Remarks on decay of correlations and Witten Laplacians III
Application to logarithmic Sobolev inequalities.

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Abstract

This is the continuation[1] of previous notes on the subject referred as [He4] and [He5]. The main application treated in Part I was a semi-classical one. The second application was more perturbative in spirit and gave very explicit explicit estimates for the lower bound of the Witten Laplacian in the case of a quartic model.
We shall develop in this third part a remark given in the second part concerning the possibility of relating our studies of the Witten Laplacian with the existence of uniform logarithmic Sobolev inequalities through a criterion of B. Zegarlinski. More precisely, we shall show how to control the decay of correlations uniformly with respect to various parameters.

[1] A first version of these notes was distributed at the end of October 97.

1 Introduction

In the recent years, a new insight has been given in the study of the decay of the correlation pairs, through the analysis of a Witten Laplacian on 1-forms ([Sj2], [He4], [NaSp], [AA2]). This gave not only a nice way to recover the Brascamp-Lieb inequality [He1] but also permits the analysis of non-convex situations ([Sj1], [BaJeSj], [He2], [He3], [He4] and [He5]). In this spirit, it is natural to analyze if this approach gives new results concerning logarithmic Sobolev inequalities in the non convex case. The main point would be to get an extension of the Bacry-Emery criterion [BaEm] in the case of non-convex situations by relating the logarithmic Sobolev best constant to the lowest eigenvalue of ths Witten Laplacian. We discuss this question in [He2] and [He3] in connection with recent studies of Antoniouk-Antoniouk [AA1], [AA2] but do not obtain decisive results outside the case when the gradient of the interaction is bounded and small.
Motivated by discussions with T. Bodineau and a talk by B. Zegarlinski in Brisbane (July 97), we consider here another approach initiated by Strook-Zegarlinski [StZe1], [StZe1] and continued by Zegarlinski [Ze] consisting in deducing uniform logarithmic Sobolev inequalities from the existence of uniform decay estimates for the correlations. This is what is obtained here through the Witten Laplacian approach.
More precisely our aim is to analyze the thermodynamic properties of the measure $\exp -\Phi^{\Lambda,\omega}(X)\, dX$ in the case when $\Phi^{\Lambda,\omega}$, which is associated with cubes $\Lambda \subset \mathbb{Z}^d$ and some $\omega \in \mathbb{R}^{\mathbb{Z}^d}$ defing the boundary condition, has the form, for $X \in \mathbb{R}^\Lambda$,

$$\Phi^{\Lambda,\omega}(X) = \sum_{j\in\Lambda} \phi(x_j) + \frac{\mathcal{J}}{2} \sum_{(\{j\}\cup\{k\})\cap\Lambda\neq\emptyset\ ,\ j\sim k} |z_j - z_k|^2 \tag{1.1}$$

where

- $X = (x_j)_{j\in\Lambda}$,
- ϕ is a one particle phase with at least quadratic increase at ∞,
- $$\begin{aligned} z_j &= x_j \quad \text{if } j \in \Lambda \\ z_j &= \omega_j \quad \text{if } j \notin \Lambda\ . \end{aligned} \tag{1.2}$$

- $j \sim k$ means that j and k are nearest neighbors[2] in $\mathbb{Z}^d$.

We shall sometimes use the following decomposition

$$\Phi^{\Lambda,\omega} = \Phi_d^{\Lambda} + \mathcal{J}\ \Phi_i^{\Lambda,\omega}\ , \tag{1.3}$$

with

$$\Phi_d^{\Lambda}(X) = \sum_{j\in\Lambda} \phi(x_j)\ , \tag{1.4}$$

and

$$\Phi_i^{\Lambda,\omega}(X) = \sum_{(\{j\}\cup\{k\})\cap\Lambda\neq\emptyset\ ,\ j\sim k} |z_j - z_k|^2\ . \tag{1.5}$$

We shall also meet the Dirichlet boundary condition which by definition corresponds to the phase

$$\Phi^{\Lambda,D} = \Phi_d^{\Lambda} + \mathcal{J}\ \Phi_i^{\Lambda,D}\ , \tag{1.6}$$

with

$$\Phi_i^{\Lambda,D}(X) = \sum_{j,k\in\Lambda\ ,\ j\sim k} |x_j - x_k|^2 \tag{1.7}$$

If necessary, the dependence on $\mathcal{J}$ will be mentioned by the notation $\Phi^{\Lambda,\omega} = \Phi^{\Lambda,\omega,\mathcal{J}}$ or $\Phi^{\Lambda,D} = \Phi^{\Lambda,D,\mathcal{J}}$.

Let us now state the assumptions on the single-spin phase ϕ. We assume that ϕ is C^∞ and convex at ∞, so there exists $C > 0$ such that

$$\phi''(x) \geq \frac{1}{C}\ ,\ \forall x \in \mathbb{R} \text{ s.t. } |x| \geq C\ . \tag{1.8}$$

We add also the technical condition that, there exists $\rho > 0$ and, for all $k \in \mathbb{N}$, C_k such that,

$$|\phi^{(k+1)}(x)| \leq C_k < \phi'(x) >^{(1-\rho k)_+}\ , \tag{1.9}$$

where, for $u \in \mathbb{R}^n$, $< u > := (1 + |u|^2)^{\frac{1}{2}}$ and, for $t \in \mathbb{R}$, $(t)_+ := \max(t, 0)$. The typical example will be

$$\phi(x) = \frac{1}{12}\lambda x^4 + \frac{1}{2}\nu x^2\ . \tag{1.10}$$

[2] In our previous studies [He4], [He5], we were mainly analyzing the case when j and k were nearest neighbors in Λ considered as a discrete torus.

where the parameters λ and ν satisfy

$$\lambda > 0 \ , \tag{1.11}$$

We would like to analyze the possibility of having any sign for ν. The sum in (1.7) is over the non-oriented pairs of $\Lambda \times \Lambda$.
Our main problem will be to analyze the properties of the measure

$$d\mu_{\Lambda,\omega} := \exp -\Phi^{\Lambda,\omega}(X)\ dX / \left(\int_{\mathbb{R}^\Lambda} \exp -\Phi^{\Lambda,\omega}(X)\ dX \right) \ , \tag{1.12}$$

or of the measure

$$d\mu_{\Lambda} := \exp -\Phi^{\Lambda,D}(X)\ dX / \left(\int_{\mathbb{R}^\Lambda} \exp -\Phi^{\Lambda,D}(X)\ dX \right) \ . \tag{1.13}$$

We shall in particular analyze the covariance associating to $f, g \in C^\infty_{temp}(\mathbb{R}^\Lambda)$

$$\mathrm{Cov}_{\Lambda,\omega}(f,g) = \langle (f - \langle f \rangle_{\Lambda,\omega})(g - \langle g \rangle_{\Lambda,\omega}) \rangle_{\Lambda,\omega} \tag{1.14}$$

where $\langle \cdot \rangle_{\Lambda,\omega}$ denotes the mean value with respect to the measure $d\mu_{\Lambda,\omega}$ and $C^\infty_{temp}(\mathbb{R}^\Lambda)$ is the space of C^∞ functions with polynomial growth.
Similarly, we associate with $\Phi^{\Lambda,D}$ and the measure $d\mu_\Lambda$ the mean value $\langle \cdot \rangle_\Lambda$ and the covariance Cov_Λ. Our main theorem is the following

Theorem 1.1 . *Let $\Phi = \Phi^{\Lambda,D,\mathcal{J}} = \Phi_d^\Lambda + \mathcal{J}\Phi_i^{\Lambda,D}$ with ϕ satisfying (1.8) and (1.9). Then there exists a constant C and $\mathcal{J}_0 > 0$ such that, for $\mathcal{J} \in [-\mathcal{J}_0, \mathcal{J}_0]$ and for any cube $\Lambda \subset \mathbb{Z}^d$, we have:*

$$\langle f \ln f \rangle_\Lambda - \langle f \rangle_\Lambda \ln \langle f \rangle_\Lambda \leq C \langle \nabla f^{\frac{1}{2}} \rangle_\Lambda \ . \tag{1.15}$$

for all non-negative function f for which the right hand side is finite.

As communicated by B. Zegarlinski, example (1.10) is also analyzed by N. Yoshida [Yo].
In this case, we say shortly that the uniform logarithmic Sobolev inequality (ULS inequality) is true.
The proof will be the conjonction of

- a criterion by Zegarlinski [Ze] relating the existence of "uniform" decay estimates with the existence of "uniform" logarithmic Sobolev inequalities,

- the proof (mainly given in [He5]), that the existence of "uniform" decay estimates may be obtained from the existence of an "uniform" lower bound for the lowest eigenvalue of a Witten Laplacian on 1-forms,

- the proof of this "uniform" lower bound in the case of a weak nearest neighbors interaction by comparison with a family of "one-particle" differential operators on $I\!R$.

Although the main techniques were already present in our previous notes. We think it is worthwhile to follow in detail this problem of the control of the uniformity with respect to Λ and ω and to show that the assumption of strict convexity at ∞ for the single spin phase is a natural condition under which this technique works.

2 Lower bound for the spectrum of the Witten Laplacian

Let us recall that the Witten Laplacian on 1-forms attached to the phase $\Phi = \Phi^{\Lambda,\omega}$ is defined as[3],

$$W_1^\Phi := \left[\sum_{j\in\Lambda}\left(-\partial/\partial x_j + \frac{1}{2}\partial\Phi/\partial x_j\right)\left(\partial/\partial x_j + \frac{1}{2}\partial\Phi/\partial x_j\right)\right]\otimes I + \text{ Hess}\,\Phi\ . \tag{2.1}$$

defined on the L^2 $1-$forms with respect to the standard Lebesgue measure on $I\!R^m$, with $m = |\Lambda|$. The aim of this section is the proof of the following theorem

Theorem 2.1 *:*
For any cube $\Lambda \subset Z\!\!Z^d$ and $\omega \in I\!R^{Z\!\!Z^d}$, let $\Phi := \Phi^{\Lambda,\omega,\mathcal{J}}$ be the phase on $I\!R^\Lambda$ with ϕ satisfying (1.8)-(1.9). There exists $\mathcal{J}_0$ and $\sigma_1 > 0$, such that the lowest eigenvalue $\lambda_1^{\Lambda,\omega,\mathcal{J}}$, of the corresponding Witten Laplacian on 1-forms W_1^Φ, satisfies, for any cube Λ, $\omega \in I\!R^{Z\!\!Z^d}$ and $\mathcal{J} \in [-\mathcal{J}_0,\mathcal{J}_0]$,

$$\lambda_1^{\Lambda,\omega,\mathcal{J}} \geq \sigma_1\ . \tag{2.2}$$

Similarly, we have also

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[3] It was denoted by $\Delta_\Phi^{(1)}$ in [Sj2].

Theorem 2.2 *:*
For any cube $\Lambda \subset \mathbb{Z}^d$, let $\Phi := \Phi^{\Lambda,D,\mathcal{J}}$ be the phase on $\mathbb{R}^\Lambda$ with ϕ satisfying (1.8)-(1.9). There exists $\mathcal{J}_0$ and $\sigma_1 > 0$, such that the lowest eigenvalue $\lambda_1^{\Lambda,D,\mathcal{J}}$, of the corresponding Witten Laplacian on 1-forms W_1^Φ, satisfies, for any cube Λ and $\mathcal{J} \in [-\mathcal{J}_0, \mathcal{J}_0]$,

$$\lambda_1^{\Lambda,D,\mathcal{J}} \geq \sigma_1 \,. \tag{2.3}$$

The starting point for the proof is the basic identity

$$\langle W_1^\Phi u \mid u\rangle_{L^2} = \sum_{j,k} ||X_k\, u_j||^2 + \sum_{j,k} \int \frac{\partial^2 \Phi}{\partial x_j\, \partial x_k} u_j\, u_k\, dX \;, \tag{2.4}$$

with

$$X_j = \frac{\partial}{\partial x_j} + \frac{1}{2}\frac{\partial \Phi}{\partial x_j} \,. \tag{2.5}$$

Let us denote by $w_j^{(0)}$ and $w_j^{(1)}$ the single-spin Witten Laplacians (respectively on 0- and 1- forms) attached to the variable x_j and the phase on $\mathbb{R}$

$$\phi_j(x_j) := \Phi^{\Lambda,\omega,\mathcal{J}}(X) \,. \tag{2.6}$$

The differential of this phase ϕ_j depends actually only on the z_ℓ with $\ell \sim j$ and recall that $z_\ell = x_\ell$ if $\ell \in \Lambda$ and $z_\ell = \omega_\ell$ if $\ell \in \mathbb{Z}^d \setminus \Lambda$. We have indeed

$$\phi_j(t) = \phi(t) + \mathcal{J} \sum_{(\{\ell\}\cup\{j\})\cap\Lambda\neq\emptyset\;,\;\ell\sim j} |t - z_\ell|^2 + \hat{\phi}(\hat{z}_j) \tag{2.7}$$

where the last term is independent of t and will be irrelevant in the discussion. More precisely, the operators $w_j^{(0)}$ and $w_j^{(1)}$ depend only on the z_ℓ with $\ell \sim j$.

It will be quite important that the estimates which will be proved are independent of these parameters.
We note the relations

$$w_j^{(0)} = X_j^*\, X_j \;, \tag{2.8}$$

and

$$w_j^{(1)} = X_j\, X_j^* = X_j^*\, X_j \; + \frac{1}{2}\frac{\partial^2 \Phi}{\partial x_j^2} \,. \tag{2.9}$$

According to the context, we shall see these identities as identities between differential operators on $L^2(I\!R^\Lambda)$ or on $L^2(I\!R_{x_j})$ (the other variables being considered as parameters).
With these conventions, we have

$$\langle W_1^\Phi u \mid v\rangle_{L^2} = \sum_{j,k\in\Lambda\,,\, j\neq k} \langle w_k^{(0)}\, u_j \mid v_j\rangle + \sum_{j\in\Lambda} \langle w_j^{(1)} u_j \mid v_j\rangle + \mathcal{J}\, \langle \mathrm{Hess}'\Phi_i\, u \mid v\rangle\ , \tag{2.10}$$

where $\Phi_i = \Phi_i^{\Lambda,\omega}$ denotes the interaction phase and Hess$'$ means that we consider only the terms outside of the diagonal of the Hessian, that is such that for $k,\ell\in\Lambda$

$$\begin{aligned} (\mathrm{Hess}'\Phi_i)_{k\ell} &= -1 \quad \text{if } k\sim\ell \\ &= 0 \quad\;\;\; \text{else}\ . \end{aligned} \tag{2.11}$$

Here we observe that $\mathrm{Hess}\,\Phi^i$ is independent of z and that $\mathcal{J}\,\mathrm{Hess}\,\Phi^i$ corresponds to a perturbation in $\mathcal{O}(\mathcal{J})$, where $\mathcal{O}$ is uniform with respect to Λ, using the Schur's Lemma.

We shall prove later the following theorem

Theorem 2.3 . *Let us assume that there exists $\rho_1 > 0$ such that, for any $j\in Z\!\!Z^d$ and any $z\in I\!R^{Z\!\!Z^d\setminus\{j\}}$, the operator*[4] *$w_j^{(1)}$ satisfies*

$$w_j^{(1)} \geq \rho_1\ , \tag{2.12}$$

then, for any $\epsilon>0$, there exists $\mathcal{J}_0>0$ such that the Witten Laplacian W_1^Φ, with $\Phi := \Phi^{\Lambda,\omega,\mathcal{J}}$ or with $\Phi := \Phi^{\Lambda,D,\mathcal{J}}$, satisfies for any Λ, $\omega\in I\!R^{Z\!\!Z^d}$ and $\mathcal{J}\in[-\mathcal{J}_0,+\mathcal{J}_0]$,

$$W_1^\Phi \geq (\rho_1-\epsilon)\ . \tag{2.13}$$

We recall that the positivity of $w_j^{(1)}$ is immediate from the definition. We also observe that it is sufficient to treat the case of a fixed j_0 all the families being unitary equivalent (after a simple change of the names of the parameters). We shall see that the strict positivity for a fixed value of the parameter z is a consequence of general arguments (see [Sj2] and [Jo]), at least for $\mathcal{J}\in[-\mathcal{J}_0,+\mathcal{J}_0]$ with $\mathcal{J}_0$ small enough. The other important point is to verify the condition of uniformity. This will be done in the next section.

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[4]For a given $j\in Z\!\!Z^d$, the effective parameters are actually the z_ℓ such that $\ell\sim j$,

3 Uniform estimates for a family of 1-dimensional Witten Laplacians.

In the preceding section, we have proved that the proof of a uniform lower bound for the Witten Laplacian W_1 can be deduced from the study of one-dimensional Witten Laplacian. In this sense, we are not far of the Dobrushin's approach as described for example in [AA2]. We want to analyze

$$w(t,\frac{d}{dt}) := -\frac{d^2}{dt^2}+\frac{1}{4}\left(\phi'(t)+2\mathcal{J}d\,t-2\mathcal{J}\sum_{0\sim k} z_k\right)^2+\frac{1}{2}(\phi''(t)+2\mathcal{J}\ d)\ . \quad (3.1)$$

As observed in [He5], it is enough to analyze the following family depending on $\mathcal{J}$:

$$w(t,\frac{d}{dt},\alpha,\mathcal{J}) := -\frac{d^2}{dt^2}+\frac{1}{4}\left(\phi'(t)+2\mathcal{J}d\,t-\alpha\right)^2+\frac{1}{2}(\phi''(t)+2\mathcal{J}d)\ , \quad (3.2)$$

with

$$\alpha = -2\mathcal{J}\sum_{0\sim k} z_k\ . \quad (3.3)$$

Of course, there is a singular situation when $\mathcal{J}=0$ because α becomes identically 0 in this case but, because we are interested in uniform properties with respect with $\mathcal{J}$ and z, we now forget the relation (3.3) and will analyze the family $w(t,\frac{d}{dt},\alpha,\mathcal{J})$ for any $(\alpha,\mathcal{J})\in I\!R\times[-\mathcal{J}_0,+\mathcal{J}_0]$.

Theorem 3.1 . *Let ϕ be a phase satisfying (1.8) and (1.9). Then, there exists $\mathcal{J}_0$, and $\rho_1>0$ such that, for any $(\alpha,\mathcal{J})\in I\!R\times[-\mathcal{J}_0,+\mathcal{J}_0]$, we have*

$$w(t,\frac{d}{dt},\alpha,\mathcal{J})\geq\rho_1\ . \quad (3.4)$$

For any fixed pair $(\alpha,\mathcal{J})$, the operator is strictly positive according to the assumptions (1.8) and (1.9) and the theorem by J. Johnsen [Jo] generalizing [Sj2]. We observe indeed that (1.9) implies their condition through

$$t\phi'(t)\geq\frac{1}{C}t^2\ ,\ \text{ for } |t|\geq C\ . \quad (3.5)$$

A similar inequality (with another constant C which depends on α and $\mathcal{J}$) holds for the phase $t\mapsto\phi_{\alpha,\mathcal{J}}(t):=\phi(t)+\mathcal{J}\,d\,t^2-\alpha t$ under the condition that

$\mathcal{J} \geq -\mathcal{J}_0$ for some small enough strictly positive $\mathcal{J}_0$. It is also clear that $|\phi'(t)| \geq \frac{|t|}{C}$, and the operator is with discrete spectrum and strictly positive for any pair $(\alpha, \mathcal{J})$ with $|\mathcal{J}|$ small enough.
Let us give an explicit proof of the injectivity (admitting that we have proved that the kernel is in $\mathcal{S}(I\!R)$) because this property is actually quite easy in dimension 1.

Lemma 3.2 . *If u is in $\mathcal{S}(I\!R)$ and solution of the equation*

$$-u''(t) + \frac{1}{4}\varphi'(t)^2\, u(t) + \frac{1}{2}\varphi''(t)\, u(t) = 0\;,$$

for a phase φ satisfying (1.8) and (1.9), then $u = 0$.

For the proof, one first remarks that this solution is also a solution of

$$u'(t) - \frac{1}{2}\varphi'(t)\, u(t) = 0\;,$$

and consequently given by $u(t) = C \exp \frac{1}{2}\varphi(t)$. But it is then necessarily identically 0 when it is assumed to be rapidly decreasing.

This lemma is then applied to $\varphi = \phi_{\alpha,\mathcal{J}}$.

The second point is that the lowest eigenvalue depends continuously on $(\alpha, \mathcal{J})$ by rather simple argument of perturbation.

So the remaining point is a problem of uniformity as $\alpha \to \pm\infty$. The specific model $\phi(t) = \frac{\lambda}{12}\, t^4 + \frac{\nu}{2}\, t^2$ was treated directly in [He5] using a semi-classical approach. We give here an easy argument which is quite general although not optimal for the specific model.
It is indeed immediate to see from (1.9) that there exists $\tau > 0$ such that, for $|\alpha| \geq \tau$ and $|\mathcal{J}| \leq \frac{1}{\tau}$ and all $t \in I\!R$,

$$\frac{1}{4}(\phi'(t) + 2\mathcal{J}\, d\, t - \alpha)^2 + \frac{1}{2}\phi''(t) + \mathcal{J}\, d \geq \frac{1}{\tau}\;. \tag{3.6}$$

This gives immediately a lower bound for the corresponding operator $w(t, \frac{d}{dt}, \alpha, \mathcal{J})$, for any $(\alpha, \mathcal{J})$ such that $|\alpha| \geq \tau$ and $\mathcal{J} \in [-\mathcal{J}_0, +\mathcal{J}_0]$ with a possibly new smaller $\mathcal{J}_0 > 0$. Combining this with the locally uniform lower bound, we have proven the theorem.

4 Uniform decay estimates

We recall the argument developed in [He4] (see also [Sj2], [NaSp] (Theorem B) or more recently [BaJeSj], [SjW]) and control its uniformity with respect to Λ and ω. The analysis in [He4] was only given for the periodic case and for $d = 1$ and we consider here other boundary conditions. Our starting point is the formula.

$$\text{Cov}_{\Lambda,\omega}(f,g) = \left(\int \left[(A_1^{-1})\nabla f\right] \cdot \nabla g \;\exp -\Phi\; dX\right) / \left(\int \exp -\Phi\; dX\right) \quad (4.1)$$

Here A_1 is unitary equivalent to W_1^Φ by conjugation by the map $U\,\omega = \exp -\frac{\Phi}{2}\,\omega$, and in particular isospectral. We have in mind to take $f = x_i$, $g = x_j$ with i and j in Λ. The idea is reminiscent of [CT] and consists in the introduction of weighted spaces on Λ, associated with strictly positive weights satisfying

$$\exp -\kappa \leq \rho(\ell)/\rho(k) \leq \exp \kappa\;, \quad (4.2)$$

where $\ell \sim k$ (this means that ℓ and k are nearest neighbors in $\mathbb{Z}^d$) and κ will be determined later.
For a given $i \in \Lambda$, the functions $\rho(\ell) = \exp -\kappa d(i,\ell)$ where d is a usual distance on $\mathbb{R}^d$ satisfy this condition.

For a given $\Lambda \subset \mathbb{Z}^d$ with $|\Lambda| = m$, let us now associate with a given weight ρ on Λ the $m \times m$ diagonal matrix M_Λ defined by

$$M_{k\ell} = \delta_{k\ell}\;\rho(\ell)\;, \quad \text{for } \ell,\, k \in \Lambda\;. \quad (4.3)$$

For any slowly increasing functions f, g, we can rewrite (4.1) in the form

$$\text{Cov}_{\Lambda,\omega}(f,g) = \frac{\left(\int \left((M^{-1}A_1^{-1}M)M^{-1}\nabla f\right) \cdot (M\nabla g) \exp -\Phi\; dX\right)}{\left(\int \exp -\Phi\; dX\right)} \quad (4.4)$$

and we deduce the estimate (we can assume that $\int \exp -\Phi\; dX = 1$ after renormalization)

$$|\text{Cov}_{\Lambda,\omega}(f,g)| \leq ||M^{-1}\,A_1^{-1}\,M|| \cdot ||M^{-1}\nabla f|| \cdot ||M\nabla g||\;. \quad (4.5)$$

We now take $f(X) = x_i$, $g(X) = x_j$ and choose $\rho = \rho_i = \exp -\kappa d(i,\cdot)$ so that (4.2) is satisfied. We immediately observe that for this choice

$$||M^{-1}\nabla f|| = 1\;,\; ||M\nabla g|| = \exp -\kappa d(i,j)\;. \quad (4.6)$$

Everything is then reduced to the control, uniformly with respect to Λ and ω, of $M^{-1}\,A_1^{-1}\,M$ in suitable L^2- norms. We have only here to analyze the effect of the "distorsion" by M. This will be done by a simple perturbation argument, once we have characterized the domain of the selfadjoint operator A_1 and verified that the domain is conserved in the distorsion. This is easily done under the assumptions (1.1)-(1.9) as proved in [Jo]. We observe that, for all $X \in I\!R^\Lambda$,

$$\begin{array}{l} ||\mathrm{Hess}\,\Phi(X) - M^{-1}\mathrm{Hess}\,\Phi(X)M||_{\mathcal{L}(\ell^2)} = \\ |\mathcal{J}|\;||\mathrm{Hess}\,\Phi_i(X) - M^{-1}\mathrm{Hess}\,\Phi_i(X)M||_{\mathcal{L}(\ell^2)}\;. \end{array} \tag{4.7}$$

In this example, observing that the coefficients of

$$\delta_M(\mathrm{Hess}\,\Phi_i) =: \mathrm{Hess}\,\Phi_i - M^{-1}\mathrm{Hess}\,\Phi_i M$$

vanish if $k \not\sim \ell$, it is immediate, using Schur's Lemma, to get that

$$\begin{array}{ll} ||\delta_M(\mathrm{Hess}\,\Phi_i)||_{\mathcal{L}(\ell^2)} & \leq 2d\,\sup_{\ell\sim k}|(1-\frac{\rho(\ell)}{\rho(k)})| \\ & \leq 2d\,\max\left((1-\exp -\kappa), (\exp\kappa - 1)\right) \\ & = 2d\,\theta\;, \end{array} \tag{4.8}$$

with,

$$\theta := (\exp\kappa - 1)\;, \tag{4.9}$$

and this is clearly uniform with respect to the lattice .
We now estimate the operator $M^{-1}\,A_1^{-1}\,M$. An immediate computation gives

$$M^{-1}A_1^{-1}M = A_1^{-1}\left[I + \mathcal{J}\delta_M(\mathrm{Hess}\,\Phi_i)A_1^{-1}\right]^{-1}\;, \tag{4.10}$$

where $\delta_M(\mathrm{Hess}\,\Phi_i)$ is now considered as an operator (of order 0) on the L^2 1-forms. According to (4.6) and (4.8), we finally obtain that, if

$$0 < 2\mathcal{J}d\,\theta \leq \frac{1}{2}\sigma_1\;,$$

then

$$||M^{-1}A_1^{-1}M|| \leq \frac{2}{\sigma_1} \tag{4.11}$$

We have obtained the following :

Theorem 4.1 . *Under the same assumptions as in Theorem 3.1, there exists $\mathcal{J}_0 > 0$, C and κ, such that, for any Λ, ω and $\mathcal{J} \in [-\mathcal{J}_0, +\mathcal{J}_0]$, the correlation pair function satisfies*

$$|\mathrm{Cov}_{\Lambda,\omega}(x_i, x_j)| \leq C \exp -\kappa d(i,j) \ , \ \forall i,j \in \Lambda \ . \tag{4.12}$$

Remark 4.2 .The proof gives actually a decay with κ such that $\exp \kappa \cdot \mathcal{J}$ is small enough. This is coherent with the Dobrushin's approach (See [He3], for a discussion about the links between the Dobrushin's approach and the Witten Laplacian approach). The transfer matrix approach (in the case $d = 1$) or the formal perturbative approach suggest also that $\kappa \asymp \ln \frac{1}{\mathcal{J}}$.

5 Proof of the logarithmic Sobolev estimates

We can just finish the proof by recalling the following theorem by B. Zegarlinski [Ze]. B. Zegarlinski considers a Gaussian measure on $I\!R^{\mathbb{Z}^d}$ with mean 0 and a covariance G whose inverse $A = G^{-1}$ satisfies that there exists $R > 0$ such that

$$A_{ij} \equiv 0 \text{ if } d(i,j) > R \ , \tag{5.1}$$

and that

$$||A|| := \sup_j \sum_i |A_{ij}| < +\infty \tag{5.2}$$

This contains our case, with $R = 1$, $A_{ii} = \mathcal{J}$ and $A_{ij} = -1$ when $i \sim j$.
The phase ϕ is a semibounded function which can be represented as the sum

$$\phi = v + w \ , \tag{5.3}$$

where w has a bounded first and second derivative and v is with non negative second derivative.
It is clear that (1.8) permits to get (5.3).

Theorem 5.1 . *Let us assume that (1.8) and (1.9) are satisfied. Suppose, that for some $\mathcal{J}$, there are constants C, $\kappa \in]0, +\infty[$ such that for any sufficiently large cube $\Lambda_0 \subset \mathbb{Z}^d$ and any $\omega \in I\!R^{\mathbb{Z}^d}$ we have*

$$|\mathrm{Cov}_{\Lambda_0,\omega}(x_i, x_j)| \leq C \exp -\kappa d(i,j) \ , \ \forall i,j \in \Lambda_0 \ . \tag{5.4}$$

Then there exists a constant $c \in]0, +\infty[$ such that for any cube $\Lambda \subset \mathbb{Z}^d$ we have

$$\langle f \ln f \rangle_\Lambda \leq 2c \langle |\nabla f^{\frac{1}{2}}| \rangle_\Lambda + \langle f \rangle_\Lambda \ \ln \langle f \rangle_\Lambda \ , \tag{5.5}$$

for all nonnegative functions f for which the right-hand side is finite.

Combining all the statements Theorems 2.1, 2.3, 3.1, 4.1 and 5.1, we get Theorem 1.1.

Remark 5.2 . The preprint [Yo] considers generalizations going in other directions. Let us mention some of them where our proof is still relevant. We can consider more generally, for any $h \in \mathbb{Z}^d$, the phases $\Phi^{\Lambda,D,\mathcal{J},h}(X) = \Phi^{\Lambda,D,\mathcal{J}}(X) + \sum_{i\in\Lambda} h_i \cdot x_i$ and get the previous result uniformly with respect to h.
We can also consider more generals A_{ij} satisfying the assumptions of Zegarlinski [Ze]. The condition on $\mathcal{J}$ is then replaced by a condition on $||A||$.
N. Yoshida mentioned also the case when, for fixed $\mathcal{J}$, one considers weak perturbations of convex situations. Here the techniques and results of our previous papers [He4] and [He5] can be used for verifying the assumptions of the Theorem 5.1 of [Ze].
On the other hand, our assumptions on the single spin phase seem weaker than in Yoshida [Yo] (see for example his Lemma 3.3). As discussed with him[5] quite recently, the use of GHS inequality seems important. This makes reasonable extension of the quartic model to other polynomials like

$$\phi(x) = a_{2n}x^{2n} + ... + a_2x^2$$

with non negative $a_4, \cdots, a_{2(n-1)} \geq 0$ and $a_{2n} > 0$.

6 The case when the one particle phase is defined on $I\!R^N$

Some of the methods for proving decay estimates use standard estimates in statistical mechanics (for example GHS) where the dimension N of the space on which the one particle phase is defined plays a role (for example the condition $N \leq 4$ is mentioned by A. Sokal in [Sok], see the discussion after

[5] Personal e-mail exchange in November 1997

his Theorem 3). We shall show here that, in the Witten Laplacian approach, the extension from $N = 1$ to general N is only a minor difficulty. Let us assume indeed that our phase ϕ is now defined on $I\!R^N$ and satisfies the two following natural conditions.

- There exists [6] C such that, $\forall x \in I\!R^N$ with $|x| \geq C$, we have
$$\text{Hess}\,\phi \geq \frac{1}{C}\,. \tag{6.1}$$
- There exists $\rho > 0$ and, for all $\beta \in I\!N^N$, a constant C_β such that
$$|\partial_x^\beta \nabla\phi(x)| \leq C_\beta < \nabla\phi(x) >^{(1-\rho|\beta|)_+} \quad . \tag{6.2}$$

The second condition is probably technical.

Theorem 6.1 . *Under these two conditions (6.1) and (6.2), there exists $\mathcal{J}_0 > 0$ such that, for $|\mathcal{J}| \leq \mathcal{J}_0$, the Witten Laplacian on the 1-forms attached to the phase Φ on $I\!R^{N|\Lambda|}$ is uniformly strictly positive with respect to the cubes Λ, the boundary conditions and $\mathcal{J} \in [-\mathcal{J}_0, +\mathcal{J}_0]$.*

The basic example satisfying these assumptions appears in the so called Lattice vector field theory (See [Fr]). We take $N = 4$, $d = 4$ and
$$\phi(x) = \frac{1}{12}\lambda|x|^4 + \frac{\nu}{2}|x|^2\,, \tag{6.3}$$
with $\lambda > 0$ and $\nu < 0$.
The proof is a consequence of the following

Lemma 6.2 . *Let ϕ satisfy (6.1) and (6.2) and $w(x, \partial_x, \alpha, \mathcal{J})$ the Witten Laplacian on 1-forms on $I\!R^N$ attached to the phase*
$$I\!R^N \ni x \mapsto \phi_{\alpha,\mathcal{J}}(x) = \phi(x) + 2\mathcal{J}d|x|^2 - \alpha\cdot x\,,$$
where $\alpha \in I\!R^N$ and $\mathcal{J} \in]-\frac{1}{C}, +\frac{1}{C}[$.
Then there exists $\mathcal{J}_0 > 0$ and $\varrho_1 > 0$ such that, for all $(\mathcal{J}, \alpha)$ in $]-\mathcal{J}_0, +\mathcal{J}_0[\times I\!R^N$, the lowest eigenvalue of $w(x, \partial_x, \alpha, \mathcal{J})$ is larger than ϱ_1.

[6] We note that this condition implies the superstability for the corresponding phase Φ defined in (1.1).

Proof.

The Witten Laplacian on 1-forms on $I\!\!R^N$ attached to the phase $\phi_{\alpha,\mathcal{J}}$ takes now the form

$$w(x,\partial_x,\alpha,\mathcal{J}) := (-\Delta + \frac{1}{4}|\nabla\phi_{\alpha,\mathcal{J}}|^2 - \frac{1}{2}\Delta\phi_{\alpha,\mathcal{J}}) \otimes I + \text{Hess}\,\phi_{\alpha,\mathcal{J}} \ .$$

We introduce a partition of unity starting of a C^∞ function $\chi(x)$ satisfying $0 \leq \chi \leq 1$, equal to 1 on $B_{I\!\!R^N}(0,1)$ and with support in $B_{I\!\!R^N}(0,2)$. We introduce also a (large) parameter R and define

$$\chi_{1;R}(x) = \chi(\frac{x}{R}) \ , \ \chi_{2;R}(x) = \sqrt{1-\chi_{1;R}(x)^2} \ .$$

We use the standard identity associated to this partition of unity

$$\langle w\, u \mid u\rangle_{L^2(I\!\!R^N)} = \langle \hat{w}\,(\chi_1\, u) \mid (\chi_1\, u)\rangle_{L^2(I\!\!R^N)} + \langle \hat{w}\,(\chi_2\, u) \mid (\chi_2\, u)\rangle_{L^2(I\!\!R^N)} \ ,$$

where $\hat{w} = w \ - \ (|\nabla\chi_1|^2 - |\nabla\chi_2|^2)\otimes I$.
We observe also that $\hat{w} \ - \ w = \mathcal{O}(\frac{1}{R^2})$. We first observe that if $R \geq C$ large enough such that $\langle w(\chi_2\, u) \mid \chi_2\, u\rangle \geq \frac{1}{C}||\chi_2 u||^2$. We can then find $R_0 \geq C$ and $0 < \mathcal{J}_0 < \frac{1}{C}$ such that, for any $\mathcal{J} \in [-\mathcal{J}_0, +\mathcal{J}_0]$, any $\alpha \in I\!\!R^N$ and $R \geq R_0$

$$\langle \hat{w}\,(\chi_2\, u) \mid (\chi_2\, u)\rangle \geq \frac{1}{2C}||\chi_2\, u||^2 \ . \tag{6.4}$$

We fix now R. It is easy to get, for any $A > 0$, the existence of D such that for all $(\alpha,\mathcal{J}) \in I\!\!R^N \times [-\mathcal{J}_0, +\mathcal{J}_0]$ such that $|\alpha| \geq D$, we have

$$\langle \hat{w}\,(\chi_1\, u) \mid (\chi_1\, u)\rangle \geq A||\chi_1\, u||^2 \ . \tag{6.5}$$

Putting (6.4) and (6.5) together, we have proved the existence of $\varrho_0 > 0$, $\mathcal{J}_0 > 0$ and $D > 0$ such that, for all $(\alpha,\mathcal{J}) \in I\!\!R^N \times [-\mathcal{J}_0, +\mathcal{J}_0]$ with $|\alpha| \geq D$, we have

$$\langle w(x,\partial_x,\alpha,\mathcal{J})\, u \mid u\rangle \geq \varrho_0||u||^2 \ . \tag{6.6}$$

We now only need the local control of the lower bound with respect to $(\alpha,\mathcal{J})$. We observe that, for any $(\alpha,\mathcal{J})$, the lowest eigenvalue of w is strictly positive (See [Jo]) and continuous (under the assumptions (6.1) and (6.2) with respect to $(\alpha,\mathcal{J}) \in I\!\!R^N \times [-\mathcal{J}_0, +\mathcal{J}_0]$. This proves the lemma.
One deduces the theorem from the lemma along the same lines as when $N = 1$.

Remark 6.3 . Under the assumptions of the theorem, one can hope to obtain the uniform logarithmic Sobolev inequality for $\mathcal{J}$ small enough and any N. Nethertheless, there would be a need of an extension of Zegarlinski's Theorem 5.1, for $N > 1$.

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