Stability of Cahn-Hilliard Fronts

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Abstract

We prove stability of the kink solution of the Cahn-Hilliard equation $\partial_t u = \partial_x^2(-\partial_x^2 u - u/2 + u^3/2)$, $x \in I\!R$. The proof is based on an inductive Renormalization Group method and we obtain detailed asymptotics of the solution as $t \to \infty$.

1 Introduction

The Cahn–Hilliard (CH) equation is given in general by

$$\partial_t u = \Delta(-\Delta u + F'(u)), \tag{1.1}$$

where $u = u(x,t)$, $x \in \Omega \in I\!R^n$ and F is a smooth function having two degenerate minima, e.g.

$$F(u) = \tfrac{1}{8}u^4 - \tfrac{1}{4}u^2 \tag{1.2}$$

which will be considered it the present paper.

This equation describes phase separation in binary alloys. When such compounds are cooled rapidly to low temperatures below the critical point they tend to form quickly inhomogenities forming a granular structure. This phenomenon is called the Spinodal Decomposition and the CH equation is used to model it, the two phases being described by the stationary solutions $u = \pm 1$ of (1.1,1.2). The Laplacean multiplying the RHS assures conservation of mass i.e. $\int u$ (with suitable boundary conditions).

The presence of the fourth order derivative in (1.1) makes the mathematical analysis of the CH equation much harder than analogous second order equations. The absence of a spectral gap in unbounded domains Ω due to the Laplacean multiplying the RHS (see Section 2) also complicates matters. Therefore mathematical results on the CH equation are rather few. One of the few results for unbounded domains is the proof [3] of the L^∞

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boundedness of the flow in $I\!R^n$. For bounded domains Ω more is known, see the references in [4] where the so called Hele-Shaw limit is constructed and also [5] and [6].

¿From the point of view of the Spinodal Decomposition a natural problem would be to characterize the time devolopment of a random (smooth, bounded) initial data or less ambitiously one describing a patchwork of regions where $u = \pm 1$ smoothly joined to each other. While this is presently too hard, an interesting question is the study of the behaviour of the CH flow in a neighbourhood of a stationary solution interpolating between the constant solutions. For $\Omega = I\!R$ and F given by (1.2) this is given by

$$u_0(x) = \tanh(x/2)$$

and in higher dimensions replacing x by some coordinate x_i on the RHS. The linearization of the CH equation around this solution is believed to have a very interesting spectrum that is reflected in peculiar scaling properties of the long wave length fluctuations observed in the interface [7]. Thus the mathematical problem to study is to prove the non-linear stability of the solution u_0 and to study the detailed long time asymptotics of the solution with initial data close to u_0. In this paper we carry out this analysis for the 1-dimensional case, $\Omega = I\!R$. We prove the stability of u_0 in a weighted L^∞ norm. We use the so-called Renormalization Group (RG) method [1] that was designed for the study of long time asymptotics of non-linear parabolic equations. The idea is explained in Section 2. A crucial part in our analysis is played by a careful study of the heat kernel of the linearization around u_0. For this we use the formalism developed by Pego and Weinstein [2] for the study of resolvents of linear differential operators.

To describe the result, let us fix a number $p > 2$ and define the Banach space

$$X := \left\{ h \in C^0(I\!R) \;\middle|\; ||h||_X := \sup_{x \in I\!R} (1 + |x|^{p+1}) |h(x)| < \infty \right\}. \tag{1.3}$$

Our main result is as follows:

Theorem 1.1. *There exists a $\delta > 0$ such that for $h \in X$, $||h||_X < \delta$, the Cauchy problem*

$$\begin{aligned} \partial_t u &= \partial_x^2(-\partial_x^2 u - u/2 + u^3/2) \quad \text{for } t > 1, & (1.4)\\ u(x,1) &= u_0(x) + h(x), & (1.5) \end{aligned}$$

where $u_0(x) = \tanh(x/2)$, has a unique smooth solution $u(x,t)$, $t \in]1, \infty[$ satisfying

$$||u(\cdot, t) - u_0(\cdot - x_0)||_\infty \to 0 \qquad \text{as } t \to \infty, \tag{1.6}$$

for some $x_0 \in I\!R$ depending only on h.

Remark. We obtain much more detailed asymptotics:

$$u(x_0 + x, t) = u_0(x) + \frac{1}{\sqrt{4\pi t}} \frac{d}{dx} (e^{-\frac{x^2}{4t}} (A u_0(x) + B)) + o(t^{-1})$$

in sup norm (see Proposition 2.1.) with A, B depending on the initial data. Moreover, $x_0 = 0$ if and only if $\int_{I\!R} h = 0$; for more details on x_0, see the beginning of the next section.

The rest of the paper is organized as follows. In Section 2 we explain the RG method and prove the Theorem given some bounds for the linearized problem (Lemmas 2.3., 2.5. and 2.9). In Section 3 these bounds are reduced to pointwise bounds for the heat kernel of the linearized problem (Proposition 3.2) which are proven in Section 4 by a careful analysis of the corresponding resolvent.

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2 The Renormalization Group

We start by transforming (1.4)–(1.5) into a form more suitable for an application of the RG–method. First, we may reduce the problem to the case where h in (1.5) is of the form

$$h = \partial_x f,$$

where $f \in C^1(\mathbf{R})$ and goes to zero at infinity. This is clearly so, if $\int_{I\!R} h = 0$. If $\int_{I\!R} h \neq 0$, it suffices to write the problem (1.4)–(1.5) for $\tilde{u}(x,t) := u(x-x_0,t)$ replacing $u(x,t)$ and $\tilde{h}(x) := h(x-x_0) + u_0(x-x_0) - u_0(x)$ replacing $h(x)$, where x_0 is chosen such that

$$\int_{I\!R} h(x-x_0) + u_0(x-x_0) - u_0(x)dx = 0$$

which for $||h||_X$ small has a unique solution $x_0(h)$ since $\int u_0' dx \neq 0$.

Writing u in the form

$$u = u_0 + \partial_x v$$

and denoting

$$\begin{aligned} \mathcal{L} &:= \partial_x(-\partial_x^2 + 1 + V)\partial_x, \\ V(x) &:= -\frac{3}{2}\cosh^{-2}(\frac{x}{2}), \end{aligned} \tag{2.1}$$

we see that (1.4)–(1.5) read in terms of v:

$$\partial_t v = \mathcal{L}v + \frac{3}{2}\partial_x(u_0(\partial_x v)^2) + \frac{1}{2}\partial_x(\partial_x v)^3, \tag{2.2}$$

$$v(x,1) = f(x). \tag{2.3}$$

To describe the asymptotics, we need some definitions. Let ϕ denote the fundamental solution of the heat equation

$$\phi(x,t) := (4\pi t)^{-\frac{1}{2}} e^{-\frac{x^2}{4t}}, \phi(x) := \phi(x,1) = (4\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{4}} \tag{2.4}$$

and define for $A, B \in I\!R$

$$\psi_{A,B}(x,t) := (Au_0(x) + B)\phi(x,t) \tag{2.5}$$

For $t \geq 1$ we define the Banach spaces X_t of continuously differentiable functions whose norm

$$||f||_t := \sup_{x\in I\!R}(1+|x|^p)|f(x)| + \sup_{x\in I\!R} t^{-1/2}(1+|x|^p)|f'(x)| \tag{2.6}$$

is finite.

Theorem 1.1 follows from the following Proposition which also gives more detailed asymptotics for the solution:

Proposition 2.1. *There exists a $\delta > 0$ such that for $f \in X_1$, $||f||_1 < \delta$, the Cauchy problem (2.2)–(2.3) has a unique global solution $v \in C([1,\infty[, X_t)$ such that*

$$\sqrt{t}||v(\sqrt{t}\cdot, t) - \psi_{A,B}(\sqrt{t}\cdot, t)||_t \to 0 \qquad \text{as } t \to \infty$$

for some $A, B \in \mathbb{R}$.

Proof of Theorem 1.1. Let $h \in X$. Then, since $f(x) = \int_{-\infty}^{x} h = -\int_{x}^{\infty} h$ (1.3) implies $|f(x)| \leq C(1+|x|)^{-p}||h||_X$. Thus $||f||_1 \leq C||h||_X$ and for $||h||_X$ small Proposition 2.1. implies that the sup-norm of $\sqrt{t}\frac{1}{\sqrt{t}}\partial_x(v(\sqrt{t}\cdot, t) - \psi_{A,B}(\sqrt{t}\cdot, t))$ goes to zero as $t \to \infty$. Hence in particular the sup-norm of $u(x,t) - u_0(x - x_0) = \partial_x v(x,t)$ goes to zero as $t \to \infty$ i.e. Theorem 1.1. Smoothness is standard. □

Before we proceed with the proofs, we will briefly explain the main intuition behind the proof.

What makes the problem difficult is that the linearized problem $\mathcal{L}$ has continuous spectrum extending to zero. This is seen by first writing $\mathcal{L} = \partial_x H \partial_x$ and noting that H is nonnegative in $L^2(\mathbb{R})$. Indeed, H is selfadjoint and bounded from below with eigenvalue 0 and eigenfunction $\partial_x u_0$ which is pointwise positive. Hence positivity of H follows from the Perron-Frobenius theorem. Thus $\mathcal{L}$ is nonpositive and from its behaviour at infinity one infers that its essential spectrum is $(-\infty, 0]$. This fact indicates that the perturbation v will tend to zero only as some inverse power of t. The RG is an inductive scheme that allows to treat such problems where the linearization has no gap in its spectrum.

Let us pick a number $L > 1$ which we will fix later in the proof. Let n be a nonnegative integer. Consider the following scaling

$$v_n(x,t) := L^n v(L^n x, L^{2n} t). \tag{2.7}$$

The equation (2.2) is then equivalent to

$$\partial_t v_n = \mathcal{L}_n v_n + \frac{3}{2} L^{-2n} \partial_x (u_n (\partial_x v_n)^2) + \frac{1}{2} L^{-4n} \partial_x (\partial_x v_n)^3, \tag{2.8}$$

where $u_n(x) = u_0(L^n x)$ and

$$\mathcal{L}_n := \partial_x(-L^{-2n}\partial_x^2 + 1 + V_n)\partial_x, \ V_n(x) := -\frac{3}{2}\cosh^{-2}\frac{L^n x}{2}$$

The scaling (2.7) is typical of parabolic problems of heat equation type. Indeed, the heat equation is invariant under that scaling and this fact can be used to obtain the usual asymptotics, also in nonlinear situations [1]. In the present situation we see that the fourth derivative picks a small constant (is "irrelevant" in the terminology of [1]) and we are left with a Laplacian and a rather singular looking fourth order operator. Note also how the nonlinear terms pick small coefficients as $n \to \infty$. The idea of the RG is to iteratively construct the solution to the original problem (2.2) on time intervals $[L^{2(n-1)}, L^{2n}]$ by reducing it there to the problem (2.8). We proceed as follows.

Let X_n be the Banach space (2.6) with t taken to be L^{2n} (with a harmless abuse of notation) i.e. the norm in X_n is given by

$$||f||_n := \sup_{x\in\mathbb{R}}(1+|x|^p)|f(x)| + \sup_{x\in\mathbb{R}} L^{-n}(1+|x|^p)|f'(x)| \tag{2.9}$$

For $f \in X_n$ let v_n solve (2.8) with initial data $v_n(x,1) = f(x)$. The n:th RGT $R_n : X_n \to X_{n+1}$ is then defined as

$$(R_n f)(x) = L v_n(Lx, L^2) \tag{2.10}$$

(Proposition 2.2. below shows that R_n is indeed well defined). It is now easy to see that if v solves (2.2), then, for $t = L^{2n}$,

$$\sqrt{t}v(\sqrt{t}x, t) = L^n v(L^n x, L^{2n}) = R_n \circ \ldots \circ R_1 f := f_n. \tag{2.11}$$

We shall proceed conversely: by controlling f_n inductively in n i.e. by controlling R_n we construct the global solution v to the original problem. Our goal will be to show that f_n in (2.11) converges to the (approximate) fixed points of R_n.

We will show that the nonlinear terms in (2.8) indeed are small perturbations as n becomes large and R_n is approximately given by the linear operator

$$f \to L(e^{(L^2-1)\mathcal{L}_n} f)(L\cdot). \tag{2.12}$$

In the case of heat equation ($\mathcal{L}_n$ replaced by ∂_x^2) this linear operator has a unique fixed point (up to a multiplicative constant), $\phi(x) = (4\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{4}}$ in spaces like X_n. In the present case, it turns out, there are *two* fixed points as n becomes large. These are approximately $u_n\phi$ and ϕ.

We write

$$f_n = \psi_n + g_n \tag{2.13}$$

where

$$\psi_n(x) = L^n \psi_{A_n,B}(L^n x, L^{2n}) = (A_n u_n(x) + B)\phi(x) \tag{2.14}$$

The constants A_n and B are chosen such that g_n contracts under the linear RG (2.12) i.e. that it has zero "projection" to the fixed point directions in X_n. Explicitely, this turns out to mean that

$$\int g_n = 0 = \int u_n g_n. \tag{2.15}$$

Since by simple calculation $\int \psi_n = B$ and $\int \psi_n u_n / \int \phi u_n^2 = A_n$, this means that we have to take $B = \int f_n$ and $A_n = \int f_n u_n / \int \phi u_n^2$ and B can be chosen n–independent since the equation (2.2) preserves the integral of v over the space:

$$B = \int f_0,$$

where f_0 is the initial data of (2.2). A_n will not be given in such a straightforward way in terms of the initial data: it will be constructed iteratively.

The following Proposition describes the properties of R_n.

Proposition 2.2. *Given $L < \infty$ there exists $\eta > 0$ such that if $f_n \in X_n$ is given by (2.13), (2.14) with (2.15) and $|A_n| + |B| + \|g_n\|_n := \eta_n \leq \eta$ then $f_{n+1} = R_{n+1} f_n$ is of the form (2.13) with*

$$|A_{n+1} - A_n| + \|g_{n+1}\|_{n+1} \leq CL^{-1}\|g_n\|_n + C(L)(L^{-n}(|A_n| + |B|) + \eta_n \|g_n\|_n). \tag{2.16}$$

Remark. Here and below, we denote by C a generic constant independent of L which may vary from place to place, even in the same equation, and by $C(L)$ a constant depending on L.

Proposition 2.2 allows us to give the

Proof of Proposition 2.1. We use the RGT introduced above. We may assume that $\eta \leq 1$. Then, if $\eta_n \leq \eta$ we have from (2.16) and $L^{-1} \leq 1$,

$$\eta_{n+1} \leq (1 + C + C(L))\eta_n. \tag{2.17}$$

Let $k = k(L)$ be the smallest integer such that $C(L)L^{-k} \leq 1$. If $\|f_0\|_0 \leq \delta$ then $\eta_0 \leq C\delta$ for some C. Thus, from (2.17) we conclude that there exists $\epsilon(L) > 0$ such that if $\delta \leq \epsilon(L)$, then $\eta_n \leq \eta/2$ for $n \leq k$.

Consider next $n > k$. Let $0 < \alpha < 1$. Choose L such that $CL^{-1} = L^{-1+\alpha}$. If $\eta_n \leq \eta$ then from (2.16)

$$|A_{n+1} - A_n| + \|g_{n+1}\|_{n+1} \leq L^{-(1-\alpha)}\|g_n\|_n + L^{-(n-k)}(|A_n| + |B|) + C(L)\eta\|g_n\|_n$$

Hence for L large enough and $\eta \leq \eta(L)$ small enough $\eta_n \leq \eta$ for all n , the sequence A_n converges and

$$\|g_n\|_n \leq C(L)L^{-n(1-\alpha)}$$

with $C(L)$ depending on the initial data.

This proves the convergence in the Proposition for sequence of times $t = L^{2n}$ as $n \to \infty$. For other times, we always replace L^2 here by τL^2 with $1 \leq \tau \leq L^2$. The bounds still hold and the result follows. □

The main input to Proposition 2.2 is the following Lemma on the contractive properties of the linearized RGT.

Lemma 2.3. *Let $L \geq 2$. Then for all $g \in X_n$ such that $\int g = 0 = \int g u_n$*

$$||(Le^{(L^2-1)\mathcal{L}_n} g)(L\cdot)||_{n+1} \leq CL^{-1}||g||_n.$$

Lemma 2.3 will be proven in Section 3. To be able to use it, we need to rewrite the equation (2.8) in a more convenient form. The RG (2.10) consists of solving (2.8) for times $t \in [1, L^2]$. We wish to extend the decomposition (2.13) of the initial data to this time interval. Define

$$\psi_n(x,t) = (A_n u_n(x) + B)\phi(x,t) \tag{2.18}$$

(recall (2.4)) and write the solution of (2.8) as

$$v_n(x,t) = \psi_n(x,t) + w_n(x,t).$$

Then w_n solves

$$\partial_t w_n = \mathcal{L}_n w_n + I_n(x,t) + M_n(x,t,w_n), \tag{2.19}$$

with initial data

$$w_n(x,1) = g_n(x)$$

(since $\psi_n(x,1) = \psi_n(x)$) and I_n and M_n are given by

$$I_n = (\mathcal{L}_n - \partial_t)\psi_n + \frac{3}{2}L^{-2n}\partial_x(u_n(\partial_x\psi_n)^2) + \frac{1}{2}L^{-4n}\partial_x(\partial_x\psi_n)^3 \tag{2.20}$$

and

$$
\begin{aligned}
M_n(w) &= 3L^{-2n}\partial_x(u_n(\partial_x\psi_n)\partial_x w) + \frac{3}{2}L^{-4n}\partial_x((\partial_x\psi_n)^2\partial_x w) \\
&+\frac{3}{2}L^{-2n}\partial_x(u_n(\partial_x w)^2) + \frac{3}{2}L^{-4n}\partial_x((\partial_x\psi_n)(\partial_x w)^2) \\
&+\frac{1}{2}L^{-4n}\partial_x(\partial_x w)^3. \qquad (2.21)
\end{aligned}
$$

Note that even if (2.20,2.21) have small coefficients, $\partial_x\psi_n$ can be as large as L^n so some care is needed. In particular the second term on the RHS of (2.20) has, from the u_n term in ψ_n in (2.18), a contribution bounded only by $L^n e^{-L^n|x|}$ which contributes $\mathcal{O}(L^n)$ to the norm of I_n and so the latter is *not* a small perturbation in our norm. The way out is to extract a subleading asymptotics from w_n as follows.

Consider the function

$$
\chi_n(x,t) = \frac{3A_n^2}{8\pi t}L^{-2n}\partial_x(u_n(\partial_x u_n)^2).
$$

This gives the leading singularity. Indeed, let

$$
i_n = \frac{3}{2}L^{-2n}\partial_x(u_n(\partial_x\psi_n)^2) - \chi_n(x,t). \qquad (2.22)
$$

Then we have

Lemma 2.4. *Let $t \in [1, L^2]$. Then*

$$
|i_n(x,t)| \le CL^{-n}(|A_n| + |B|)^2(e^{-L^n|x|/2} + L^{-n}e^{-\frac{x^2}{L^2}}).
$$

Proof. Using (2.18) we write

$$
i_n = \frac{3A_n^2}{2}L^{-2n}\partial_x\Big(u_n(\partial_x u_n)^2(\phi^2 - \frac{1}{4\pi t})
$$

$$
+2(A_n^2 u_n + A_n B)\partial_x u_n\phi\partial_x\phi + (B^2 + 2A_n B u_n + A_n^2 u_n^2)(\partial_x\phi)^2).
$$

Using, for $j \neq 0$,

$$
|x^i\partial_x^j u_n(x)| \le CL^{(j-i)n}e^{-L^n|x|/2}, \qquad (2.23)
$$

$\partial_x\phi(x,t) = -\frac{x}{2t}\phi(x,t)$, and $(\phi^2 - \frac{1}{4\pi t}) \le Cx^2$ the claim follows by inspection. □

Next we note that, since $u_0(x) = \tanh(x/2)$ is a stationary solution of the CH equation (1.4), we have $\partial_x^2(-L^{-2n}\partial_x^2 u_n - u_n/2 + u_n^3/2) = 0$ and then, after commuting a derivative, we get $\mathcal{L}_n\partial_x u_n = -\partial_x(3u_n(\partial_x u_n)^2)$. Thus

$$
\chi_n = -\mathcal{L}_n\omega_n,\ \omega_n := \frac{A_n^2 L^{-2n}}{8\pi t}\partial_x u_n \qquad (2.24)
$$

and the equation (2.19) reads

$$
\partial_t(w_n - \omega_n) = \mathcal{L}_n(w_n - \omega_n) + J_n(x,t) + M_n(x,t,w_n), \qquad (2.25)
$$

where

$$J_n = (\mathcal{L}_n - \partial_t)\psi_n + i_n + \frac{1}{2}L^{-4n}\partial_x(\partial_x\psi_n)^3 - \partial_t\omega_n. \tag{2.26}$$

To proceed, we write (2.25) as an integral equation

$$w_n(t) = e^{(t-1)\mathcal{L}_n}g_n + j_n(t) + m_n(w_n, t) \tag{2.27}$$

where

$$j_n = \omega_n(t) - e^{(t-1)\mathcal{L}_n}\omega_n(1) + \int_0^{t-1} d\tau \int_{I\!R} d\xi K_n(x,\xi,\tau)J_n(\xi, t-\tau) \tag{2.28}$$

$$m_n = \int_0^{t-1} d\tau \int_{I\!R} d\xi K_n(x,\xi,\tau)M_n(\xi, t-\tau, w_n(t-\tau)) \tag{2.29}$$

and where we defined the integral kernel by

$$K_n(x,\xi,t) := e^{t\mathcal{L}_n}(x,\xi). \tag{2.30}$$

To solve (2.27) introduce the space $C([1,L^2], X_n)$ with the norm

$$|||w|||_n = \sup_{t\in[1,L^2]}\|w(\cdot,t)\|_n.$$

We need a couple of Lemmas.

Lemma 2.5. *Let $g \in X_n$. Then,*

$$|||e^{(t-1)\mathcal{L}_n}g|||_n \leq C(L)\|g\|_n$$

Lemma 2.6. *Assume that $|A_n| \leq 1$, $|B| \leq 1$. Then,*

$$|||j_n|||_n \leq C(L)L^{-n}(|A_n| + |B|)$$

Lemma 2.7. *Let $w \in C([1,L^2], X_n)$. Assume that $|A_n| \leq 1$, $|B| \leq 1$, $|||w|||_n \leq 1$. Then,*

$$|||m_n(w)|||_n \leq C(L)\lambda(w)|||w|||_n, \tag{2.31}$$

where $\lambda(w) := |A_n| + |B| + |||w|||_n$ and, for $w^i \in C([1,L^2], X_n)$, $i = 1, 2$,

$$|||m_n(w^1) - m_n(w^2)|||_n \leq C(L)(\lambda(w_1) + \lambda(w_2))|||w^1 - w^2|||_n. \tag{2.32}$$

We can now give the

Proof of Proposition 2.2. We solve (2.27) by using the contraction mapping principle. Consider the ball $\mathbf{B}$ in $C([1,L^2], X_n)$ consisting of functions w satisfying

$$|||w - e^{(t-1)\mathcal{L}_n}g_n - j_n|||_n \leq L^{-n}(|A_n| + |B|) + \|g_n\|_n.$$

By Lemmas 2.5 and 2.6, $w \in \mathbf{B}$ is bounded by $(C(L)+1)(L^{-n}(|A_n|+|B|)+\|g_n\|_n)$ so for η small (2.27) maps $\mathbf{B}$ into itself by (2.31) and is a contraction there by (2.32). Hence, by the contraction mapping principle, there is a unique solution that, moreover, satisfies

$$|||j_n + m_n(w)|||_n \leq C(L)((L^{-n}(|A_n| + |B|) + \eta_n\|g_n\|_n) \tag{2.33}$$

To prove (2.16), let us introduce the notation $P_n(f) := \int u_n f (\int u_n^2 \phi)^{-1}$ for $f \in X_n$. Then $A_n = P_n(v_n(\cdot, 1))$ and

$$A_{n+1} = LP_{n+1}(v_n(L\cdot, L^2)) = LP_{n+1}(\psi_n(L\cdot, L^2) + w_n(L\cdot, L^2)) = A_n + LP_{n+1}(w_n(L\cdot, L^2))$$

where the last equality follows from $L\psi_n(Lx, L^2) = (A_n u_{n+1}(x) + B)\phi(x)$ since $L\phi(Lx, L^2) = \phi(x)$, $u_n(Lx) = u_{n+1}(x)$ and $\int u_{n+1}(x)\phi(x)dx = 0$. Thus, by Lemma 2.3 and (2.33),

$$\begin{aligned} |A_{n+1} - A_n| \leq \; & C\|Lw_n(L\cdot, L^2)\|_{n+1} \\ & \leq C\|Le^{(L^2-1)\mathcal{L}_n} g_n(L\cdot)\|_{n+1} + \|L(j_n + m_n)(w, L\cdot, L^2)\|_{n+1} \\ & \leq CL^{-1}\|g_n\|_n + C(L)((L^{-n}(|A_n| + |B|) + \eta_n\|g_n\|_n). \end{aligned} \tag{2.34}$$

for some $C(L)$. As for g_{n+1}, we have

$$g_{n+1}(x) = L(e^{(L^2-1)\mathcal{L}_n} g_n)(Lx) + L(j_n + m_n)(w, Lx, L^2) + L\psi_n(Lx, L^2) - \psi_{n+1}(x, 1)$$

and so from Lemma 2.3 and (2.33)

$$\begin{aligned} \|g_{n+1}\|_{n+1} \leq \; & CL^{-1}\|g_n\|_n + C(L)((L^{-n}(|A_n| + |B|) + \eta_n\|g_n\|_n) \\ & +|A_{n+1} - A_n|\|u_{n+1}\phi\|_{n+1}. \end{aligned} \tag{2.35}$$

Since $|u_{n+1}| \leq 1$ and $|\partial_x u_{n+1}| \leq L^n$, we see from (2.9) that $\|u_{n+1}\phi\|_{n+1} \leq C\sup_x |e^{-x^2/4}(1+|x|^p) \leq C$. (2.16) follows from (2.34),(2.35). □

Thus only Lemmas 2.3 and 2.5-2.7 remain to be proven. Lemmas 2.3 and 2.5 are proven in Section 3. Lemmas 2.6 and 2.7 will be deduced from the following estimates for K_n which are also proven in Section 3:

Lemma 2.9. *The kernel K_n defined by (2.30) satisfies the following bounds:*

$$\int_0^{t-1} d\tau \int d\xi |\partial_x^i K_n(x, \xi, \tau)| e^{-L^n|\xi|/2} \leq \frac{C(L)L^{-n(1-i)}}{1 + |x|^p} \tag{2.36}$$

$$\int_0^{t-1} d\tau \int d\xi |\partial_x^i K_n(x, \xi, \tau)|(1 + |\xi|^p)^{-1} \leq \frac{C(L)L^{ni}}{1 + |x|^p} \tag{2.37}$$

$$\int_0^{t-1} d\tau \int d\xi |\partial_x^i \partial_\xi K_n(x, \xi, \tau)|(1 + |\xi|^p)^{-1} \leq \frac{C(L)L^{ni}}{1 + |x|^p} \tag{2.38}$$

for $t \leq L^2$ and $i = 0, 1$.

Proof of Lemma 2.6. We need to estimate j_n given in (2.28) with J_n given by (2.26) . i_n is bounded in Lemma 2.4. and, using the bounds (2.23), we get, for all $t \in [1, L^2]$,

$$|\frac{1}{2}L^{-4n}\partial_x(\partial_x\psi_n)^3| \leq C(|A_n| + |B|)^3(e^{-L^n|x|/2} + L^{-4n}e^{-\frac{x^2}{4L^2}}) \tag{2.39}$$

$$L^{-n}|\partial_x\omega_n(x, t)| + |\partial_t\omega_n(x, t)| + |\omega_n(x, t)| \leq CL^{-n}|A_n|^2 e^{-L^n|x|/2}. \tag{2.40}$$

We use (2.40) to bound $\partial_t \omega_n$ in (2.26) and $\omega_n(t)$ in (2.28) (note that we need a bound on $L^{-n}\partial_x \omega_n$ to control $\|\omega_n\|_n$). And we use (2.40) and Lemma 2.5 to bound $e^{(t-1)\mathcal{L}_n}\omega_n(1)$ in (2.28). Thus, only $(\mathcal{L}_n - \partial_t)\psi_n$ remains to be bounded. From (2.18) and the facts that $\mathcal{L}_n u_n = 0$, $(\partial_x^2 - \partial_t)\phi = 0$ we obtain

$$
\begin{aligned}
(\mathcal{L}_n - \partial_t)\psi_n = & \ (\mathcal{L}_n - \partial_t)(A_n u_n \phi + B\phi) \\
= & \ -L^{-2n} A_n \sum_{j=0}^{3} \binom{4}{j} \partial_x^j u_n \partial_x^{4-j}\phi + 2A_n \partial_x u_n \partial_x \phi \\
+ & \ V_n A_n \partial_x u_n \partial_x \phi + \partial_x (V_n (A_n u_n + B))\partial_x \phi \\
+ & \ V_n (A_n u_n + B)\partial_x^2 \phi + L^{-2n} B \partial_x^4 \phi.
\end{aligned}
$$

Using the estimates (2.23) and noting that $\partial_x \phi(x,t) = -\frac{x}{2t}\phi(x,t)$, we see that, for $t \in [1, L^2]$,

$$|(\mathcal{L}_n - \partial_t)\psi_n| \leq C(|A_n| + |B|)(e^{-L^n|x|/2} + L^{-2n} e^{-x^2/(4L^2)}).$$

As we saw in (2.39, 2.40), the other terms in (2.26) fit to this same bound. Using $e^{-x^2/(4L^2)} \leq C(L)(1+|x|^p)^{-1}$ Lemma 2.6 follows from (2.36, 2.37). □

Proof of Lemma 2.7. Note that M_n (2.21) is of the form $M_n = \partial_x \tilde{M}_n$. Integrating by parts, we have

$$m_n = -\int_0^{t-1} dt \int_{I\!R} d\xi \partial_\xi K_n(x,\xi,\tau)\tilde{M}_n(\xi,\tau,w_n(\tau)).$$

Using $|u_n| \leq 1$, $|\partial_x \psi_n| \leq C(|A_n| + |B|)L^n$, $|A_n| \leq 1$, $|B| \leq 1$ and $|\partial_x w_n| \leq CL^n \|w_n\|_n (1 + |x|^p)^{-1}$ we get

$$|\tilde{M}_n(\xi,\tau,w_n(\tau))| \leq C(L)(|A_n| + |B| + |||w_n|||_n)|||w_n|||_n (1 + |\xi|^p)^{-1}$$

and (2.31) follows from (2.38). (2.32) is similar. □

3 Estimates for the semigroup

We will now turn to the study of the integral kernel

$$K(x,\xi,t) = e^{t\mathcal{L}}(x,\xi). \tag{3.1}$$

The crux of our analysis is a detailed study of K for t tending to ∞. However, we also need a small time result which shows that K is a perturbation of the constant coefficient problem where $V = 0$. For this let $\mathcal{L}_\infty := -\partial_x^4 + \partial_x^2$ and consider the corresponding semigroup

$$e^{t\mathcal{L}_\infty}(x,\xi) = \int_{I\!R} e^{-t(p^4+p^2)} e^{ip(x-\xi)} \frac{dp}{2\pi} =: K_\infty(x-\xi,t).$$

A straightforward shift of contour gives, for $i,j \in \{0,1\}$,

$$|\partial_x^i \partial_\xi^j K_\infty(x,t)| \leq Ct^{-(1+i+j)/4} e^{-t^{-1/4}|x-\xi|}. \tag{3.2}$$

Define $\tilde{K}(x,\xi,t)$ by

$$K(x,\xi,t) = K_\infty(x-\xi,t) + \tilde{K}(x,\xi,t). \tag{3.3}$$

Then we have the easy

Proposition 3.1. *There exists $t_0 > 0$ such that for $0 < t < t_0$ and $i, j \in \{0,1\}$*

$$|\partial_x^i \partial_\xi^j \tilde{K}(x,\xi,t)| \leq Ct^{-(i+j)/4} e^{-t^{-1/4}|x-\xi|}. \tag{3.4}$$

Proof. We expand

$$\begin{aligned} e^{t\mathcal{L}} = \; & e^{t(\mathcal{L}_\infty + \partial_x V \partial_x)} = e^{t\mathcal{L}_\infty} + \sum_{k=1}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{k-1}} dt_k e^{(t-t_1)\mathcal{L}_\infty} \\ & \cdot \partial_x V \partial_x e^{(t_1-t_2)\mathcal{L}_\infty} \partial_x V \partial_x \ldots \partial_x V \partial_x e^{t_k \mathcal{L}_\infty}, \end{aligned} \tag{3.5}$$

where, as we will see, the series for the corresponding kernels converges for small enough t uniformly in x and y with bounds such that the series itself converges in the space of bounded linear operators $X \to X$. Integrating by parts the first derivative of $\partial_x V \partial_x$ we find that the kth term of the series expansion of $\partial_x^i \partial_\xi^j e^{t\mathcal{L}}(x,\xi)$ is

$$\begin{aligned} J_k := \; & \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{k-1}} dt_k \int_{I\!R} d\xi_1 \ldots \int_{I\!R} d\xi_k \partial_x^i \partial_{\xi_1} K_\infty(x-\xi_1, t-t_1) \\ & V(\xi_1) \partial_{\xi_1} \partial_{\xi_2} K_\infty(\xi_1 - \xi_2, t_1 - t_2) V(\xi_2) \ldots \partial_{\xi_k} \partial_\xi^j K_\infty(\xi_k, \xi, t_k). \end{aligned} \tag{3.6}$$

Inserting now the bound (3.2) and using

$$\frac{|x-\xi_1|}{(t-t_1)^{1/4}} + \frac{|\xi_1 - \xi_2|}{(t_1-t_2)^{1/4}} + \ldots \frac{|\xi_k - \xi|}{t_k^{1/4}} \geq \frac{|x-\xi|}{t^{1/4}}$$

together with

$$|V(x)| \leq Ce^{-|x|}, \tag{3.7}$$

we get

$$\begin{aligned} |J_k| \leq \; & C^k e^{-t^{-1/4}|x-\xi|} \int (t-t_1)^{-\frac{1}{4}(2+i)} (t_1 - t_2)^{-\frac{3}{4}} \ldots t_k^{-\frac{1}{4}(2+j)} \\ \leq \; & C^k e^{-t^{-1/4}|x-\xi|} t^{-\frac{1}{4}(1+i+j)} t^{k/4}, \end{aligned} \tag{3.8}$$

where we used $\int_0^t (t-\tau)^{-\frac{1}{4}(2+p)} \tau^q d\tau \leq Ct^{q+\frac{1}{2}-p/4}$ for C independent of $q \geq -\frac{3}{4}$, $p = 0, 1$. Hence,

$$|J_k| \leq t^{-(1+i+j)/4} e^{-t^{-1/4}|x-\xi|} (\frac{t}{2t_0})^{k/4} \tag{3.9}$$

where $t_0 > 0$ is independent on k. Thus for $0 \leq t \leq t_0$, we get

$$|\partial_x^i \partial_\xi^j \tilde{K}(x,\xi,t)| \leq t^{-(1+i+j)/4} e^{-t^{-1/4}|x-\xi|} \sum_{k=1}^{\infty} (\frac{t}{2t_0})^{k/4} \leq Ct^{-(i+j)/4} e^{-t^{-1/4}|x-\xi|}$$

which is our claim. □

The larger times require a more detailed analysis of the resolvent, which yields:

Proposition 3.2. *There exists* $\epsilon > 0$ *such that for all* $t \geq t_0$

$$e^{t\mathcal{L}}(x,\xi) = K^*(x,\xi,t) + k(x,\xi,t) \tag{3.10}$$

where for $x \geq 0$

$$K^*(x,\xi,t) = \frac{1}{\sqrt{\pi t}} \begin{cases} 2y_1(x)y_1(-\xi) - y_1(x)e^{-\frac{\xi^2}{4t}} - y_1(-\xi)e^{-\frac{x^2}{4t}} & \xi \leq 0 \\ -2y_1(x)y_1(\xi) + y_1(x)e^{-\frac{\xi^2}{4t}} + y_1(\xi)e^{-\frac{x^2}{4t}} - e^{-\frac{(x+\xi)^2}{4t}} & \xi \geq 0 \end{cases} \tag{3.11}$$

with $y_1(x) = \frac{1}{2}(1 - \tanh(x/2))$ *and, for* $x \leq 0$, $K^*(x,\xi,t) = K^*(-x,-\xi,t)$, *while for* $i,j \in \{0,1\}$,

$$|\partial_x^i \partial_\xi^j k(x,\xi,t)| \leq \frac{C}{t} e^{-\frac{\epsilon}{\sqrt{t}}|x-\xi|} \tag{3.12}$$

This Proposition will be proven in the next section. Now, we shall use the two Propositions above to prove the Lemmas that were not proven in Section 2, i.e. Lemmas 2.3, 2.5 and 2.9. To prove the Lemmas, recall that

$$K_n(x,\xi,t) = L^n K(L^n x, L^n \xi, L^{2n} t) =: K_n^* + k_n, \tag{3.13}$$

where, from (3.11),

$$K_n^*(x,\xi,t) = \frac{1}{\sqrt{\pi t}} \begin{cases} 2y_1(L^n x)y_1(-L^n\xi) - y_1(L^n x)e^{-\frac{\xi^2}{4t}} - y_1(-L^n\xi)e^{-\frac{x^2}{4t}} & \xi \leq 0 \\ -2y_1(L^n x)y_1(L^n\xi) + y_1(L^n x)e^{-\frac{\xi^2}{4t}} + y_1(L^n\xi)e^{-\frac{x^2}{4t}} - e^{-\frac{(x+\xi)^2}{4t}} & \xi \geq 0 \end{cases} \tag{3.14}$$

and, from (3.12),

$$|\partial_x^i \partial_\xi^j k_n(x,\xi,t)| \leq \frac{CL^{-n}}{t} e^{-\frac{\epsilon}{\sqrt{t}}|x-\xi|} \tag{3.15}$$

Moreover, from (3.11), (3.12), we get for $L^2 \geq t \geq L^{-2n} t_0 =: t_n$, using $|\partial_x y_1(L^n x)| \leq CL^n e^{-L^n|x|}$, $|\partial_x e^{-x^2/4t}| = \frac{|x|}{2t} e^{-\frac{x^2}{4t}} \leq \frac{C}{\sqrt{t}} e^{-\frac{|x|}{2\sqrt{t}}}$:

$$\begin{aligned} |\partial_x^i \partial_\xi^j K_n(x,\xi,\tau)| \leq \frac{C}{\sqrt{t}} (&L^{n(i+j)} e^{-L^n(|x|+|\xi|)} + L^{ni} t^{-\frac{1}{2}j} e^{-L^n|x|} e^{-\frac{|\xi|}{2\sqrt{t}}} \\ &+ L^{nj} t^{-\frac{1}{2}i} e^{-L^n|\xi|} e^{-\frac{|x|}{2\sqrt{t}}} + t^{-\frac{1}{2}(i+j)} e^{-\frac{|x|+|\xi|}{2\sqrt{t}}} + L^{-n} t^{-\frac{1}{2}} e^{-\frac{\epsilon|x-\xi|}{\sqrt{t}}}). \end{aligned} \tag{3.16}$$

And we have, from (3.2) and (3.4), for $0 < t \leq t_n$,

$$|\partial_x^i \partial_\xi^j K_n(x,\xi,t)| \leq CL^{\frac{n}{2}(1+i+j)} t^{-(1+i+j)/4} e^{-t^{-1/4} L^{n/2}|x-\xi|}. \tag{3.17}$$

This bound also holds by (3.2) for $K_{\infty n}(x,\xi,t) := L^n K_\infty(L^n x, L^n \xi, L^{2n} t)$.

Proof of Lemma 2.3. Use (3.13), and consider first

$$L \int K_n^*(Lx,\xi,L^2-1) g(\xi) d\xi.$$

First, observe that our assumptions on g, i.e. $\int g(\xi) d\xi = \int g(\xi) \tanh(L^n\xi/2) d\xi = 0$, imply $\int g(\xi) y_1(L^n\xi) d\xi = 0$ since $y_1(L^n\xi) = \frac{1}{2}(1 - \tanh(L^n\xi/2))$. Using this, one computes that

$$L \int K_n'(Lx,\xi,L^2-1) g(\xi) d\xi = 0$$

where K'_n is obtained from K^*_n by setting ξ equal to zero in the exponentials in (3.11):

$$K'_n(x,\xi,t) = \frac{1}{\sqrt{\pi t}}(y_1(L^n x)\tanh(L^n\xi/2) - y_1(-L^n\xi)e^{\frac{-x^2}{4t}}.$$

So, it suffices to prove the bound for (3.11) with $e^{-\frac{\xi^2}{4(L^2-1)}}$ replaced by $e^{-\frac{\xi^2}{4(L^2-1)}} - 1$, and $e^{-\frac{(x+\xi)^2}{4(L^2-1)}}$ replaced by $e^{-\frac{(x+\xi)^2}{4(L^2-1)}} - e^{-\frac{x^2}{4(L^2-1)}}$.

Then, since $|g(\xi)| \le ||g||_n(1+|\xi|^p)^{-1}$ and $|y_1(L^n x)| \le C(1+|x|^p)^{-1}$ for $x \ge 0$, it is enough to observe that

$$\frac{L}{\sqrt{L^2-1}}\int |e^{-\frac{\xi^2}{4(L^2-1)}} - 1|(1+|\xi|^p)^{-1}d\xi \le CL^{-1}, \tag{3.18}$$

(for $p>2$) which can be seen by dividing the integral into $|\xi|\le L$ and $|\xi|\ge L$ and using $|e^{-\frac{\xi^2}{4(L^2-1)}} - 1| \le C\frac{|\xi|^2}{L^2}$ in the first integral, and $|e^{-\frac{\xi^2}{4(L^2-1)}} - 1| \le 2$ in the second.

Similarly, we can bound, by dividing again the integral into $|\xi|\le L$ and $|\xi|\ge L$,

$$\frac{L}{\sqrt{L^2-1}}\int |e^{-\frac{(x+\xi)^2}{4t}} - e^{-\frac{(x)^2}{4t}}|(1+|\xi|^p)^{-1}d\xi \le CL^{-1}. \tag{3.19}$$

The terms with derivatives can be bounded similarly. On the other hand, the term $L\int k_n(Lx,\xi,L^2-1)g(\xi)d\xi$ is, using (3.15), of order L^{-n}. $\square$

Proof of Lemma 2.5. We distinguish between $t\le t_n$ and $t>t_n$. For $t\le t_n$, we write $\partial_x^i K_n(x,\xi,t) := \partial_x^i K_{\infty n}(x,\xi,t) + \partial_x^i \tilde{K}_n(x,\xi,t)$ and, for the first term, we write

$$\begin{aligned} |\int \partial_x^i K_{\infty n}(x,\xi,t)g(\xi)d\xi| &= |\int\, K_{\infty n}(x,\xi,t)\partial_x^i g(\xi)d\xi| \\ &\le ||g||_n L^{ni}L^{\frac{n}{2}}t^{-\frac{1}{4}}\int e^{-t^{-1/4}L^{n/2}|x-\xi|}(1+|\xi|^p)^{-1}d\xi \end{aligned} \tag{3.20}$$

where we used $|\partial_\xi^i g(\xi)| \le ||g||_n L^{ni}(1+|\xi|^p)^{-1}$, and (3.17) with $i=j=0$, which holds for $K_{\infty n}$ (see (3.2)). Finally, (3.20) is bounded by $C||g||_n L^{ni}(1+|x|^p)^{-1}$ using

$$\int e^{-\rho|x-\xi|}(1+|\xi|^p)^{-1}d\xi \le \frac{C}{\rho}(1+|x|^p)^{-1} \tag{3.21}$$

with $\rho = t^{-1/4}L^{n/2}$.

We also have, from scaling (3.4),

$$|\partial_x^i\partial_\xi^j \tilde{K}_n(x,\xi,t)| \le CL^{n(1+\frac{i}{2}+\frac{j}{2})}t^{-(i+j)/4}e^{-t^{-1/4}L^{\frac{n}{2}}|x-\xi|}. \tag{3.22}$$

which implies, for $j=0$,

$$\begin{aligned} &|\int \partial_x^i \tilde{K}_n(x,\xi,t)g(\xi)d\xi| \\ &\le ||g||_n L^{n(1+\frac{i}{2})}t^{-\frac{i}{4}}\int e^{-t^{-1/4}L^{n/2}|x-\xi|}(1+|\xi|^p)^{-1}d\xi \\ &\le CL^{ni}||g||_n(1+|x|^p)^{-1} \end{aligned} \tag{3.23}$$

where we again used (3.21) with $\rho = t^{-1/4}L^{n/2}$ and, for $i=0$, $L^n t^{1/4}L^{-n/2} \le t_n^{1/4}L^{n/2} = t_0^{1/4}$, while, for $i=1$, $L^{3n/2}L^{-n/2} = L^{ni}$.

For $t > t_n$, we use (3.16) with $j = 0$ and $|g(\xi)| \leq ||g||_n(1 + |\xi|^p)^{-1}$. Then, combine $t^{-\frac{1}{2}} \leq CL^n$ (with $C = t_0^{-\frac{1}{2}}$), $e^{-\frac{\epsilon}{\sqrt{t}}|x|} \leq C(L)(1+|x|^p)^{-1}$, $e^{-L^n|x|} \leq C(1+|x|^p)^{-1}$ and (3.21) with $\rho = \epsilon/\sqrt{t}$ or L^n, or the similar bound

$$\int e^{-\frac{|\xi|}{2\sqrt{t}}} d\xi \leq C\sqrt{t}, \tag{3.24}$$

to finish the proof. □

Proof of Lemma 2.9. Write $\int_0^{t-1} = \int_0^{t_n} + \int_{t_n}^{t-1}$. Consider first the $t \geq t_n$ integrals, starting with (2.36). Use (3.16) with $j = 0$, together with $\int_0^{t-1} \tau^{-1/2} d\tau < C$, $t^{-\frac{1}{2}} \leq CL^n$, $e^{-|x|/(2\sqrt{t})} \leq C(L)(1 + |x|^p)^{-1}$, $e^{-L^n|x|} \leq C(1 + |x|^p)^{-1}$,

$$\int e^{-L^n|\xi|/2}|f(\xi)|d\xi \leq CL^{-n} \tag{3.25}$$

for $f(\xi)$ bounded, and

$$\int e^{-|x-\xi|/(2\sqrt{t})} e^{-L^n|\xi|/2} d\xi \leq C(L)L^{-n}(1 + |x|^p)^{-1}. \tag{3.26}$$

The proof of (2.37) is similar, using (3.21) with $\rho = \epsilon/\sqrt{t}$ or L^n, instead of (3.26). For (2.38), we use (3.16) with $j = 1$ and (3.24) to control one power of $t^{-\frac{1}{2}}$.

For $t \leq t_n$, we use (3.17) and (3.21) with $\rho = t^{-1/4}L^{n/2}$, and we use

$$t_n^{1/4} L^{-n/2} L^{\frac{n}{2}(1+i+j)} \int_0^{t_n} \tau^{-(1+i+j)/4} d\tau \leq CL^{n(i+j)-2n} \tag{3.27}$$

to obtain the L^{-n} factor in (2.36). □

4 Analysis of the resolvent

We are left with the proofs of Proposition 3.2. For $t \geq t_0$ we will reduce the study of the semigroup of $\mathcal{L}$ to that of the resolvent $1/(\mathcal{L} + z)$, $z \in \mathbb{C}$ via the formula

$$e^{t\mathcal{L}}(x, \xi) = \int_S e^{-tz}(\mathcal{L} + z)^{-1}(x, \xi)\frac{dz}{2\pi i} \tag{4.1}$$

where the integration contour S goes around the spectrum of $-\mathcal{L}$ i.e. around the positive real axis (directed starting from negative imaginary parts). Writing $z = k^2$ and defining

$$R(x, \xi, k) = \frac{k}{\pi i}(\mathcal{L} + k^2)^{-1}(x, \xi), \tag{4.2}$$

R is analytic in the upper half plane and (4.1) can be written as

$$e^{t\mathcal{L}}(x, \xi) = \int_{S'} e^{-tk^2} R(x, \xi, k)dk \tag{4.3}$$

with an appropriate contour S' in the upper half plane. Note that the large t behaviour of K is controlled by the small k behaviour of R. The main part of this section is devoted to the study of that behaviour. Before turning to that, we note an easy estimate for R if $|\mathrm{Im}k^2|$ is large:

Proposition 4.1. *There exists an $r < \infty$ such that for $|\mathrm{Im}k^2| \geq r$ and $i, j \in \{0, 1\}$*

$$|\partial_x^i \partial_\xi^j R(x, \xi, k)| \leq e^{-|x-\xi|}. \tag{4.4}$$

Proof. Let

$$R_\infty(x, \xi, k) = (\mathcal{L}_\infty + k^2)^{-1}(x, \xi) = \int_{I\!R} \frac{e^{ip(x-\xi)}}{(k^2 - p^4 - p^2)} \frac{dp}{2\pi}.$$

Again, a straightforward shift of contour gives, for $i, j \in \{0, 1\}$,

$$|\partial_x^i \partial_\xi^j R_\infty(x, \xi, k)| \leq C r^{-(3-i-j)/4} e^{-|x-\xi|} \leq C r^{-1/4} e^{-|x-\xi|}, \tag{4.5}$$

for $|\mathrm{Im}k^2| \geq r$, r large enough. Now expand

$$(\mathcal{L} + k^2)^{-1} = \sum_{l=0}^{\infty} (\mathcal{L}_\infty + k^2)^{-1} (\partial_x V \partial_x (\mathcal{L}_\infty + k^2)^{-1})^l. \tag{4.6}$$

Using (4.5) and (3.7) (this bound also holds for $|V'(x)|$) the claim readily follows. □

We will now turn to the analysis of the resolvent for $|\mathrm{Im}k^2|$ bounded. It will be constructed from the solutions of the homogenous equation

$$(\mathcal{L} + k^2) y = 0, \tag{4.7}$$

which we will now study in detail. We shall write explicitely the resolvent in terms of solutions of (4.7) that have a given decay at infinity, see (4.88) below. The method is similar to the one of the more familiar Wronskian, see [2] for the general theory.

Equation (4.7) is a fourth order linear ODE having four linearly independent solutions. They are most conveniently studied by transforming it into a first order system. Recall from (2.1) that $\mathcal{L} := \partial_x(-\partial_x^2 + 1 + V)\partial_x$ where $V(x) := -\frac{3}{2}\cosh^{-2}\frac{x}{2}$. (4.7) is then equivalent to the system

$$\partial_x \mathbf{y} = A \mathbf{y}, \tag{4.8}$$

where $\mathbf{y} := (y, \partial_x y, \partial_x^2 y, \partial_x^3 y)^T$ and

$$A = A(x, k) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ k^2 & \partial_x V & 1 + V & 0 \end{bmatrix}.$$

Let us also define

$$A_\infty = A_\infty(k) = \lim_{|x| \to \infty} A(k) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ k^2 & 0 & 1 & 0 \end{bmatrix}, \tag{4.9}$$

and $R := A - A_\infty$. Notice that A is a perturbation of A_∞ in the sense that $\|R(x)\|$ (the usual norm of the matrix R) is rapidly decreasing as $|x| \to \infty$:

$$\|R(x)\| \leq C e^{-|x|}. \tag{4.10}$$

We construct the solutions of (4.8) as perturbations of those of the equation $\partial_x \mathbf{y} = A_\infty \mathbf{y}$, which we will now study.

The eigenvalues of A_∞ are solutions of the equation

$$k^2 - \mu^4 + \mu^2 = 0. \tag{4.11}$$

There are four solutions $\mu_j = \mu_j(k)$, $j = 1, 2, 3, 4$, which we label as follows

$$\mu_1 = -\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{1+4k^2}},\ \mu_2 = i\sqrt{\frac{1}{2}\sqrt{1+4k^2} - \frac{1}{2}},\ \mu_3 = -\mu_2,\ \mu_4 = -\mu_1. \tag{4.12}$$

The signs of the square roots are unambiguous in the upper half plane cut along the positive imaginary axis from $\frac{i}{2}$ to $+i\infty$. Furthermore, for small k in this region,

$$\mu_1 = -1 - \frac{1}{2}k^2 + \mathcal{O}(k^4),\ \ \mu_2 = ik + \mathcal{O}(k^3). \tag{4.13}$$

and μ_i are analytic in a neighbourhood of 0 and can be extended to analytic functions in the whole plane cut along positive and negative imaginary axis from $\pm\frac{i}{2}$.

In the integral (4.3) we need to pick a contour that passes close to $k = 0$ (for t large) and eventually reaches the region $|\mathrm{Im}k^2| > r$ where R has the good bound (4.4). For this it is convenient to consider the following region C_r where the μ_i are readily bounded:

$$C_r = \{|k| < r^{-2})\} \cup \{k \mid \mathrm{Im}k > 0,\ \ |\mathrm{Im}k| < r^{-2}|\mathrm{Re}k|,\ \ |\mathrm{Re}k| < r^{\frac{3}{2}}\}. \tag{4.14}$$

We then have the following elementary

Lemma 4.2. *For $k \in C_r$, and r large enough*

$$\begin{aligned} |\mathrm{Re}\mu_2| &\le 2|\mathrm{Im}k| \le 2r^{-\frac{1}{2}} && (4.15)\\ \mathrm{Re}\mu_2 &\ge -C(r)\mathrm{Im}k && (4.16)\\ \mathrm{Re}\mu_1 &\le -1 + \epsilon(r) && (4.17)\end{aligned}$$

where $C(r) > 0$ and $\epsilon(r) \to 0$ as $r \to \infty$. Moreover, Re$\mu_2 < 0$ if Im$k > 0$, $k \in C_r$.

Proof. Write $k = k_0 + ik_1$ and note that (4.13) implies the claims for $|k| \le r^{-2}$. For the other part of C_r, we have for r large enough $\mathrm{Re}(\sqrt{1+4k^2} - 1) \ge Ck_0^2$, for $|k_0| \le 1$, and $\mathrm{Re}(\sqrt{1+4k^2} - 1)$ is bounded away from zero, if $|k_0| > 1$, $k \in C_r$; $\mathrm{Im}(\sqrt{1+4k^2})$ is to leading order, given by $\frac{4k_0k_1}{\sqrt{1+4(k_0^2-k_1^2)}}$, which is small for $k \in C_r$ and r large. Moreover, $|\frac{\mathrm{Im}(\sqrt{1+4k^2})}{\mathrm{Re}(\sqrt{1+4k^2}-1)}|$ is also small, so that μ_2 can be computed by expanding the square root. Using this, the claims readily follow. □

Thus, in $C_r \cap \{\mathrm{Im}z > 0\}$ we also have the ordering of eigenvalues as

$$\mathrm{Re}\mu_1 < \mathrm{Re}\mu_2 \le 0 \le \mathrm{Re}\mu_3 < \mathrm{Re}\mu_4. \tag{4.18}$$

¿From now on r is assumed large enough so that these relations hold.

Eigenvectors corresponding to μ_j are given by

$$\mathbf{v}_j^\infty(k) := (1, \mu_j, \mu_j^2, \mu_j^3)^T. \tag{4.19}$$

Note that at the point $k = 0$, where $\mu_2 = \mu_3 = 0$, $\mathbf{v}_2^\infty = \mathbf{v}_3^\infty$ i.e. these eigenvectors are not independent.

Thus $e^{\mu_i(k)x}\mathbf{v}_i^\infty(k)$ are solutions of

$$\partial_x \mathbf{y} = A_\infty \mathbf{y} \quad . \tag{4.20}$$

They are linearly independent provided $k \neq 0$. To cover $k = 0$ too, we define

$$\mathbf{y}_i^\infty(x,k) = e^{\mu_i(k)x}\mathbf{v}_i^\infty(k),\ i \neq 3,\ \mathbf{y}_3^\infty(x,k) = \frac{1}{2\mu_3(k)}(e^{\mu_3(k)x}\mathbf{v}_3^\infty(k) - e^{\mu_2(k)x}\mathbf{v}_2^\infty(k)) \tag{4.21}$$

which provide a basis of solutions for all $k \in C_r$. Note that $\mathbf{y}_3^\infty(x,0) = (x,1,0,0)^T$ is linearly increasing at ∞.

We will now construct solutions $\mathbf{y}_i(x,k)$ to (4.8) that approach $\mathbf{y}_i^\infty(x,k)$ as $x \to \infty$. For this let us first consider $k = 0$ and solve

$$\mathcal{L}y = \partial_x(-\partial_x^2 + 1 + V)\partial_x y = 0. \tag{4.22}$$

First, the functions

$$y_1(x) = (1 - \tanh(x/2))/2\ ,\quad y_2(x) = 1 \tag{4.23}$$

are solutions to the equation $(-\partial_x^2 + 1 + V)\partial_x y = 0$, hence, they also solve (4.22).

Although we do not need the exact formuli for the two remaining solutions we give them here for completeness. One is gotten by finding a second solution $\tilde{y}$ of the equation $(-\partial_x^2 + 1 + V)y = 0$ by variation of constants starting from the solution $y_0 := 2\partial_x \tanh(x/2) = \cosh^{-2}(\frac{x}{2})$ and integrating the result. After some computation one arrives at the following solution:

$$u = \sinh^2\frac{x}{2} + \frac{3x}{2}\tanh\frac{x}{2}. \tag{4.24}$$

For the remaining fourth solution, find a special solution $\bar{y}$ to $(-\partial_x^2+1+V)\bar{y} = 1$ again by trying $\bar{y} = ay_0$. After some calculation one gets $\bar{y} = -\cosh^2\frac{x}{2}$ and then a fourth solution

$$w = \sinh\frac{x}{2}\cosh\frac{x}{2} + \frac{x}{2}. \tag{4.25}$$

We set

$$y_3(x) = u(x) - w(x) = x - \frac{1}{2} + \frac{1}{2}e^{-x} - \frac{3x}{2}(1 - \tanh\frac{x}{2}) \tag{4.26}$$

$$y_4(x) = 2(u(x) + w(x)) = e^x - 1 + 4x - 3x(1 - \tanh\frac{x}{2}). \tag{4.27}$$

Finally, we denote by $\mathbf{y}_i(x)$ the corresponding solutions of (4.8).

Let us now find the solutions $\mathbf{y}_i(x,k)$ at nonzero k. We look for $\mathbf{y}_i(x,k)$ having the asymptotics $\mathbf{y}_i^\infty(x,k)$ at $x = \infty$ and the $k = 0$ behaviour described above. While this is elementary, some care has to be taken because of the degeneracy at $k = 0$.

Proposition 4.3. *There exists an $r < \infty$ such that the equation*

$$\partial_x \mathbf{y} = A\mathbf{y} \tag{4.28}$$

has four solutions $\mathbf{y}_i(x,k)$, $i = 1,\ldots,4$ satisfying *a)* $\mathbf{y}_i$ *are analytic in* $k \in C_r$*, smooth in* x *and linearly independent.*

b) There exist constants $a_{ij} \in \mathbf{R}$ *such that*

$$|\partial_x^n(\mathbf{y}_i(x,k) - \mathbf{y}_i^\infty(x,k) - \sum_{j=1}^{i-1} a_{ij}\mathbf{y}_j^\infty(x,k))| \le Ce^{-\frac{1}{2}x}e^{\mathrm{Re}\mu_i x}, \tag{4.29}$$

where $|\cdot|$ *denotes the norm in* $\mathbb{C}^4$*, for* $k \in C_r$*,* $x \ge 0$*,* $n = 0, 1$*.*

c) We have $\mathbf{y}_i(x,0) = \mathbf{y}_i(x)$ *and moreover*

$$y_1(x,k) = y_1(x) + u_1(x,k),\ y_2(x,k) = e^{ikx} + u_2(x,k) \tag{4.30}$$

with

$$|\partial_x^n u_1(x,k)| \le C|k|e^{-\frac{1}{2}x},\ |\partial_x^n u_2(x,k)| \le C\varrho(k)e^{c\mathrm{Re}\mu_2 x}, \tag{4.31}$$

where

$$\varrho := |k| + |k|^3/|\mathrm{Im}k|, \tag{4.32}$$

for some $c < \infty$*, for all* $x \ge 0$*,* $n = 0, 1$*.*

d) There exist functions $a_i(k)$ *and* $b_i(k)$*, analytic in* C_r *and bounded there by*

$$|a_i(k)|,\ |b_i(k)| \le C|k| \tag{4.33}$$

such that, for $x > 0$*,*

$$\begin{aligned} y_1(-x,k) = &\ \ y_2(x,k) - y_1(x,k) + \sum_{i=1}^4 a_i(k)y_i(x,k) \\ y_2(-x,k) = &\ \ y_2(x,k) + \sum_{i=1}^4 b_i(k)y_i(x,k). \end{aligned} \tag{4.34}$$

Moreover, for Im $k > 0$*,*

$$|a_3(k)| + |a_4(k)| \ne 0 \ne |b_3(k)| + |b_4(k)|. \tag{4.35}$$

Proof. a) and b) Let us define $\mathbf{v}_i(x,k)$ by

$$\mathbf{y}_i(x,k) = e^{\mu_i x}\mathbf{v}_i(x,k) \tag{4.36}$$

and $B_i = A_\infty - \mu_i$ where we recall that we wrote $A = A_\infty + R$. Thus $\mathbf{v}_i$ is a solution of

$$\partial_x \mathbf{v}_i(x,k) = (B_i + R)\mathbf{v}_i(x,k). \tag{4.37}$$

We shall construct solutions of (4.37) as perturbations of solutions of

$$\partial_x \mathbf{u}_i = B_i \mathbf{u}_i. \tag{4.38}$$

In order to do this, let us write the R term in (4.37) more explicitely. First note that, writing $\mathbf{v}(x,k) = (v^{(1)}(x,k), \ldots, v^{(4)}(x,k))$, we have

$$R\mathbf{v}(x,k) = (\partial_x V(x)v^{(2)}(x,k) + V(x)v^{(3)}(x,k))(0,0,0,1)^T. \tag{4.39}$$

Then, using (4.19) we write

$$(0,0,0,1)^T = \frac{1}{2(\mu_1^2(k)-\mu_2^2(k))}\left(\frac{1}{\mu_1(k)}(\mathbf{v}_1^\infty(k)-\mathbf{v}_4^\infty(k)) - \frac{1}{\mu_2(k)}(\mathbf{v}_2^\infty(k)-\mathbf{v}_3^\infty(k))\right) \quad (4.40)$$

Notice that $\mu_1^2(k)-\mu_2^2(k)=\sqrt{1+4k^2}$, which is bounded away from 0 for $k\in C_r$. Combining (4.39) and (4.40) we can thus write

$$R\mathbf{v}(x,k) = r_1(x,k,\mathbf{v})\mathbf{v}_1^\infty(k) + r_2(x,k,\mathbf{v})\mathbf{v}_2'(k) + r_4(x,k,\mathbf{v})\mathbf{v}_4^\infty(k) \quad (4.41)$$

where

$$\mathbf{v}_2'(k) = \frac{1}{\mu_2(k)}(\mathbf{v}_2^\infty(k)-\mathbf{v}_3^\infty(k)) \quad (4.42)$$

and $r_i = r_i(x,k,\mathbf{v})$ are linear in $\mathbf{v}$ and satisfy the bounds

$$|r_i| \le Ce^{-|x|}|\mathbf{v}| \quad (4.43)$$

where we used (4.10).

Since we are interested in bounds for $x\ge 0$ (see b)), it will be enough to construct solutions in the interval $[x_0,\infty)$, for x_0 large, satisfying (4.29) there. Indeed, since the equation (4.37) is linear, a solution defined in that interval extends to a unique solution in $\mathbf{R}$, and (4.29) will still hold, for all $x\ge 0$, possibly with a larger C. Thus, let $\mathcal{B}$ be the Banach space of functions $\mathbf{v}(x,k)$ defined in $[x_0,\infty)\times C_r$, with values in $\mathbf{C}^4$, which are C^1 in x, analytic in k in C_r and continuous in its closure $\overline{C}_r$ with the norm:

$$||\mathbf{v}|| = \sup_{k\in\overline{C}_r}\sup_{x\ge x_0} e^{-\frac{1}{4}x}|\mathbf{v}(x,k)| + \sup_{k\in\overline{C}_r}\sup_{x\ge x_0} e^{-\frac{1}{4}x}|\partial_x\mathbf{v}(x,k)|. \quad (4.44)$$

Consider now the integral equation

$$\mathbf{v} = \mathbf{u} + \mathcal{H}_i\mathbf{v} \quad (4.45)$$

where $\mathbf{u}$ solves (4.38) and the linear operators $\mathcal{H}_i$ are defined on $\mathcal{B}$ as follows.

$$(\mathcal{H}_1\mathbf{v})(x,k) = -\int_x^\infty e^{(x-\xi)B_1}(\mathbf{v}_1^\infty(k)r_1(\xi,k,\mathbf{v}) + \mathbf{v}_2'(k)r_2(\xi,k,\mathbf{v}) + \mathbf{v}_4^\infty(k)r_4(\xi,k,\mathbf{v}))d\xi \quad (4.46)$$

$$\begin{aligned}(\mathcal{H}_i\mathbf{v})(x) &= \int_{x_0}^x e^{(x-\xi)B_i}\mathbf{v}_1^\infty(k)r_1(\xi,k,\mathbf{v})d\xi \\ &\quad - \int_x^\infty e^{(x-\xi)B_i}(\mathbf{v}_2'(k)r_2(\xi,k,\mathbf{v}) + \mathbf{v}_4^\infty(k)r_4(\xi,k,\mathbf{v}))d\xi, \quad i=2,3 \end{aligned} \quad (4.47)$$

$$(\mathcal{H}_4\mathbf{v})(x) = \int_{x_0}^x e^{(x-\xi)B_4}(\mathbf{v}_1^\infty(k)r_1(\xi,k,\mathbf{v}) + \mathbf{v}_2'(k)r_2(\xi,k,\mathbf{v}) + \mathbf{v}_4^\infty(k)r_4(\xi,k,\mathbf{v}))d\xi \quad (4.48)$$

Suppose now that $\mathbf{v}$ solves (4.45) and that $\mathbf{u}$ is a solution of (4.38). Then, by a simple calculation, one sees that $\mathbf{v}$ solves (4.37). We shall prove that, for x_0 large enough, $||\mathcal{H}_i||<1$ whereby (4.45) has a unique solution given a $\mathbf{u}\in\mathcal{B}$. We shall then choose four suitable $\mathbf{u}_i\in\mathcal{B}$ and the functions $\mathbf{y}_i(x,k)$ in the Proposition will be obtained from the corresponding solutions $\mathbf{v}_i$ through (4.36).

To estimate the norms of $\mathcal{H}_i$, use first (4.43) and (4.44) to get:

$$|r_i(x,k,\mathbf{v})| \le Ce^{-\frac{3}{4}x}||\mathbf{v}||. \tag{4.49}$$

Then, observe that

$$|e^{(x-\xi)B_i}\mathbf{v}_j^\infty| \le \; Ce^{(x-\xi)\mathrm{Re}(\mu_j-\mu_i)} \tag{4.50}$$

$$\begin{aligned} |e^{(x-\xi)B_i}\mathbf{v}_2'| = \; & |\frac{1}{\mu_2}(e^{(x-\xi)(\mu_2-\mu_i)}\mathbf{v}_2^\infty - e^{(x-\xi)(\mu_3-\mu_i)}\mathbf{v}_3^\infty)| \\ & \le C(1+|x-\xi|)(e^{(x-\xi)\mathrm{Re}(\mu_2-\mu_i)} + e^{(x-\xi)\mathrm{Re}(\mu_3-\mu_i)}) \end{aligned} \tag{4.51}$$

where we bound the difference by the sum for k not too small, and we use $\mu_2 = -\mu_3$ for k small.

Let us now bound $\|\mathcal{H}_i\|$. For $i=1$, we get, using (4.49), (4.50) and $\mathrm{Re}(\mu_j - \mu_1) \ge 0$,

$$|(\mathcal{H}_1\mathbf{v})(x,k)| \le C||\mathbf{v}||e^{-\frac{1}{2}x}. \tag{4.52}$$

For $i = 2,3$ note that $\mathrm{Re}(\mu_1 \pm \mu_2) \le -1+\delta$ and $\mathrm{Re}(\mu_j - \mu_i) \ge -2\delta$ for $i = 2,3$ and $j \ge 2$, so for small δ (4.52) holds for $\mathcal{H}_i$ with $i = 2,3$ too.

For $i = 4$, $\mathrm{Re}(\mu_i - \mu_4) \le 0$ gives

$$|(\mathcal{H}_4\mathbf{v})(x,k)| \le C||\mathbf{v}||e^{-\frac{1}{2}x_0}. \tag{4.53}$$

We have $e^{-\frac{1}{2}x_0}$ instead of $e^{-\frac{1}{2}x}$ here because of the last term, with $\mathbf{v}_4^\infty$, in (4.48). For the derivatives $\partial_x(\mathcal{H}_i\mathbf{u})(x,k)$ we get the bounds (4.52), (4.53) multiplied with μ_i. Altogether, these bounds prove that $||\mathcal{H}_i|| < 1$ for x_0 large, and therefore the equations (4.45) have unique solutions $\mathbf{v}_i(\mathbf{u})$ in $\mathcal{B}$ given a $\mathbf{u}$ in that space. Also, we have

$$||\mathbf{v}|| \le C||\mathbf{u}||. \tag{4.54}$$

Now, we shall choose $\mathbf{u}$ appropriately to get the right asymptotics. We consider the different i's separately.

i=1. Take $\mathbf{u} = \mathbf{v}_1^\infty(k)$ (i.e. constant in x), and by (4.45), (4.36), we obtain a solution $\mathbf{y}_1(x,k)$. Using (4.21), (4.54), we get that (4.29) follows from (4.45), (4.52). Since $\mathbf{y}_1(x,0)$ solves $\mathbf{y}' = A(x,0)\mathbf{y}$ and has, since $\mu_1(0) = -1$, the same asymptotics as the solution $\mathbf{y}_1(x)$ obtained from $y_1(x)$ (see (4.23)), they have to agree since the latter is the unique solution with such asymptotics.

i=2. Taking $\mathbf{u} = \mathbf{v}_2^\infty(k)$ we get a solution $\mathbf{y}(x,k)$ with (4.29) holding for any a_{21}, since $y_1^\infty(x,k)$ tends to zero faster than $e^{-\frac{1}{2}x+\mathrm{Re}\mu_2 x}$ as $x \to \infty$. At $k = 0$, $\mathbf{y}(x,0)$ is a solution of $\mathbf{y}' = A(x,0)\mathbf{y}$ which is bounded as $x \to \infty$ and so has to be a linear combination

$$a\mathbf{y}_1(x) + b\mathbf{y}_2(x)$$

with $b = 1$ because of the asymptotics. Thus, taking $\mathbf{y}_2(x,k) = \mathbf{y}(x,k) - a\mathbf{y}_1(x,k)$ and $a_{21} = a$, we are done.

i=3. We take

$$\mathbf{u} = \frac{1}{2\mu_2}(\mathbf{v}_3^\infty(k) - e^{2\mu_2 x}\mathbf{v}_2^\infty(k)). \tag{4.55}$$

$\mathbf{u}$ is not bounded, but belongs to $\mathcal{B}$ for r large, as one sees by using (4.15). So, (4.29) holds for $\mathbf{y}(x,k) = e^{\mu_3 x}\mathbf{v}(x,k)$ since $e^{\mu_3 x}\mathbf{u}(x,k) = e^{-\mu_2 x}\mathbf{u}(x,k) = \mathbf{y}_3^\infty(x,k)$. At $k = 0$ $\mathbf{y}$

solves $\mathbf{y}' = A(x,0)\mathbf{y}$ with $|\mathbf{y}(x,0) - (x,1,0,0)^T|$ bounded as $x \to \infty$. Hence $\mathbf{y}_3(x,0) = \mathbf{y}(x,0) + a_{31}\mathbf{y}_1(x,0) + a_{32}\mathbf{y}_2(x,0) = \mathbf{y}_3(x)$ for suitable a_{3j}, $j = 1,2$.

i=4. The first two terms in (4.48) are bounded by (4.52) since $\mathrm{Re}(\mu_i - \mu_4) \leq -1 + \delta$ for $i < 4$. The third term equals

$$(\int_{x_0}^{\infty} r_4(\xi)d\xi - \int_{x}^{\infty} r_4(\xi)d\xi)\mathbf{v}_4^{\infty}. \tag{4.56}$$

The second term satisfies (4.52) again and we can write

$$(\mathcal{H}_4\mathbf{v})(x) = \mathbf{v}_4^{\infty}(k)\ell(\mathbf{v}) + \mathbf{r}(\mathbf{v})(x) \tag{4.57}$$

where $\ell(\mathbf{v}) := \int_{x_0}^{\infty} r_4(\xi)d\xi$ and the linear maps ℓ and $\mathbf{r}$ satisfy

$$|\ell(\mathbf{v})| \leq \; Ce^{-\frac{1}{2}x_0}||\mathbf{v}|| \tag{4.58}$$

$$|\mathbf{r}(\mathbf{v})(x)| \leq \; Ce^{-\frac{1}{2}x}||\mathbf{v}||. \tag{4.59}$$

Hence, we take

$$\mathbf{u} = (1-\alpha)\mathbf{v}_4^{\infty} \tag{4.60}$$

and determine α from the equation

$$\alpha = \ell(\mathbf{v}(\alpha)) \tag{4.61}$$

where $\mathbf{v}(\alpha)$ is the solution of $\mathbf{v} = \mathbf{u} + \mathcal{H}_4\mathbf{v}$. (4.61) has a solution since it is affine in α and $|\frac{d}{d\alpha}\ell(\mathbf{v}(\alpha))| < 1$ for x_0 large enough. Thus $\mathbf{y}(x,k) = e^{\mu_4 x}\mathbf{v}(x,k)$ satisfies (4.29) for $i = 4$. Again, considering $k = 0$, we see that $\mathbf{y}_4(x,k) = e^{\mu_4 x}\mathbf{v}(x,k) + a_{41}\mathbf{y}_1(x,k) + a_{42}\mathbf{y}_2(x,k) + a_{43}\mathbf{y}_3(x,k)$ with appropriate a_{4j}.

c) Note that (4.29) and (4.17) imply that, for $n = 0,1$,

$$|\partial_x^n y_1(x,k)| \leq Ce^{-\frac{1}{2}x} \tag{4.62}$$

for $x \geq 0$, $k \in C_r$. Since $y_1(x) = (1 - \tanh(x/2))/2$ also satisfies this bound, the bound (4.31) holds for $u_1(x,k)$, for $|k| \geq \frac{\delta}{2}$. And, for $|k| \leq \frac{\delta}{2}$, one may use (4.62) and a Cauchy estimate to bound $|\partial_k(\partial_x^n y_1(x,k))| \leq \frac{C}{\delta}e^{-\frac{1}{2}x}$.

The bound on u_2 is similar: using (4.29) as above and $\mathrm{Re}\mu_1 < \mathrm{Re}\mu_2$, $-\frac{1}{2} < \mathrm{Re}\mu_2$ (following from (4.18), (4.15)), one gets

$$|y_2(x,k)| \leq Ce^{\mathrm{Re}\mu_2 x}. \tag{4.63}$$

On the other hand, $|\partial_x^n e^{ikx}| = |k|^n e^{-\mathrm{Im}kx} \leq |k|^n e^{c\mathrm{Re}\mu_2 x}$, since, using (4.16) and letting $c = C(r)^{-1}$, $-\mathrm{Im}k \leq c\mathrm{Re}\mu_2$ for $k \in C_r$. Hence, the bound (4.31) holds for $u_2(x,k)$, for $|k| \geq \frac{\delta}{2}$. For $|k| \leq \frac{\delta}{2}$, we write

$$|\partial_x^n(y_2(x,k) - e^{ikx})| \leq |\partial_x^n(y_2(x,k) - y_2^{\infty}(x,k))| + |\partial_x^n(y_2^{\infty}(x,k) - e^{ikx})|. \tag{4.64}$$

The first term is bounded using (4.29), a Cauchy estimate and $y_2(x,0) = y_2^{\infty}(x,0) = 1$, by $C|k|e^{\mathrm{Re}\mu_2 x - \frac{1}{2}x}$. The second term is bounded using the explicit formula $y_2^{\infty}(x,k) = e^{\mu_2(k)x}$ and the bound (see (4.13), (4.16))

$$|e^{ikx}(e^{(\mu_2(k)-ik)x} - 1)| \leq Ce^{-\mathrm{Im}kx}x|k|^3 \leq C\frac{|k|^3}{|\mathrm{Im}k|}e^{c\mathrm{Re}\mu_2 x} \tag{4.65}$$

d) Let us first show that

$$y_1(-x,k)=\sum_{i=1}^{4}\tilde{a}_i(k)y_i(x,k) \tag{4.66}$$

with $\tilde{a}_i(k)$ analytic and bounded in C_r. Let J be the matrix

$$J=\mathrm{diag}(1,-1,1,-1). \tag{4.67}$$

Since $V(x)=V(-x)$, $J\mathbf{y}_1(-x,k)$ is also a solution of (4.28). But, for each $k\in C_r$ and each x large enough, one gets from (4.29) that the set of vectors $(\mathbf{y}_i(x,k))_{i=1}^4$ forms a basis of $\mathbf{C}^4$. Since the evolution (in x) corresponding to equation (4.28) is nonsingular, the same holds for all x. Hence, choosing $x=0$, we can write:

$$J\mathbf{y}_1(0,k)=\sum_{i=1}^{4}\tilde{a}_i(k)\mathbf{y}_i(0,k), \tag{4.68}$$

where the coefficients $\tilde{a}_i(k)$ are analytic in C_r because $J\mathbf{y}_i(0,k)$ is analytic in k and the set $(\mathbf{y}_i(0,k))_{i=1}^4$ forms an analytic basis. We claim that

$$J\mathbf{y}_1(-x,k)=\sum_{i=1}^{4}\tilde{a}_i(k)\mathbf{y}_i(x,k) \tag{4.69}$$

with the same $\tilde{a}_i(k)$; indeed, both sides of (4.69) are solutions of (4.28), hence (4.69) holds because (4.28) has a unique solution with a given value at $x=0$. The expansion (4.66) follows by considering the first component of (4.69).

Now, for $k=0$, we write $y_1(-x,0)=(1+\tanh(x/2))/2=1-(1-\tanh(x/2))/2=y_2(x,0)-y_1(x,0)$, which implies that, in (4.66), $\tilde{a}_1(0)=-1$, $\tilde{a}_2(0)=1$, $\tilde{a}_3(0)=\tilde{a}_4(0)=0$. Defining $a_i(k)=\tilde{a}_i(k)-\tilde{a}_i(0)$ and using, for small k, the analyticity of $\tilde{a}_i(k)$ finishes the proof of (4.33, 4.34) for y_1. The proof for y_2 is similar.

To prove (4.35), observe that, if $a_3(k)=a_4(k)=0$ for some k, then, combining (4.29) and (4.34), $y_1(x,k)$ would decay exponentially as $x\to\pm\infty$, hence would belong to $L^2(\mathbf{R})$ (recall that $\mathrm{Re}\mu_2<0$ if $\mathrm{Im}k<0$; hence $y_2^\infty(x,k)$ decays exponentially as $x\to\infty$); but, for $k\notin\mathbf{R}$ equation (4.7) cannot have such solutions since the spectrum of $\mathcal{L}$ is $(-\infty,0]$. We can use the same argument for $b_3(k)$, $b_4(k)$. □

We turn now to the study of the resolvent of $\mathcal{L}$, which can be expressed in terms of the solutions $\mathbf{y}_j$ (for the general theory, see [2]). It will be convenient to use another basis of solutions for this purpose. Recall from the proof of Proposition 4.3 (d) that

$$\mathbf{y}'_j(x,k):=J\mathbf{y}_j(-x,k), \tag{4.70}$$

where J was defined in(4.67), also are solutions of (4.28). The set $(\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}'_1,\mathbf{y}'_2)$ is a linearly independent set of solutions for (4.28) if $\mathrm{Im}k>0$. Indeed, $\mathbf{y}_1$ and $\mathbf{y}_2$ are independent from $\mathbf{y}'_1$ and $\mathbf{y}'_2$ since by (4.35) they increase exponentially as $x\to-\infty$ whereas $\mathbf{y}'_1$ and $\mathbf{y}'_2$ both tend to zero as $x\to-\infty$.

We need also solutions of the transposed equation

$$\partial\mathbf{z}=-\mathbf{z}A. \tag{4.71}$$

For this, let $H=-\partial^2+1+V$ and consider the differential operator $\mathbf{D}$:

$$\mathbf{D}y:=(H\partial_x y,-Hy,-\partial_x y,y). \tag{4.72}$$

Suppose now that y solves (4.7). Then a simple computation shows that $\mathbf{z} = \mathbf{D}y$ solves (4.71).

Denote now by $y_j(x,k)$ the first component of $\mathbf{y}_j(x,k)$ and $y'_j(x,k) = y_j(-x,k)$. We define

$$\mathbf{z}_j(x,k) = \mathbf{D}y'_j(x,k)\ j = 1,2. \tag{4.73}$$

(note the prime!) and

$$\mathbf{z}'_j(x,k) = \mathbf{D}y_j(x,k). \tag{4.74}$$

The set $(\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}'_1, \mathbf{z}'_2)$ is a linearly independent set of solutions for (4.71) if $k \notin \mathbf{R}$, $\mathrm{Im}k > 0$, such that $\mathbf{z}_i$ decrease as $x \to -\infty$ and $\mathbf{z}'_i$ decrease at $x \to \infty$. To describe their precise asymptotic behaviour we need to introduce the corresponding solutions of the constant coefficient system

$$\partial_x \mathbf{z} = -\mathbf{z}A_\infty. \tag{4.75}$$

These are obtained by considering the left eigenvectors of $\mathbf{A}_\infty$. They are

$$\mathbf{w}_j^\infty(k) := ((\mu_j^2 - 1)\mu_j, \mu_j^2 - 1, \mu_j, 1). \tag{4.76}$$

One has the "orthogonality" property

$$\mathbf{w}_i^\infty(k)\mathbf{v}_j^\infty(k) = 2\delta_{ij}(2\mu_i^3 - \mu_i) \tag{4.77}$$

for $i,j \in \{1,2\}$. Note that $2\mu_i^3 - \mu_i = \mu_i^{-1}(1 + 4k^2 \pm \sqrt{1+4k^2})$ which vanishes only at $k = 0,\ \pm\frac{i}{2}$.

Solutions of (4.75) are then given by

$$\mathbf{z}_i^\infty(x,k) = e^{-\mu_i(k)x}\mathbf{w}_i^\infty(k). \tag{4.78}$$

The asymptotics of the $\mathbf{z}_i$ and $\mathbf{z}'_i$ is then given by

Lemma 4.4. *The $\mathbf{z}_i$ satisfy*

$$|\partial_x^n(\mathbf{z}_i(x,k) - \mathbf{z}_i^\infty(x,k) - \sum_{j=1}^{i-1} a_{ij}\mathbf{z}_j^\infty(x,k))| \le Ce^{\frac{1}{2}x}e^{-\mathrm{Re}\mu_i x} \tag{4.79}$$

for $k \in C_r$, $x \le 0$, $n = 0,1$ and the a_{ij}'s are the same as in (4.29).

Proof. From (4.72) and (4.73) we get that $\mathbf{z}(x,k) = -(\partial_x^3 y, \partial_x^2 y, \partial_x y, y)(-x,k) + (1 + V(x))(\partial_x y, y, 0, 0) =: \mathbf{y}(-x,k)^T M(x)$. Let $M_\infty = \lim_{x\to\infty} M(x)$ so by (3.7) $||\partial_x^n(M(x) - M_\infty)|| \le Ce^{-|x|}$. Since $\mathbf{z}_i^\infty(x,k) = \mathbf{y}_i^\infty(-x,k)^T M_\infty$, the claim follows immediately from (4.29). □

The resolvent will be expressed in terms of the following 2×2–matrix

$$\Omega = \Omega(k) = \begin{bmatrix} \mathbf{z}_1\mathbf{y}_1 & \mathbf{z}_1\mathbf{y}_2 \\ \mathbf{z}_2\mathbf{y}_1 & \mathbf{z}_2\mathbf{y}_2 \end{bmatrix}, \tag{4.80}$$

which is defined for all $k \in C_r$. (We do not require $\mathrm{Im}k > 0$ here.) The entries of Ω are analytic functions of k *independent* of x. This is so because $\partial_x(\mathbf{z}_i\mathbf{y}_j) = (\partial_x\mathbf{z}_i)\mathbf{y}_j + \mathbf{z}_i\partial_x\mathbf{y}_j = -\mathbf{z}_i A\mathbf{y}_j + \mathbf{z}_i A\mathbf{y}_j = 0$ for all $i,\ j$.

We have the

Lemma 4.5. *$\Omega(k)$ has an analytic inverse for $k \in C_r$, $\mathrm{Im} k > 0$.*

Proof. Suppose $\Omega(k)(\alpha, \beta)^T = 0$. Then

$$\mathbf{z}_i(x,k)(\alpha \mathbf{y}_1(x,k) + \beta \mathbf{y}_2(x,k)) = 0 \tag{4.81}$$

for $i = 1, 2$. Let us consider the limit $x \to -\infty$. By (4.34) in Proposition 4.3

$$\alpha \mathbf{y}_1(x,k) + \beta \mathbf{y}_2(x,k) = \sum_{i=1}^{4} \lambda_i(k) J \mathbf{y}_i(-x,k). \tag{4.82}$$

Since the vectors $J\mathbf{y}_i(-x,k)$ and $\mathbf{z}_i(x,k)$, $i = 1, 2$ are rapidly decreasing as $x \to -\infty$ and since $J\mathbf{y}_i(-x,k) = e^{-\mu_i(k)x} J\mathbf{v}_i^\infty(k)(1 + \mathcal{O}(e^{-\mathrm{Re}\mu_2 x}))$ for $i = 3, 4$ and $\mu_4 = -\mu_1$, $\mu_3 = -\mu_2$, $J\mathbf{v}_4^\infty = \mathbf{v}_1^\infty$, $J\mathbf{v}_3^\infty = \mathbf{v}_2^\infty$, we get, using (4.77),

$$\begin{aligned}
& \mathbf{z}_1(x,k)(\alpha \mathbf{y}_1(x,k) + \beta \mathbf{y}_2(x,k)) \\
= & \left(e^{-\mu_1(k)x} \mathbf{w}_1^\infty(k)(1 + \mathcal{O}(e^{x/2}))\right) \lambda_4(k) e^{\mu_1(k)x} \left(\mathbf{v}_1^\infty(k) + \mathcal{O}(e^{-\mathrm{Re}\mu_2 x})\right) \\
= & \ 2\lambda_4(k)(2\mu_1^3 - \mu_1) + \mathcal{O}(e^{-\mathrm{Re}\mu_2 x}).
\end{aligned}$$

We thus get a contradiction with (4.81) unless $\lambda_4 = 0$. In the same way one then proves that $\lambda_3 = 0$. Thus (4.82) and (4.29) imply that $\mathbf{y} = \alpha \mathbf{y}_1 + \beta \mathbf{y}_2$ is a solution of $\mathcal{L}\mathbf{y} = -k^2 \mathbf{y}$ decaying exponentially at $\pm\infty$ i.e. in $L^2(\mathbb{R})$. This is impossible since $-k^2$ with Im $k \neq 0$ is not in the spectrum $]-\infty, 0]$ of $\mathcal{L}$. $\square$

The resolvent is now gotten from the following

Proposition 4.6. *Given a continuous function $\mathbf{f} : \mathbb{R} \to \mathbb{C}^4$ with compact support, the function*

$$\begin{aligned}
\mathbf{y}(x,k) = & \int_{-\infty}^{x} (\mathbf{y}_1(x,k), \mathbf{y}_2(x,k)) \Omega(k)^{-1} (\mathbf{z}_1(\xi,k), \mathbf{z}_2(\xi,k))^T \mathbf{f}(\xi) d\xi \\
& + \int_{x}^{\infty} (\mathbf{y}_1'(x,k), \mathbf{y}_2'(x,k)) \Omega(k)^{-1} (\mathbf{z}_1'(\xi,k), \mathbf{z}_2'(\xi,k))^T \mathbf{f}(\xi) d\xi
\end{aligned} \tag{4.83}$$

is a solution of

$$\partial_x \mathbf{y} = A\mathbf{y} + \mathbf{f} \tag{4.84}$$

for all $k \in C_r$, $\mathrm{Im} k > 0$.

Proof. Differentiating the right hand side, one obtains A times the right hand side plus two other terms, namely

$$\begin{aligned}
& (\mathbf{y}_1(x,k), \mathbf{y}_2(x,k)) \Omega^{-1} (\mathbf{z}_1(x,k), \mathbf{z}_2(x,k))^T \mathbf{f}(x) \\
- & (\mathbf{y}_1'(x,k), \mathbf{y}_2'(x,k)) \Omega^{-1} (\mathbf{z}_1'(x,k), \mathbf{z}_2'(x,k))^T \mathbf{f}(x).
\end{aligned} \tag{4.85}$$

But here for all x the 2×4 matrix $P_1 := (\mathbf{y}_1(x), \mathbf{y}_2(x)) \Omega^{-1} (\mathbf{z}_1(x), \mathbf{z}_2(x))^T$ is a projection from $\mathbb{C}^4$ onto the subspace spanned by the vectors $\mathbf{y}_1(x)$ and $\mathbf{y}_2(x)$ (P_1 is onto since the $\mathbf{y}_i$ and the $\mathbf{z}_i$ are independent), and for the second term in (4.85) we note that $\mathbf{z}\mathbf{y} = -\mathbf{z}'\mathbf{y}'$ whereby $\Omega = -\Omega'$ where the latter is defined with the primed solutions. Thus the second term is also a projection, P_2, acting on $\mathbf{f}$, and P_2 projects onto the subspace spanned by the vectors $\mathbf{y}'_1(x)$ and $\mathbf{y}'_2(x)$. So, (4.85) $= (P_1 + P_2)\mathbf{f}$. But these projections satisfy $P_1 P_2 = 0 = P_2 P_1$ since

$$\mathbf{z}'_i(x)\mathbf{y}_j(x) = 0 = \mathbf{z}_i(x)\mathbf{y}'_j(x) \tag{4.86}$$

for $i, j \in \{1, 2\}$. The reason is, that z' and y both tend to zero as $x \to \infty$, but $\mathbf{z}'\mathbf{y}$ is constant in x. Similarily for y' and z, this time as $x \to -\infty$. Hence $P_1 + P_2$ is a projection onto $\mathbf{C}^4$ and therefore equals the identity. The claim follows. □

Now consider equation $(\mathcal{L}+k^2)y = f$ which is equivalent to (4.84) with $\mathbf{f} = (0, 0, 0, f)^T$. Then $(\mathbf{z}_1(\xi, k), \mathbf{z}_2(\xi, k))^T \mathbf{f}(\xi) = (y_1(-\xi, k), y_2(-\xi, k))^T f(\xi)$ and we get the following resolvent formula for the equation $(\mathcal{L} + k^2)y = f$:

$$\begin{aligned} y(x,k) = & \int_{-\infty}^{x} (y_1(x,k), y_2(x,k))\Omega(k)^{-1}(y_1(-\xi,k), y_2(-\xi,k))^T f(\xi) d\xi \\ & + \int_{x}^{\infty} (y_1(-x,k), y_2(-x,k))\Omega(k)^{-1}(y_1(\xi,k), y_2(\xi,k))^T f(\xi) d\xi \end{aligned} \tag{4.87}$$

which implies

$$\begin{aligned} (\mathcal{L} + k^2)^{-1}(x, \xi) = & \sum_{i,j=1}^{2} \theta(x - \xi)\omega_{ij}(k) y_i(x,k) y_j(-\xi,k) \\ & + \sum_{i,j=1}^{2} \theta(\xi - x)\omega_{ij}(k) y_i(\xi,k) y_j(-x,k) \end{aligned} \tag{4.88}$$

where

$$\omega_{ij}(k) = (\Omega(k)^{-1})_{ij} \tag{4.89}$$

are analytic functions in C_r. Notice that although the resolvent formula is used only for $\mathrm{Im}k > 0$, the matrix Ω is defined in the whole C_r and its properties can be understood via the following

Lemma 4.7.

$$\omega(k) = \frac{i}{k} \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix} + \tilde{\omega}(k), \tag{4.90}$$

where $\tilde{\omega}$ is analytic on C_r.

Proof. We will calculate $\Omega(0)$ and $\partial_k \Omega(0)$.

Recall that, by (4.23), (4.72), (4.73),

$$\mathbf{y}_2(x, 0) = (1, 0, 0, 0)^T, \quad \mathbf{z}_2(x, 0) = (0, -1 - V(x), 0, 1) \tag{4.91}$$

and so $\Omega_{22}(0) = 0$. Since $\mathbf{y}_1(x, 0)$ and $\mathbf{z}_1(x, 0)$ are bounded with $\lim_{x\to\infty} \mathbf{y}_1(x, 0) = 0$ $\lim_{x\to-\infty} \mathbf{z}_1(x, 0) = 0$ we get $\Omega_{11}(0) = \lim_{x\to\infty} \mathbf{z}_1(x, 0)\mathbf{y}_1(x, 0) = 0$ and in the same way $\Omega_{12}(0) = \lim_{x\to-\infty} \mathbf{z}_1(x, 0)\mathbf{y}_2(x, 0) = 0$, $\Omega_{21}(0) = \lim_{x\to\infty} \mathbf{z}_2(x, 0)\mathbf{y}_1(x, 0) = 0$. Thus

$$\Omega(0) = 0\ . \tag{4.92}$$

To compute $\partial_k\Omega(0)$, note first that from the definition

$$\partial_k\Omega_{ij}(0) = \partial_k\mathbf{z}_i(x,0)\mathbf{y}_j(x,0) + \mathbf{z}_i(x,0)\partial_k\mathbf{y}_j(x,0). \tag{4.93}$$

Since $\partial_k A(x,0) = 0$, $\partial_k\mathbf{z}_i(x,0)$ satisfies the equation

$$\partial_x\partial_k\mathbf{z}_i(x,0) = -\partial_k\mathbf{z}_i(x,0)A(x,0) \tag{4.94}$$

and $\partial_k\mathbf{y}_j(x,0)$ the equation

$$\partial_x\partial_k\mathbf{y}_j(x,0) = A(x,0)\partial_k\mathbf{y}_j(x,0), \tag{4.95}$$

we see that both of the terms on the RHS of (4.93) are independent of x. Hence, we may write

$$\partial_k\Omega_{ij}(0) = \partial_k\mathbf{z}_i(-x,0)\mathbf{y}_j(-x,0) + \mathbf{z}_i(x,0)\partial_k\mathbf{y}_j(x,0) \tag{4.96}$$

and we can calculate this now as $x\to\infty$. We consider now the various cases:

a) $\partial_k\Omega_{11}(0)$. By (4.79) and (4.29) and a Cauchy estimate, $\partial_k\mathbf{z}_1(-x,0)$ and $\partial_k\mathbf{y}_1(x,0)$ decay exponentially as $x\to\infty$ whereas $\mathbf{y}_1(-x,0)$ and $\mathbf{z}_1(x,0)$ are bounded. Hence taking $x\to\infty$ in (4.96) we get $\partial_k\Omega_{11}(0) = 0$.

b) $\partial_k\Omega_{22}(0)$. From (4.79) we infer, for $x\geq 0$,

$$\partial_k\mathbf{z}_2(-x,0) = \partial_k[e^{\mu_2(k)x}\mathbf{w}_2^\infty(k)]|_{k=0} + \mathcal{O}(e^{-x/2}) \tag{4.97}$$

and from (4.76) we have $\mathbf{w}_2^\infty(0) = (0,-1,0,1)$ $\partial_k\mathbf{w}_2^\infty(0) = (-i,0,+i,0)$. Combining with (4.91) we have

$$\lim_{x\to\infty}\partial_k\mathbf{z}_2(-x,0)\mathbf{y}_2(-x,0) = -i. \tag{4.98}$$

Similarly,

$$\partial_k\mathbf{y}_2(x,0) = \partial_k[e^{\mu_2(k)x}\mathbf{v}_2^\infty(k)]|_{k=0} + \mathcal{O}(e^{-x/2}) \tag{4.99}$$

and from (4.19) we have $\mathbf{v}_2^\infty(0) = (1,0,0,0)^T$, $\partial_k\mathbf{v}_2^\infty(0) = (0,i,0,0)^T$,

$$\partial_k\mathbf{y}_2(x,0) = (ix,i,0,0)^T + \mathcal{O}(e^{-x/2}). \tag{4.100}$$

Since, from (4.91), $\mathbf{z}_2(x,0) = (0,-1,0,1) + \mathcal{O}(e^{-x/2})$ we have

$$\lim_{x\to\infty}\mathbf{z}_2(x,0)\partial_k\mathbf{y}_2(x,0) = -i \tag{4.101}$$

and thus $\partial_k\Omega_{22}(0) = -2i$.

c) $\partial_k\Omega_{12}(0)$. Since $\partial_k\mathbf{z}_1(-x,0)$ vanishes exponentially as $x\to\infty$ and $\mathbf{y}_2(-x,0)$ is bounded we have

$$\partial_k\Omega_{12}(0) = \lim_{x\to\infty}\mathbf{z}_1(x,0)\partial_k\mathbf{y}_2(x,0). \tag{4.102}$$

¿From (4.73), (4.72) and $|\partial_x^n\mathbf{y}_1(x,0)|\leq Ce^{-|x|}$ we get $\mathbf{z}_1(x,0) = (0,-1,0,+1) + \mathcal{O}(e^{-x})$ so, combining with (4.100), we infer $\partial_k\Omega_{12}(0) = -i$. By symmetry $\partial_k\Omega_{21}(0) = -i$.

Thus, we see that in a neighbourhood of 0

$$\Omega(k) = -ik\begin{bmatrix}0 & 1\\ 1 & 2\end{bmatrix} + \hat{\omega}(k), \tag{4.103}$$

where $\hat{\omega}$ is $\mathcal{O}(k^2)$. Combining (4.103) with Lemma 4.5, the claim follows provided r is taken large enough. $\square$

We can now bound the kernel $R(x,\xi,k)$ of (4.2):

Proposition 4.8. *Let $k\in C_r$, $\mathrm{Im}k>0$. Then*

$$R(x,\xi,k)=R^*(x,\xi,k)+r(x,\xi,k) \tag{4.104}$$

where R^* is given, for $x\geq 0$ by

$$R^*(x,\xi,k)=\frac{1}{\pi}\begin{cases} 2y_1(x)y_1(-\xi)-y_1(x)e^{-ik\xi}-y_1(-\xi)e^{ikx} & \xi\leq 0\\ -2y_1(x)y_1(\xi)+y_1(x)e^{ik\xi}+y_1(\xi)e^{ikx}-e^{ik(x+\xi)} & \xi\geq 0\end{cases} \tag{4.105}$$

with $y_1(x)$ defined in (4.23), and for $x\leq 0$ $R^*(x,\xi,k)=R^*(-x,-\xi,k)$. r is bounded by

$$|\partial_x^m\partial_\xi^n r(x,\xi,k)|\leq C\varrho(k)e^{-\mu(k)|x-\xi|} \tag{4.106}$$

with $\varrho(k)=|k|+|k|^3/|\mathrm{Im}k|$, for $m,n\in\{0,1\}$ where $\mu(k)=\min\{c|\mathrm{Im}k|,1\}$, for a constant $c>0$.

Proof. Let $x\geq 0$. Consider first $\xi\leq 0$. Use (4.2) and (4.88); insert in (4.88) the decompositions (4.30) for $y_i(x,k)$ and $y_i(-\xi,k)$ and (4.90) for ω_{ij}. The terms coming from the first terms in (4.30) and (4.90) give rise to R^*. Now, using (4.29) and (4.31), one sees that all the other terms are bounded by $C\varrho(k)$ ($k\tilde{\omega}$ is of order k by Lemma 4.7) and decay exponentially in x and ξ with rate at least μ. So, we get ($-\xi\geq 0$):

$$|\partial_x^m\partial_\xi^n r(x,\xi,k)|\leq C\varrho(k)e^{-\mu x+\mu\xi}\leq C\varrho(k)e^{-\mu|x-\xi|} \tag{4.107}$$

which fits the bound (4.106).

Next, let $x\geq\xi\geq 0$. We insert the expansion (4.34) into (4.88). Collecting terms we have

$$\begin{aligned}\pi R(x,\xi,k)&=2y_1(x,k)(y_2(\xi,k)-y_1(\xi,k))-y_1(x,k)y_2(\xi,k)-y_2(x,k)\\ &\cdot(y_2(\xi,k)-y_1(\xi,k))+y_1(x,k)\sum_{i=1}^4\alpha_i(k)y_i(\xi,k)+y_2(x,k)\sum_{i=1}^4\beta_i(k)y_i(\xi,k)\end{aligned} \tag{4.108}$$

where α_i and β_i are obtained by adding the contributions of $\tilde{\omega}$ and from a_i and b_i in (4.34); α_i and β_i are analytic in C_r, and bounded by $C\varrho(k)$ there. Note that, using (4.29), all the terms in (4.108) are bounded, for $x\geq\xi$, by C except $y_2(x,k)\beta_4(k)y_4(\xi,k)$ which satisfies (using (4.29) again)

$$|y_2(x,k)\beta_4(k)y_4(\xi,k)-\beta_4(k)e^{\mu_2x+\mu_4\xi}|\leq Ce^{-\mu|x-\xi|}. \tag{4.109}$$

Let χ_N be the characteristic function of $[N,N+1]$. Then, (4.109) implies that

$$||R\chi_N||_2\geq\beta_4(k)e^{\frac{1}{2}N}$$

for N large. But, since R defines a bounded operator in L^2 ($\mathcal{L}\leq 0$), $\beta_4(k)=0$.

Thus, in (4.108) the second sum runs from 1 to 3. We may now proceed as in the $\xi\leq 0$ case, i.e. we insert (4.30) into the first three terms of (4.108) and get R^* from the terms coming from the first terms in (4.30). All other terms, as well as the sums in (4.108) are bounded by $C\varrho(k)$ times functions that decay at least like $e^{-\mu(x+\xi)}$, except $y_1(x,k)\alpha_4(k)y_4(\xi,k)$ and $y_2(x,k)\beta_3(k)y_3(\xi,k)$, which are bounded by $C|k|e^{\mu(x-\xi)}\leq C|k|e^{\mu|x-\xi|}$, since $x\geq\xi$. Finally, for $\xi\geq x\geq 0$ we use $R(x,\xi,k)=R(\xi,x,k)$. Since the second line in (4.105) and (4.106) are symmetric under the exchange of x and ξ, (4.105), (4.106) hold also in this region. $\square$

Proof of Proposition 3.2. Proposition 4.8 can be used to bound (4.1). By a change of variables,

$$e^{t\mathcal{L}}(x,\xi) = (\frac{t_0}{t})^{\frac{1}{2}} \int_{S_0} e^{-t_0 k^2} R(x,\xi,(\frac{t_0}{t})^{\frac{1}{2}}k)dk. \tag{4.110}$$

We choose the contour S_0 to be the piecewise linear curve formed by the lines $\mathrm{Im}k = \frac{1}{2}r^{-2}|\mathrm{Re}k|$ and $\mathrm{Im}k = \frac{1}{2}r^{-4}$ (see (4.14)). The bound on $R(x,\xi,(\frac{t_0}{t})^{\frac{1}{2}}k)$ comes from a combination of Propositions 4.1 and 4.8. Is is convenient to observe that actually (4.104), (4.105) and (4.106) hold everywhere on S_0 and not only in $S_0 \cap C_r$.

Indeed, on the boundary of $S_0 \cap C_r$ we have

$$\mathrm{Im}k = \frac{r^{-2}}{2}|\mathrm{Re}k| = \frac{r^{-1/2}}{2} \tag{4.111}$$

and $|\mathrm{Im}k^2| = 2|\mathrm{Im}k||\mathrm{Re}k| = r$ so that we can bound

$$|\partial_x^m \partial_\xi^n r(x,\xi,k)| \leq |\partial_x^m \partial_\xi^n R^*(x,\xi,k)| + |\partial_x^m \partial_\xi^n R(x,\xi,k)| \tag{4.112}$$

for $k \in S_0 \cap C_r$ i.e. for $|\mathrm{Im}k^2| > r$ and use Proposition 4.1 to bound the second term and the explicit formula (4.104) to bound the first:

$$|\partial_x^m \partial_\xi^n R^*(x,\xi,k)| \leq Ce^{-c(|x|+|\xi|)} \tag{4.113}$$

for $\mathrm{Im}k > \frac{r^{-1/2}}{2}$. Hence, (4.106) holds everywhere on S_0.

The integration of R^* gives rise to K^* in (3.11) and we note that $\varrho(k) \leq |k|$ for $k \in S_0$, which upon integration over k gives the $\frac{1}{t}$ factor in (3.12) while the exponential decay follows from the fact that $\mathrm{Im}k > \epsilon$ for $k \in S_0$.

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