beta\Gamma_0}e^{-\beta(2k_0-\gamma_0)})} \leq e^{-\frac{1}{2}e^{\beta(k-k_0)}}. \tag{3.13}$$

On the other hand, from Lemma 1 we have $\forall\, y \in Q_{L_0}$:

$$\mathbb{P}\left(\exists\, x \in Q_{L_0} : \tilde{\sigma}_t^{\eta_0^y}(x) = -1\right) \le e^{-C_1 e^{\beta(k-k_0)}} \tag{3.14}$$

¿From (2.9), (2.10), (3.13), (3.14) we have that $\exists\, \bar{C}_1 > 0$:

$$\mathbb{P}\left(\exists\, x \in Q_{L_0} : \sigma_{e^{\beta k}}(x) = -1\right) \le e^{-\bar{C}_1 e^{\beta(k-k_0)}}. \tag{3.15}$$

Now, if $L_1 = e^{\beta(k_1 - \frac{\gamma_1}{2})} > L_0$ we can use the super-exponentially fast decay in β of the above estimate by tiling Q_{L_1} with the translates of Q_{L_0}; if L_{max} is the maximum between L_0 and L_1 we have that the probability that $Q_{L_{max}}$ contains minuses at time t goes to 0 as $\beta \to \infty$. We can repeat the above arguments based on inequalities (3.13), (3.14) for the transition $0 \to 1$ to conclude the proof.

■

Let, now $\gamma := \gamma_1 \vee \gamma_0$.

In order to prove Theorem 3, we give an upper bound for τ_{0+} in the following way: starting from a single zero at the origin, we consider a square of suitable side L^+ and we wait until no one of the spins of Q_{L^+} is minus. Then we wait until a plus appears and finally we wait until the droplet originated by this plus grows and fills up of pluses the whole Q_{L^+}.

For any L^+, we have

$$\tau_{0+} \le \tau_g^o(L^+) + \tau_b^1(L^+) + \tau_g^1(L^+), \tag{3.16}$$

where $\tau_g^o(L^+)$ is defined by (3.4) while

$$\tau_b^1(L^+) := \min\{t : \exists\, x \text{ such that } \sigma_t^{\underline{o}}(x) = +1\}$$

is the time needed by a $+1$ to appear in Q_{L^+} if Q_{L^+} is initially full of zeroes, and

$$\tau_g^1(L^+) := \max_{y \in Q_{L^+}} \min\{t; \bar{\sigma}_t^{\eta_1^y}(x) = +1\ \forall\, x \in Q_{L^+}\}$$

is the maximum over y of the time needed by the $+1$ droplet emerging from y to invade the whole Q_{L^+}.

Since $\tau_g^0(L^+)$ and $\tau_g^1(L^+)$ are typically smaller than $\max\{L^+ e^{\beta\frac{\gamma}{2}}, e^{\beta\gamma}\}$ (see Lemma 1) and $\tau_b^1(L^+)$ is typically smaller than $e^{\beta\Gamma_1}/(L^+)^2$ (see (3.13)), the minimum of $\tau_g^o(L^+) + \tau_b^1(L^+) + \tau_g^1(L^+)$ is achieved when

$$L^+ = e^{\beta M^+} \text{ where } M^+ := \begin{cases} \frac{\Gamma_1 - \frac{\gamma}{2}}{3} & \text{if } \Gamma_1 \ge 2\gamma \\ \gamma/2 & \text{if } \Gamma_1 < 2\gamma \end{cases} \tag{3.17}$$

By (3.17) and (2.14), we have that:

$$k^+ = M^+ + \frac{\gamma}{2} = \begin{cases} \frac{\Gamma_1+\gamma}{3} & \text{if } \Gamma_1 \geq 2\gamma \\ \gamma & \text{if } \Gamma_1 < 2\gamma \end{cases}$$

On the basis of (3.16), in the following proposition we will give an upper bound on τ_{0+}.

Proposition 1

Let hypothesis H_0 be satisfied.
$\forall\, k > k^+ \;\; \exists\, C, C' > 0:$

$$\mathbb{P}\left(\sigma_{\tau_0+e^{\beta k}}(0) \neq +1\right) \leq C e^{-C' e^{\beta(k-k^+)}}$$

Proof
Using inequality (2.9)

$$\mathbb{P}(\tau_{0+} > e^{\beta k}) \leq \mathbb{P}(\tau_g^o(L^+) > \frac{1}{3}e^{\beta k}) + \mathbb{P}(\tau_b^1(L^+) > \frac{1}{3}e^{\beta k}) + \mathbb{P}(\tau_g^1(L^+) > \frac{1}{3}e^{\beta k})$$

The first and the third terms are estimated by using Lemma 1 and its analogous for the zero to plus transition, while the second one is estimated in the same way as in (3.13)

■

Proof of Theorem 3
We can conclude the proof of Theorem 3 by observing that this is a corollary of Proposition 1 since $k_0 > k^+$ if and only if the hypotheses of Theorem 3 are satisfied.

■

Section 4. Proof of theorems 3 and 4.

To prove Theorem 4 we will make use of the following

Lemma 2

Let $R_{a,b}$ be a rectangle of sides $\lfloor e^{\beta a} \rfloor$ and $\lfloor e^{\beta b} \rfloor$ with $a \geq b > 0$, and let us consider an initial configuration only containing zeroes and minuses, such that there exists a

**-connected set of zeroes (in the sense of site percolation) inside $R_{a,b}$ connecting both pairs of opposite sides of the rectangle.*

Let

$$\tau_R := \min\{t : \sigma_t(x) \neq -1 \ \forall\, x \in R_{a,b}\}$$

Then $\forall\, k > a$ there is a constant C such that

$$\mathbb{P}\left(\tau_R > e^{\beta k}\right) \leq e^{-Ce^{\beta k}}$$

Proof

It is sufficient to consider the Ising-like system with only minuses and zeroes obtained by neglecting the zero-plus transitions and freezing the spins outside $R_{a,b}$ to be minus (state space $\equiv \{-1,0\}^{R_{a,b}}$).

By hypothesis there exists a *-connected path of zeroes that divides $R_{a,b}$ into (at most) four regions.

Let SE be the South-Eastern of these regions. The time to fill SE is the same as the one needed to fill SE starting from the configuration η given by $\{\eta(x) = 0$ for $x \in R_{a,b}\backslash SE,\ \eta(x) = -1$ elsewhere$\}$; by (2.9), this is less than the time needed to fill the whole $R_{a,b}$ starting from the configuration in which the sides N and W are occupied by zeroes while the rest of $R_{a,b}$ is full of minuses.

Using again (2.9) we start from the NW corner and allow only "diagonal growth": given $i \in \mathbb{Z}$, we consider sets of the form

$$A_i := \{x : x_2 = x_1 + i\} \cap R_{a,b}$$

Let i' and i'', respectively, be the smallest and the largest i such that A_i is non-empty (so that $i'' = i' + \lfloor e^{\beta a} \rfloor + \lfloor e^{\beta b} \rfloor - 1$) and define

$$\xi_i := \max_{x \in A_i} \zeta_0^1(x)$$

if A_{i-1} is full of zeroes, all sites of A_i have two zero neighbours and can be filled at rate 1, so that taking the sum over i of ξ_i corresponds to fill $R_{a,b}$ by filling the A_i consecutively.

Then

$$\mathbb{P}\left(\tau_R > e^{\beta k}\right) \leq \mathbb{P}\left(4 \sum_{i=i'}^{i''} \xi_i > e^{\beta k}\right) \leq$$

$$\leq \left(e^{\beta b}\right)^{2e^{\beta a}} \mathbb{P}\left(\sum_{i=1}^{\lfloor e^{\beta a} \rfloor + \lfloor e^{\beta b} \rfloor} \zeta_0^1(x_i) > \frac{1}{4} e^{\beta k}\right) \leq e^{-Ce^{\beta k}}$$

where we have used arguments similar to the ones used to prove Lemma 1 and the fact that $|A_i| \le e^{\beta b}$.

■

Proof of Theorem 4

We will consider the process at times $\hat{t}$, t', t'' defined as $\hat{t} := t - e^{\beta(k-r)}$, $t' = \hat{t} + c_1 e^{\beta(k-\frac{\gamma_0}{2}+\varepsilon)}$, $t'' := t' + c_2 e^{\beta(k_1+\varepsilon)} = \hat{t} + c_1 e^{\beta(k-\frac{\gamma_0}{2}+\varepsilon)} + c_2 e^{\beta(k_1+\varepsilon)}$ (notice that $\hat{t} < t' < t''$) and show that $R^0_{ext}(\hat{t})$ becomes free of minuses within the time t' and becomes filled of pluses within the time t''. Then we will show that $t'' < t$ for suitable ε and large β.

By basic estimates on the speed of growth (see Theorem 1.1 in [KS]) $\exists c_3, c_4 > 0$ such that

$$\mathbb{P}(\mathrm{diam}(R^0_{ext}(\hat{t})) \ge c_3 e^{\beta(k-\frac{\gamma_0}{2})}) \to 1$$

and

$$\mathbb{P}\left(\mathrm{diam}(R^0_{ext}(t)) - \mathrm{diam}(R^0_{ext}(\hat{t})) \le c_4 e^{\beta(k-\frac{\gamma_0}{2}-r)}\right) \to 1$$

Using Lemma 2, we can show that $\forall\, \varepsilon > 0$, looking at the process at time t', we have

$$\mathbb{P}(\forall\, x \in R^0_{ext}(\hat{t}) : \sigma_{t'}(x) \neq -1) \to 1$$

We now tile $R^0_{ext}(\hat{t})$ with squares of side $e^{\beta(k_1-\frac{\gamma_1}{2})}$. These tiles are less than $e^{2\beta k}$. Using the same argument as in the proof of Proposition 1, we can show that $\exists c_5, c_6 > 0$ such that $\forall\, \varepsilon > 0$, looking at the process at time t'', we have:

$$\mathbb{P}\left(\exists\, x \in R^0_{ext}(\hat{t}) : \sigma_{t''}(x) \neq 1\right) \le c_5 e^{2\beta k} e^{-c_6 e^{\beta\varepsilon}} \to 0$$

By choosing $\varepsilon = \frac{\hat{r}-r}{2}$ such that $t'' < t$, Theorem 4 is then proven.

■

Proof of theorem 5

Let us consider the system at time $\tau' := (\tau_0 - e^{\beta(k^+ + \frac{\varepsilon}{2})}) \vee 0$. Recalling that $\check{L} = e^{\beta(k+\varepsilon)}$ let

$$\Xi := \{x \in Q_{\check{L}} : \sigma_{\tau'}(x) \neq -1\}$$

With high probability as $\beta \to \infty$ we have: since the speed of growth of a droplet of 0 is at most one, the droplet that reaches the origin is not coming from outside $Q_{\check{L}}$ and since $\Gamma_0 \ge \Gamma_1 + \gamma_0$, it is not originated inside $Q_{\check{L}}$ after τ', so that $\mathbb{P}\{\Xi \neq \emptyset\} \to 1$.

On the other hand, by using the same argument as the one used in the proof of Theorem 3, we can show that if $x \in \Xi$ at time τ', then

$$\mathbb{P}\left(\sigma_{\tau'+e^{\beta(k^+ + \frac{\varepsilon}{3})}}(x) = 1\right) \to 1$$

that concludes the proof.

■

Remark. We can also prove that calling $\hat{\Xi} := \bigcup_{x\in\Xi} Q_{L^+}(x)$,
because of ferromagnetic inequality (2.9)

$$\mathbb{P}\left(\exists x \in \hat{\Xi} : \sigma_{\tau'+e^{\beta(k^+ + \frac{\varepsilon}{3})}}(x) \neq 1\right) \leq$$

$$\leq c_1 e^{2\beta k^+} e^{-c_2 e^{\beta \frac{\varepsilon}{2}}} \to 0$$

We remark once more that this estimate works only if $\kappa^+ \leq \kappa_0 - \frac{\gamma_0}{2}$, but this is the case in the neighbourhood of the triple point of Blume-Capel model.

Section 5. Discussion.

Let us make some further remarks on the hypotheses of Theorems 3 and 5.

Blume-Capel parameters $\Gamma_0^{BC}, \Gamma_1^{BC}, \gamma_0^{BC}$ and γ_1^{BC} either fulfil hypoteses of theorem 1 for $h > -\lambda > 0$ or hypotheses of theorems 2, 3 ,4 for $h > \lambda > 0$; hypoteses of theorem 5 are fulfilled in the interesting case $h > \lambda > 0$, $h \to 0$. Indeed for Blume-Capel parameters $\Gamma_0^{BC} > \Gamma_1^{BC}$ iff $\gamma_0^{BC} > \gamma_1^{BC}$.

Equation (2.23) and its counterpart (2.25) state that the parameter controlling the "dynamical phase transition" is $k_0 - k_1$, that is, neither the nucleation rate nor the speed rate but their sum controls the transition.

Theorem 1 and its counterpart theorem 3 improve the above results giving information on τ_{0+}, but theorem 3 needs stronger hypoteses than theorem 2. The hypothesis $\Gamma_0 > 2\gamma_0$ of theorem 3 (see (2.26)) is only needed to have a strict inequality $C' > 0$ since otherwise the droplet of zeroes that reaches the origin does not achieve its asymptotic speed of growth.

The physical meaning of the condition $\Gamma_0 > \Gamma_1$ in (2.26) is that a single droplet of zeroes becomes large enough to allow nucleation of the pluses in its interior before τ_0. Indeed the volume of the space-time droplet of zeroes at time τ_0 is by definition (see (1.1) and (2.12)) the inverse of the nucleation rate of the zeroes so that the probability of nucleation of pluses inside the droplet of zeroes is bounded by $e^{-\beta[\Gamma_1-\Gamma_0]} \wedge 1$. In the other case $\Gamma_0 < \Gamma_1$, even if the speed of growth of the pluses is very large, there is no way for a plus droplet to reach the origin by a further time τ_{0+} typically smaller than $e^{\beta k_0}$, unless a large scale coalescence of the zeroes occurs connecting with the origin a nucleus of pluses that is very far from it. The only way for a plus to be originated in the cluster of zeroes containing the origin at time $\tau_0 + e^{\beta k^+}$ is that this cluster becomes large enough from the coalescence of many droplets of typical size $e^{\beta(k_0+\frac{\gamma_0}{2})}$.

Our difficulties in taking into account the coalescence of the droplets of zeroes at time τ_0 arise from the fact that we are exactly at the percolation threshold in the sense of [AL] and we cannot use the methods of Aizenman and Lebowitz as it was done in [DSch1]. Indeed even if we know from Theorem DS that for $k < k_0$ coalescence has a negligible effect on the average speed of growth and on the other hand from Theorem 2 that, for $k > k_0$, coalescence happens on super-exponential scale, we cannot say anything about the case $k = k_0$.

¿From heuristic analysis we can argue that the probability that the cluster of zeroes including the origin has a large enough volume to allow the nucleation of the pluses is separated from zero and one and hence the second of (2.26) is necessary for Theorem 3.

A similar problem we have in order to prove Theorem 5, where to give the right distance from the origin of the pluses we would like to rule out the possibility that the origin becomes occupied because of a large-scale bootstrap percolation phenomenon (see [AL] and [DSch1]).

Finally we remark that the estimate on τ_{0+} given by Theorem 3 is far from being optimal. In order to give the right asymptotics for τ_{0+}, it is necessary to study the interface between zeroes and minuses from a "mesoscopic" point of view. Indeed to deal with the pursuit of the pluses it is crucial to have this interface regular on scale $e^{\beta k_1}$ in order for the droplet of pluses inside the droplet of zeroes to have enough room to reach its asymptoptic speed of growth. We can argue, on heuristic grounds, that this interface is regular on scale $e^{\beta\frac{\gamma_0}{2}}$. This result would also improve the estimates of Theorems 4 and 5.

Aknowledgements

We want to express thanks to Roberto Schonmann who made available his results with P. Dehghanpour before publication. This work has been partially supported by the grant CHRX-CT93-0411 and CIPA-CT92-4016 of the Commission at European Communities.

References

[AL] M. Aizenman and J. L. Lebowitz " Metastability effects in bootstrap percolation" J. Phys. A **21** (1988), 3801–3813

[BC] G. Ben Arous, R. Cerf " Metastability of the three-dimensional Ising model on a torus at very low temperature". Preprint. (1995)

[CCO] D. Capocaccia, M. Cassandro, E. Olivieri: " A study of metastability in the Ising model" Commun. Math. Phys. **39** 185-205, (1974)

[CGOV] M.Cassandro, A.Galves, E.Olivieri, M.E.Vares, "Metastable behavior of stochastic dynamics: a pathwise approach", J.Stat.Phys.**35**, 603-634 (1984)

[CiL] E.Cirillo, J. Lebowitz "Metastability in the two dimensional ising model with free boundary conditions." Preprint. Rutgers, 1997

[CiO] E.Cirillo, E.Olivieri,"Metastability and nucleation for the Blume -Capel model: different mechanisms of transition" Journ. Stat. Phys. **83**, 473-571 (1996)

[DSch1] P. Dehghanpour, R. Schonmann, " A Nucleation-and-growth model", Preprint (1996)

[DSch2] P. Dehghanpour, R. Schonmann," Metropolis dynamics relaxation via nucleation and growth" Preprint (1996)

[FW] M. Freidlin, A. Wentzell, " Random perturbations of Dynamical Systems", Springer, (1984)

[KO1] R.Kotecky, E.Olivieri, "Droplet dynamics for asymmetric Ising model" Journ. Stat. Phys. **70** , 1121-1148, (1993).

[KO2] R.Kotecky, E.Olivieri "Shapes of growing droplets - a model of escape from a metastable phase" Journ. Stat. Phys. **75** , 409-507, (1994).

[KS] H. , Kesten, R. Schonmann, " On some growth models with a small parameter", Probab. Theory Relat. Fields,**101**,434-468, (1995)

[Li] T.M.Liggett, Interacting particle systems, Springer Verlag (1985)

[MOS1] F.Martinelli, E.Olivieri, E.Scoppola, "On the Swendsen and Wang dynamics II: Critical droplets and homogeneous nucleation at low temperature for the two dimensional Ising model", J.Stat.Phys. **62**, 135-159 (1991)

[MOS2] F.Martinelli, E.Olivieri, E.Scoppola, "Metastability and exponential approach to equilibrium for low temperature stochastic Ising models", J.Stat.Phys **61**, 1105 (1990)

[NaO] F.R.Nardi, E.Olivieri,"Low temperature stochastic dynamics for an Ising model with alternating field."Markov Proc. Rel. Field **2**, 117-166 (1996)

[N] E.J. Neves," A discrete variational problem related to Ising droplets at low temperatures", Journ. Stat. Phys. **80**, 103-123, (1995)

[NSch1] E.J. Neves, R.H. Schonmann, "Behaviour of droplets for a class of Glauber dynamics at very low temperatures" Comm. Math.Phys. **137**, 209 (1991).

[NSch2] E.J. Neves, R.H. Schonmann, "Critical Droplets and Metastability for a Glauber Dynamics at Very Low Temperatures", Prob.Theor.Rel.Fields **91**, 331 (1992)

[OS1] E. Olivieri, E. Scoppola, " Markov chains with exponentially small transition probabilities: First exit problem from a general domain -I. The reversible case" JSP **79**, 613 (1995)

[OS2] E. Olivieri, E. Scoppola, " Markov chains with exponentially small transition probabilities: First exit problem from a general domain -II The general case". In press on JSP

[OS3] E. Olivieri, E. Scoppola, " Metastability and typical exit paths in stochastic dynamics" Preprint (1997)

[PL1] O.Penrose, J.L.Lebowitz "Rigorous treatment of metastable states in van der Waals theory", J.Stat.Phys. **3**, 211-236 (1971)

[PL2] O.Penrose, J.L.Lebowitz "Towards a rigorous molecular theory of metastability" In Fluctuation Phenomena (second ed.) E. W.Montroll, J.L.Lebowitz ed., North Holland Publ. (1987)

[RTMSi] P. Rikvold, H. Tomita, S. Miyashita, and S. Sides, " Metastable lifetimes in a kinetic Ising model: dependence on field and system sizes, Phys. Rev. E**49**, 5080-5090, (1994)

[Sch1] R.Schonmann, "The pattern of escape from metastability of a stochastic Ising model", Comm. Math. Phys. **147**, 231-240 (1992)

[Sch2] R.Schonmann, "Slow droplet driven relaxation of stochastic Ising models in the vicinity of the phase coexistence region", Comm. Math. Phys. **161**, 1-49 (1994)

[Sch3] R.Schonmann, "Theorems and conjectures on the droplet driven relaxation of stochastic Ising model" in Probability and Phase Transition, Ed. G.Grimmett, NATO ASI Series, Kluwer Ac. Publ.pg. 265-301 (1994)

[SchSh] R.Schonmann, S. Shlosman, "Wulff droplets and the metastable relaxation of kinetic Ising model" Preprint (1997)

[S1] E.Scoppola " Renormalization group for Markov chains and application to metastability", Jour. Stat. Phys. **73**, 83 (1993)

[S2] E.Scoppola, "Metastability for Markov chains: a general procedure based on renormalization group ideas" in Probability and Phase Transition, Ed. G.Grimmett, NATO ASI Series, Kluwer Ac. Publ. (1994)

[ToMi] H.Tomita, S.Miyashita "Statistical properties of the relaxation process of metastable states in the kinetic Ising model", Phys.Rev.B. Condensed Matter **46**, 8886-8893 (1992)