Schrödinger operators with strong magnetic fields: Quasi-periodicity of spectral orbits and topology

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Abstract

We investigate the large λ behavior of $\sigma((p-\lambda A)^2)$ when the zero set of $B = dA$ has a non-empty interior. With certain technical hypotheses we show that if either B is bounded away from zero for large $|x|$ or periodic and certain quotients of standard homology groups are finite rank, then $\sigma((p-\lambda A)^2)$ approaches a quasi-periodic orbit in the space of subsets of $[0,\infty)$.

1 Introduction

We consider the Schrödinger operator

$$H(\lambda) = (p-\lambda A)^2 \qquad \text{in } L^2(\mathbf{R}^n),$$

where $p = -i\nabla$, $A(x)$ is the vector potential (which we assume as smooth as necessary—at least in C^1), and $B = dA$ is the vector potential. (Here we think of A as a 1-form,

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$A(x) = \sum_j A_j(x)dx_j$.) The strength of the field is measured by the coupling constant λ. We are interested in the limiting behavior of $H(\lambda)$, or rather its resolvent, as $\lambda \to \infty$.

In recent papers ([HH1], [HH2]), Hempel and Herbst studied this limiting behavior in the situation where the sets

$$M = \{x \in \mathbf{R}^n : B(x) = 0\},$$

and

$$M_A = \{x \in \mathbf{R}^n : A(x) = 0\}$$

are equal up to a set of Lebesgue measure zero. They showed that with this hypothesis, $H(\lambda)$ converges in the strong resolvent sense to $-\Delta_M$, the Laplacian with "Dirichlet boundary conditions" on the (closed) set M. (This operator will be defined later.) Under additional conditions on the field, B, this strong resolvent convergence could be improved to norm resolvent convergence. It was also shown that the gauge dependent hypothesis $|M \setminus M_A| = 0$ ($| \cdot |$ indicates Lebesgue measure) is almost necessary if one is to have norm resolvent convergence. For if $M \setminus M_A$ contains an open set, $(H(\lambda)+1)^{-1}$ cannot converge to any operator in norm. (Note that it is always true that $|M_A \setminus M| = 0$.) In another recent paper ([N]), Nakamura studied the rate of approach of the resolvent $(H(\lambda)+1)^{-1}$ to a limit in order to learn about the limiting behavior of $\sigma(H(\lambda))$.

The condition that there exists a gauge A such that $|M \setminus M_A| = 0$ is a topological condition. We will mostly be interested in situations where this condition is violated. We will show that in many situations of interest, the unitary equivalence class, $\mathcal{U}_\lambda$, of $H(\lambda)$ approaches a periodic or almost periodic orbit in the space of such classes as $\lambda \to \infty$. In particular this is true of the various unitary invariants of $H(\lambda)$, including the spectrum, $\sigma(H(\lambda))$.

In the remainder of this section we formulate a particularly simple result in dimension two. For this purpose we introduce a metric in the space of closed subsets of $\mathbf{C}$. Let $d(\,\cdot\,,\,\cdot\,)$ be the Hausdorff distance

$$d(\Gamma, \Gamma') = \max\left(\sup_{x\in\Gamma} d(x,\Gamma'), \sup_{x\in\Gamma'} d(x,\Gamma)\right)$$

for $\Gamma, \Gamma' \subset \mathbf{C}$. Here $d(x,\Gamma) = \inf_{y\in\Gamma} |x-y|$. The relevance of this distance to spectral theory is evident from the following:

Lemma 1.1 *Let A and B be normal operators on a Hilbert space. Then*

$$d(\sigma(A), \sigma(B)) \le \|A - B\|.$$

For a proof, see Appendix A.

The notion of convergence provided by the metric $d(\,\cdot\,,\,\cdot\,)$ is too restrictive for our purposes. Closeness of $\sigma(A)$ and $\sigma(B)$ in this metric means that these spectra look nearly the same uniformly on all parts of the real axis, a situation not guaranteed by norm resolvent convergence. (It is, however, guaranteed for self-adjoint A and B if $\sup_{\lambda\in\mathbf{R}} \|(\lambda+i-A)^{-1}-(\lambda+i-B)^{-1}\|$ is small; see Appendix A.) We will use the resolvent to define the convergence of spectra for unbounded self-adjoint operators.

In this paper we will be dealing with closed subsets, σ, of $[0,\infty)$. If σ and η are two such subsets, we introduce the metric

$$\bar{d}(\sigma,\eta)=d\left((\sigma+1)^{-1},(\eta+1)^{-1}\right),$$

where $(\sigma+1)^{-1}=\{(x+1)^{-1}:\, x\in\sigma\}$. It is easy to see that

$$\bar{d}(\sigma,\eta)\le d(\sigma,\eta).$$

All of our notions of convergence and closeness of spectra will be with respect to the $\bar{d}$ metric. Thus if A_n and A are non-negative, self-adjoint operators,

$$\sigma(A_n)\to\sigma(A)\Leftrightarrow \bar{d}(\sigma(A_n),\sigma(A))\to 0.$$

This type of convergence is discussed further in Appendix A.

We now discuss conditions we will impose on the set M. (See [HZ] and references therein for most of the following material.)

Let $P\subset\mathbf{R}^n$ be measurable and define

$$\tilde{W}^{1,2}(P)=W^{1,2}(\mathbf{R}^n)\cap L^2(P),$$

where we use the notation

$$L^2(P)=\{f\in L^2(\mathbf{R}^n):\, f(x)=0 \text{ a.e. in } P^c\},$$

and

$$W^{1,2}(\mathbf{R}^n)=\{f\in L^2(\mathbf{R}^n):\, \int|\nabla f(x)|^2<\infty\}.$$

Here $P^c=\mathbf{R}^n\setminus P$. Let 1_P be the indicator function of the set P. The sequence of operators $\{-\Delta+m1_{P^c}\}_{m=1}^{\infty}$ converges in the strong resolvent sense:

$$(-\Delta+m1_{P^c}+1)^{-1}\xrightarrow{s}(-\Delta_P+1)^{-1}\oplus 0, \tag{1.1}$$

where $-\Delta_P$ is a self-adjoint operator with form domain $\tilde{W}^{1,2}(P)$. There is a measurable set $Q\subset P$ so that the closure of $\tilde{W}^{1,2}(P)$ in $L^2(\mathbf{R}^n)$ is $L^2(Q)$ and thus the direct sum decomposition in (1.1) is with respect to the decomposition $L^2(\mathbf{R}^n) = L^2(Q) \oplus L^2(Q^c)$. The quadratic form associated with $-\Delta_P$ is of course $f \mapsto \int |\nabla f|^2$ for $f \in \tilde{W}^{1,2}(P)$.

If the Lebesgue measure, $|\partial M|$ of ∂M, is zero, then if $\Omega = M^0 = \text{interior}(M)$ we have $\tilde{W}^{1,2}(M) = \tilde{W}^{1,2}(\Omega)$ and $\Delta_M = \Delta_\Omega$. We will almost always assume

$$\tilde{W}^{1,2}(\Omega) = \overset{\circ}{W}{}^{1,2}(\Omega), \tag{1.2}$$

where $\overset{\circ}{W}{}^{1,2}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the norm $(\int(|\nabla f|^2 + |f|^2))^{\frac{1}{2}}$. This is a rather mild condition on $\partial\Omega$. Note that since $\Omega = M^0$, $\partial\Omega = \partial\bar{\Omega} \subset \partial M$. (If $\partial\Omega = \partial\bar{\Omega}$ is not assumed there are simple examples of open sets Ω with $\tilde{W}^{1,2}(\Omega) \neq \overset{\circ}{W}{}^{1,2}(\Omega)$.) Condition (1.2) guarantees that Δ_Ω is the usual Dirichlet Laplacian on Ω.

Let P^x be the probability measure for Brownian motion $X(\,\cdot\,)$ starting at $x \in \mathbf{R}^n$. For $P \subset \mathbf{R}^n$, let

$$\tau_P = \inf\{t > 0 : \, X(t) \in P^c\},$$

the first exit time of Brownian motion from P. Then, if we assume $|\partial M| = 0$, (1.2) holds if and only if

$$\tau_{\bar{\Omega}} = \tau_\Omega; \;\; P^x \text{ almost surely for each } x \in \Omega. \tag{1.3}$$

A sufficient condition for (1.3) is that the set

$$\{x \in \partial\Omega : \, P^x(\tau_{\bar{\Omega}} > 0) = 1\}$$

is polar. (The set $\{x : \, P^x(\tau_{\bar{\Omega}} > 0) = 1\}$ is the set of irregular points for $(\bar{\Omega})^c$. A Borel set P is polar if $P^x(X(t) \in P$ for some $t > 0) = 0$ for all $x \in \mathbf{R}^N$. See [PS] for material on this subject.)

If we already know (1.2) for each connected component Ω_j of Ω, then one approach to proving (1.2) for $\Omega = \bigcup_j \Omega_j$ is to show

$$\tilde{W}^{1,2}(\Omega) = \bigoplus_j \tilde{W}^{1,2}(\Omega_j). \tag{1.4}$$

Conditions for (1.4) are given in Appendix C. We remark that if an open set D satisfies the segment condition, then ([HZ])

$$\tilde{W}^{1,2}(\bar{D}) = \tilde{W}^{1,2}(D) = \overset{\circ}{W}{}^{1,2}(D).$$

Before stating a simple result in dimension two we recall the definition of quasi-periodicity: a function f of a real variable is quasi-periodic if for some $N \geq 1$ there exists a function g of N real variables, periodic with period 1 in each variable, and real numbers $\omega_1, \ldots, \omega_N$ such that

$$f(\lambda) = g(\lambda\omega_1, \lambda\omega_2, \ldots, \lambda\omega_N).$$

The notation $H_1(\Omega; \mathbf{Z}) = H_1(\Omega)$ indicates the first homology group of Ω (1-cycles/ boundaries) with integer coefficients.

Theorem 1.2 *Suppose $n = 2$, $A \in C^\infty(\mathbf{R}^n)$, and $\liminf_{|x|\to\infty} |B(x)| > 0$. Suppose $|\partial M| = 0$, $\tilde{W}^{1,2}(\Omega) = \overset{\circ}{W}{}^{1,2}(\Omega)$, and that Ω is non-empty with finitely many holes (i.e., $H_1(\Omega)$ is finitely generated). Then the spectral orbit $\lambda \mapsto \sigma(H(\lambda))$ approaches a periodic or quasi-periodic orbit in the space of subsets of $[0, \infty)$. More specifically, there exists a quasi-periodic function f on $\mathbf{R}$ with values in the closed subsets of $[0, \infty)$, continuous in the topology given by the metric $\bar{d}$, such that*

$$\bar{d}(\sigma(H(\lambda)), f(\lambda)) \to 0 \text{ as } \lambda \to \infty.$$

In addition:

(i) $f(\lambda)$ is constant $\iff$ the flux of B through each hole of Ω is zero (i.e., $\int_\gamma A = 0$ for each 1-cycle γ in Ω);

(ii) if $H_1(\Omega)$ has one generator, then $f(\lambda)$ is periodic.

This theorem is a corollary of Theorem 3.4.

The plan of the rest of this paper is as follows:

Section 2, Convergence: An operator family $\{K(\lambda)\}$ is defined (roughly speaking $K(\lambda)$ is $H(\lambda)$ with Dirichlet boundary conditions on ∂M). It is shown that in two favorable situations, $(H(\lambda)+1)^{-1} - (K(\lambda)+1)^{-1} \oplus 0$ converges to zero in norm as $\lambda \to \infty$.

Section 3, Topology and quasi-periodicity: If either $B = dA$ is bounded away from zero for large $|x|$ or B is periodic, and if certain quotients of standard homology groups are finitely generated, it is shown that $f(\lambda) = \sigma(K(\lambda))$ is quasi-periodic.

Appendix A, The Hausdorff distance between spectra: The meaning of convergence of spectra is discussed and continuity of $\sigma(K(\lambda))$ as a function of λ is proved.

Appendix B, Lower bounds for $H(\lambda)$: Lower bounds are proved if the field B is bounded away from zero for large $|x|$ and does not "twist" too much.

Appendix C, Decoupling of disjoint regions: For two disjoint open sets $\mathcal{O}_1$ and $\mathcal{O}_2$, it may happen that $\tilde{W}^{1,2}(\mathcal{O}_1 \cup \mathcal{O}_2) \neq \tilde{W}^{1,2}(\mathcal{O}_1) \oplus \tilde{W}^{1,2}(\mathcal{O}_2)$. We investigate conditions which guarantee equality.

We remark that as a simple corollary of the methods of this paper, identical results hold for the two-dimensional Pauli operator *as long as $B = \partial_1 A_2 - \partial_2 A_1$ has one sign.* (See Theorems 1.2 and 3.4.)

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2 Convergence

In this section, after carefully defining the relevant operators, we show that there is an operator family $\{K(\lambda)\}$ associated with the set M so that in certain interesting cases $(H(\lambda)+1)^{-1} - (K(\lambda)+1)^{-1} \oplus 0 \to 0$ in norm as $\lambda \to \infty$. (No assumptions are made about M.) The function $f(\lambda)$ of Theorem 1.2 will later be defined as $\sigma(K(\lambda))$.

We suppose $A_j \in C^2(\mathbf{R}^n)$, $j = 1, 2, \ldots, n$ and define the self-adjoint operator

$$H(\lambda) = (p - \lambda A(x))^2$$

by giving the associated closed quadratic form:

$$Q(\lambda)(\psi, \phi) = \int \overline{(i\nabla + \lambda A(x))\psi(x)} \cdot (i\nabla + \lambda A(x))\phi(x)dx,$$

with form domain

$$\mathcal{D}(Q(\lambda)) = \{\psi \in L^2(\mathbf{R}^n) : \psi \in W^{1,2}_{\mathrm{loc}}(\mathbf{R}^n), (i\nabla + \lambda A(x))\psi(x) \in L^2(\mathbf{R}^n)\}.$$

Here $W^{1,2}_{\mathrm{loc}}(\mathbf{R}^n)$ is the set of all measurable functions on $\mathbf{R}^n$ such that $\chi\psi \in W^{1,2}(\mathbf{R}^n)$ for all $\chi \in C_0^\infty(\mathbf{R}^n)$. To define the operator $K(\lambda)$ we first give its form domain $\mathcal{D}(Q_M(\lambda))$:

$$\mathcal{D}(Q_M(\lambda)) = \mathcal{D}(Q(\lambda)) \cap L^2(M),$$

and define the associated quadratic form $Q_M(\lambda)$ to be $Q(\lambda)$ restricted to $\mathcal{D}(Q_M(\lambda))$. It follows that $K(0) = -\Delta_M$, an operator already defined in Section 1.

We first give some conditions which guarantee norm resolvent convergence.

Lemma 2.1 *Suppose there is a sequence of open sets $G_m \subset M^c$ satisfying*

(i) Given $\epsilon > 0$ and m, there is a $\lambda(\epsilon) > 0$ so that if $\lambda \geq \lambda(\epsilon)$,

$$H(\lambda) + 1 \geq \epsilon^{-1} 1_{G_m},$$

(ii) $\|(-\Delta + 1)^{-1}(1_{M^c} - 1_{G_m})\| \to 0$ *as* $m \to \infty$;

(iii) $\left\|(-\Delta + m1_{M^c} + 1)^{-1} - (-\Delta_M + 1)^{-1} \oplus 0\right\| \to 0$ *as* $m \to \infty$.

Then

$$\left\|(H(\lambda)+1)^{-1} - (K(\lambda)+1)^{-1} \oplus 0\right\| \to 0 \text{ as } \lambda \to \infty.$$

Proof. This is essentially proved in [HH1] so we leave out some details.

The estimate in (i) gives $\left\|(H(\lambda)+1)^{-\frac{1}{2}}1_{G_m}\right\| \leq \sqrt{\epsilon}$ for large λ so for these λ,

$$\begin{aligned}\left\|(H(\lambda)+1)^{-\frac{1}{2}}1_{M^c}\right\| &\leq \sqrt{\epsilon}+\left\|(H(\lambda)+1)^{-1}\left(1_{M^c}-1_{G_m}\right)\right\| \\ &\leq \sqrt{\epsilon}+\left\|(-\Delta+1)^{-1}\left(1_{M^c}-1_{G_m}\right)\right\|.\end{aligned}$$

Thus from (ii),

$$\left\|(H(\lambda)+1)^{-\frac{1}{2}}1_{M^c}\right\| \to 0 \text{ as } \lambda \to \infty.$$

It follows that we can choose $m(\lambda) \in \mathbf{Z},\ m(\lambda) \uparrow \infty$, such that

$$\lim_{\lambda\to\infty} m(\lambda)\left\|(H(\lambda)+1)^{-\frac{1}{2}}1_{M^c}\right\| = 0,$$

which implies (using the resolvent equation) that

$$\left\|(H(\lambda)+m(\lambda)1_{M^c}+1)^{-1}-(H(\lambda)+1)^{-1}\right\| \to 0 \text{ as } \lambda \to \infty.$$

But using the Feynman-Kac-Ito formula we obtain

$$\begin{aligned}&\left\|\left\{(H(\lambda)+m(\lambda)1_{M^c}+1)^{-1}-(K(\lambda)+1)^{-1}\oplus 0\right\}\phi\right\| \\ &\quad\leq \left\|\left\{(-\Delta+m(\lambda)1_{M^c}+1)^{-1}-(-\Delta_M+1)^{-1}\oplus 0\right\}|\phi|\right\|,\end{aligned}$$

and thus by (iii),

$$\begin{aligned}\left\|(H(\lambda)+1)^{-1}-(K(\lambda)+1)^{-1}\oplus 0\right\| &\leq \left\|(H(\lambda)+1)^{-1}-(H(\lambda)+m(\lambda)1_{M^c}+1)^{-1}\right\| \\ &\quad+\left\|(-\Delta+m(\lambda)1_{M^c}+1)^{-1}-(-\Delta_M+1)^{-1}\oplus 0\right\| \to 0,\end{aligned}$$

as $\lambda \to \infty$. □

We now give two interesting conditions on the geometry of B, each of which guarantees that the conditions of Lemma 2.1 are fulfilled.

Theorem 2.2 *Suppose $A \in C^2(\mathbf{R}^n)$ and that $B = dA = \sum_{i<j} B_{ij}(x)dx_i \wedge dx_j$ satisfies either*

(i) there is a $c > 0$ so that for $|x|$ large,

$$|B(x)| = \left(\sum_{i<j}|B_{ij}(x)|^2\right)^{\frac{1}{2}} \geq c,$$

and

$$\left| \nabla \left(\frac{B_{ij}(x)}{|B(x)|} \right) \right| \leq c^{-1} \quad \textit{for all } i < j,$$

or

(ii) $B(x)$ *is periodic in each* $x_j,\ j = 1, 2, \ldots, n$.

Then

$$\left\| (H(\lambda) + 1)^{-1} - (K(\lambda) + 1)^{-1} \oplus 0 \right\| \to 0 \quad \textit{as } \lambda \to \infty.$$

Note: If $n = 2$, and $|B(x)| \geq c > 0$ for $|x|$ large, the rest of (i) is automatic.

Proof. Again, much of the proof is given in [HH1] so we only provide a sketch.

In case (i), the first step is to find a sequence of open sets $G_m \uparrow M^c$ so that for any $\epsilon > 0$ there is a $\lambda(\epsilon)$ with

$$H(\lambda) + 1 \geq \epsilon^{-1} 1_{G_m} \quad \text{if } \lambda \geq \lambda(\epsilon).$$

In addition, we require that G_m^c is compact. If G is a compact subset of M^c, it is easy to show using the inequality $H(\lambda) \geq \lambda B_{ij}(x)$ of [AHS] (see Appendix B) and a partition of unity, that

$$H(\lambda) \geq \lambda c_G 1_G - d_G \tag{2.1}$$

for some positive constants c_G and d_G. From Appendix B, it follows that for some positive constants c_1 and d_1 and R large enough,

$$H(\lambda) \geq \lambda c_1 1_{B_R^c} - d_1, \tag{2.2}$$

where B_R is the ball of radius R centered at the origin. By choosing an appropriate sequence of G's and adding (2.1) and (2.2), we easily obtain open $G_m \uparrow M^c$ so that G_m^c is compact.

In the periodic case we will produce a sequence $G_m \uparrow M^c$ such that each G_m is invariant under translation by the periods of $B(\,\cdot\,)$ (which we assume for simplicity are given by the lattice $\mathbf{Z}^n$). Let C_0 be the open cube of side 2, centered at the origin and suppose $x_0 \in C_0 \cap M^c$. Then there exists an $r > 0$ and a pair (i,j) so that for $x \in B_{2r}(x_0)$, $B_{ij}(x) \geq \alpha > 0$ (we define $B_{ij}(x) = -B_{ji}(x)$ if $j < i$). If $\eta \in C_0^\infty(B_{2r}(x_0))$ with $0 \leq \eta \leq 1$ and $\eta = 1$ on $B_r(x_0)$, let $\eta_a(x) = \eta(x - a)$ for $a \in \mathbf{Z}^n$. We clearly have

$$\eta_a H(\lambda) \eta_a \geq \lambda \alpha \eta_a^2$$

by the AHS inequality. Let $\xi^2 = 1 + \sum_a \eta_a^2$. Then with $\gamma_a = \xi^{-1}\eta_a$ and $\tilde{\gamma} = \xi^{-1}$ we have

$$H(\lambda) \geq \left(\sum_a \gamma_a^2 \right) \lambda \alpha - |\nabla \tilde{\gamma}|^2 - \sum_a |\nabla \gamma_a|^2.$$

It follows that for some $\beta,\ \beta' > 0$,

$$H(\lambda) \geq \lambda\beta\, 1_{\tilde{B}_r(x_0)} - \beta',$$

where $\tilde{B}_r(x_0) = \bigcup_{a\in\mathbf{Z}^n} B_r(x_0 + a)$. Since we can cover an arbitrary compact subset of $C_0 \cap M^c$ with a finite set of open balls $B_{r_\ell}(x_\ell)$, with $B_{i_\ell, j_\ell}(x) \geq \alpha_\ell > 0$ in $B_{2r_\ell}(x_\ell)$, the required open *periodic* G_m can be found with $G_m \uparrow M^c$.

We now show that the remainder of the requirements of Lemma 2.1 are fulfilled. Let

$$\begin{aligned} D_m &= (-\Delta + 1)^{-1} (1_{M^c} - 1_{G_m}) (-\Delta + 1)^{-1} \\ E_m &= (-\Delta + m 1_{M^c} + 1)^{-1} - (-\Delta_M + 1)^{-1} \oplus 0, \end{aligned}$$

and for $a \in \mathbf{Z}^n$ write $\Gamma_a = \frac{1}{2} C_0 + a$. It is clear that for some $k > 0$,

$$\|1_{\Gamma_a} E_m 1_{\Gamma_b}\| + \|1_{\Gamma_a} D_m 1_{\Gamma_b}\| \leq \mathrm{const} \cdot e^{-k|a-b|},$$

because this is true if E_m and D_m are replaced by $(-\Delta+1)^{-1}$ and $(-\Delta+1)^{-2}$.

According to Proposition 3.7 of [HH1] we must only show that

$$\mu_{m,r} \equiv \sup\left\{\|1_{\Gamma_a} E_m 1_{\Gamma_b}\| + \|1_{\Gamma_a} D_m 1_{\Gamma_b}\| :\ |a-b| < r\right\} \to 0$$

as $m \to \infty$ for each $r > 0$. Consider case (i) where M is compact. Then because $\|E_m 1_{M^c}\| \leq m^{-\frac{1}{2}}$ and $\|D_m 1_{\Gamma_a}\| \leq \mathrm{const} \cdot e^{-k|a|}$ for some $k > 0$, it follows we need only show

$$\|E_m 1_{\Gamma_a}\| + \|D_m 1_{\Gamma_a}\| \to 0 \ \text{ as } m \to \infty$$

for each $a \in \mathbf{Z}^n$. But we obtain this easily from the strong convergence $E_m \xrightarrow{s} 0,\ D_m \xrightarrow{s} 0$ and a compactness argument (see [HH1]).

In the periodic case, if $|b - a| \leq r$,

$$\begin{aligned} \|1_{\Gamma_a} E_m 1_{\Gamma_b}\| + \|1_{\Gamma_a} D_m 1_{\Gamma_b}\| &= \left\|1_{\Gamma_0} E_m 1_{\Gamma_{b-a}}\right\| + \left\|1_{\Gamma_0} D_m 1_{\Gamma_{b-a}}\right\| \\ &\leq \|1_C E_m 1_C\| + \|1_C D_m 1_C\|, \end{aligned}$$

where

$$C = \bigcup_{|a|<r} \Gamma_a.$$

The convergence again follows from the strong convergence $E_m \xrightarrow{s} 0,\ D_m \xrightarrow{s} 0$, and a compactness argument. $\square$

We state the following corollary of the proofs of Lemma 2.1 and Theorem 2.2 which will be of use in Appendix A:

Corollary 2.3 *Suppose $A \in C^1(\mathbf{R}^n)$, and either M is compact or invariant under $\mathbf{Z}^n$. Then*

$$\lim_{m\to\infty} \left\| (H(\lambda) + m1_{M^c} + 1)^{-1} - (K(\lambda)+1)^{-1} \oplus 0 \right\| = 0.$$

3 Topology and quasi-periodicity

In this section we suppose $A_j \in C^\infty(\mathbf{R}^n)$, $|\partial M| = 0$, and $\tilde{W}^{1,2}(\Omega) = \overset{\circ}{W}{}^{1,2}(\Omega)$. We consider the operator $K(\lambda)$ defined in Section 2. Our assumptions make the computation of the operator $\eta^{-1}K(\lambda)\eta$ simple when $\eta(x) \in C^\infty(\Omega)$ with $|\eta(x)| = 1$. Let $\Gamma(x) = -i\nabla\eta/\eta$. Then it is easy to see that Γ is $\mathbf{R}^n$-valued and in $C^\infty(\Omega)$.

Lemma 3.1 *With the assumptions and notations above, $\eta^{-1}K(\lambda)\eta$ is a self-adjoint operator with quadratic form*

$$f \mapsto Q_\Gamma(f,f) = \int_\Omega |(i\nabla + A(x) - \Gamma(x))f(x)|^2 \, dx,$$

and form domain equal to the closure of $C_0^\infty(\Omega)$ in the norm

$$\left(\|f\|^2 + Q_\Gamma(f,f)\right)^{\frac{1}{2}}.$$

We omit the easy proof.

Note that while $d\Gamma = 0$, it is certainly not always true that $\Gamma = d\phi$. (Take $e^{i\theta} = \eta$ on $\Omega = \mathbf{R}^2 \setminus \{x : |x| \le 1\}$.) But we have

$$\int_\gamma \Gamma = 2\pi \text{ (integer), for any smooth 1-cycle in } \Omega. \tag{3.1}$$

(We will only work with smooth chains but this will not affect the relevant homology or cohomology groups. See [B].) To prove (3.1), note that γ is homologous to a sum of loops so we can assume γ is a loop; $\gamma : [0,1] \to \mathbf{R}^n$, $\gamma(0) = \gamma(1) = x_0$. In a small ball around any point in Ω, we can choose a branch of the logarithm so that $i\Gamma = d(\log \eta)$. Let $\gamma_{t,s} = \gamma|[t,s]$. Then for small $|t-s|$,

$$e^{i\int_{\gamma_{t,s}}\Gamma} = \frac{\eta(\gamma(s))}{\eta(\gamma(t))},$$

which shows $e^{i\int_\gamma \Gamma} = 1$. This proves (3.1).

We use this circle of ideas to give a simple proof of a theorem of Helffer ([H], Theorem 7.2.1.1).

Proposition 3.2 *Suppose $\bar{\Omega}$ is compact along with our standing assumptions for this section. Then $E(1) = \inf \sigma(K(1)) \geq \inf \sigma(-\Delta_\Omega) = E(0)$ with equality if and only if*

$$\int_\gamma A = 2\pi\,(\text{integer}) \quad \text{for every 1-cycle in } \Omega,$$

in which case

$$A = -\frac{id\eta}{\eta},$$

for some $\eta \in C^\infty(\Omega)$ with $|\eta(x)| = 1$. If $E(\lambda) = \inf \sigma(K(\lambda)) = E(0)$ for λ_1 and $\lambda_2 \in \mathbf{R} \setminus \{0\}$ with λ_1/λ_2 irrational, then

$$A = d\phi$$

for some $\phi \in C^\infty(\Omega)$.

Proof: The inequality $E(\lambda) \geq E(0)$ is the well-known diamagnetic inequality [CFKS]. If $\bar{\Omega}$ is compact, the resolvents of $K(\lambda)$ are compact for all λ so the bottom of the spectrum is an eigenvalue. Suppose $K(\lambda)\psi_\lambda = E(\lambda)\psi_\lambda$ with $\|\psi_\lambda\| = 1$ and $\psi_0 > 0$. Then by the diamagnetic inequality

$$E(1) = (\psi_1, K(1)\psi_1) \geq \|\nabla|\psi_1|\|^2 \geq \|\nabla\psi_0\|^2 = E(0).$$

If $E(1) = E(0)$ it follows that $|\psi_1| = \psi_0$ because $E(0)$ is non-degenerate. By elliptic regularity, ψ_1 and ψ_0 are in $C^\infty(\Omega)$ and thus so is

$$\eta(x) = \frac{\psi_1(x)}{\psi_0(x)}.$$

With $\Gamma = -id\eta/\eta$ we obtain

$$(p - A)\psi_1(x) = -\eta(id\psi_0 + (A - \Gamma)\psi_0),$$

so that

$$\int_\Omega |A - \Gamma|^2 \psi_0^2 = 0.$$

It follows that $\Gamma(x) = A(x)$ for all $x \in \Omega$, and by (3.1),

$$\int_\gamma A = 2\pi(\text{integer}), \text{ all 1-cycles } \gamma \text{ in } \Omega.$$

Conversely, if this is true we can define

$$\eta(x) = e^{i \int_{\gamma_x} A}$$

in each component of Ω by choosing some point and a path γ_x from that point to x. η is well defined, and in fact,

$$-i\,\frac{d\eta}{\eta} = A.$$

It follows that $\eta^{-1}K(1)\eta = K(0) = -\Delta_\Omega$ so that $E(0) = E(1)$.

If $E(\lambda_1) = E(\lambda_2) = E(0)$ for $\lambda_1, \lambda_2 \in \mathbf{R} \setminus \{0\}$, then from what we have proved above, $\lambda_j \int_\gamma A = 2\pi(\text{integer})$, for each j and each 1-cycle γ in Ω. If λ_1/λ_2 is irrational, this implies

$$\int_\gamma A = 0, \quad \text{for each 1-cycle } \gamma \text{ in } \Omega,$$

which implies $A = d\phi$ for some $\phi \in C^\infty(\Omega)$. □

It is clear from what we have seen so far that the topology of Ω is important for certain qualitative features of the spectrum of $K(\lambda)$. We denote by $H_1(\Omega; R)$ the first homology group of Ω with coefficients from the ring R.

For our purposes one of the natural groups to consider is

$$H_1^B(\Omega; \mathbf{Z}) = H_1(\Omega; \mathbf{Z})/\mathcal{N}_B,$$

where

$$\mathcal{N}_B = \left\{[\gamma] \in H_1(\Omega; \mathbf{Z}) : \int_\gamma A = 0\right\}.$$

Here, of course, $[\gamma]$ is the equivalence class of all 1-cycles in Ω which differ from γ by the boundary $\partial\mu$ of a 2-chain μ in Ω. Notice that $\mathcal{N}_B$ actually contains the torsion subgroup, $\mathrm{Tor}_1(\Omega)$, defined by

$$\mathrm{Tor}_1(\Omega) = \{[\gamma] \in H_1(\Omega; \mathbf{Z}) : m[\gamma] = 0 \text{ for some } m \in \mathbf{Z} \setminus \{0\}\}.$$

Of course, if $[\gamma] \in \mathrm{Tor}_1(\Omega)$,

$$m \int_{[\gamma]} A = 0 \text{ so } \int_\gamma A = 0 \text{ for any } A \text{ with } dA = 0 \text{ in } \Omega.$$

If $[\gamma] \notin \mathrm{Tor}_1(\Omega)$ and $[\gamma] \neq 0$ but $\int_\gamma A = 0$, then there must be a rational dependence between the flux of $B = dA$ through the "holes" in Ω. To see this, note that $\partial\gamma = 0$ implies $\gamma = \partial\mu$ where μ is a 2-chain in $\mathbf{R}^n$ (not in Ω) and Stokes' theorem implies

$$\int_\mu B = \int_\mu dA = \int_{\partial\mu} A = \int_\gamma A = 0.$$

One simple result shows the importance of the group $H_1^B(\Omega; \mathbf{Z})$.

Theorem 3.3 *With the standing assumptions of this section,* $H_1^B(\Omega;\mathbf{Z}) = 0$ *if and only if* $A = d\phi$ *for some* $\phi \in C^\infty(\Omega)$ *in which case*

$$e^{-i\lambda\phi}K(\lambda)e^{i\lambda\phi} = -\Delta_\Omega.$$

Proof: By definition $H_1^B(\Omega;\mathbf{Z}) = 0$ if and only if $\int_\gamma A = 0$ for every 1-cycle γ in Ω. This is the case if and only if $A = d\phi$ and the rest follows by Lemma 3.1. $\square$

The next best thing to being zero is being finitely generated.

Theorem 3.4 *With the standing assumptions of this section, suppose* $H_1^B(\Omega;\mathbf{Z})$ *is finitely generated. Suppose, in addition, either*

(i) there is a $c > 0$ *so that for large* $|x|$

$$|B(x)| = \left(\sum_{i<j} |B_{ij}(x)|^2\right)^{\frac{1}{2}} \geq c,$$

and

$$\left|\nabla\left(\frac{B_{ij}(x)}{|B(x)|}\right)\right| \leq c^{-1} \textit{ for all } i,j,$$

or

(ii) $B(x)$ *is periodic in each variable.*

Then there is a continuous quasi-periodic function $f:\ \mathbf{R} \to \{$*closed subsets of* $[0,\infty)\}$ *such that*

(a) $\bar{d}(f(\lambda), \sigma(H(\lambda))) \to 0$ *as* $\lambda \to \infty$*;*

(b) if $H_1^B(\Omega;\mathbf{Z})$ *has one generator, then* f *is periodic;*

(c) if $\bar{\Omega}$ *is compact, then* f *is constant* $\Longleftrightarrow H_1^B(\Omega;\mathbf{Z}) = 0$.

Proof: Since $H_1^B(\Omega;\mathbf{Z})$ is finitely generated and has no torsion we can write

$$H_1^B(\Omega;\mathbf{Z}) = \mathbf{Z}\langle\gamma_1\rangle \oplus \mathbf{Z}\langle\gamma_2\rangle \oplus \cdots \oplus \mathbf{Z}\langle\gamma_N\rangle \cong \mathbf{Z}^N,$$

where $\langle\gamma\rangle = [\gamma] + \mathcal{N}_B$, and $\mathbf{Z}\langle\gamma_j\rangle$ is the infinite cyclic group generated by $\langle\gamma_j\rangle$. Let $\mathbf{P}$ be the projection $\mathbf{P}:\ H_1(\Omega;\mathbf{Z}) \to H_1^B(\Omega;\mathbf{Z})$ given by $\mathbf{P}[\gamma] = \langle\gamma\rangle$ and define

$$\lambda_1\left(\sum_{j=1}^N n_j\langle\gamma_j\rangle\right) = 2\pi n_1.$$

Here $n_j \in \mathbf{Z}$ for $j = 1, 2, \ldots, N$. $\lambda_1 \in \mathrm{Hom}(H_1^B(\Omega; \mathbf{Z}); \mathbf{R})$ (Hom indicates homomorphism), and thus $\lambda_1 \circ \mathbf{P} \in \mathrm{Hom}(H_1(\Omega; \mathbf{Z}); \mathbf{R})$. According to de Rham's theorem ([B], [Fl]) there exists a C^∞ 1-form A_1 with $dA_1 = 0$ so that for any 1-cycle γ in Ω,

$$(\lambda_1 \circ \mathbf{P})([\gamma]) = \int_\gamma A_1.$$

Since $\lambda_1(0) = 0$, it follows that

$$\int_\gamma A_1 = 0 \text{ for all } \gamma \text{ with } [\gamma] \in \mathcal{N}_B.$$

If γ is any 1-cycle in Ω, we can write

$$\langle \gamma \rangle = \sum_{j=1}^N n_j \langle \gamma_j \rangle \text{ or } \left\langle \gamma - \sum_{j=1}^N n_j \gamma_j \right\rangle = 0.$$

This means that

$$[\gamma] = \sum_{j=1}^N n_j [\gamma_j] + [\mu]; \quad [\mu] \in \mathcal{N}_B,$$

and therefore

$$\int_\gamma A_1 = 2\pi n_1,$$

i.e., the integral of A_1 over any 1-cycle is 2π(integer). Proceeding similarly for $j \geq 2$, we find C^∞ 1-forms A_j with $dA_j = 0$ such that

$$\begin{aligned} \int_{\gamma_j} A_k &= 2\pi \delta_{jk}; \quad 1 \leq j \leq k \leq N; \\ \int_\gamma A_k &= 0 \text{ if } [\gamma] \in \mathcal{N}_B; \\ \int_\gamma A_k &= 2\pi(\text{integer}) \text{ for any 1-cycle } \gamma \text{ in } \Omega. \end{aligned}$$

Consider the 1-form A appearing in $H(\lambda)$. When restricted to Ω we have $dA = 0$ and by definition, $\int_\gamma A = 0$ if $[\gamma] \in \mathcal{N}_B$. Define the normalized fluxes

$$\omega_j = (2\pi)^{-1} \int_{\gamma_j} A.$$

We see that

$$\int_\gamma \left(A - \sum_{j=1}^N \omega_j A_j \right) = 0$$

for $\gamma = \gamma_1, \gamma_2, \ldots, \gamma_N$ and thus for all 1-cycles γ in Ω. It follows that

$$A = \sum_{j=1}^{N} \omega_j A_j + d\phi$$

for some $\phi \in C^\infty(\Omega)$. Let

$$g(\xi_1, \xi_2, \ldots, \xi_N) = \sigma(T(\xi_1, \xi_2, \ldots, \xi_N)),$$

where $T(\xi_1, \ldots, \xi_N)$ is the self-adjoint operator with quadratic form

$$f \mapsto \int_\Omega \left| \left(i\nabla + \sum_{j=1}^{N} \xi_j A_j(x) \right) f(x) \right|^2 \equiv Q_{\xi_1, \ldots, \xi_N}(f, f),$$

and form domain equal to the closure of $C_0^\infty(\Omega)$ in the norm

$$\left(\|f\|^2 + Q_{\xi_1, \ldots, \xi_N}(f, f) \right)^{\frac{1}{2}}.$$

We let

$$\eta_j(x) = \exp\left(-i \int_{\gamma_x} A_j \right),$$

where in each connected component of Ω, γ_x is a path starting from a fixed point and ending at x. Since $\int_\gamma A_j = 2\pi(\text{integer})$ for each 1-cycle γ in Ω, η_j is well defined and smooth. As in Lemma 3.1,

$$\eta_j^{-1} T(\xi_1, \ldots, \xi_n) \eta_j = T(\xi_1, \ldots, \xi_j + 1, \ldots, \xi_N),$$

so that g is periodic with period 1 in each variable. Similarly, after a gauge transformation implemented by $e^{i\lambda\phi}$, we obtain

$$\sigma(K(\lambda)) = g(\lambda\omega_1, \ldots, \lambda\omega_N).$$

Thus we define

$$f(\lambda) = \sigma(K(\lambda)),$$

and note that f is quasi-periodic. It follows from Theorem 2.2 and Lemma 1.1 that

$$\bar{d}(\sigma(H(\lambda)), f(\lambda)) \to 0 \text{ as } \lambda \to \infty.$$

This proves part (a) of the theorem. Note that if $H_1(\Omega; \mathbf{Z})/\mathcal{N}_B$ has one generator, then $N = 1$ above, so f is periodic. This is part (b). Part (c) is a corollary of Theorem 3.3.

Finally, the continuity of f is proved in Proposition A.4 of Appendix A. $\square$

Theorem 3.4 singles out the quotient $H_1^B(\Omega;\mathbf{Z}) = H_1(\Omega;\mathbf{Z})/\mathcal{N}_B$ which is natural for our problem. In the periodic case it is interesting to ask for a result which depends only on the topology of Ω and not our particular magnetic field B (except that B is zero in Ω). For any C^∞ periodic, closed 2-form F on $\mathbf{R}^n$ which vanishes in Ω we can define

$$\mathcal{N}_F = \left\{ [\gamma] \in H_1(\Omega;\mathbf{Z}) : \int_\gamma A_F = 0 \right\},$$

where $dA_F = F$. The group

$$\mathcal{N}_\Omega = \bigcap_F \mathcal{N}_F,$$

where the intersection is over all C^∞, $\mathbf{Z}^n$-periodic, closed 2-forms F on $\mathbf{R}^n$ which vanish on Ω clearly depends only on the open set $\Omega \subset \mathbf{R}^n$. But since this group may be difficult to analyze, we give a sufficient condition for $H_1(\Omega;\mathbf{Z})/\mathcal{N}_\Omega$ (and thus $H_1^B(\Omega;\mathbf{Z})$) to be finitely generated which only involves familiar topological notions.

We first introduce some additional notation. Let $\pi : \mathbf{R}^n \to \mathbf{R}^n/\mathbf{Z}^n$ be the natural projection onto the n-torus T^n. Let $\tilde{\pi}$ be its restriction to Ω and denote $\pi(\Omega) = \tilde{\Omega} \subset T^n$. For a k-chain μ in $\mathbf{R}^n$, $\pi \circ \mu$ will denote the k-chain in T^n defined by linearity from its natural meaning on (smooth) singular k-simplexes. Similarly π_* will denote the induced action on homology. Similar definitions hold for $\tilde{\pi} \circ \mu$ and $\tilde{\pi}_*$.

Let $\mathrm{Tor}_1(\Omega)$ = Torsion subgroup of $H_1(\Omega;\mathbf{Z})$ and $\ker \tilde{\pi}_*$ the kernel of the map $\tilde{\pi}_* : H_1(\Omega;\mathbf{Z}) \to H_1(\tilde{\Omega};\mathbf{Z})$. Similarly, let $\mathrm{im}\ \tilde{\pi}_*$ be the image of this map.

For $0 < m \in \mathbf{Z}$, let

$$K_m = \{[\gamma] \in H_1(\Omega;\mathbf{Z}) : m\tilde{\pi}_*([\gamma]) = 0\}.$$

Note that if we define T_m as the m-torsion of $\mathrm{im}\ \tilde{\pi}_*$,

$$T_m = \{[\tilde{\gamma}] \in \mathrm{im}\ \tilde{\pi}_* : \ m[\tilde{\gamma}] = 0\},$$

then $K_m = \tilde{\pi}_*^{-1}(T_m)$.

Theorem 3.5 *If any of the groups*

(i) $H_1(\Omega;\mathbf{Z})/(\mathit{Tor}_1(\Omega) + K_m)$,

(ii) *im* $\tilde{\pi}_*$, *or*

(iii) $H_1(\tilde{\Omega};\mathbf{Z})$,

is finitely generated, then so is $H_1(\Omega;\mathbf{Z})/\mathcal{N}_\Omega$.

To prove Theorem 3.5, we will make use of the following lemma:

Lemma 3.6 *$(K_m + \mathcal{N}_\Omega)/\mathcal{N}_\Omega$ is finitely generated.*

Proof: If $[\gamma] \in K_m$, $\tilde{\pi} \circ (m\gamma) = \partial \tilde{c}$ for some 2-chain $\tilde{c}$ in $\tilde{\Omega}$. Since $\tilde{\pi}$ is a covering map, we can lift $\tilde{c}$ to a 2-chain c in Ω satisfying $\tilde{\pi} \circ c = \tilde{c}$. This gives $\pi \circ (m\gamma - \partial c) = 0$. Since $m\gamma - \partial c$ is a 1-cycle in $\mathbf{R}^n$ and $H_1(\mathbf{R}^n; \mathbf{Z}) = 0$, we can find a 2-chain μ in $\mathbf{R}^n$ with

$$m\gamma - \partial c = \partial \mu, \text{ and } \pi \circ \partial \mu \quad = \quad \partial(\pi \circ \mu) = 0.$$

Since the rank of $H_2(T^n; \mathbf{Z}) = n(n-1)/2$, we can find a basis $[\tilde{c}_i]$; $i = 1, \ldots, n(n-1)/2$ for this group and thus integers n_j so that

$$\pi \circ \mu = \sum_j n_j \tilde{c}_j + \partial \tilde{d},$$

where $\tilde{d}$ is a 3-chain in T^n. The totality of possible integer combinations $\sum_j n_j \tilde{c}_j$ formed in this way is a subgroup $G_\Omega^{(m)}$ of $\{\sum_j n_j \tilde{c}_j \,:\, n_j \in \mathbf{Z}\}$. Since $G_\Omega^{(m)}$ is a subgroup of a free abelian group of rank $n(n-1)/2$ it is also free abelian and of rank $\leq n(n-1)/2$. We let $\{\tilde{e}_j : j = 1, \ldots, N\}$, $(N \leq n(n-1)/2)$, be a basis, and rewrite $\pi \circ \mu$ in terms of this basis:

$$\pi \circ \mu = \sum_j m_j \tilde{e}_j + \partial \tilde{d}.$$

We can find $[\gamma_j] \in K_m$, so that this process results in $\tilde{e}_j$. In other words,

$$\pi \circ (m\gamma_j - \partial c_j) = 0;$$

$$m\gamma_j - \partial c_j = \partial \mu_j;$$

$$\pi \circ \mu_j = \tilde{e}_j + \partial \tilde{d}_j.$$

For $[\gamma] \in K_m$ as above, let

$$\gamma' = \sum_j m_j \gamma_j,$$

so that

$$m\gamma' - \partial \left(\sum_j m_j c_j \right) \quad = \quad \partial \left(\sum_j m_j \mu_j \right)$$

$$\pi \circ \left(\sum_j m_j \mu_j \right) \quad = \quad \sum m_j \tilde{e}_j + \partial \left(\sum_j m_j \tilde{d}_j \right).$$

Then

$$m\gamma - m\gamma' - \partial h = \partial \lambda;$$

$$\lambda = \mu - \sum_j m_j \mu_j, \;\; h = c - \sum_j m_j c_j;$$
$$\pi \circ \lambda = \partial k, \;\; k = \tilde{d} - \sum_j m_j \tilde{d}_j.$$

If F is a (smooth) closed periodic 2-form in $\mathbf{R}^n$ which vanishes in Ω and A_F is a smooth 1-form in $\mathbf{R}^n$ with $dA_F = F$, then

$$m \int_{\gamma - \gamma'} A_F = \int_{\partial\lambda} A_F = \int_\lambda F = \int_{\pi\circ\lambda} F^{\text{torus}} = \int_{\partial k} F^{\text{torus}} = 0,$$

so that $[\gamma] - [\gamma'] \in \mathcal{N}_\Omega$. Here $F^{\text{torus}}(\pi(x)) = F(x)$. The first equality above follows from Stokes' theorem and the fact that h is a 2-chain in Ω. The lemma follows if we note that we can use $[\gamma_j] + \mathcal{N}_\Omega$, $j = 1, \ldots, N$ as a set of generators for $(K_m + \mathcal{N}_\Omega)/\mathcal{N}_\Omega$. $\square$

Proof of Theorem 3.5: Let $J = \mathrm{Tor}_1(\Omega) + K_m$ and suppose $H_1(\Omega; \mathbf{Z})/J$ has generators $[\lambda_1] + J, \ldots, [\lambda_{N'}] + J$. Suppose $[\gamma] \in H_1(\Omega; \mathbf{Z})$. Then we can find integers m_j so that

$$[\gamma] - \sum_{j=1}^{N'} m_j [\lambda_j] \equiv [\gamma'] \in J.$$

So there exist $[\lambda'] \in \mathrm{Tor}_1(\Omega)$ and $[\xi] \in K_m$ with $[\gamma'] = [\lambda'] + [\xi]$. By Lemma 3.6 there are integers n_j and $[\xi'] \in \mathcal{N}_\Omega$ so that

$$[\xi] = \sum_{j=1}^{N} n_j [\gamma_j] + [\xi'].$$

It follows that

$$[\gamma] - \sum_{j=1}^{N'} m_j [\lambda_j] - \sum_{j=1}^{N} n_j [\gamma_j] = [\lambda'] + [\xi'] \in \mathcal{N}_\Omega,$$

which proves that if $H_1(\Omega; \mathbf{Z})/J$ has finite rank, so does $H_1(\Omega; \mathbf{Z})/\mathcal{N}_\Omega$. The group im $\tilde{\pi}_*$ in (ii) is isomorphic to $H_1(\Omega; \mathbf{Z})/\ker \tilde{\pi}_* = H_1(\Omega; \mathbf{Z})/K_1$, and thus if im $\tilde{\pi}_*$ is finitely generated so is $H_1(\Omega; \mathbf{Z})/(\mathrm{Tor}_1(\Omega) + K_1)$. Finally, im $\tilde{\pi}_*$ is a subgroup of $H_1(\tilde{\Omega}; \mathbf{Z})$ and since a subgroup of a finitely generated abelian group is finitely generated, if the group in (iii) has finite rank so does the group in (ii). $\square$

We remark that if

$$K = \bigcup_{m=1}^{\infty} K_m,$$

then $K \supset \mathrm{Tor}_1(\Omega)$ and $H_1(\Omega; \mathbf{Z})/K$ is isomorphic to im $\tilde{\pi}_*/\mathrm{Torsion}(\text{im } \tilde{\pi}_*)$. It would be interesting to know whether the condition that this group be finite rank is sufficient for $H_1(\Omega; \mathbf{Z})/\mathcal{N}_\Omega$ to be finite rank.

Appendix A:
The Hausdorff distance between spectra

In this appendix we examine the Hausdorff distance between the spectra of two operators.

Lemma A.1 *Let A and B be normal operators on a Hilbert space $\mathcal{H}$. Then $d(\sigma(A), \sigma(B)) \leq \|A - B\|$. (If $A - B$ is not a bounded operator, we set $\|A - B\| = \infty$.)*

Proof. From the spectral theorem it follows that

$$\|(z - A)^{-1}\| = \frac{1}{d(z, \sigma(A))}, \text{ where } d(z, S) = \inf_{s \in S} |z - S|,$$

so that for any $\phi \in \mathcal{D}(A)$ with $\|\phi\| = 1$ we have

$$\|(z - A)\phi\| \geq d(z, \sigma(A)).$$

We can assume $\|A - B\| < \infty$ so that $\mathcal{D}(B) = \mathcal{D}(A)$. Assuming $z \in \sigma(B)$, choose $\phi_\epsilon \in \mathcal{D}(B)$ such that $\|\phi_\epsilon\| = 1$ and

$$\|(z - B)\phi_\epsilon\| < \epsilon.$$

Then,

$$d(z, \sigma(A)) \leq \|[(z - B) + (B - A)]\phi_\epsilon\| < \epsilon + \|B - A\|.$$

It follows that

$$\sup_{z \in \sigma(B)} d(z, \sigma(A)) \leq \|A - B\|.$$

A similar inequality with A and B reversed proves the result. $\square$

We are mostly interested in closed unbounded subsets of $[0, \infty)$ which occur as the spectra of non-negative, unbounded, self-adjoint operators, and thus we have introduced the metric

$$\overline{d}(\sigma, \mu) = d\left((\sigma + 1)^{-1}, (\mu + 1)^{-1}\right).$$

With the $\overline{d}$ metric, two spectra can be close even if they differ significantly at high spectral parameter. To understand the meaning of the statement $\overline{d}(\sigma_n, \sigma) \to 0$, we introduce two conditions.

Condition 1:
Given $\lambda_0 \in \sigma$, there is $\lambda_n \in \sigma_n$ with $\lambda_n \to \lambda_0$.

Condition 2:

Given $\lambda_0 \in \sigma^c$, there is a neighborhood $\mathcal{O}$ of λ_0 so that $\mathcal{O} \cap \sigma_n = \emptyset$ for large n.

Lemma A.2 *Suppose σ_n and σ are closed unbounded subsets of $[0, \infty)$. Then $\overline{d}(\sigma_n, \sigma) \to 0$ if and only if both Conditions 1 and 2 hold.*

Proof. Suppose both conditions are true but $\overline{d}(\sigma_n, \sigma) \not\to 0$. By going to a subsequence we can assume $\overline{d}(\sigma_n, \sigma) \geq \gamma > 0$. We have two possibilities. Either

(a) $d\left((t_n+1)^{-1}, (\sigma+1)^{-1}\right) \geq \frac{\gamma}{2}$ for some $t_n \in \sigma_n$, or

(b) $d\left((s_n+1)^{-1}, (\sigma_n+1)^{-1}\right) \geq \frac{\gamma}{2}$ for some $s_n \in \sigma$.

Here, if necessary, we have renamed a subsequence. Since σ and σ_n are unbounded, we see that $d\left((t_n+1)^{-1}, (\sigma+1)^{-1}\right) \leq (t_n+1)^{-1}$ and $d\left((s_n+1)^{-1}, (\sigma_n+1)^{-1}\right) \leq (s_n+1)^{-1}$. It also follows that t_n and s_n are bounded so we can assume, by renaming a subsequence if necessary, that $t_n \to t_0,\ s_n \to s_0$.

If (a) is true we obtain $d\left((t_0+1)^{-1}, (\sigma+1)^{-1}\right) \geq \frac{\gamma}{2}$ so that $t_0 \in \sigma^c$. This contradicts Condition 2. If (b) is true, then $d(s_n, \sigma_n) \geq \frac{\gamma}{2}$ and thus $d(s, \sigma_n) \geq \frac{\gamma}{4}$ for large n. But since σ is closed, $s_0 \in \sigma$, and this contradicts Condition 1.

Conversely, suppose $\overline{d}(\sigma_n, \sigma) \to 0$. If $\lambda_0 \in \sigma$, then $d\left((\lambda_0+1)^{-1}, (\sigma_n+1)^{-1}\right) \to 0$ so that $d(\lambda_0, \sigma_n) \to 0$. This implies Condition 1. If $\lambda_0 \in \sigma^c$, choose $\delta > 0$ so that $(\lambda_0 - \delta, \lambda_0 + \delta) \cap \sigma = \emptyset$. If $\lambda_n \in (\lambda_0 - \frac{1}{2}\delta, \lambda_0 + \frac{1}{2}\delta) \cap \sigma_n$, then

$$d\left((\lambda_n+1)^{-1}, (\sigma+1)^{-1}\right) = \inf_{\lambda \in \sigma} \frac{|\lambda_n - \lambda|}{(\lambda_n+1)(\lambda+1)} \geq \frac{\frac{\delta}{2}}{(\lambda_0 + \frac{\delta}{2} + 1)(\lambda_0 + \delta + 1)},$$

so that if this is true for infinitely many n, $\overline{d}(\sigma_n, \sigma) \not\to 0$. Thus, $(\lambda_0 - \frac{1}{2}\delta, \lambda_0 + \frac{1}{2}\delta) \cap \sigma_n = \emptyset$ for large n. This is Condition 2. $\square$

Let us investigate the consequences of norm resolvent convergence "uniformly in the spectral parameter."

Lemma A.3 *Suppose A and B are self-adjoint, and*

$$\sup_{\lambda \in \mathbf{R}} \left\| (\lambda + i - A)^{-1} - (\lambda + i - B)^{-1} \right\| \leq \gamma < 1.$$

Then

$$d(\sigma(A), \sigma(B)) \leq \frac{\gamma}{\sqrt{1-\gamma^2}}.$$

Proof. Assuming $\lambda \in \sigma(B)$, choose $\phi_\epsilon \in \mathcal{D}(B)$ with $\|\phi_\epsilon\| = 1$, and $\|(\lambda - B)\phi_\epsilon\| < \epsilon$ as in the proof of Lemma A.1. Then $\|(\lambda - B)(\lambda + i - B)^{-1}\phi_\epsilon\| < \epsilon$. We also have

$$\left\|(\lambda - A)(\lambda + i - A)^{-1}\phi_\epsilon\right\| \geq \inf_{x\in\sigma(A)} \left|\frac{\lambda - x}{\lambda + i - x}\right| = \frac{d(\lambda, \sigma(A))}{\sqrt{1 + (d(\lambda, \sigma(A)))^2}}$$

so that for any $\epsilon > 0$,

$$\frac{d(\lambda, \sigma(A))}{\sqrt{1 + (d(\lambda, \sigma(A)))^2}} \leq \left\|\left((\lambda - A)(\lambda + i - A)^{-1} - (\lambda - B)(\lambda + i - B)^{-1}\right)\phi_\epsilon\right\| + \epsilon \leq \gamma + \epsilon.$$

It follows that $d(\lambda, \sigma(A)) \leq \gamma/\sqrt{1 - \gamma^2}$. A similar result holds with A replaced by B, and thus the lemma is proved. $\square$

It clearly follows that if A_n, A are self-adjoint and

$$\sup_{\lambda\in\mathbf{R}} \left\|(\lambda + i - A)^{-1} - (\lambda + i - A_n)^{-1}\right\| \to 0 \text{ as } n \to \infty,$$

then

$$d(\sigma(A_n), \sigma(A)) \to 0 \text{ as } n \to \infty.$$

Finally, let us apply Lemmas A.1 and A.2 to prove the continuity of the spectra of $H(\lambda)$ and $K(\lambda)$ in certain situations (the operator $K(\lambda)$ is defined in Section 2).

Proposition A.4 *Suppose $A \in C^2(\mathbf{R}^n)$ and $B = dA$ is bounded along with its first derivatives. Then the function $t \mapsto \sigma(H(t))$ is continuous in the metric $\bar{d}$. Suppose $A \in C^2(\mathbf{R}^n)$ and satisfies either (i) or (ii) of Theorem 2.2. Then the function $t \mapsto \sigma(K(t))$ is continuous.*

Proof: Let $H_m(t) = H(t) + m1_{M^c}$. Then from Corollary 2.3, if A satisfies the assumptions of Theorem 2.2,

$$\lim_{m\to\infty} \left\|(H_m(t) + 1)^{-1} - (K(t) + 1)^{-1} \oplus 0\right\| = 0.$$

It follows by a simple argument that if $\sigma(H_m(t))$ is continuous for each $m \in \mathbf{Z}$, then $\sigma(K(t))$ is also continuous. To prove the continuity of $\sigma(H_m(t))$ we assume $t_\ell \to t_0$ as $\ell \to \infty$ and show that $\sigma_\ell \equiv \sigma(H_m(t_\ell)) \to \sigma \equiv \sigma(H_m(t_0))$. We do this by proving that Conditions 1 and 2 introduced above are satisfied. To prove Condition 1 is satisfied, it is enough to show that $H_m(t_\ell)$ converges to $H_m(t_0)$ in the strong resolvent sense. For if $\lambda_0 \in \sigma$, then for $\epsilon > 0$,

$$\left\|(H_m(t_0) - \lambda_0 - i\epsilon/3)^{-1}\right\| = 3\epsilon^{-1},$$

and if we have strong resolvent convergence,

$$\liminf_{\ell\to\infty} \left\|(H_m(t_\ell) - \lambda_0 - i\epsilon/3)^{-1}\right\| \geq \left\|(H_m(t_0) - \lambda_0 - i\epsilon/3)^{-1}\right\|.$$

It follows that for large enough ℓ,

$$d(\lambda_0, \sigma(H_m(t_\ell))) < \epsilon,$$

and thus we can find $\lambda_\ell \in \sigma_\ell$ with $\lambda_\ell \to \lambda_0$.

To prove the strong convergence we use the Feynman-Kac-Ito formula

$$e^{-tH_m(s)} f(x) = E^x e^{-is\Phi(X(\,\cdot\,),t)} e^{-\int_0^t m1_{M^c}(X(\tau))d\tau} f(X(t)),$$

where

$$\Phi(X(\,\cdot\,), t) = \frac{1}{2}\int_0^t \nabla\cdot A(X(\tau))d\tau + \int_0^t A(X(\tau))\cdot dX(\tau).$$

Since $A \in C^1(\mathbf{R}^n)$, $\Phi(X(\cdot),t)$ is clearly finite almost surely and hence for $f \in C_0^\infty(\mathbf{R}^n)$, $e^{-tH_m(s)}f(x)$ is continuous in s for fixed x. It is trivial to see that $e^{-tH_m(s)}f$ is continuous in $L^2(\mathbf{R}^n)$, and hence $e^{-tH_m(s)}$ is strongly continuous in s. This proves that the resolvent is also strongly continuous.

To prove that Condition 2 is satisfied, we appeal to [I]. Here it is shown (in particular) that Condition 2 is satisfied if B is bounded along with all first partial derivatives. Thus the continuity of $t \mapsto \sigma(H(t))$, and $t \mapsto \sigma(K(t))$ are proved when B is bounded. We need to prove the continuity of $t \mapsto \sigma(K(t))$ when M is compact. Instead of considering $H_m(t)$ where $B = dA$ may be unbounded near infinity, we choose $\eta \in C_0^\infty(\mathbf{R}^n)$ with $0 \leq \eta \leq 1$, and $\eta = 1$ in a neighborhood of M. If instead of A we consider $A_\eta = \eta A$ and the resulting operator $H_{m,\eta}(t) = (p - tA_\eta)^2 + m1_{M^c}$, we note that

$$\lim_{m\to\infty} \left\|(H_{m,\eta}(t)+1)^{-1} - (K(t)+1)^{-1} \oplus 0\right\| = 0$$

(see Corollary 2.3) so that it is enough to know that the spectrum of $H_{m,\eta}(t)$ is continuous in t for each $m \in \mathbf{Z}$. But the magnetic field dA_η is C^1 with compact support. The proof now proceeds as before. $\square$

Appendix B: Lower Bounds for $H(\lambda)$

Let

$$|B(x)| = \left(\sum_{i<j} |B_{ij}(x)|^2\right)^{\frac{1}{2}},$$

and

$$\beta_{ij}(x) = \frac{B_{ij}(x)}{|B(x)|}.$$

The quantity

$$\gamma(x) = \left(\sum_{i<j} |\nabla \beta_{ij}(x)|^2 \right)^{\frac{1}{2}}$$

gives a measure of the "twisting" of a non-zero magnetic field. The main result of this appendix is the following:

Proposition B.1 *Suppose that for $|x| \geq R$ we have $|B(x)| > 0$ and $\gamma(x) \leq r^{-1}$. Then for each $s \in (0, r)$,*

$$H(\lambda) \geq \left(1 - \frac{s}{r}\right) \lambda |B| \, 1_{(B_{R+2s})^c} - \frac{c_0}{s^2}, \tag{B.1}$$

where

$$c_0 = \inf \left\{ \int |\nabla \psi|^2 d^n x : \ \psi \in C_0^\infty(B_1), \|\psi\|_{L^2} = 1 \right\}.$$

Here B_t is the open ball of radius t centered at zero, and B_t^c is its complement.

We will use the elementary

Lemma B.2 *Suppose $\beta = \{\beta_{ij}\}_{i,j=1}^n$ is a real anti-symmetric matrix. Then*

$$\|\beta\| \leq \left(\sum_{i<j} |\beta_{ij}|^2 \right)^{\frac{1}{2}}. \tag{B.2}$$

Here $\|\beta\|$ is the norm of β as an operator on $\mathbf{C}^n$. The inequality in (B.2) is an equality if $n = 2$ or 3.

Proof. Since $(i\beta)^* = i\beta$, and $\det(\lambda - \beta) = \det(\lambda + \beta)$, β has an orthonormal basis of eigenvectors, with pure imaginary eigenvalues λ_α appearing in complex conjugate pairs (except possibly for zero eigenvalues). It follows that

$$\sum_{i<j} |\beta_{ij}|^2 = \frac{1}{2} \mathrm{tr} \beta^* \beta = \frac{1}{2} \sum_\alpha |\lambda_\alpha|^2 \geq \left(\operatorname*{Max}_\alpha |\lambda_\alpha| \right)^2 = \|\beta\|^2.$$

In dimension 2 or 3 there are at most two non-zero eigenvalues, equal in magnitude, so that the last inequality above is an equality. $\square$

We will also need the following (slight) generalization of a result of [AHS]:

Lemma B.3 *Suppose ξ_{ij} is real and anti-symmetric with $\sum_{i<j} |\xi_{ij}|^2 = 1$. Then*

$$H(\lambda) \geq \sum_{\ell<m} \lambda \xi_{\ell m} B_{\ell m}(x).$$

Proof. We write

$$\xi_{ij} = \sum_{\alpha} \lambda_\alpha (v_\alpha)_i (\overline{v}_\alpha)_j,$$

where $\{v_\alpha : 1 \leq \alpha \leq n\}$ is an orthonormal basis of eigenvectors of ξ in $\mathbf{C}^n$, and $\{\lambda_\alpha : 1 \leq \alpha \leq n\}$ are the corresponding eigenvalues. Let

$$\Pi_j = -i\nabla_j - \lambda A_j, \;\; v_\alpha \cdot \Pi = \sum_j (v_\alpha)_j \Pi_j, \;\; \bar{v}_\alpha \cdot \Pi = \sum_j (\bar{v}_\alpha)_j \Pi_j.$$

Then

$$\begin{aligned}
\sum_{\ell<m} (\psi, \lambda \xi_{\ell m} B_{\ell m} \psi) &= \sum_{\ell<m} (\psi, -i[\Pi_\ell, \Pi_m]\psi) \xi_{\ell m} \\
&= \frac{1}{2} \sum_\alpha (-i\lambda_\alpha)(\psi, [v_\alpha \cdot \Pi, \bar{v}_\alpha \cdot \Pi]\psi) \\
&\leq \frac{1}{2} \sum_\alpha |\lambda_\alpha| \left(\|v_\alpha \cdot \Pi\psi\|^2 + \|\bar{v}_\alpha \cdot \Pi\psi\|^2 \right) \\
&\leq \frac{1}{2} \|\xi\| \sum_\alpha \left(\|v_\alpha \cdot \Pi\psi\|^2 + \|\bar{v}_\alpha \cdot \Pi\psi\|^2 \right) \\
&= \|\xi\| (\psi, H(\lambda)\psi) \leq (\psi, H(\lambda)\psi).
\end{aligned}$$

Here we have used

$$\|\xi\| = \max_\alpha |\lambda_\alpha|, \;\; \sum_\alpha \|v_\alpha \cdot \Pi\psi\|^2 = \sum_j \|\Pi_j \psi\|^2,$$

where the latter is true for any orthonormal basis of $\mathbf{C}^n$. The last inequality above uses Lemma B.2. □

Proof of Proposition B.1. Choose $\mu \in C_0^\infty(B_1)$ with μ real and $\|\mu\|_{L^2(\mathbf{R}^n)} = 1$. Note that with

$$\mu_{s,x_0}(x) = s^{-\frac{n}{2}} \mu(s^{-1}(x - x_0)),$$

we have

$$\int |\mu_{s,x_0}(x)|^2 dx_0 = 1, \quad \int |\nabla \mu_{s,x_0}(x)|^2 dx_0 = s^{-2} \int |\nabla \mu(y)|^2 dy.$$

Choose $|x_0| \geq R+s$, and let

$$\xi_{\ell m} = \frac{B_{\ell m}(x_0)}{|B(x_0)|}.$$

Then

$$\begin{aligned}
\sum_{\ell<m} \xi_{\ell m} B_{\ell m}(x) &= |B(x)| + \sum_{\ell<m} (\xi_{\ell m} - \beta_{\ell m}(x)) B_{\ell m}(x) \\
&\geq \left(1 - \left(\sum_{\ell<m} |\beta_{\ell m}(x_0) - \beta_{\ell m}(x)|^2\right)^{\frac{1}{2}}\right) |B(x)|.
\end{aligned}$$

If we assume in addition to $|x_0| \geq R+s$ that $|x-x_0| \leq s \leq r$, then from

$$\beta_{\ell m}(x) - \beta_{\ell m}(x_0) = \int_0^1 \nabla \beta_{\ell m}(tx + (1-t)x_0) \cdot (x - x_0) dt,$$

it follows that

$$\begin{aligned}
\left(\sum_{\ell<m} |\beta_{\ell m}(x) - \beta_{\ell m}(x_0)|^2\right)^{\frac{1}{2}} &\leq |x - x_0| \left(\int_0^1 |\gamma(tx + (1-t)x_0)|^2 \, dt\right)^{\frac{1}{2}} \\
&\leq \frac{|x-x_0|}{r} \leq \frac{s}{r}.
\end{aligned}$$

Thus under these conditions

$$\sum_{\ell<m} \xi_{\ell m} B_{\ell m}(x) \geq \left(1 - \frac{s}{r}\right) |B(x)|.$$

We use the well-known formula

$$\frac{1}{2}\left(\mu_{s,x_0}^2 H(\lambda) + H(\lambda)\mu_{s,x_0}^2\right) = \mu_{s,x_0} H(\lambda) \mu_{s,x_0} - |\nabla \mu_{s,x_0}|^2,$$

and integrate over x_0 to obtain

$$H(\lambda) \geq \int \mu_{s,x_0} H(\lambda) \mu_{s,x_0} dx_0 - s^{-2} \int |\nabla \mu(y)|^2 dy.$$

We estimate

$$\begin{aligned}
\int \mu_{s,x_0} H(\lambda) \mu_{s,x_0} dx_0 &\geq \int_{|x_0| \geq R+s} \mu_{s,x_0} H(\lambda) \mu_{s,x_0} dx_0 \\
&\geq \lambda \left(1 - \frac{s}{r}\right) |B| \int_{|x_0| \geq R+s} |\mu_{s,x_0}|^2 dx_0.
\end{aligned}$$

But

$$\begin{aligned}\int_{|x_0|\geq R+s} |\mu_{s,x_0}(x)|^2 dx_0 &= 1 - \int_{|x_0|\leq R+s} |\mu_{s,x_0}(x)|^2 dx_0 \\ &\geq 1_{(B_{R+2s})^c}(x).\end{aligned}$$

The result now follows by minimizing $\int |\nabla \mu|^2(y)dy$ subject to $\|\mu\|_{L^2(\mathbf{R}^n)} = 1$, $\mu \in C_0^\infty(B_1)$. $\square$

Appendix C: Decoupling of Disjoint Regions

Suppose $\mathcal{O}_1$ and $\mathcal{O}_2$ are disjoint open sets in $\mathbf{R}^n$ with boundaries $\partial\mathcal{O}_j$ of measure zero. We went to investigate conditions under which

$$\tilde{W}^{1,2}(\mathcal{O}_1 \cup \mathcal{O}_2) = \tilde{W}^{1,2}(\mathcal{O}_1) \oplus \tilde{W}^{1,2}(\mathcal{O}_2). \tag{C.1}$$

It is easy to see that $\tilde{W}^{1,2}(\mathcal{O}_1) \cap \tilde{W}^{1,2}(\mathcal{O}_2) = \{0\}$ so that (C.1) makes sense. In general,

$$\tilde{W}^{1,2}(\mathcal{O}_1 \cup \mathcal{O}_2) \supset \tilde{W}^{1,2}(\mathcal{O}_1) \oplus \tilde{W}^{1,2}(\mathcal{O}_2),$$

and this inequality can be strict (take $\mathcal{O}_1 = \mathbf{R}^2_+$, $\mathcal{O}_2 = \mathbf{R}^2_-$).

In the following, $X(\,\cdot\,)$ will signify Brownian motion in $\mathbf{R}^n$, with probability $P^x(A)$ that $X(\,\cdot\,) \in A$ if $X(0) = x$. Similarly, $E^x f$ is the expectation of f with the measure $P^x(\,\cdot\,)$. For a subset $S \subset \mathbf{R}^n$, let τ_s be the first exist time of Brownian motion from S:

$$\tau_s = \inf\{t > 0 :\ X(t) \notin S\}.$$

As discussed in Section 1, $\tilde{W}^{1,2}(P)$ is the form domain of $-\Delta_p$. Since $\partial\mathcal{O}_j$ has measure zero we have $\Delta_{\mathcal{O}_j} = \Delta_{\bar{\mathcal{O}}_j}$ and $\Delta_{\mathcal{O}_1\cup\mathcal{O}_2} = \Delta_{\overline{\mathcal{O}_1\cup\mathcal{O}_2}}$. Brownian motion is a useful concept in investigating C.1 because of the formula

$$e^{t\Delta_Q} f(x) = E^x(f(X(t));\, \tau_Q > t) \text{ a.e.} \tag{C.2}$$

for any closed subset $Q \subset \mathbf{R}^n$. (See [HZ] for example.)

Proposition C.1 *Suppose $\mathcal{O}_1$ and $\mathcal{O}_2$ are disjoint open sets in $\mathbf{R}^n$ with boundaries $\partial\mathcal{O}_j$ of measure zero. Then (C.1) holds if and only if*

$$\tau_{\overline{\mathcal{O}_1\cup\mathcal{O}_2}} = \tau_{\overline{\mathcal{O}}_1} \quad a.s.\ P^x \text{ for } x \in \mathcal{O}_1,$$

and

$$\tau_{\overline{\mathcal{O}_1\cup\mathcal{O}_2}} = \tau_{\overline{\mathcal{O}}_2} \quad a.s.\ P^x \text{ for } x \in \mathcal{O}_2,$$

Proof: (C.1) holds if and only if for each $f \in L^2(\mathbf{R}^n)$ and each $t > 0$,

$$e^{t\Delta_{\overline{\mathcal{O}_1 \cup \mathcal{O}_2}}} f = e^{t\Delta_{\overline{\mathcal{O}_1}}} f + e^{t\Delta_{\overline{\mathcal{O}_2}}} f.$$

¿From (C.2), (C.1) is equivalent to

$$E^x \left(f(X(t)); \tau_{\overline{\mathcal{O}_1 \cup \mathcal{O}_2}} > t \right) = E^x(f(X(t)); \tau_{\overline{\mathcal{O}_1}} > t) \quad \text{a.e. } x \in \mathcal{O}_1,$$

and

$$E^x \left(f(X(t)); \tau_{\overline{\mathcal{O}_1 \cup \mathcal{O}_2}} > t \right) = E^x(f(X(t)); \tau_{\overline{\mathcal{O}_2}} > t) \quad \text{a.e. } x \in \mathcal{O}_2,$$

for all $f \in L^2(\mathbf{R}^n)$ and all $t > 0$. The first equation is true for a.e. $x \in \mathcal{O}_1$ and all $f \in L^2(\mathbf{R}^n)$ if and only if

$$P^x \left(\tau_{\overline{\mathcal{O}_1 \cup \mathcal{O}_2}} > t \geq \tau_{\overline{\mathcal{O}_1}} \right) = 0 \quad \text{a.e. } x \in \mathcal{O}_1, \tag{C.3}$$

for all $t > 0$. (C.3) is in turn true for each $t > 0$ if and only if

$$P^x \left(\tau_{\overline{\mathcal{O}_1 \cup \mathcal{O}_2}} = \tau_{\overline{\mathcal{O}_1}} \right) = 1 \quad \text{all } x \in \mathcal{O}_1,$$

for if

$$P^{x_0} \left(\tau_{\overline{\mathcal{O}_1 \cup \mathcal{O}_2}} > \tau_{\overline{\mathcal{O}_1}} \right) > 0 \quad \text{for some } x_0 \in \mathcal{O}_1,$$

then by countable additivity and continuity of paths

$$P^{x_0} \left(\tau_{\overline{\mathcal{O}_1 \cup \mathcal{O}_2}} > t \geq \tau_{\overline{\mathcal{O}_1}} > r \right) > 0,$$

for some rational t and r with $t > r > 0$. The Markov property implies that for $x \in \mathcal{O}_1$,

$$P^x \left(\tau_{\overline{\mathcal{O}_1 \cup \mathcal{O}_2}} > t \geq \tau_{\overline{\mathcal{O}_1}} > r \right) = E^x \left(P^{X(r/2)} \left(\tau_{\overline{\mathcal{O}_1 \cup \mathcal{O}_2}} > t - \frac{r}{2} \geq \tau_{\overline{\mathcal{O}_1}} > \frac{r}{2} \right); \tau_{\overline{\mathcal{O}_1}} > \frac{r}{2} \right),$$

and thus we can write

$$P^x \left(\tau_{\overline{\mathcal{O}_1 \cup \mathcal{O}_2}} > t \geq \tau_{\overline{\mathcal{O}_1}} > r \right) = E^x \left(g(X(r/2)); \tau_{\overline{\mathcal{O}_1}} > \frac{r}{2} \right),$$

where g is a bounded, non-negative function. Since the kernel $Q_{\overline{\mathcal{O}_1}}(x, y; t)$ of $e^{t\Delta_{\overline{\mathcal{O}_1}}}$ can be taken continuous for $x, y \in \mathcal{O}_1$ (see [PS], p. 36), and since $\partial\mathcal{O}_1$ has measure zero we see (using Fatou's lemma) that

$$E^x \left(g \left(X \left(\frac{r}{2} \right) \right); \tau_{\overline{\mathcal{O}_1}} > \frac{r}{2} \right) = \int_{\mathcal{O}_1} Q_{\overline{\mathcal{O}_1}} \left(x, y; \frac{r}{2} \right) g(y) dy$$

is lower semi-continuous. This proves $P^x \left(\tau_{\overline{\mathcal{O}_1 \cup \mathcal{O}_2}} > t \geq \tau_{\overline{\mathcal{O}_1}} > r \right) > 0$ on an open set containing x_0 which contradicts (C.3) for some $t > 0$. Similarly $P^x \left(\tau_{\overline{\mathcal{O}_1 \cup \mathcal{O}_2}} = \tau_{\overline{\mathcal{O}_2}} \right) = 1$ if $x \in \mathcal{O}_2$. □

We state a corollary of the proof of Proposition C.1.

Proposition C.2 *Suppose $\mathcal{O} \subset \mathbf{R}^n$ is open and $\{\mathcal{O}_j : j = 1, 2, \ldots\}$ are the connected components of $\mathcal{O}$. Suppose $\partial\mathcal{O}$ has measure zero. Then*

$$\tilde{W}^{1,2}(\mathcal{O}) = \bigoplus_j \tilde{W}^{1,2}(\mathcal{O}_j) \tag{C.4}$$

if and only if for each j,

$$\tau_{\overline{\mathcal{O}}} = \tau_{\overline{\mathcal{O}_j}} \quad a.s.\ P^x \text{ for } x \in \mathcal{O}_j.$$

With probability one if a Brownian path starting at $x \in \mathcal{O}_j$ has $\tau_{\overline{\mathcal{O}}} > \tau_{\overline{\mathcal{O}_j}}$, then $X(\cdot)$ must exit $\overline{\mathcal{O}_j}$ through a point of $\partial\overline{\mathcal{O}}_j \cap \partial(\overline{\mathcal{O} \setminus \mathcal{O}_j})$ (here we use that with probability one the time spent in $\partial\mathcal{O}_j$ has Lebesgue measure zero). One way to guarantee that $P^x\left(\tau_{\overline{\mathcal{O}}} > \tau_{\overline{\mathcal{O}_j}}\right) = 0$ for $x \in \mathcal{O}_j$ is to demand that $\partial\mathcal{O}_j \cap \partial(\mathcal{O} \setminus \mathcal{O}_j)$ (and hence $\partial\overline{\mathcal{O}_j} \cap \partial\mathcal{O}_j \cap \partial(\overline{\mathcal{O} \setminus \mathcal{O}_j})$) is polar:

Proposition C.3 *Suppose $\mathcal{O} \subset \mathbf{R}^n$ is open and $\{\mathcal{O}_j : j = 1, 2, \ldots\}$ are the connected components of $\mathcal{O}$. Suppose $\partial\mathcal{O}$ has measure zero. Then (C.4) holds if for each j, $\partial\mathcal{O}_j \cap \partial(\mathcal{O} \setminus \mathcal{O}_j)$ is polar.*

In the situation we deal with in the text we have $\mathcal{O} = \Omega = M^0$ and $\mathcal{O}_j = \Omega_j$. The fact that Ω is the interior of a closed set provides some regularity:

Lemma C.4 *If $S = \bigcup_{j \in \mathcal{J}} \Omega_j$ is a union of some of the connected components of Ω, then $\bar{S}^0 = S$ or equivalently $\partial S = \partial\bar{S}$.*

We omit the easy proof. Note that one way for $\tau_{\overline{\mathcal{O}}} > \tau_{\overline{\mathcal{O}_j}}$ with $X(0) \in \mathcal{O}_j$ to be possible is if adding a subset Q of $\partial\overline{\mathcal{O}_j} \cap \partial(\overline{\mathcal{O} \setminus \mathcal{O}_j})$ to $\mathcal{O}$ we get a strictly larger open set $\mathcal{O} \cup Q$ which connects $\mathcal{O}_j$ to $\mathcal{O} \setminus \mathcal{O}_j$. This cannot occur when $\Omega = \mathcal{O} = M^0$ since if $\Omega \cup Q$ is open

$$\Omega = \overline{\Omega}^0 \supset (\Omega \cup Q)^0 = \Omega \cup Q.$$

We end by giving a sufficient condition for polarity:

Lemma C.5 *The following are sufficient conditions for a closed set $Q \subset \mathbf{R}^n$ to be polar:*

(a) if $n = 2$, Q is countable;

(b) if $n > 2$, Q has finite $n-2$ dimensional Hausdorff measure.

Proof: For the $n = 2$ statement, see [PS], p. 20–21 and for $n > 2$ see [F].

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