Small denominators and anomalous behaviour in the uncommensurate Hubbard-Holstein model

V. Mastropietro*
Dipartimento di Matematica, Università di Roma "Tor Vergata"
Via della Ricerca Scientifica, I-00133, Roma

ABSTRACT. *A small denominator problem arising in the perturbative expansion for the Schwinger functions of a quantum model of interacting fermions in one dimension is solved by renormalization group techniques. We show that the Schwinger functions asymptotic behaviour is anomalous and described in terms of critical indices.*

1. Introduction.

1.1 Peierls [P] and Fröhlich [F] noted in the 50s that one dimensional metals are unstable at low temperature, in the sense that they can lower their energy by a periodic distorsions of the lattice with period $\frac{2\pi}{2p_F}$, if p_F is the Fermi momentum. Such a distorsion is called *Charge Density Wave* (CDW), as both the lattice and the electrons charge density form a new periodic structure with period possibly bigger than the original lattice period a. Fröhlich suggested that, if the period of the CDW is *incommensurate* with the original period of the lattice, the CDW can carry an electric current. The important role of uncommensurate CDW in conductivity was confirmed in [LRA]. Starting from the early 70s many experimental properties of anysotropic compounds were explained qualitively in terms of CDW. The CDW were observed in TTF-TCNQ by X-rays and the anomalous behaviour of conductivity in many compounds was explained in terms of them (see for instance [L] for a rewiew).

The most simple model to understand the formation of a CDW is the *Holstein model* [H], whose hamiltonian is given by a thigth binding hamiltonian, describing fermions on a lattice, and an interaction with a classical phonon field. The interaction between electrons is completely neglected. Numerical simulations show, see for instance [AAR], that for small phonon fields the energy in such a model is minimized if the phonon field is a periodic function with period $\frac{2\pi}{2p_F}$, either if the Fermi momentum is commensurate or incommensurate with the lattice periodicity *i.e.* if $\frac{p_F}{\pi a}$ is a rational or irrational number. A mathematical proof of the formation of a CDW in the Holstein model is up to now available only in the commensurate case, [KL], [BGM2]. Assuming the existence of CDW and their modelization by a periodic function with period $\frac{2\pi}{2p_F}$, a quantitative comparation with the experiments requires in any case the computation of the Holstein Schwinger functions, in terms of which the physical properties of the model can be obtained. The Schwinger functions properties could be obtained studying the solutions of the Schroedinger equation on a lattice with commensurate or uncommensurate periodic potential. This second case is much more difficult as it is technically equivalent to the study of the Schroedinger equation in the continuum with a quasi-periodic potential, which is characterized by a *small divisor problem* and it is generally faced by KAM iterative techniques. Many properties are known about the solutions of the Schroedinger equation in a quasi periodic potential (see for instance [DS],[BLT],[JM] and the review [PF]), but the Schwinger functions properties of the Holstein model do not seem an immediate consequence of them. A different approach was developed then in [BGM1] for studying the Schwinger functions of the Holstein model using functional integration and renormalization group techniques. It was found that the Schwinger function decays for large distances faster than any power, and that there is a spectral gap at the Fermi surface.

* Supported by MURST, Italy.

Of course in a realistic model for anysotropic compounds with a CDW one has to take into account also the interaction between electrons, as it is well known that in one dimension the interaction changes deeply the properties of the system. So one can consider the *Holstein-Hubbard model*, which is equal to the Holstein model but a term describing the short range interaction between fermions is added. Such a model is considered relevant for the understanding of the high-T_c superconductivity, see [A].

We study the Holstein-Hubbard model in the incommensurate case, but for the (much easier) commensurate case similar results follow from the analysis (for a sligthly different model) in [BM2], [BM3].

1.2 The Schwinger functions of the Holstein-Hubbard model are studied writing them as a *functional Grassmanian integral* which is expressed by a erturbative series. The small denominator problem affecting this series is controlled, like in [BGM1], by using a sort of *Bryuno Lemma* [B] (see sec. 3.6 below) and using suitable *cancellations* to face the problem of the *resonances*. Such cancellations are implemented by Renormalization group techniques. The approach we follow is then very closely related to the "direct" methods developed in recent years for proving the KAM theorem by showing the convergence of the *Lindstedt series* expressing the invariant tori, see [E1],[G],[CF],[GM]. The main technical novelty of the present paper is that in the above quoted papers (including [BGM1]) the perturbative expansion can be expressed in terms of *Feynmann graphs* which are only *tree graphs i.e.* with no loops: this is not surprising as the KAM theorem is a classical problem and the Holstein model is a non interacting model *i.e.* bilinear in the fields.

On the contrary the perturbative series for the Schwinger functions of the Holstein-Hubbard model are expressed in terms of Feynmann graphs with loops. So as far as we known this is the first case in which a small denominator problem in a fully interacting quantum theory is solved, and it is not clear to us if one can solve it also by the traditional iterative KAM techniques. The main idea is to combine such direct methods to study KAM problems with the Renormalization Group techniques developed to study interacting fermions starting from [BG] (similar techniques were introduced also in [FMRT]).

1.3 The hamiltonian of the *Holstein-Hubbard model* is

$$H = H_0 + uP + \lambda V + \nu N \ , \tag{1.1}$$

$$H_0 = \sum_{x,y\in\Lambda} t_{xy}\, \psi^+_x \psi^-_y - \mu \sum_{x\in\Lambda} \psi^+_x \psi^-_x$$

$$P = \sum_{x\in\Lambda} \varphi_x \psi^+_x \psi^-_x \qquad N = \sum_{x\in\Lambda} \psi^+_x \psi^-_x$$

$$V = \sum_{x,y\in\Lambda} v(x-y) \psi^+_x \psi^-_x \psi^-_y \psi^+_y$$

where x, y are points on the one-dimensional lattice Λ with unit spacing, length L and periodic boundary conditions; we shall identify Λ with $\{x \in Z : \ -[L/2] \le x \le [(L-1)/2]\}$. Moreover the matrix t_{xy} is defined as $t_{xy} = \delta_{x,y} - (1/2)[\delta_{x,y+1} + \delta_{x,y-1}]$, where $\delta_{x,y}$ is the Kronecker delta, and μ is the chemical potential. The fields $\psi^\pm_x$ are creation $(+)$ and annihilation $(-)$ fermionic fields, satisfying periodic boundary conditions: $\psi^\pm_x = \psi^\pm_{x+L}$. We define also $\psi^\pm_{\mathbf{x}} = e^{tH}\psi^\pm_x e^{-Ht}$, with $\mathbf{x} = (x,t)$, $-\beta/2 \le t \le \beta/2$ for some $\beta > 0$; on t antiperiodic boundary conditions are imposed.

The term P rapresents the interaction of the fermions with a classical phonon field. We are interested in studying potentials which, in the limit $L \to \infty$, are of the form $\varphi_x = \bar\varphi(2px)$, where $\bar\varphi$ is a real function on the real line 2π-periodic and p/π is an irrational number, so that the phonon field has a period which is incommensurate with the period of the lattice.

We also impose that $\bar{\varphi}(u)$ is of mean zero (its mean value can be absorbed in the chemical potential), even and analytic in u, so that

$$\bar{\varphi}(u) = \sum_{0\neq n\in Z} \hat{\varphi}_n \, e^{inu} \;, \qquad |\hat{\varphi}_n| \leq F_0 \, e^{-\xi|n|} \;, \qquad \hat{\varphi}_n = \hat{\varphi}_{-n} = \hat{\varphi}_n^* \;. \tag{1.2}$$

At finite volume we need a potential satisfying periodic boundary conditions; hence, at finite L, we approximate φ_x by

$$\varphi_x^{(L)} = \sum_{n=-[L/2]}^{[(L-1)/2]} \hat{\varphi}_n \, e^{2inp_L x} \;, \tag{1.3}$$

where p_L tends to p as $L \to \infty$ and is of the form $p_L = n_L \pi / L$, with n_L an integer, relatively prime with respect to L. The definition of p_L implies that $2np_L$ is an allowed momentum (modulo 2π), for any n, and that the sum in (1.3) is indeed a sum over all allowed values of k, except $k = 0$. For technical reasons we need p_L verifying the *Diophantine property* (1.4) uniformly in L. In [BGM1], App. 1, a sequence of numbers verifying (1.4) is constructed.

The term V in the hamiltonian rapresents the interaction of the fermions by a short range two body potential; in particular we assume $|v(x-y)| \leq Ce^{-\kappa|x-y|}$ for suitable values of the costants C, κ, and $\hat{v}(0) - \hat{v}(2p_F) \geq 0$. Finally ν is a *counterterm* which will be fixed to a λ, u-dependent value in order to fix the Fermi momentum *i.e.* we fix ν so that the Fermi momentum in the interacting $\lambda \neq 0$, with chemical potential $\mu + \nu$, is equal to the Fermi momentum of the free $\lambda = 0$ system with chemical potential μ. We assume that u is positive while λ can be positive or negative.

1.3 Denote by $\|\alpha - \beta\|_{T^1}$ the distance on T^1 of $\alpha, \beta \in T^1$, and, for $\mathbf{x} = (x, x_0), \mathbf{y} = (y, y_0) \in R^2$, by $|\mathbf{x} - \mathbf{y}|$ the distance $|\mathbf{x} - \mathbf{y}| = \sqrt{(x-y)^2 + (x_0 - y_0)^2}$. The definition of the two-point Schwinger function is standard and it will be recalled below, see (2.8). If E_0^n is the *ground state energy i.e.* the minimum value of H over the states $|\psi_n >$ with n particles $N|\psi_n >= n|\psi_n >$, the *spectral gap* is $\Delta = E_0^{n+1} + E_0^{n-1} - 2E_0^n$.

With the above definitions we shall prove the following theorem.

THEOREM *Let us consider a sequence L_i, $i \in Z^+$, such that*

$$\lim_{i\to\infty} L_i = \infty \;, \qquad \lim_{i\to\infty} p_{L_i} = p \;,$$

and let $S^{L_i,\beta}(\mathbf{x};\mathbf{y})$ be the two-points Schwinger function at finite temperature β^{-1}. Suppose also that the fermions are spinless and that there is a positive integer m such that $p_F = mp_{L_i} \, (\mathrm{mod}\, 2\pi)$, $\hat{\varphi}_m \neq 0$ and p_{L_i} satisfies the diophantine condition

$$\|2np_{L_i}\|_{T^1} \geq C_0 |n|^{-\tau} \;, \qquad 0 \neq n \in Z \qquad |n| \leq \frac{L_i}{2}, \tag{1.4}$$

for some positive constants C_0 and τ independent of i. Then it is possible to choose an $\varepsilon_0 > 0$ and a function $\nu \equiv \nu(\lambda, u)$ such that if $|u|, |\lambda| \leq \varepsilon_0$, the following sentences are true:

(i) There exists the limit $\lim_{\substack{\beta\to\infty \\ i\to\infty}} S^{L_i,\beta}(\mathbf{x};\mathbf{y}) = S(\mathbf{x};\mathbf{y})$

(ii) $S(\mathbf{x};\mathbf{y})$ is continuous as a function of λ, u and there exist three constants K_1, K_2, K_3 and, for any $N > 1$, a constant C_N, such that, if $|\mathbf{x} - \mathbf{y}| \geq K_3 |\hat{u}(p_F)|^{-1}$

$$|S(\mathbf{x},\mathbf{y})| \leq \frac{1}{\hat{Z}(p_F)} K_1 |\hat{u}(p_F)| \frac{C_N}{1 + (|\hat{u}(p_F)| \, |\mathbf{x} - \mathbf{y}|)^N} \tag{1.5}$$

while for $|\mathbf{x} - \mathbf{y}| \leq K_3 |\hat{u}(p_F)|^{-1}$ one has

$$|S(\mathbf{x},\mathbf{y})| \leq \frac{K_2}{1 + |\mathbf{x} - \mathbf{y}|^{1+\eta_3}} \tag{1.6}$$

with

$$\hat{u}(p_F) = u^{1+\eta_2} \qquad \hat{Z}(p_F) = u^{-\eta_1} \tag{1.7}$$

and $\eta_1 \equiv \eta_1(\lambda, u), \eta_2 \equiv \eta_2(\lambda, u), \eta_3 \equiv \eta_1(\lambda, u)(1+\eta_2(\lambda, u))^{-1}$ *continuous in their arguments for* $|\lambda| \leq \varepsilon_0$ *and such that* $\eta_1 \equiv \eta_1(\lambda, u) = \beta_1\lambda^2 + O(\lambda^3)$ *and* $\eta_2 \equiv \eta_2(\lambda, u) = \beta_2\lambda + O(\lambda^2)$, *with* β_1, β_2 *positive non vanishing constants.*

(iii) We can write

$$S(\mathbf{x}; \mathbf{y}) = S_1(\mathbf{x}; \mathbf{y}) + \tilde{\varepsilon} S_2(\mathbf{x}; \mathbf{y}) \;, \tag{1.8}$$

where

$$S_1(\mathbf{x}; \mathbf{y}) = g^{(1)}(\mathbf{x}; \mathbf{y}) + \int \frac{d\mathbf{k}}{(2\pi)^2} [1 - \hat{f}_1(\mathbf{k})]\, \phi(k, x, \sigma)\, \phi^*(k, y, \sigma)\, \frac{e^{-ik_0(x_0 - y_0)}}{-ik_0 + \varepsilon(k, \sigma)}$$

with

$$\begin{aligned} g^{(1)}(\mathbf{x}; \mathbf{y}) &= \int \frac{d\mathbf{k}}{(2\pi)^2} \frac{\hat{f_1(\mathbf{k})}}{\hat{Z}(k)} \frac{e^{-ik_0(x_0 - y_0)}}{-ik_0 + 1 - \cos k - \mu} \;, \\ \varepsilon(k, \sigma) &= [1 - \cos(|k| - p_F)] \cos p_F \\ &\quad + \operatorname{sign}(|k| - p_F) \sqrt{[\sin(|k| - p_F) \sin p_F]^2 + \hat{u}(k)^2} \;, \\ \phi(k, x, \hat{u}(k)) &= e^{-ikx} U(k, x, \hat{u}(k)) \;, \\ U(k, x, \hat{u}(k)) &= e^{i \operatorname{sign}(k) p_F x} \left[\cos(p_F x) \sqrt{1 - \frac{\operatorname{sign}(|k| - p_F) \hat{u}(k)}{\sqrt{(\sin(|k| - p_F) \sin p_F)^2 + \hat{u}(k)^2}}} \right. \\ &\quad \left. -i \operatorname{sign}(k) \sin(p_F x) \sqrt{1 + \frac{\operatorname{sign}(|k| - p_F) \hat{u}(k)}{\sqrt{(\sin(|k| - p_F) \sin p_F)^2 + \hat{u}(k)^2}}} \right] . \end{aligned} \tag{1.9}$$

Here $\hat{f}_1(\mathbf{k})$ *denotes a cutoff function with support far enough from the two singular points* $\mathbf{k} = (\pm p_F, 0)$, *and* $\hat{u}(k), \hat{Z}(k)$ *are two regular functions such that* $|\hat{u}(k) - u| = O(u\lambda)|$, $|\hat{Z}(k)^{-1} - 1| = O(\lambda)$ *for* $||k| - p_F| > \frac{p_F}{2}$, *and* $\hat{u}(p_F), \hat{Z}(p_F)$ *are given by (1.7); moreover if* $\lambda = 0$ *then* $\hat{u}(k) = u$, $Z(\mathbf{k}) = 1$. *Finally* $\tilde{\varepsilon} = Max(|\lambda|, u, \hat{u}(p_F))$ *and* $S_1(\mathbf{x}, \mathbf{y}), S_2(\mathbf{x}, \mathbf{y})$ *obeys to the same bound (1.6). Moreover if* $\lim_{\lambda \to 0, u \to 0} S(\mathbf{x}, \mathbf{y}) = g(\mathbf{x}, \mathbf{y})$ *then*

$$|S_1(\mathbf{x}; \mathbf{y}) - \frac{g(\mathbf{x}; \mathbf{y})}{|\mathbf{x} - \mathbf{y}|^{\eta_3}}| \leq K_2 \frac{Max(u, \hat{u}(p_F))}{\hat{Z}(p_F)} \log \hat{u}(p_F) \tag{1.10}$$

(iv)For any i *there is a spectral gap* Δ *verifying*

$$\Delta \geq \frac{\hat{u}(p_F)}{2} \tag{1.11}$$

1.4 Let us comment the quite elaborated theorem above. The point (ii) says that, like in the non interacting $\lambda = 0$ case, one can distinguish two regions in the large distance behavior of the Schwinger function, discriminated by an intrinsic length which is $O(u^{1+\eta_2})$ instead $O(u)$. In the second region (larger distances) the decay is faster than any power but, contrary to the free case, the *decay rate* is renormalized by the interaction as it is not $O(u)$ but $O(u^{1+\eta_2})$. Moreover there is a factor $\hat{Z}(p_F)$ in the bound which can be interpretated as a *wave function renormalization*, which can be very large if u is small ($\hat{Z}(p_F) \to 0$ if $u \to 0$). In the non interacting case of course $\hat{Z}(p_F) = 1$. In the first region (smaller distances) the slower bound (1.6) is found for the Schwinger function decay. Note that combing (1.6) with (1.10) we can conclude that for $|\mathbf{x} - \mathbf{y}|$ less than $O(u^{\frac{-1}{2}})$ (say) the Schwinger function decays as the free ($\lambda = u = 0$) one divided by $\frac{1}{|\mathbf{x} - \mathbf{y}|^{\eta_3}}$, in the sense that $|S(\mathbf{x}, \mathbf{y}) - \frac{g(\mathbf{x}; \mathbf{y})}{|\mathbf{x} - \mathbf{y}|^{\eta_3}}| \leq \frac{C}{|\mathbf{x} - \mathbf{y}|^{1+\eta_3}}$ and noting that $\limsup_{|\mathbf{x} - \mathbf{y}| \to \infty} |(\mathbf{x} - \mathbf{y}) g(\mathbf{x}; \mathbf{y})| > 0$.

The faster than any power decay and the modification of the decay rate suggest the presence of an *anomalous gap i.e.* $\Delta \simeq u^{1+\eta_2}$. This is confirmed by (iv) in which a lower bound for the gap is obtained; note that , if the interaction is attractive ($\lambda \leq 0$), (1.11) says that the ratio between the bare and interacting gap is $<< 1$ and diverging as $u \to 0$. In the limit $u \to 0$ it is known [BGL] that the Schwinger function decays as the Luttinger model one, and this is in fact what we find in the limit $u \to 0$. At the end the interaction produces an anomalous gap and an anomalous wave function renormalization, and the anomaly is related to the appearance of critical indices, explicitely computed by convergent series.

1.5 The point iii) in the theorem allows us to make clearer the relationship of our results with the literature. In the $\lambda = 0$ case the hamiltonian is the sum of many single particle hamiltonians and the eigenfunctions of the many body problem are a product of the eigenfunctions of the *finite difference Schroedinger equation*

$$-\psi(x+1) - \psi(x-1) + u\phi_x\psi_x = E\psi_x$$

where $\phi_x = \bar{\phi}(2px)$, $\bar{\phi}(u)$ 2π-periodic and p Diophantine. By a number of works on the spectrum it is known that for certain values of E the eigenfunctions are *quasi Bloch waves* of the form $\phi(k, x, u) = e^{ikx}U(x)$, with $k = k(E)$ and $U(x) = \bar{U}(px)$ and $\bar{U}$ 2π-periodic. In particular this is true for $k = mp$, for λ small enough, and the spectrum has a gap $O(u)$ in corrispondence of this E (this was proved by [JM],[MP],[E] for the case of the Schoedinger equation in the continuum with a quasi periodic potential; but one can extend these results to the finite difference Schroedinger equation, see [BLT]). ¿From the decomposition (1.8) we see that S_1 for $\lambda = 0$ is essentially the Schwinger function corresponding to a quasi Bloch wave $\phi(k, x, u)$ with a gap $O(u)$ in the spectrum and, as we can expect that S_1 is the dominant part for large distances *i.e.* for $k \simeq p_F = mp$, this is in agreement with the results about Schroedinger equation. If $\lambda \neq 0$ we find that the Schwinger function has the same structure with the main difference that u is replaced by $\hat{u}(k)$ and there is a factor $\frac{1}{\hat{Z}(\mathbf{k})}$ more. It is natural to interpretate this result saying that the fermions for $k \simeq mp$ are *quasi particles* which are *interacting quasi Bloch waves* $\frac{1}{\sqrt{\hat{Z}(\mathbf{k})}}\phi(k, x, \hat{u}(k))$. Of course to really prove this one has to prove that $S_1(\mathbf{x}; \mathbf{y})$ is really the dominant term.

1.6 The Schwinger function behaviour we find is not peculiar of the Holstein-Hubbard model in the uncommensurate case; one can prove similar statements in the commensurate case *i.e.* large distance behaviour of the system is independent from the periodic or quasi periodic nature of the phonon field ϕ_x. The commensurate case was studied in [BM2],[BM3] (there was studied the case of interacting fermions in the continuum space, but an adaptation to this case is trivial). The large distances Schwinger function behaviour we find is typical to a wide class of models (in statystical physics or particle physics), like the massive Thirring model or the Yukawa2 model (see [BM3],[BM4] and references therein). A renormalization group analysis shows in fact that such models are equivalent in the sense that the relevant part of their Beta function is equal. Finally some comments about the spin; the discussion is essentially identical to the one in [BM2] and we do not repeat it. In the spinning case the number of running coupling constants is larger than in the spinless case and the renormalization group flow is more complex. For repulsive interaction ($\lambda > 0$) things do not change while in the opposite case the flow is umbounded and no conclusions can be drawn (except if $u > e^{\frac{-1}{k\lambda^2}}$, the trivial case); so our results are valid for spinning fermions only if $\lambda > 0$.

The paper is self-consistent but we have omitted the proofs of some technical lemmas which can be found in the literature (especially in [BGPS]) in order to save space.

2. Multiscale decomposition and anomalous integration

2.1 As it is well known, the Schwinger functions can be written as power series in λ, convergent for $|\lambda| \le \varepsilon_\beta$, for some constant ε_β (the only trivial bound of ε_β goes to zero, as $\beta \to \infty$). This power expansion is constructed in the usual way in terms of Feynman graphs, by using as *free propagator* the function

$$
\begin{aligned}
g^{L,\beta}(\mathbf{x};\mathbf{y}) \equiv g^{L,\beta}(\mathbf{x}-\mathbf{y}) &= \frac{\mathrm{Tr}\left[e^{-\beta H_0}\mathbf{T}(\psi^-_{\mathbf{x}}\psi^+_{\mathbf{y}})\right]}{\mathrm{Tr}[e^{-\beta H_0}]} = \\
&= \frac{1}{L}\sum_{k\in\mathcal{D}_L} e^{-ik(x-y)}\left\{\frac{e^{-\tau e(k)}}{1+e^{-\beta e(k)}}\mathbf{1}(\tau>0) - \frac{e^{-(\beta+\tau)e(k)}}{1+e^{-\beta e(k)}}\mathbf{1}(\tau\le 0)\right\} ,
\end{aligned}
\tag{2.1}
$$

where $\tau = x_0 - y_0$, $\mathbf{1}(E)$ denotes the indicator function ($\mathbf{1}(E)=1$, if E is true, $\mathbf{1}(E)=0$ otherwise), $e(k) = 1-\cos k - \mu$ and $\mathcal{D}_L \equiv \{k = 2\pi n/L, n\in Z, -[L/2] \le n \le [(L-1)/2]\}$.

It is easy to prove that, if $x_0 \ne y_0$,

$$
g^{L,\beta}(\mathbf{x}-\mathbf{y}) = \lim_{M\to\infty}\frac{1}{L\beta}\sum_{\mathbf{k}\in\mathcal{D}_{L,\beta}}\frac{e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{-ik_0+\cos p_F - \cos k} ,
\tag{2.2}
$$

where $\mathbf{k} = (k,k_0)$, $\mathbf{k}\cdot\mathbf{x} = k_0x_0 + kx$, $\mathcal{D}_{L,\beta} \equiv \mathcal{D}_L \times \mathcal{D}_\beta$, $\mathcal{D}_\beta \equiv \{k_0 = 2(n+1/2)\pi/\beta, n \in Z, -M \le n \le M-1\}$ and p_F is the *Fermi momentum*, defined so that $\cos p_F = 1-\mu$ and $0 \le p_F \le \pi$.

Hence, if we introduce a finite set of Grassmanian variables $\{\psi^\pm_{\mathbf{k}}\}$, one for each of the allowed $\mathbf{k}$ values, and a linear functional $P(d\psi)$ on the generated Grassmanian algebra, such that

$$
\int P(d\psi)\psi^-_{\mathbf{k}_1}\psi^+_{\mathbf{k}_2} = L\beta\delta_{\mathbf{k}_1,\mathbf{k}_2}\hat g_{\mathbf{k}_1} , \quad \hat g_{\mathbf{k}} = \frac{1}{-ik_0+\cos p_F-\cos k} ,
\tag{2.3}
$$

we have

$$
\frac{1}{L\beta}\sum_{\mathbf{k}\in\mathcal{D}_{L,\beta}} e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\,\hat g_{\mathbf{k}} = \int P(d\psi)\psi^-_{\mathbf{x}}\psi^+_{\mathbf{y}} \equiv g^{L,\beta}(\mathbf{x};\mathbf{y}) ,
\tag{2.4}
$$

where the *Grassmanian field* $\psi_{\mathbf{x}}$ is defined by

$$
\psi^\pm_{\mathbf{x}} = \frac{1}{L\beta}\sum_{\mathbf{k}\in\mathcal{D}_{L,\beta}}\psi^\pm_{\mathbf{k}}e^{\pm i\mathbf{k}\cdot\mathbf{x}} .
\tag{2.5}
$$

The "Gaussian measure" $P(d\psi)$ has a simple representation in terms of the "Lebesgue Grassmanian measure" $d\psi d\bar\psi$, defined as the linear functional on the Grassmanian algebra, such that, given a monomial $Q(\psi,\bar\psi)$ in the variables $\psi_{\mathbf{k}},\bar\psi_{\mathbf{k}}$,

$$
\int d\psi d\bar\psi Q(\psi,\bar\psi) = \begin{cases} 1 & \text{if } Q(\psi,\bar\psi) = \prod_{\mathbf{k}}\psi_{\mathbf{k}}\bar\psi_{\mathbf{k}} , \\ 0 & \text{otherwise} . \end{cases}
\tag{2.6}
$$

We have

$$
P(d\psi) = \Big\{\prod_{\mathbf{k}}(L\beta\hat g_{\mathbf{k}})\Big\}\exp\Big\{-\sum_{\mathbf{k}}(L\beta\hat g_{\mathbf{k}})^{-1}\bar\psi_{\mathbf{k}}\psi_{\mathbf{k}}\Big\}d\psi d\bar\psi .
\tag{2.7}
$$

Note that, since $\psi^2_{\mathbf{k}} = \bar\psi^2_{\mathbf{k}} = 0$, $e^{-z\bar\psi_{\mathbf{k}}\psi_{\mathbf{k}}} = 1 - z\bar\psi_{\mathbf{k}}\psi_{\mathbf{k}}$, for any complex z.

By using standard arguments (see, for example, [NO], where a different regularization of the propagator is used), one can show that the Schwinger functions can be calculated as expectations of suitable functions of the Grassmanian field with respect to the "Gaussian measure" $P(d\psi)$. In particular, the two-point Schwinger function can be written, if $x_0 \ne y_0$, as

$$
S^{L,\beta}(\mathbf{x};\mathbf{y}) = \lim_{M\to\infty}\frac{\int P(d\psi)\,e^{-\mathcal{V}(\psi)}\,\psi^-_{\mathbf{x}}\psi^+_{\mathbf{y}}}{\int P(d\psi)\,e^{-\mathcal{V}(\psi)}} ,
\tag{2.8}
$$

where $\mathcal{V}(\psi)=uP(\psi)+\lambda V(\psi)+\nu N(\psi)$ with

$$V(\psi)=\sum_{x,y\in\Lambda}\int_{-\beta/2}^{\beta/2}dx_0\int_{-\beta/2}^{\beta/2}dy_0 v(x-y)\delta(x_0-y_0)\psi^+_{\mathbf{x}}\psi^-_{\mathbf{x}}\psi^-_{\mathbf{y}}\psi^+_{\mathbf{y}}$$

$$P(\psi)=\sum_{x\in\Lambda}\int_{-\beta/2}^{\beta/2}dx_0\left[\varphi_x\psi^+_{\mathbf{x}}\psi^-_{\mathbf{x}}\right]\qquad N(\psi)=\sum_{x\in\Lambda}\int_{-\beta/2}^{\beta/2}dx_0\psi^+_{\mathbf{x}}\psi^-_{\mathbf{x}}\ . \tag{2.9}$$

If $x_0=y_0$, $S^{L,\beta}(\mathbf{x};\mathbf{y})$ must be defined as the limit of (2.8) as $x_0-y_0\to 0^-$, as we shall understand always in the following.

2.2 We assume from now on $p_F=p_L$ *i.e.* $m=1$ (this assumption is of course not restrictive) and start by evalutating the partition function*i.e.* the denominator of (2.8)

$$\int P(d\psi)e^{-\mathcal{V}(\psi)}\ , \tag{2.10}$$

It is convenient to decompose the Grassmanian integration $P(d\psi)$ into a finite product of independent integrations:

$$P(d\psi)=\prod_{h=h_\beta}^{1}P(d\psi^{(h)})\ , \tag{2.11}$$

where $h_\beta>-\infty$ will be defined below (before (2.19)) This can be done by setting

$$\psi^{\pm}_{\mathbf{k}}=\bigoplus_{h=h_\beta}^{1}\psi^{(h)\pm}_{\mathbf{k}}\ ,\qquad \hat g_{\mathbf{k}}=\sum_{h=h_\beta}^{1}\hat g^{(h)}_{\mathbf{k}}\ , \tag{2.12}$$

where $\psi^{(h)\pm}_{\mathbf{k}}$ are families of Grassmanian fields with propagators $\hat g^{(h)}_{\mathbf{k}}$ which are defined in the following way.

We introduce a *scaling parameter* $\gamma>1$ and a function $\chi(\mathbf{k}')\in C^\infty(T^1\times R)$, $\mathbf{k}'=(k',k_0)$, such that, if $|\mathbf{k}'|\equiv\sqrt{k_0^2+||k'||^2_{T^1}}$:

$$\chi(\mathbf{k}')=\chi(-\mathbf{k}')=\begin{cases}1 & \text{if } |\mathbf{k}'|<t_0\equiv a_0/\gamma\ ,\\ 0 & \text{if } |\mathbf{k}'|>a_0\ ,\end{cases} \tag{2.13}$$

where $a_0=\min\{p_F/2,(\pi-p_F)/2\}$. This definition is such that the supports of $\chi(k-p_F,k_0)$ and $\chi(k+p_F,k_0)$ are disjoint and the C^∞ function on $T^1\times R$

$$\hat f_1(\mathbf{k})\equiv 1-\chi(k-p_F,k_0)-\chi(k+p_F,k_0) \tag{2.14}$$

is equal to 0, if $||k|-p_F||^2_{T^1}+k_0^2<t_0^2$.

We define also, for any integer $h\le 0$,

$$f_h(\mathbf{k}')=\chi(\gamma^{-h}\mathbf{k}')-\chi(\gamma^{-h+1}\mathbf{k}')\ ; \tag{2.15}$$

we have, for any $\bar h<0$,

$$\chi(\mathbf{k}')=\sum_{h=\bar h+1}^{0}f_h(\mathbf{k}')+\chi(\gamma^{-\bar h}\mathbf{k}')\ . \tag{2.16}$$

Note that, if $h\le 0$, $f_h(\mathbf{k}')=0$ for $|\mathbf{k}'|<t_0\gamma^{h-1}$ or $|\mathbf{k}'|>t_0\gamma^{h+1}$, and $f_h(\mathbf{k}')=1$, if $|\mathbf{k}'|=t_0\gamma^h$.

We finally define, for any $h\le 0$:

$$\hat f_h(\mathbf{k})=f_h(k-p_F,k_0)+f_h(k+p_F,k_0)\ , \tag{2.17}$$

$$\hat{g}^{(h)}_{\mathbf{k}} \equiv \frac{\hat{f}_h(\mathbf{k})}{-ik_0 + \cos p_F - \cos k} \; . \tag{2.18}$$

Note that, if $\mathbf{k} \in \mathcal{D}_{L,\beta}$, then $|k_0| \geq \pi/\beta$, implying that $\hat{f}_h(\mathbf{k}) = 0$ for any $h < h_\beta = \min\{h : t_0\gamma^{h+1} > \pi/\beta\}$. Hence, if $\mathbf{k} \in \mathcal{D}_{L,\beta}$, the definitions (2.14) and (2.17), together with the identity (2.16), imply that

$$1 = \sum_{h=h_\beta}^{1} \hat{f}_h(\mathbf{k}) \; . \tag{2.19}$$

The definition (2.17) implies also that, if $h \leq 0$, the support of $\hat{f}_h(\mathbf{k})$ is the union of two disjoint sets, A_h^+ and A_h^-. In A_h^+, k is strictly positive and $||k - p_F||_{T^1} \leq a_0\gamma^h \leq a_0$, while, in A_h^-, k is strictly negative and $||k+p_F||_{T^1} \leq a_0\gamma^h$. Therefore, if $h \leq 0$, we can write $\psi^{(h)\pm}_{\mathbf{k}}$ as the sum of two independent Grassmanian variables $\psi^{(h)\pm}_{\mathbf{k},\omega}$ with propagator

$$\int P(d\psi^{(h)})\, \psi^{(h)-}_{\mathbf{k}_1,\omega_1} \psi^{(h)+}_{\mathbf{k}_2,\omega_2} = L\beta\delta_{\mathbf{k}_1,\mathbf{k}_2}\, \delta_{\omega_1,\omega_2}\, \hat{g}^{(h)}_{\omega_1}(\mathbf{k}_1) \; , \tag{2.20}$$

so that

$$\psi^{(h)\pm}_{\mathbf{k}} = \bigoplus_{\omega=\pm1} \psi^{(h)\pm}_{\mathbf{k},\omega} \; , \qquad \hat{g}^{(h)}_{\mathbf{k}} = \sum_{\omega=\pm1} \hat{g}^{(h)}_{\omega}(\mathbf{k}) \; , \tag{2.21}$$

$$\hat{g}^{(h)}_{\omega}(\mathbf{k}) = \frac{\theta(\omega k)\, \hat{f}_h(\mathbf{k})}{-ik_0 + \cos p_F - \cos k} \; , \tag{2.22}$$

where $\theta(k)$ is the (periodic) step function. If $\omega k > 0$, we will write in the following $k = k' + \omega p_F$, where k' is the *momentum measured from the Fermi surface* and we shall define, if $h \leq 0$,

$$\tilde{g}^{(h)}_{\omega}(\mathbf{k}') \equiv \hat{g}^{(h)}_{\omega}(\mathbf{k}) = \frac{f_h(\mathbf{k}')}{-ik_0 + v_0\omega \sin k' + (1-\cos k')\cos p_F} \; , \tag{2.23}$$

where $v_0 = \sin p_F$. We call moreover $\psi^{(\leq k)\pm}_{\mathbf{k},\omega} = \bigoplus_{h=h_\beta}^{k} \psi^{(h)\pm}_{\mathbf{k},\omega}$ and $g^{\leq k}_{\mathbf{k},\omega} = \sum_{h=h_\beta}^{k} \hat{g}^{(h)}_{\mathbf{k},\omega}$

This kind of decomposition is completely standard in the theory of the $d = 1$ Fermi system; we repeat it here only for clarity.

We define

$$e^{-\mathcal{V}^{(0)}(\psi^{(\leq 0)})} = \int P(d\psi^{(1)}) e^{-\mathcal{V}^{(0)}(\psi^{(1)} + \psi^{(\leq 0)})}$$

It is possible to prove , see [BGL], [BGPS] that

$$\mathcal{V}^{(0)}(\psi^{(\leq 0)}) = \lambda \frac{1}{(L\beta)^4} \sum_{\mathbf{k}_1,\ldots,\mathbf{k}_4 \in \mathcal{D}_{L,\beta}} \hat{v}(\mathbf{k}_1 - \mathbf{k}_2) \psi^{(\leq 0)+}_{\mathbf{k}_1} \psi^{(\leq 0)-}_{\mathbf{k}_2} \psi^{(\leq 0)+}_{\mathbf{k}_3} \psi^{(\leq 0)-}_{\mathbf{k}_4} \delta(\mathbf{k}_1 + \mathbf{k}_3 - \mathbf{k}_2 - \mathbf{k}_4)$$

$$+ \frac{1}{L\beta} \sum_{\mathbf{k} \in \mathcal{D}_{L,\beta}} (\nu + F(\mathbf{k})) \psi^{(\leq 0)+}_{\mathbf{k}} \psi^{(\leq 0)-}_{\mathbf{k}} + u \sum_{m=1}^{\infty} \hat{\phi}_m \frac{1}{L\beta} \sum_{\mathbf{k} \in \mathcal{D}_{L,\beta}} \psi^{(\leq 0)+}_{\mathbf{k}} \psi^{(\leq 0)-}_{\mathbf{k}+2m\mathbf{p}_L} + \psi^{(\leq 0)+}_{\mathbf{k}} \psi^{(\leq 0)-}_{\mathbf{k}-2m\mathbf{p}_L}$$

$$+ \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{1}{(L\beta)^m} \sum_{\mathbf{k}_1,\ldots,\mathbf{k}_m \in \mathcal{D}_{L,\beta}} \psi^{(\leq 0)\sigma_1}_{\mathbf{k}_1} \ldots \psi^{(\leq 0)\sigma_m}_{\mathbf{k}_m} W^{\leq 0}_{m,n}(\mathbf{k}_1, \ldots, \mathbf{k}_m; z) \delta(\sum_{i=1}^{n} \sigma_i \mathbf{k}_i + 2n\mathbf{p}_L)$$

where $\sigma_i = \pm$, $|F(\mathbf{k})| \leq C|\lambda|$ and the kernels $W^{\leq 0}_{m,n}(\mathbf{k}_1, \ldots, \mathbf{k}_m; z)$ are C^∞ bounded functions such that $W_{m,n} = W_{m,-n}$ and $|W_{m,n}| \leq C^m z^{\max(2,m/2-1)}$ if $z = Max(|\lambda|, u, |\nu|)$; moreover $\delta(\mathbf{k}) = L\beta\delta_{k_0}\delta_k$.

2.2 We perform now the infrared integration

$$\int P(d\psi^{(\leq 0)}) e^{-\mathcal{V}^{(0)}(\psi^{(\leq 0)})} \tag{2.24}$$

It is convenient, for reasons which will be clear below, to split $\mathcal{V}$ in a *relevant* and *irrelevant* part $\mathcal{V} = \mathcal{L}\mathcal{V} + \mathcal{R}\mathcal{V}$ where $\mathcal{L}$, the *localization operator*, a linear operator defined in the following way:.

1) If $m > 4$ then

$$\mathcal{L}\{\frac{1}{(L\beta)^m} \sum_{\mathbf{k}'_1,\ldots,\mathbf{k}'_m \in \mathcal{D}_{L,\beta}} W^{\leq 0}_{m,n}(\mathbf{k}'_1+\omega_1 p_F, \ldots)[\prod_{i=1}^m \psi^{(\leq 0)\sigma_i}_{\mathbf{k}'_i+\omega_i p_F,\omega_i}]\delta(\sum_{i=1}^m \sigma_i(\mathbf{k}'_i+\omega_i p_F)+2np_L)\} = 0$$

2) If $m = 4$ then

$$\mathcal{L}\{\frac{1}{(L\beta)^4} \sum_{\mathbf{k}'_1,\ldots,\mathbf{k}'_4 \in \mathcal{D}_{L,\beta}} W^{\leq 0}_{4,n}(\mathbf{k}'_1+\omega_1 p_F, \ldots, \mathbf{k}'_4+\omega_4 p_F)$$

$$[\prod_{i=1}^4 \psi^{(\leq 0)\sigma_i}_{\mathbf{k}'_i+\omega_i p_F,\omega_i}]\delta(\sum_{i=1}^4 \sigma_i(\mathbf{k}'_i+\omega_i p_F)+2np_L)\}$$

$$= \delta_{\sum_{i=1}^4 \sigma_i\omega_i p_F+2np_L} \frac{1}{(L\beta)^4} \sum_{\mathbf{k}'_1,\ldots,\mathbf{k}'_4 \in \mathcal{D}_{L,\beta}} W^{\leq 0}_{m,n}(\omega_1 p_F, \ldots, \omega_4 p_F)[\prod_{i=1}^4 \psi^{(\leq 0)\sigma_i}_{\mathbf{k}'_i+\omega_i p_F,\omega_i}]\delta(\sum_{i=1}^4 \sigma_i \mathbf{k}'_i) \tag{2.25}$$

3)If $m = 2$ then

$$\mathcal{L}\{\frac{1}{(L\beta)^2} \sum_{\mathbf{k}'_1,\mathbf{k}'_2 \in \mathcal{D}_{L,\beta}} W^{\leq 0}_{2,n}(\mathbf{k}'_1+\omega_1 p_F, \mathbf{k}'_2+\omega_2 p_F)[\prod_{i=1}^2 \psi^{(\leq 0)\sigma_i}_{\mathbf{k}'_i+\omega_i p_F,\omega_i}]\delta(\sum_{i=1}^2 \sigma_i(\mathbf{k}'_i+\omega_i p_F)+2np_L)\}$$

$$= \delta_{(\omega_1-\omega_2)p_F+2np_L} \frac{1}{(L\beta)} \sum_{\mathbf{k}' \in \mathcal{D}_{L,\beta}} [W^{\leq 0}_{2,n}(\omega_1 p_F, \omega_2 p_F) + \tag{2.26}$$

$$+\omega_1 E(k'+\omega_1 p_F)\partial_k W^{\leq 0}_{2,n}(\omega_1 p_F, \omega_2 p_F) + k^0 \partial_{k_0} W^{\leq 0}_{2,n}(\omega_1 p_F, \omega_2 p_F)][\prod_{i=1}^2 \psi^{(\leq 0)\sigma_i}_{\mathbf{k}'_i+\omega_i p_F,\omega_i}]$$

where $E(k'+\omega p_F) = v_0\omega \sin k' + (1-\cos k')\cos p_F$ (the symbol $\partial_k, \partial_{k_0}$ means discrete derivativs).

We can write then the relevant part of the effective potential in the following way:

$$\mathcal{L}\mathcal{V}^{(0)} = n_0 F^{(\leq 0)}_\nu + s_0 F^{(\leq 0)}_\sigma + z_0 F^{(\leq 0)}_\zeta + a_0 F^{(\leq 0)}_\alpha + i_0 F^{(\leq 0)}_\iota + t_0 F^{(\leq 0)}_\tau + l_0 F^{(\leq 0)}_\lambda \tag{2.27}$$

where

$$F^{(\leq 0)}_\nu = \sum_\omega \frac{1}{(L\beta)} \sum_{\mathbf{k}' \in \mathcal{D}_{L,\beta}} \psi^{(\leq 0)+}_{\mathbf{k}'+\omega p_F,\omega} \psi^{(\leq 0)-}_{\mathbf{k}'+\omega p_F,\omega}$$

$$F^{(\leq 0)}_\sigma = \sum_\omega \frac{1}{(L\beta)} \sum_{\mathbf{k}' \in \mathcal{D}_{L,\beta}} \psi^{(\leq 0)+}_{\mathbf{k}'+\omega p_F,\omega} \psi^{(\leq 0)-}_{\mathbf{k}'-\omega p_F,-\omega}$$

$$F^{(\leq 0)}_\alpha = \sum_\omega \frac{1}{(L\beta)} \sum_{\mathbf{k}' \in \mathcal{D}_{L,\beta}} E(k'+\omega p_F) \psi^{(\leq 0)+}_{\mathbf{k}'+\omega p_F,\omega} \psi^{(\leq 0)-}_{\mathbf{k}'+\omega p_F,\omega}$$

$$F_\zeta^{(\le 0)} = \sum_\omega \frac{1}{(L\beta)} \sum_{\mathbf{k}'\in\mathcal{D}_{L,\beta}} (-ik_0)\psi^{(\le 0)+}_{\mathbf{k}'+\omega p_F,\omega}\psi^{(\le 0)-}_{\mathbf{k}'+\omega p_F,\omega}$$

$$F_\iota^{(\le 0)} = \sum_\omega \frac{1}{(L\beta)} \sum_{\mathbf{k}'\in\mathcal{D}_{L,\beta}} E(k'+\omega p_F)\psi^{(\le 0)+}_{\mathbf{k}'+\omega p_F,\omega}\psi^{(\le 0)-}_{\mathbf{k}'-\omega p_F,-\omega}$$

$$F_\tau^{(\le 0)} = \sum_\omega \frac{1}{(L\beta)} \sum_{\mathbf{k}'\in\mathcal{D}_{L,\beta}} (-ik_0)\psi^{(\le 0)+}_{\mathbf{k}'+\omega p_F,\omega}\psi^{(\le 0)-}_{\mathbf{k}'-\omega p_F,-\omega}$$

$$F_\lambda^{(\le 0)} = \frac{1}{(L\beta)^4} \sum_{\mathbf{k}'_1,\ldots,\mathbf{k}'_4\in\mathcal{D}_{L,\beta}} \psi^{(\le 0)+}_{\mathbf{k}'_1+p_F,1}\psi^{(\le 0)+}_{\mathbf{k}'_1-p_F,-1}\psi^{(\le 0)-}_{\mathbf{k}'_3+p_F,1}\psi^{(\le 0)-}_{\mathbf{k}'_4-p_F,-1}\delta(\sum_{i=1}^4 \sigma_i \mathbf{k}_i)$$

and $\lambda_0 = \lambda(\hat v(0) - \hat v(2p_F)) + O(\lambda^2)$, $s_0 = u + O(u\lambda)$, $t_0, i_0 = O(u\lambda)$, $a_0.z_0 = O(\lambda)$, $n_0 = \nu + O(\lambda)$.

We write (2.24) as

$$\int P_{Z_0}(d\psi^{\le 0})\, e^{-\mathcal{V}^{(0)}(\sqrt{Z_0}\psi^{\le 0})}\ , \tag{2.28}$$

where $Z_0 = 1$ and $P_{Z_0}(d\psi^{(\le 0)})$ is the Grassmanian integration with propagator

$$\begin{aligned} &g^{(\le 0)}(\mathbf{x};\mathbf{y}) = \sum_{\omega,\omega'=\pm1} \frac{1}{L\beta} \sum_{\mathbf{k}'\in\mathcal{D}_{L,\beta}} \frac{1}{Z_0} e^{-i\mathbf{k}'\cdot(\mathbf{x}-\mathbf{y})}\, e^{-i(\omega x - \omega' y)p_F}\ \tilde g^{(\le 0)}_{\omega,\omega'}(\mathbf{k}')\ ,\\ &\tilde g^{(\le 0)}_{\omega,\omega'}(\mathbf{k}') = \delta_{\omega,\omega'}\ \tilde g^{(\le 0)}_\omega(\mathbf{k}')\ , \end{aligned} \tag{2.29}$$

with

$$\tilde g^{(\le 0)}_\omega(\mathbf{k}') = \frac{C_0^{-1}(\mathbf{k}')}{-ik_0 + (1-\cos k')\cos p_F + v_0\omega\sin k'}\ , \qquad C_h^{-1}(\mathbf{k}') = \sum_{j=h_\beta}^{h} f_j(\mathbf{k}')\ , \tag{2.30}$$

see (2.22), (2.23).

We write

$$\int P_{Z_0}(d\psi^{(\le 0)})\, e^{-\mathcal{V}^0(\sqrt{Z_0}\psi^{(\le 0)})} = \frac{1}{\mathcal{N}_0}\int \tilde P_{Z_{-1}}(d\psi^{(\le 0)})\, e^{-\tilde{\mathcal{V}}^{(0)}(\sqrt{Z_0}\psi^{\le 0})}\ , \tag{2.31}$$

where $\mathcal{N}_0$ is a suitable constant and, again up to a constant,

$$\begin{aligned} &\tilde P_{Z_{-1}}(d\psi^{(\le 0)}) = \prod_{\mathbf{k}}\prod_{\omega=\pm1} d\psi^{(\le 0)+}_{\mathbf{k}'+\omega\mathbf{p}_F,\omega} d\psi^{(\le 0)-}_{\mathbf{k}'+\omega\mathbf{p}_F,\omega}\\ &\exp\Big\{-\sum_{\omega=\pm1}\frac{1}{L\beta}\sum_{\mathbf{k}'\in\mathcal{D}_{L,\beta}} C_0(\mathbf{k}')Z_{-1}(\mathbf{k}')\Big[\Big(-ik_0 - (\cos k'-1)\cos p_F + \omega v_0 \sin k'\Big)\\ &\psi^{(\le 0)+}_{\mathbf{k}'+\omega\mathbf{p}_F,\omega}\psi^{(\le 0)-}_{\mathbf{k}'+\omega\mathbf{p}_F,\omega} + \sigma_{-1}(\mathbf{k}')\,\psi^{(\le 0)+}_{\mathbf{k}'+\omega\mathbf{p}_F,\omega}\psi^{(\le 0)-}_{\mathbf{k}'-\omega\mathbf{p}_F,-\omega}\Big]\Big\}\ , \end{aligned} \tag{2.32}$$

with $Z_{-1}(\mathbf{k}')\sigma_{-1}(\mathbf{k}') = C_0^{-1}(\mathbf{k}')\, s_0$, $Z_{-1}(\mathbf{k}') = Z_0(1 + C_0^{-1}(\mathbf{k}')z_0)$ and $\tilde{\mathcal{V}}^{(0)} = \mathcal{L}\tilde{\mathcal{V}}^{(0)} + (1-\mathcal{L})\mathcal{V}^{(0)}$, if

$$\mathcal{L}\tilde{\mathcal{V}}^{(0)}(\sqrt{Z_0}\psi) = n_0 F_\nu^{(\le 0)} + (a_0 - z_0)F_\alpha^{(\le 0)} + i_0 F_\iota^{(\le 0)} + t_0 F_\tau^{(\le 0)} + l_0 F_\lambda^{(\le 0)}\ . \tag{2.33}$$

The r.h.s of (2.31) can be written as

$$\frac{1}{\mathcal{N}_0}\int P_{Z_{-1}}(d\psi^{(\le -1)})\int \tilde P_{Z_{-1}}(d\psi^{(0)})\, e^{-\tilde{\mathcal{V}}^{(0)}(\sqrt{Z_0}\psi^{(\le 0)})}\ , \tag{2.34}$$

where $P_{Z_{-1}}(d\psi^{(\le -1)})$ and $\tilde P_{Z_{-1}}(d\psi^{(0)})$ are given by (2.32) with $Z_{-1}(\mathbf{k}')$ replaced by

$Z_{-1}(0) \equiv Z_{-1}$ and $C_0(\mathbf{k}')$ replaced with $C_{-1}(\mathbf{k}')$ and $\tilde f_0^{-1}(\mathbf{k}')$ respectively, if

$$\tilde f_0(\mathbf{k}') = Z_{-1}[\frac{C_0^{-1}(\mathbf{k}')}{Z_{-1}(\mathbf{k}')} - \frac{C_{-1}^{-1}(\mathbf{k}')}{Z_{-1}}]$$

and $\psi^{(\le 0)}$ replaced with $\psi^{(\le -1)}$ and $\psi^{(0)}$ respectively.

The Grassmanian integration $\tilde P_{Z_{-1}}(d\psi^{(0)})$ has propagator

$$g^{(0)}(\mathbf{x};\mathbf{y}) = \sum_{\omega,\omega'=\pm 1} e^{-i(\omega x - \omega' y)p_F}\, g^{(0)}_{\omega,\omega'}(\mathbf{x};\mathbf{y})\ , \tag{2.35}$$

if

$$g^{(0)}_{\omega,\omega'}(\mathbf{x};\mathbf{y}) \equiv \int \tilde P_{Z_{-1}}(d\psi^{(0)})\, \psi^{(0)-}_{\mathbf{x},\omega}\psi^{(0)+}_{\mathbf{y},\omega'} \tag{2.36}$$

is given by

$$g^{(0)}_{\omega,\omega'}(\mathbf{x};\mathbf{y}) = \frac{1}{L\beta}\sum_{\mathbf{k}'\in\mathcal{D}_{L,\beta}} e^{-i\mathbf{k}'\cdot(\mathbf{x}-\mathbf{y})}\frac{\tilde f_0(\mathbf{k}')}{Z_{-1}}[T_0^{-1}(\mathbf{k}')]_{\omega,\omega'}\ , \tag{2.37}$$

where the 2×2 matrix $T_0(\mathbf{k}')$ has elements

$$\begin{cases} [T_0(\mathbf{k}')]_{1,1} = (-ik_0 - (\cos k' - 1)\cos p_F + v_0 \sin k')\ , \\ [T_0(\mathbf{k}')]_{1,2} = [T_0(\mathbf{k}')]_{2,1} = \sigma_{-1}(\mathbf{k}')\ , \\ [T_0(\mathbf{k}')]_{2,2} = (-ik_0 - (\cos k' - 1)\cos p_F - v_0 \sin k')\ , \end{cases} \tag{2.38}$$

which is well defined on the support of $f_0(\mathbf{k}')$, so that, if we set

$$A_0(\mathbf{k}') = \det T_0(\mathbf{k}') = [-ik_0 - (\cos k' - 1)\cos p_F]^2 - (v_0\sin k')^2 - [\sigma_{-1}(\mathbf{k}')]^2\ , \tag{2.39}$$

then

$$T_0^{-1}(\mathbf{k}') = \frac{1}{A_0(\mathbf{k}')}\begin{pmatrix} [\tau_0(\mathbf{k}')]_{1,1} & [\tau_0(\mathbf{k}')]_{1,2} \\ [\tau_0(\mathbf{k}')]_{2,1} & [\tau_0(\mathbf{k}')]_{2,2} \end{pmatrix}\ , \tag{2.40}$$

with

$$\begin{cases} [\tau_0(\mathbf{k}')]_{1,1} = [-ik_0 - (\cos k' - 1)\cos p_F - v_0 \sin k']\ , \\ [\tau_0(\mathbf{k}')]_{1,2} = [\tau_0(\mathbf{k}')]_{2,1} = -\sigma_{-1}(\mathbf{k}')\ , \\ [\tau_0(\mathbf{k}')]_{2,2} = [-ik_0 - (\cos k' - 1)\cos p_F + v_0 \sin k']\ . \end{cases} \tag{2.41}$$

We *rescale* the fields so that

$$\frac{1}{\mathcal{N}_0}\int P_{Z_{-1}}(d\psi^{(\le -1)})\int \tilde P_{Z_{-1}}(d\psi^{(0)})\, e^{-\hat{\mathcal{V}}^{(0)}(\sqrt{Z_{-1}}\psi^{(\le 0)})}$$

so that, if the operator $\mathcal{L}$ is defined as in (2.25),(2.26)

$$\mathcal{L}\hat{\mathcal{V}}^{(0)}(\psi) = \nu_0 F_\nu^{(\le 0)} + \delta_0 F_\alpha^{(\le 0)} + \iota_0 F_\iota^{(\le 0)} + \tau_0 F_\tau^{(\le 0)} + \lambda_0 F_\lambda^{(\le 0)}\ . \tag{2.42}$$

where by definition

$$\nu_o = \frac{Z_0}{Z_{-1}}n_0;\quad \delta_0 = \frac{Z_0}{Z_{-1}}(a_0 - z_0);\quad \tau_0 = \frac{Z_0}{Z_{-1}}t_0\quad \iota_0 = \frac{Z_0}{Z_{-1}}i_0;\quad \lambda_0 = (\frac{Z_0}{Z_{-1}})^2 l_0$$

Then we perform the integration

$$\int \tilde P_{Z_{-1}}(d\psi^{(0)})\, e^{-\hat{\mathcal{V}}^{(0)}(\sqrt{Z_{-1}}\psi^{(\le 0)})} = e^{-\mathcal{V}^{-1}(\sqrt{Z_{-1}}\psi^{(\le -1)}) + \tilde E_0}\ , \tag{2.43}$$

where $\tilde E_0$ is a suitable constant and

$$\mathcal{L}\mathcal{V}^{(-1)}(\psi) = \gamma^{-1}\nu_{-1}F_\nu^{(-1)} + s_{-1}F_\sigma^{(-1)} + a_{-1}F_\alpha^{(\le -1)} + z_{-1}F_\zeta^{(\le -1)} +$$

$$i_{-1}F_\iota^{(\le -1)}+\tau_{-1}F_\tau^{(\le -1)}+l_{-1}F_\lambda^{(\le -1)} \tag{2.44}$$

2.3 The procedure can be iterated, and at each step one has to perform the integration

$$\int P_{Z_h}(d\psi^{(\le h)})\, e^{-\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\le h)})} \tag{2.45}$$

with

$$P_{Z_h}(d\psi^{(\le h)})=\prod_{\mathbf{k}}\prod_{\omega=\pm1} d\psi^{(\le h)+}_{\mathbf{k}'+\omega\mathbf{p}_F,\omega}d\psi^{(\le h)-}_{\mathbf{k}'+\omega\mathbf{p}_F,\omega}$$
$$\exp\Big\{-\sum_{\omega=\pm1}\frac{1}{L\beta}\sum_{\mathbf{k}'\in\mathcal{D}_{L,\beta}} C_0(\mathbf{k}')Z_h\Big[\Big(-ik_0-(\cos k'-1)\cos p_F+\omega v_0\sin k'\Big)$$
$$\psi^{(\le 0)+}_{\mathbf{k}'+\omega\mathbf{p}_F,\omega}\psi^{(\le 0)-}_{\mathbf{k}'+\omega\mathbf{p}_F,\omega}+\sigma_h(\mathbf{k}')\,\psi^{(\le 0)+}_{\mathbf{k}'+\omega\mathbf{p}_F,\omega}\psi^{(\le 0)-}_{\mathbf{k}'-\omega\mathbf{p}_F,-\omega}\Big]\Big\}\;, \tag{2.46}$$

Moreover, with notations analogues to (2.25),(2.26) (2.47) we can write

$$\mathcal{L}\mathcal{V}^{(h)}(\psi)=\gamma^h n_h F_\nu^{(\le h)}+s_hF_\sigma^{(\le h)}+z_hF_\zeta^{(\le h)}+a_hF_\alpha^{(\le h)}+i_hF_\iota^{(\le h)}+t_hF_\tau^{(\le h)}+l_hF_\lambda^{(\le h)}\;. \tag{2.48}$$

We write

$$\int P_{Z_h}(d\psi^{(\le h)})\, e^{-\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\le h)})}=\int \tilde P_{Z_{h-1}}(d\psi^{(\le h)})\, e^{-\tilde{\mathcal{V}}^{(h)}(\sqrt{Z_h}\psi^{(\le h)})} \tag{2.49}$$

where

$$\tilde P_{Z_{h-1}}(d\psi^{(\le h)})=\prod_{\mathbf{k}}\prod_{\omega=\pm1} d\psi^{(\le h)+}_{\mathbf{k}'+\omega\mathbf{p}_F,\omega}d\psi^{(\le h)-}_{\mathbf{k}'+\omega\mathbf{p}_F,\omega}$$
$$\exp\Big\{-\sum_{\omega=\pm1}\frac{1}{L\beta}\sum_{\mathbf{k}'\in\mathcal{D}_{L,\beta}} C_0(\mathbf{k}')Z_{h-1}(\mathbf{k}')\Big[\Big(-ik_0-(\cos k'-1)\cos p_F+\omega v_0\sin k'\Big)$$
$$\psi^{(\le 0)+}_{\mathbf{k}'+\omega\mathbf{p}_F,\omega}\psi^{(\le 0)-}_{\mathbf{k}'+\omega\mathbf{p}_F,\omega}+\sigma_{h-1}(\mathbf{k}')\,\psi^{(\le 0)+}_{\mathbf{k}'+\omega\mathbf{p}_F,\omega}\psi^{(\le 0)-}_{\mathbf{k}'-\omega\mathbf{p}_F,-\omega}\Big]\Big\}\;, \tag{2.50}$$

with $Z_{h-1}(\mathbf{k}')=Z_h(1+C_h^{-1}(\mathbf{k}')z_h)$, $Z_{h-1}(\mathbf{k}')\sigma_{h-1}(\mathbf{k}')=Z_h(\sigma_h(\mathbf{k}')+C_h^{-1}(\mathbf{k}')s_h)$ and $\tilde{\mathcal{V}}=\mathcal{L}\tilde{\mathcal{V}}+(1-\mathcal{L})\mathcal{V}$ with

$$\mathcal{L}\tilde{\mathcal{V}}^{(h)}(\psi)=\gamma^h n_h F_\nu^{(\le h)}+(a_h-z_h)F_\alpha^{(\le h)}+i_hF_\iota^{(\le h)}+t_hF_\tau^{(\le h)}+l_hF_\lambda^{(\le h)}\;. \tag{2.51}$$

The r.h.s of (2.49) can be written as

$$\frac{1}{\mathcal{N}_h}\int P_{Z_{h-1}}(d\psi^{(\le h-1)})\int \tilde P_{Z_{h-1}}(d\psi^{(h)})\, e^{-\tilde{\mathcal{V}}^{(h)}(\sqrt{Z_h}\psi^{(\le h)})}\;, \tag{2.52}$$

where $P_{Z_{h-1}}(d\psi^{(\le h-1)})$ and $\tilde P_{Z_{h-1}}(d\psi^{(h)})$ are given by (2.50) with $Z_{h-1}(\mathbf{k}')$ replaced by $Z_{h-1}(0)\equiv Z_{h-1}$ and $C_h(\mathbf{k}')$ replaced with $C_{h-1}(\mathbf{k}')$ and $\tilde f_h^{-1}(\mathbf{k}')$ respectively, if

$$\tilde f_h(\mathbf{k}')=Z_{h-1}\Big[\frac{C_h^{-1}(\mathbf{k}')}{Z_{h-1}(\mathbf{k}')}-\frac{C_{h-1}^{-1}(\mathbf{k}')}{Z_{h-1}}\Big]$$

and $\psi^{(\le h)}$ replaced with $\psi^{(\le h-1)}$ and $\psi^{(h)}$ respectively. Note that $\tilde f_h(\mathbf{k}')$ is a compact support function, with support $O(\gamma^h)$.

The Grassmanian integration $\tilde P_{Z_{h-1}}(d\psi^{(h)})$ has propagator

$$\frac{g^{(h)}(\mathbf{x};\mathbf{y})}{Z_{h-1}}=\sum_{\omega,\omega'=\pm1} e^{-i(\omega x-\omega' y)p_F}\,\frac{g^{(h)}_{\omega,\omega'}(\mathbf{x};\mathbf{y})}{Z_{h-1}}\;, \tag{2.53}$$

if

$$g^{(h)}_{\omega,\omega'}(\mathbf{x};\mathbf{y}) \equiv \int \tilde P_{Z_{h-1}}(d\psi^{(h)})\, \psi^{(h)-}_{\mathbf{x},\omega}\psi^{(h)+}_{\mathbf{y},\omega'} \tag{2.54}$$

is given by

$$g^{(h)}_{\omega,\omega'}(\mathbf{x};\mathbf{y}) = \frac{1}{L\beta}\sum_{\mathbf{k}'\in\mathcal{D}_{L,\beta}} e^{-i\mathbf{k}'\cdot(\mathbf{x}-\mathbf{y})}\tilde f_h(\mathbf{k}')[T_h^{-1}(\mathbf{k}')]_{\omega,\omega'}\ , \tag{2.55}$$

where the 2×2 matrix $T_h(\mathbf{k}')$ has elements

$$\begin{cases} [T_h(\mathbf{k}')]_{1,1} = (-ik_0 - (\cos k' - 1)\cos p_F + v_0 \sin k')\ ,\\ [T_h(\mathbf{k}')]_{1,2} = [T_h(\mathbf{k}')]_{2,1} = \sigma_{h-1}(\mathbf{k}')\ ,\\ [T_h(\mathbf{k}')]_{2,2} = (-ik_0 - (\cos k' - 1)\cos p_F - v_0 \sin k')\ , \end{cases} \tag{2.56}$$

which is well defined on the support of $f_0(\mathbf{k}')$, so that, if we set

$$A_h(\mathbf{k}') = \det T_h(\mathbf{k}') = [-ik_0 - (\cos k' - 1)\cos p_F]^2 - (v_0\sin k')^2 - [\sigma_{h-1}(\mathbf{k}')]^2\ , \tag{2.57}$$

then

$$T_h^{-1}(\mathbf{k}') = \frac{1}{A_h(\mathbf{k}')}\begin{pmatrix} [\tau_h(\mathbf{k}')]_{1,1} & [\tau_h(\mathbf{k}')]_{1,2} \\ [\tau_h(\mathbf{k}')]_{2,1} & [\tau_h(\mathbf{k}')]_{2,2}\end{pmatrix}\ , \tag{2.58}$$

with

$$\begin{cases} [\tau_h(\mathbf{k}')]_{1,1} = [-ik_0 - (\cos k' - 1)\cos p_F - v_0 \sin k']\ ,\\ [\tau_h(\mathbf{k}')]_{1,2} = [\tau_h(\mathbf{k}')]_{2,1} = -\sigma_{h-1}(\mathbf{k}')\ ,\\ [\tau_h(\mathbf{k}')]_{2,2} = [-ik_0 - (\cos k' - 1)\cos p_F + v_0 \sin k']\ . \end{cases} \tag{2.59}$$

Note that $\sigma_h(\mathbf{k}')$ is a smooth function on $T^1\times R$ and, if $\mathbf{k}'$ varies in the support of $C_h^{-1}(\mathbf{k}')$ then there exists two positive constants c_1, c_2 such that:

$$c_1\sigma_h \le \sigma_h(\mathbf{k}') \le c_2\sigma_h \tag{2.60}$$

if $\sigma_h \equiv \sigma_h(0)$, as $C_k^{-1}(\mathbf{k}') = 1$ for $k\ge h+1$.

The large distance behaviour of the propagator (2.53) is given by the following lemma, see [BM2]:

2.4LEMMA *The propagator $g^h_{\omega,\omega'}(\mathbf{x}-\mathbf{y})$ can be written as:*

$$g^h_{\omega,\omega}(\mathbf{x}-\mathbf{y}) = g^h_{L;\omega}(\mathbf{x}-\mathbf{y}) + C_1(\mathbf{x}-\mathbf{y}) + C_2(\mathbf{x}-\mathbf{y}) \tag{2.61}$$

with

$$g^h_{L;\omega}(\mathbf{x}-\mathbf{y}) = \int d\mathbf{k}\frac{e^{i\mathbf{k}\mathbf{x}}}{-ik_0 + \omega v_0 \mathbf{k}'}\tilde f_h(\mathbf{k}')$$

For any integer $N>1$ and for $|x-y|\le \frac{L}{2}$, $|x_0-y_0|\le\frac{\beta}{2}$ it holds $|C_1(\mathbf{x}-\mathbf{y})| \le \frac{\gamma^{2h}C_N}{1+(\gamma^h(\mathbf{x}-\mathbf{y}))^N}$ and $|C_2(\mathbf{x}-\mathbf{y})|\le |\frac{\sigma^h}{\gamma^h}|^2\frac{\gamma^h C_N}{1+(\gamma^h(\mathbf{x}-\mathbf{y}))^N}$.

Moreover

$$|g^h_{\omega,-\omega}(\mathbf{x}-\mathbf{y})| \le |\frac{\sigma^h}{\gamma^h}|\frac{\gamma^h C_N}{1+(\gamma^h(\mathbf{x}-\mathbf{y}))^N}$$

Note that $g^h_{L;\omega}(\mathbf{x}-\mathbf{y})$ coincides with the propagator "at scale γ^h" of the Luttinger model, see [BeGM]. This remark will be crucial for studying the Renormalization group flow, see sec.4.

2.5 We *rescale* the fields so that

$$\frac{1}{\mathcal{N}_h}\int P_{Z_{h-1}}(d\psi^{(\le h-1)})\int \tilde P_{Z_{h-1}}(d\psi^{(h)})\, e^{-\hat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)})}$$

so that

$$\mathcal{L}\hat{\mathcal{V}}^{(h)}(\psi)=\gamma^h\nu_h F_\nu^{(\le h)}+\delta_h F_\alpha^{(\le h)}+\iota_h F_\iota^{(\le h)}+\tau_h F_\tau^{(\le h)}+\lambda_h F_\lambda^{(\le h)}\,. \tag{2.62}$$

where by definition

$$\nu_h=\frac{Z_h}{Z_{h-1}}n_0;\quad \delta_h=\frac{Z_h}{Z_{h-1}}(a_h-z_h);\quad \tau_h=\frac{Z_h}{Z_{h-1}}t_h;\quad \iota_h=\frac{Z_h}{Z_{h-1}}i_0;\quad \lambda_h=(\frac{Z_h}{Z_{h-1}})^2 l_h$$

We call $\vec{v}_h=(\nu_h,\delta_h,\tau_h,\iota_h,\lambda_h)$.

We perform the integration

$$\int \tilde{P}_{Z_{h-1}}(d\psi^{(h)})\,e^{-\hat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)})}=e^{-\mathcal{V}^{h-1}(\sqrt{Z_{h-1}}\psi^{(\le h-1)})+\tilde{E}_h}\,, \tag{2.63}$$

where $\tilde{E}_h$ is a suitable constant and

$$\mathcal{L}\mathcal{V}^{(h-1)}(\psi)=\gamma^{h-1}n_{h-1}F_\nu^{(h-1)}+s_{h-1}F_\sigma^{(h-1)}+a_{h-1}F_\alpha^{(\le h-1)}+$$

$$z_{h-1}F_\zeta^{(\le h-1)}+i_{h-1}F_\iota^{(\le h-1)}+\tau_{h-1}F_\tau^{(\le h-1)}+l_{h-1}F_\lambda^{(\le -1)}\,, \tag{2.64}$$

Note that the above procedure allows us to write the running coupling constants $\vec{v}_h$ in terms of $\vec{v}_k$, $k\ge h-1$

$$\vec{v}_h=\vec{\beta}(\vec{v}_{h+1},...,\vec{v}_0) \tag{2.65}$$

The function $\vec{\beta}(\vec{v}_{h+1},...,\vec{v}_0)$ is called *Beta function.*

The *effective potential* $V^h(\psi)$ is a sum of terms of the form

$$\frac{1}{(L\beta)^m}\sum_{\mathbf{k}'_1,\dots,\mathbf{k}'_m\in\mathcal{D}_{L,\beta}}\delta(\sum_{i=1}^n\sigma_i(\mathbf{k}'_i+\omega_i p_F)+2mp)\hat{W}^{\le h}_{n,m}(\mathbf{k}'_1,..,\mathbf{k}'_n;\{\omega\})\prod_{i=1}^n\psi^{\sigma_i(\le h)}_{\mathbf{k}'_1+\omega_i p_F,\omega_i} \tag{2.66}$$

where $\hat{W}^h_{n,m}(\mathbf{k}'_1,..,\mathbf{k}'_n;\{\omega\})=W^h_{n,m}(\mathbf{k}'_1+\omega_1 p_F,..)$.

The following lemmas will be useful in the future.

2.6 LEMMA *Assume that $\sum_{i=1}^n\sigma_i\omega_i p_F+2mp_L\ne 0$ and $m\ne 0$. Then (2.66) is identically vanishing unless*

$$|m|\ge C_1[\gamma^{\frac{-h}{\tau}}-n] \tag{2.67}$$

PROOF Remembering the compact support of the Grassmanian operator $\psi^{\sigma_i(\le h)}_{\mathbf{k}'_i+\omega_i p_F,\omega_i}$ we can write, using (1.4)

$$a_0 n\gamma^h\ge|\sum_{i=1}^n\sigma_i\mathbf{k}'_i|\ge|2mp_L+\sum_{i=1}^n\sigma_i\omega_i p_F|_{T^1}\ge C_0(2|m|p_L+np_F)^{-\tau}$$

from which (2.67) holds. ■

2.7 LEMMA *Assume that $\sum_{i=1}^n\sigma_i\omega_i p_F+2mp_L\ne 0$ and $m=0$. Then (2.66) is identically vanishing unless $h\ge\bar{h}$, if $\bar{h}$ is a suitable constant.*

PROOF Using again compact support of the Grassmanian operator we can write

$$a_0 n\gamma^h\ge|\sum_{i=1}^n\sigma_i\omega_i p_F|_{T^1}\ge 4p_F \qquad ■$$

Let be

$$h^* = \inf\{h \geq h_\beta : a_0\gamma^{h+1} \geq 4|\sigma_h|\} \tag{2.68}$$

¿From Lemma 2.4 it follows trivially:

2.8 LEMMA *For $h > h^*$ for any integer $N > 1$ it is possible to find a C_N such that, for $|x-y| \leq \frac{L}{2}$, $|x_0 - y_0| \leq \frac{\beta}{2}$*

$$|g^h_{\omega,\omega'}(\mathbf{x}-\mathbf{y})| \leq \frac{C_N\gamma^h}{1+(\gamma^h|\mathbf{x}-\mathbf{y}|)^N}$$

2.9 In the next section we will see that, using the above lemmas and assuming that the running coupling constants are bounded, the integrations (2.52) are well defined for $0 \geq h \geq h^*$.

The integration of the scale from h^* to h_β can be performed "in a single step"

$$\int P_{Z_{h^*}}(d\psi^{(\leq h^*)})e^{-\mathcal{V}^{h^*}(\sqrt{Z_{h^*}}\psi^{(\leq h^*)}} = \int \tilde{P}_{Z_{h^*-1}}(d\psi^{(\leq h^*)})e^{-\tilde{\mathcal{V}}^{h^*}(\sqrt{Z_{h^*}}\psi^{(\leq h^*)}} \tag{2.69}$$

Calling $\int \tilde{P}_{Z_{h^*-1}}(d\psi^{(\leq h^*)})\psi^{+(\leq h^*)}_{\mathbf{x}}\psi^{-(\leq h^*)}_{\mathbf{y}} = \frac{g^{(\leq h^*)}(\mathbf{x}-\mathbf{y})}{Z_{h^*-1}}$, we prove in the next section that the integration in the r.h.s. in (2.69) is well defined. This will be proved by using the following lemma.

2.10LEMMA *Assume that h^* is finite uniformly in L,β so that $|\frac{\sigma_{h^*-1}}{\gamma^{h^*}}| \geq \kappa$, if κ is a constant. Then it is possible to find a constant C_N such that, for $|x-y| \leq \frac{L}{2}$, $|x_0-y_0| \leq \frac{\beta}{2}$*

$$|g^{\leq h^*}_{\omega,\omega'}(\mathbf{x}-\mathbf{y})| \leq \frac{C_N\gamma^{h^*}}{1+(\gamma^{h^*}|\mathbf{x}-\mathbf{y}|)^N}$$

Comparing Lemma 2.8 and Lemma 2.10 we see that the propagator of the integration of all the scale between h^* and h_β has the same bound as the propagator of the integration of a single scale greater than h^*.

3. Analiticity of the effective potential

3.1 We found convenient in order to discuss the analiticity of the effective potential $\mathcal{V}^h$ to pass to the coordinate rapresentation.

We use then the fields

$$\psi^{(\leq h)\sigma}_{\mathbf{x},\omega} = \frac{1}{L\beta}\sum_{k' \in \mathcal{D}_{L,\beta}} e^{-i\sigma\mathbf{k}'\mathbf{x}}\psi^{(\leq h)\sigma}_{\mathbf{k}'+\omega p_F,\omega}$$

and we can write the analogue of (2.25) (2.26) in coordinate space (we write the action of $\mathcal{R} = 1 - \mathcal{L}$):

1)if $n>4$

$$\mathcal{R}\{\sum_{\mathbf{x}_1,\ldots,\mathbf{x}_m\in\Lambda}\prod_{i=1}^n \psi^{(\le h)\sigma_i}_{\mathbf{x}_i,\omega_i} W^{\le h}_{n,m}\}=\sum_{\mathbf{x}_1,\ldots,\mathbf{x}_m\in\Lambda}\prod_{i=1}^n \psi^{(\le h)\sigma_i}_{\mathbf{x}_i,\omega_i} W^{\le h}_{n,m}$$

2) if $n=4$ and $\sum_{i=1}^4 \sigma_i\omega_i p_F+2np_L\neq 0$ one has $\mathcal{R}=I$ otherwise

$$\mathcal{R}\sum_{\mathbf{x}_1,\ldots,\mathbf{x}_4\in\Lambda}\prod_{i=1}^4 \psi^{(\le h)\sigma_i}_{\mathbf{x}_i,\omega_i} W^{\le h}_{4,m}(\mathbf{x}_1-\mathbf{x}_4,\mathbf{x}_2-\mathbf{x}_4,\mathbf{x}_3-\mathbf{x}_4;\{\omega\})= \tag{3.1}$$

$$\sum_{\mathbf{x}_1,\ldots,\mathbf{x}_4\in\Lambda}\prod_{i=1}^4 \psi^{(\le h)\sigma_i}_{\mathbf{x}_i,\omega_i} W^{\le h}_{4,m}(\mathbf{x}_1-\mathbf{x}_4,\mathbf{x}_2-\mathbf{x}_4,\mathbf{x}_3-\mathbf{x}_4;\{\omega\})-$$

$$-\delta(x_1-x_2)\delta(x_2-x_3)\delta(x_3-x_4)\sum_{\mathbf{t}_1,\mathbf{t}_2,\mathbf{t}_3\in\Lambda} W^{\le h}_{4,m}(\mathbf{t}_1,\mathbf{t}_2,\mathbf{t}_3;\{\omega\})\}$$

where we have used that if $\sum_{i=1}^4 \sigma_i\omega_i p_F+2np_L=0$ the kernels $W^{\le h}_{4,m}$ are *traslation invariant.*

3)in $n=2$ and $\sum_{i=1}^2 \sigma_i\omega_i p_F+2np_L\neq 0$ then $\mathcal{R}=I$ otherwise

$$\mathcal{R}\{\sum_{\mathbf{x}_1,\mathbf{x}_2\in\Lambda}\prod_{i=1}^2 \psi^{(\le h)\sigma_i}_{\mathbf{x}_i,\omega_i} W^{\le h}_{2,m}(\mathbf{x}_1-\mathbf{x}_2;\{\omega\})\}=\sum_{\mathbf{x}_1,\mathbf{x}_2\in\Lambda}\prod_{i=1}^2 \psi^{(\le h)\sigma_i}_{\mathbf{x}_i,\omega_i}[W^{\le h}_{2,0}(\mathbf{x}_1-\mathbf{x}_2;\{\omega\})-$$

$$\delta(\mathbf{x}_1-\mathbf{x}_2)\sum_{\mathbf{t}\in\Lambda} W^{\le h}_{2,m}(\mathbf{t};\{\omega\})+ \tag{3.2}$$

$$\partial_{x_2^0}\delta(\mathbf{x}_1-\mathbf{x}_2)\sum_{\mathbf{t}\in\Lambda} t_0 W^{\le h}_{2,m}(\mathbf{t};\{\omega\})+\partial_{x_2}\delta(\mathbf{x}_1-\mathbf{x}_2)\sum_{\mathbf{t}\in\Lambda}\omega t W^{\le h}_{2,m}(\mathbf{t};\{\omega\})]+$$

$$\sum_{\mathbf{x}_1\in\Lambda}\psi^{(\le h)+}_{\mathbf{x},\omega_1}\Delta\psi^{(\le h)-}_{\mathbf{x},\omega_2}\sum_{\mathbf{t}\in\Lambda}\omega t W^{\le h}_{2,m}(\mathbf{t};\{\omega\})]$$

and again we have used that when the $\mathcal{R}$ operation acts the kernels are traslation invariant; moreover if $\omega E(k'+\omega p_F)=2p_F k'+\bar f(k')$ we have that $\Delta\psi^{(\le h)\sigma}_{\mathbf{x},\omega}=\frac{1}{L\beta}\sum_{k'\in\mathcal{D}_{L,\beta}} e^{-i\sigma\mathbf{k}'\mathbf{x}}\bar f(k')\psi^{(\le h)\sigma}_{\mathbf{k}'+\omega p_F,\omega}$.

3.2 The following identities will be useful in the following

$$(\mathbf{x}_i-\mathbf{x}_j)^a\mathcal{R}\{\sum_{\mathbf{x}_1,\ldots,\mathbf{x}_4\in\Lambda}\prod_{i=1}^4 \psi^{(\le h)\sigma_i}_{\mathbf{x}_i,\omega_i}(W^{\le h}_{4,0}(\mathbf{x}_1-\mathbf{x}_4,\mathbf{x}_2-\mathbf{x}_4,\mathbf{x}_3-\mathbf{x}_4;\{\omega\})\}= \tag{3.3}$$

$$(\mathbf{x}_i-\mathbf{x}_j)^a\int\sum_{\mathbf{x}_1,\ldots,\mathbf{x}_4\in\Lambda}\prod_{i=1}^4 \psi^{(\le h)\sigma_i}_{\mathbf{x}_i,\omega_i}(W^{\le h}_{4,0}(\mathbf{x}_1-\mathbf{x}_4,\mathbf{x}_2-\mathbf{x}_4,\mathbf{x}_3-\mathbf{x}_4;\{\omega\})$$

if a is any integer ≥ 1 and i,j can take any value between 1 and 4.

$$(\mathbf{x}_1-\mathbf{x}_2)^a\mathcal{R}\{\int\sum_{\mathbf{x}_1,\mathbf{x}_2\in\Lambda}\prod_{i=1}^2 \psi^{(\le h)\sigma_i}_{\mathbf{x}_i,\omega_i} W^{\le h}_{2,0}(\mathbf{x}_1-\mathbf{x}_2;\{\omega\})\}=, \tag{3.4}$$

$$\sum_{\mathbf{x}_1,\mathbf{x}_2\in\Lambda}\prod_{i=1}^{2}\psi^{(\le h)\sigma_i}_{\mathbf{x}_i,\omega_i}[(\mathbf{x}_1-\mathbf{x}_2)^a W^h_{2,0}(\mathbf{x}_1-\mathbf{x}_2;\{\omega\})]+$$

$$(\mathbf{x}_1-\mathbf{x}_2)^a\psi^{(\le h)+}_{\mathbf{x},\omega_1}\Delta\psi^{(\le h)-}_{\mathbf{x},\omega_2}\sum_{\mathbf{t}\in\Lambda}\omega t W^{\le h}_{2,0}(\mathbf{t};\{\omega\})$$

if a is any integer ≥ 2. Finally

$$(x_1-x_2)\mathcal{R}\{\sum_{\mathbf{x}_1,\mathbf{x}_2\in\Lambda}\prod_{i=1}^{2}\psi^{(\le h)\sigma_i}_{\mathbf{x}_i,\omega_i}W^{\le h}_{2,0}(\mathbf{x}_1-\mathbf{x}_2;\{\omega\})\}= \tag{3.5}$$

$$\sum_{\mathbf{x}_1,\mathbf{x}_2\in\Lambda}\prod_{i=1}^{2}\psi^{(\le h)\sigma_i}_{\mathbf{x}_i,\omega_i}[(x_1-x_2)W^h_{2,0}(\mathbf{x}_1-\mathbf{x}_2;\{\omega\})-\delta(\mathbf{x}_1-\mathbf{x}_2)\sum_{\mathbf{t}\in\Lambda}tW^{\le h}(\mathbf{t};\{\omega\})]$$

$$-(x_1-x_2)\psi^{(\le h)+}_{\mathbf{x},\omega_1}\Delta\psi^{(\le h)-}_{\mathbf{x},\omega_2}\sum_{\mathbf{t}\in\Lambda}\omega t W^h_{2,0}(\mathbf{t};\{\omega\})$$

and a similar equation if (x_1-x_2) is replaced by $(x_1^0-x_2^0)$.

Of course one can integrate the δ functions in (3.1) (3.2) so obtaining

$$\sum_{\mathbf{x}_1,\dots,\mathbf{x}_4\in\Lambda}\prod_{i=1}^{4}\psi^{(\le h)\sigma_i}_{\mathbf{x}_1,\omega_i}\mathcal{R}(W^{\le h}_{4,m}(\mathbf{x}_1-\mathbf{x}_4,\mathbf{x}_2-\mathbf{x}_4,\mathbf{x}_3-\mathbf{x}_4;\{\omega\})=, \tag{3.6}$$

$$\sum_{\mathbf{x}_1,\dots,\mathbf{x}_4\in\Lambda}W^{\le h}_{4,m}(\mathbf{x}_1-\mathbf{x}_2,\mathbf{x}_2-\mathbf{x}_4,\mathbf{x}_3-\mathbf{x}_4;\{\omega\})$$

$$[\psi^{(\le h)\sigma_1}_{\mathbf{x}_1,\omega_1}\psi^{(\le h)\sigma_2}_{\mathbf{x}_2,\omega_2}\psi^{(\le h)\sigma_3}_{\mathbf{x}_3,\omega_3}\psi^{(\le h)\sigma_4}_{\mathbf{x}_4,\omega_4}-\psi^{(\le h)\sigma_1}_{\mathbf{x}_1,\omega_1}\psi^{(\le h)\sigma_2}_{\mathbf{x}_1,\omega_2}\psi^{(\le h)\sigma_3}_{\mathbf{x}_1,\omega_3}\psi^{(\le h)\sigma_4}_{\mathbf{x}_1,\omega_4}]$$

The square brakets in the above equation can be written as

$$\psi^{(\le h)\sigma_1}_{\mathbf{x}_1,\omega_1}D^{(\le h)\sigma_2}_{\mathbf{x}_{2,1},\omega_2}\psi^{(\le h)\sigma_2}_{\mathbf{x}_3,\omega_3}\psi^{(\le h)\sigma_4}_{\mathbf{x}_4,\omega_4}$$

$$+\psi^{(\le h)\sigma_1}_{\mathbf{x}_1,\omega_1}\psi^{(\le h)\sigma_2}_{\mathbf{x}_1,\omega_2}D^{(\le h)\sigma_3}_{\mathbf{x}_{3,1},\omega_3}\psi^{(\le h)\sigma_4}_{\mathbf{x}_4,\omega_4}+\psi^{(\le h)\sigma_1}_{\mathbf{x}_1,\omega_1}\psi^{(\le h)\sigma_2}_{\mathbf{x}_1,\omega_2}\psi^{(\le h)\sigma_3}_{\mathbf{x}_1,\omega_3}D^{(\le h)\sigma_4}_{\mathbf{x}_{1,4},\omega_4}$$

so that the effect of $\mathcal{R}$ is essentially to change the coordinate of the $\psi^{(\le h)\sigma_i}_{\mathbf{x}_i,\omega}$ fields and to replace a field $\psi^{(\le h)\sigma_i}_{\mathbf{x}_i,\omega}$ with $D^{(\le h)\sigma_i}_{\mathbf{x}_{ij}}$ where

$$D^{(\le h)\sigma_i}_{\mathbf{x}_{ij}}=\psi^{(\le h)\sigma_i}_{\mathbf{x}_i,\omega_i}-\psi^{(\le h)\sigma_i}_{\mathbf{x}_j,\omega_i}=(\mathbf{x}_i-\mathbf{x}_j)\int_0^1 du\vec{\partial}\psi^{(\le h)\sigma_i}_{\mathbf{x}_{ji}(u),\omega_i}\ , \tag{3.7}$$

with $\mathbf{x}_{ji}(u)=u\mathbf{x}_j+(1-u)\mathbf{x}_i$ and $\vec{\partial}=(\partial_x,\partial_{x_0})$. In the same time integrating the δ's in (3.2) one can see that the effect of $\mathcal{R}$ is to change the coordinate of the $\psi^{(\le h)\sigma}_{x_i,\omega}$ fields and to replace a field $\psi(\le h)\sigma$ with a field $A^{(\le h)\sigma}$ wich can be written as:

$$(\mathbf{x}_2-\mathbf{x}_1)^2\int_0^1 dt_1\int_0^{t_1}dt_2\partial^2\psi(\le h)\sigma_{x_{21}(t_2)}+(x_2-x_1)\Delta\psi^{(\le h)\sigma}_{x_2}\ , \tag{3.8}$$

Note the two different ways to write the $\mathcal{R}$ operation: in (3.1),(3.2) the $\mathcal{R}$ acts on the kernels of the effective potential *i.e.* it consists in replacing $W^h_{n,m}$ with $\mathcal{R}W^h_{n,m}$ (the definition of $\mathcal{R}W^h_{n,m}$ is evident from (1.5),(1.6)); on the other hand writing as in (3.6) the $\mathcal{R}$ operation act on the ψ fields, leaving the kernel untouched.

3.3 From (2.63) we have that, if E^T_{h+1} denotes the *truncated expectation* with propagator (2.53):

$$\mathcal{V}^h(\sqrt{Z_h}\psi^{\leq h}) = \sum_{n=1}^{\infty}\frac{1}{n!}(-1)^{n+1}E^T_{h+1}(\hat{\mathcal{V}}^{h+1}(\sqrt{Z_h}\psi^{\leq h+1},...)$$

where $\hat{\mathcal{V}}^{h+1}$ is obtained from $\mathcal{V}^{h+1}$ following the operations described in sec.2. Iterating the above equation we obtain that the effective potential can be written in term of a *tree expansion.*

A tree τ consists of a family of lines arranged to connect a partially ordered set of points (*vertices*). Every line has two vertices at the extreme except the first one which has only one vertex, the first vertex v_0. The other extreme r is the *root* and it is not a vertex. From each vertex v start s_v lines; if $s_v = 0$ the vertices is called *end-point*, if $s_v = 1$ is called *trivial vertex* and if $s_v > 1$ is called *non trivial vertex.* To each vertex v we associate a *scale* h_v. The vertices of τ are naturally (partially) ordered. Giving to the tree an orientation from left to right we write $v_1 < v_2$ (so that $h_{v_1} < h_{v_2}$) if v_1 is before v_2 on the tree.

To each end point v with scale h_v we associate either one of the addens in (2.62) or one of the terms of $\mathcal{R}V^0$; this second case is possible only if the scale of the end point is 1. For semplicity of notations we assume in this section

$$\mathcal{R}V^0 = \sum_{n\neq 0,\pm 1} e^{impx}\hat{\phi}_m\psi^{(\leq h)+}_{\mathbf{x}}\psi^{(\leq h)-}_{\mathbf{x}} \tag{3.9}$$

It will be clear from the following consideration that the general case is completely equivalent. Between two non trivial vertices v_1, v_2 there are all the vertices v with scales $h_{v_1} < h_v < h_{v_2}$ If $n \geq 2$ and the scale of an end point is different from 1 there is no trivial vertex between the end-point and the non trivial vertex v immediately preceding it on the tree, so that the scale of the end point is $h_v + 1$ and the running coupling constants associated to it is $\vec{v}_{h_v}$. If $n = 1$ and to the end point is associated a running coupling constants there is only a vertex v_0 on the tree, besides the end-point, and the scale of the running coupling constant is $h_{v_0} = h_{k+1}$. Each tree τ take a label distinguishing which term is associated to the end points. Each trivial or non trivial vertex carry a label $\mathcal{R}$ except v_0 which can carry an $\mathcal{R}$ or $\mathcal{L}$ operation. The set of all the trees with n end points with root with scale k will be denoted by $\tau_{n,k}$.

A *cluster* L_v with frequeny h_v is the set of the end points (possibly only one) reachable from the vertex v, and the tree provides an organization of end points into a hierarchy of clusters. Given a cluster L_v (with scale h_v), we define

$$N_v = \sum_{i\in L_v} n_i \tag{3.10}$$

where the index i denotes the end-points contained in L_v, $n_i = 0$ if to the end-point i is associated a running coupling constant not of the kind τ, ι, $n_i = \pm 1$ if to the end-point i is associated a running couplng constant of the kind τ, ι and finally if to the end point i is associated one of the irrelevant terms (3.9) $n_i = m$. By definition $N_v = N_{v^1} + ... + N_{v^{s_v}}$.

Following standard arguments (see [G1]) the effective potential can be written in the following way

$$\mathcal{V}^k(\sqrt{Z_k}\psi^{(\leq k)}) = \sum_{n=1}^{\infty}\sum_{\tau\in\tau_{k,n}} V^k(\tau,\sqrt{Z_k}\psi^{(\leq k)}) \tag{3.11}$$

If v_0 is the first vertex after the root, $\tau_1,..,\tau_{s_{v_0}}$ are the subtrees starting after the vertex v_0, then $V^k(\tau,\sqrt{Z_k}\psi^{(\leq k)})$ is defined inductively by the relation

$$V^k(\tau,\sqrt{Z_k}\psi^{(\leq k)}) = \frac{1}{s_{v_0}!}E^T_{k+1}[\bar V^{k+1}(\tau_1,\sqrt{Z_k}\psi^{(\leq k+1)}),..,\bar V^{k+1}(\tau_{s_{v_0}},\sqrt{Z_k}\psi^{(\leq k+1)}))] \tag{3.12}$$

where $\bar V^{k+1}(\tau_i,\sqrt{Z_k}\psi^{(\leq k+1)})$

1)is equal to $\mathcal{R}\hat V^{k+1}(\tau_i,\sqrt{Z_k}\psi^{\leq k+1})$ if the subtree τ_i is not trivial *i.e.* if the first vertex of τ_i is not an and-point,

2)if the first vertex of τ_i is an end-point $\bar V^{k+1}(\tau_i,\sqrt{Z_k}\psi^{(\leq k+1)})$ is equal to one addend of $\mathcal{L}\hat V^{k+1}(\tau_i,\sqrt{Z_k}\psi^{\leq k+1})$ (2.62) or, if $k=0$, to one term of $V^0(\tau_i,\psi^{\leq k+1})$.

Let we assume that the effective potential can be written as:

$$V^k(\tau,\sqrt{Z_k}\psi^{(\leq k)}) = \int_\Lambda dx_{v_0}\sum_{P_{v_0}}\sqrt{Z_k}^{|P_{v_0}|}\tilde\psi^{(\leq k)}(P_{v_0})W^k(\tau,P_{v_0},\mathbf{x}_{v_0}) \tag{3.13}$$

where P_{v_0} is a non empty set of I_{v_0}, the field labels associated with the end-points reachable from v_0 (*i.e.* all of them), $\sum_{P_{v_0}}$ is the sum over such subsets, x_{v_0} are the coordinates associated with the tree and:

$$\tilde\psi^{\leq k}(P_{v_0}) = \prod_{f\in P_v}\psi^{\leq k\varepsilon(f)}_{\mathbf{x}(f),\omega(f)} \tag{3.14}$$

The above assunption is proved by induction assuming that (3.13) holds also for the subtrees τ_i and using (3.12). In fact by the identity

$$\tilde\psi^{(\leq k+1)}(P) = \sum_{Q\subset P}\tilde\psi^{(<k+1)}(Q)\tilde\psi^{(k+1)}(P/Q)$$

we obtain

$$V^k(\tau,\sqrt{Z_k}\psi^{(\leq k)}) = \frac{1}{s_{v_0}!}\sum_{P_{v_0}}\sqrt{Z_k}^{|P_{v_0}|}\tilde\psi^{\leq k}(P_{v_0})\sum_{P_{v_0^1},..,P_{v_0^{s_{v_0}}}}\sum_{Q_{v_0^1},..,Q_{v_0^{s_{v_0}}}}$$

$$\mathcal{E}^T_{k+1}[\int d\mathbf{x}_{v_0^1}\tilde\psi^{k+1}(P_{v_0^1}/Q_{v_0^1})\bar W^{k+1}(\tau_1,P_{v_0^1},\mathbf{x}_{v_0^1}),...,$$

$$\int d\mathbf{x}_{v_0^{s_{v_0}}}\tilde\psi^{k+1}(P_{v_0^{s_{v_0}}}/Q_{v_0^{s_{v_0}}})\bar W^{k+1}(\tau_{s_{v_0}},P_{v_0^{s_{v_0}}},\mathbf{x}_{v_0^{s_{v_0}}})]$$

where $\mathcal{E}^T_h$ denotes the truncated expectation with respect to the propagator g^h (2.55), $P_{v_0}\cup_i Q_{v_0^i}$, $Q_{v_0^i}\subset P_{v_0^i}$, v_0^i is the first vertex of the subtree τ_i, and $\bar W^{k+1}(\tau,P_{v_0^1},\mathbf{x}_{v_0^i}) = \mathcal{R}\bar W^{k+1}(\tau,P_{v_0^1},\mathbf{x}_{v_0^i})$ if τ_i is not an end-point, $\bar W(\tau,P_{v_0^1},\mathbf{x}_{v_0^1}) = \vec v_{k+1}$ if it is an end-point but $k+1\neq 0$ and if it is an end-point with $k+1=0$ is equal to $\vec v_0$ or to $\hat\phi_n e^{inpx}$. The above expression of course proves (3.13).

We prove the following theorem, if $\vec v_h$ are the running coupling constants in (2.62)

3.4THEOREM *Let be $k \geq h^*$ given by (2.68). There exists a constant $\bar\varepsilon_k$ such that, if* $\sup_{h>k}|\vec v_h| \leq \bar\varepsilon_k$ *and* $\sup_{h>k}|\frac{Z_h}{Z_{h-1}}| \leq e^{c_1\bar\varepsilon_k^2}$, *chosen* $\gamma^{\frac{1}{\tau}}/2 \geq 1$, *then*

$$|\int d\mathbf{x}^{P_{v_0}} \sum_{\tau\in\mathcal{T}_{n,k}} |W^k(\tau, P_{v_0}, \mathbf{x}^{P_{v0}})|(1+\gamma^k d(P_{v_0})^N \leq |\Lambda|\gamma^{-kD(P_{v_0})}(C_N\bar\varepsilon_k)^N|e^{\frac{-\gamma^{-k/\tau}c_2 s_{P_v}}{2^{-k+3}}}$$

where $s_{P_v}=1$ *if* $N_v \neq 0$, $|P_v|=2$ *and* 0 *otherwise,* N *is a positive integer,* $x^{P_{v^0}}$ *is the set of points associated to* P_{v_0}, c_1, c_2, C_N *are constants (not depending on* L,β*),* $d(P_{v_0})$ *is the length of the shortest tree connecting the set of points* $\mathbf{x}^{P_{v_0}}$, $|\Lambda| = L\beta$ *and*

$$D(P_{v_0}) = -2 + \sum_{f\in P_{v_0}} (1/2 + m_f)$$

where m_f *is the order of the derivative applied to the fields of label* f, *see (2.53)*

3.5 PROOF In order to write in an explicit form (3.12) it is convenient to start studying $\mathcal{R}\hat V^h$:

$$\mathcal{R}\hat V^k(\tau, \sqrt{Z_{k-1}}\psi^{(\leq k)}) = \frac{1}{s_{v_0}!}\sum_{P_{v_0}}(\sqrt{Z_{k-1}})^{|P_{v_0}|}\mathcal{R}\{\tilde\psi^{(\leq k)}(P_{v_0}) \sum_{P_{v_0^1},..,P_{v_0 s_{v_0}}} \sqrt{\frac{Z_k}{Z_{k-1}}}^{|P_{v_0}|}$$

$$\sum_{Q_{v_0^1},..,Q_{v_0^{s_{v_0}}}} \mathcal{E}^T_{k+1}[\int d\mathbf{x}_{v_0^1}\tilde\psi^{(k+1)}(P_{v_0^1}/Q_{v_0^1})\bar W^{k+1}(\tau, P_{v_0^1}, \mathbf{x}_{v_0^1}), ...,$$

$$\sum_{x_{v_0^{s_{v_0}}}\in\Lambda} \tilde\psi^{(k+1)}(P_{v_0^{s_{v_0}}}/Q_{v_0^{s_{v_0}}})\bar W^{k+1}(\tau, P_{v_0^{s_{v_0}}}, \mathbf{x}_{v_0^{s_{v_0}}})] \tag{3.15}$$

¿From (3.6)-(3.8) we know that the effect of $\mathcal{R}$ is simply to replace $\tilde\psi^{(\leq k)}(P_{v_0})$ with

$$\sum^*_{i,j,b}(\mathbf{x}_i - \mathbf{x}_j)_b^{a_{P_{v_0}}}\hat\psi^{\leq k}(P_{v_0}) \tag{3.16}$$

where $b=0,1$ is an index distinguishing the space ot time component of $\mathbf{x}$ (in the following this index and $\sum^*_{i,j,b}$ will be omitted), $\mathbf{x}_i, \mathbf{x}_j$ are two coordinates in $\mathbf{x}^{P_{v_0}}$ and

1)$a_{P_{v_0}} = 0$ and $\hat\psi^k(P_{v_0}) = \tilde\psi^k(P_{v_0})$ if $|P_{v_0}| > 4$ or the field index do not verify verify the Kronecker δ in (2.25), (2.26)

2)if $|P_{v_0}| = 4$ and the field index verifies the Kronecker δ in (2.25) then $a_{P_{v_0}} = 1$ and $\hat\psi^k(P_{v_0})$ differs from $\tilde\psi^k(P_{v_0})$ becouse a field $\int_0^1 dt\partial\psi^\varepsilon_{x_{ij}(t)}$ replaces a ψ field, and the remaning ψ fields are applied on different coordinates among $\mathbf{x}^{P_{v_0}}$, see (3.6);

3)if $|P_{v_0}| = 2$ and the field index verify the Kronecker δ in (2.26), $a=2$ and $\hat\psi^k(P_{v_0})$ differs from $\tilde\psi^k(P_{v_0})$ becouse a field $\int_0^1 dt_1\int_0^{t_1} dt_2\partial^2\psi_{x_{ij}(t_2)}$ or a $\frac{\Delta\psi}{(x_1-x_2)}$ replaces a ψ field, and the remaning ψ fields are applied on different coordinates among $\mathbf{x}^{P_{v_0}}$.

Let we call:

$$\hat\psi(P) = \prod_{f\in P}\partial^{q(f)}_{\mathbf{x}(f)}\psi^{\varepsilon(f)}_{\mathbf{x}(f)} \tag{3.17}$$

where $q=0,1,2,3$, $\partial^0 = 1$, $\partial^1 = \partial_{\mathbf{x}}$, $\partial^2_{\mathbf{x}} = \partial_{\mathbf{x}}\partial_{\mathbf{x}}$, $\partial^3 = \frac{1}{(x_1-x_2)}\Delta$.

We remember the well known expansion of truncated expectation in term of interpolating parameters s_t, $t=1,..,k-1$:

$$\mathcal{E}_h^T(\hat{\psi}(P_1),...,\hat{\psi}(P_k))=\sum_T\prod_{l\in T}\partial^{q_l}_{\mathbf{x}_l}\partial^{q'_l}_{\mathbf{y}_l}g^h(\mathbf{x}_l-\mathbf{y}_l)\int dP_T(s)detG^T(s) \tag{3.18}$$

where T is a set of lines forming an *anchored tree graph* between the cluster of vertices from which the fields labeled with $P_1,..,P_k$ emerge: this means that T is a set of lines connecting two points in different clusters, which becomes a tree graph if one identifies all the points in the same cluster; if $l\in T$ $\mathbf{x}_l,\mathbf{y}_l$ are the end-points of the line and are such that $\mathbf{x}_l=\mathbf{x}_{ij}$ or $\mathbf{y}_l=\mathbf{x}_{i'j'}$, where $\mathbf{x}_{ij}$ denotes the coordinate of the i-th field of the monomial $\tilde{\psi}(P_j)$. If f is a field variable such that $\mathbf{x}(f)\equiv\mathbf{x}_l\equiv\mathbf{x}_{ij}$, we write $q(f)\equiv q_l\equiv q_{ij}$. In the same way if $\bar{f}$ is such that $\mathbf{x}(\bar{f})\equiv\mathbf{y}_l\equiv\mathbf{x}_{i'j'}$ we say $q(\bar{f})\equiv q'_l$. $G^T(s)$ is a $(n-k+1)\times(n-k+1)$ matrix (if n is the total number of fields), whose elements are $G^T_{jij'i'}=S_{jj'}\partial^{q_{ij}}\partial^{q_{i'j'}}g^h(x_{ij}-x_{i'j'})$ with $x_{ij}-x_{i'j'}$ non belonging to T, $S_{jj'}=\prod_{i=j}^{j'-1}s_t$ and $dP_T(s)$ is a normalized measure which depends on s_t and T (for the well known explicit formula of dP_T and for its derivation, one can look for istance [Le],[BGPS]).

We write in an explicit way the action of $\mathcal{R}$ in (3.15) obtaining

$$\mathcal{R}V^k(\tau,\sqrt{Z_{k-1}}\psi^{(\le k)})=\frac{1}{s_{v_0}!}\sum_{P_{v_0}}\sqrt{Z_{k-1}}^{|P_{v_0}|}\hat{\psi}^{(\le k)}(P_{v_0})\sum_{P_{v_0^1},..,P_{v_0^{s_{v_0}}}}\sqrt{\frac{Z_k}{Z_{k-1}}}^{|P_{v_0}|}$$

$$\sum_{Q_{v_0^1},..,Q_{v_0^{s_{v_0}}}}\sum_{\mathbf{x}_{v_0}\in\Lambda}[(\mathbf{x}_i-\mathbf{y}_j)^{a(P_{v_0})}\mathcal{E}^T_{k+1}(\tilde{\psi}^{k+1}(P_{v_0^1}/Q_{v_0^1}),...,\tilde{\psi}^{k+1}(P_{v_0^{s_{v_0}}}/Q_{v_0^{s_{v_0}}}))] \tag{3.19}$$

$$\bar{W}^{k+1}(\tau_1,P_{v_0^1},\mathbf{x}_{v_0^1})...\bar{W}^{k+1}(\tau_{s_{v_0}},P_{v_0^{s_{v_0}}},\mathbf{x}_{v_0^{s_{v_0}}})]$$

We can write, for any anchored tree graph T_{v_0} connecting the clusters $L_{v_0^1}...L_{v_0^{s_{v_0}}}$:

$$(\mathbf{x}_i-\mathbf{y}_j)=\sum_{l\in T}(\mathbf{x}_l-\mathbf{y}_l)+\sum^*_{iji'j'}(\mathbf{x}_{ij}-\mathbf{y}_{i'j'}) \tag{3.20}$$

where $\sum^*_{iji'j'}$ is a sum such that there is no $l\in T_{v_0}$ such that $\mathbf{x}_{ij}=\mathbf{x}_l,\mathbf{y}_{i'j'}=\mathbf{y}_l$.

So we can write:

$$\sum^*_{ij}(\mathbf{x}_{ij}-\mathbf{y}_{ij})^{a(P_{v_0})}\mathcal{E}^T_{k+1}(\tilde{\psi}^{k+1}(P_{v_0^1}/Q_{v_0^1}),...)]\bar{W}^{k+1}(\tau_1,P_{v_0^1},\mathbf{x}_{v_0^1})...\bar{W}^{k+1}(\tau_{s_{v_0}},P_{v_0^{s_{v_0}}},\mathbf{x}_{v_0^{s_{v_0}}})]=$$

$$[\sum_{T_{v_0}}\sum^*_{\{a\}}\{\prod_{l\in T_{v_0}}(\mathbf{x}_l-\mathbf{y}_l)^{a_l}g^{k+1}(\mathbf{x}_l-\mathbf{y}_l)][\int dP_T(s)detG^T(s)]$$

$$|\mathbf{x}^{P_{v_0^1}}|^{a_{v_0^1}}\bar{W}^{k+1}(\tau_1,P_{v_0^1},\mathbf{x}_{v_0^1})...|\mathbf{x}_{v_0^{s_{v_0}}}|^{a_{v_0^{s_{v_0}}}}\bar{W}^{k+1}(\tau_{s_{v_0}},P_{v_0^{s_{v_0}}},\mathbf{x}_{v_0^{s_{v_0}}})] \tag{3.21}$$

where $|\mathbf{x}^{P_{v_0^i}}|$ is the difference among two coordinates in $\mathbf{x}^{P_{v_0^i}}$, and $\sum^*_{\{a\}}$ is the sum over the indices $a_l+a_{v_0^1}+..$ with the constraint that $a_l+a_{v_0^1}+...+a_{v_0^{s_{v_0}}}=a(P_{v_0})$. We are now in position to iterate the above procedure studying each $|\mathbf{x}^{P_{v_0^i}}|^{a_{v_0^i}}\bar{W}^{k+1}(\tau_i,P_{v_0^i},\mathbf{x}^{P_{v_0^i}})$. We can distinguish several cases

1)if τ^i is a trivial tree, $a_{v_0^i}=0$ and $|\mathbf{x}^{P_v}|^{a_v}\bar{W}^{k+1}=\vec{v}_{k+1}$;

2)if τ^i is not a trivial tree, there are the following possibilityes:

2a)$a_{v_0^i}=0$. In this case we repeat word by word the analysis for W^k (3.15),(3.19), with the trivial substitution $k\to k+1$, $v_0\to v_0^i$. Note that if $\mathcal{R}\neq 0$ the effect of the renormalization is of replacing in the truncated expectation in (3.19) $\tilde\psi(P_{v_0^i}/Q_{v_0^i})$ with $\hat\psi(P_{v_0^i}/Q_{v_0^i})$, defined in an analogous way as $\hat\psi(P_{v_0})$ (3.16), and to produce a factor $(\mathbf{x}_i-\mathbf{x}_j)^{a_{P_{v_0^i}}}$, $ij\in P_{v_0^i}$ which is written as in (3.17).

2b)$a_{v_0^i}\neq 0$. In this case we have to use (3.3) (3.4) (3.5) and we do as in (3.15),(3.19). Note that the propagators belonging $T_{v_0^i}$ are of the form $(\mathbf{x}_l-\mathbf{x}_l)^a g(\mathbf{x}_l-\mathbf{x}_l)$ with $a<2$.

Iterating this procedure we find at the end

$$\hat V^k(\tau,\sqrt{Z_{k-1}}\psi^{\le k})=\sum_{\mathbf{x}_{v0}\in\Lambda}\sum_{P_{v0}}\sqrt{Z_{k-1}}^{|P_{v_0}|}\tilde\psi^{\le k}(P_{v_0})\sum_{\{P_v\}}\{\int dt\}\prod_{v\,not\,e.p.}|\frac{Z_{h_v-1}}{Z_{h_v-2}}|^{|P_v|/2}\frac{1}{s_v!}$$

$$\sum_{\{q\}}^{*}[\prod_{v\,not\,e.p.}\frac{1}{s_v!}\tilde{\mathcal{E}}^T_{h_v}(\hat\psi(P_{v^1}/Q_{v^1}),...,\hat\psi(P_{v^{s_v}}/Q_{v^{s_v}}))][\prod_{i\in V_1}(\vec v_{h_i})_a e^{2ip_F x_i\gamma_a}\gamma^{h_i\delta_a}][\prod_{i\in V_2}\hat\varphi_{n_i}e^{in_i p\mathbf{x}_i}] \tag{3.22}$$

where:

1)The symbol $\sum_{\{P_v\}}$ denotes the sum over all the compatible choices of the sets P_v on all the vertex of τ except v_0; moreover $Q_v\subset P_v$, $P_v=\bigcup_i Q_{v^i}$.

2)$\hat\psi(P_v/Q_v)=\prod_{f\in P_v/Q_v}\partial^{q_f}\psi^{\varepsilon(f)}_{\mathbf{x}_{ij}(t)}$ where $q_f=0,1,2,3$ with the same conventions as in (3.17). Moreover $\mathbf{x}_{ij}(t)=\sum_{i'}\varepsilon_{i'j}(t)\mathbf{x}_{i'j}$ with $\sum_{i'}\varepsilon_{i'j(t)}=1$ and $\varepsilon_{i'j}(0)=\delta_{i,i'}$. Finally

$$\tilde{\mathcal{E}}^T_h(\hat\psi(P_1),...,\hat\psi(P_k))=\sum_{T_v}\sum_{\{a\}}^{*}[\prod_{l\in T_v}\partial^{q_l}_{\mathbf{x}_l}\partial^{q'_l}_{\mathbf{y}_l}(\mathbf{x}_l-\mathbf{y}_l)^{a_l}g^h(\mathbf{x}_l-\mathbf{y}_l)]\int dP_T(s)detG^T(s)$$

where $\sum^*_{\{q\}}$ and $\sum^*_{\{a\}}$ have the following constraints. If v is such that $|P_v|=4$ and the Kronecker δ of (2.25) is verified then $\hat\psi(P_v)$ contains a field $\partial\psi_{x'_{ij}}$ and there is a line $\bar l\in T_{\bar v}$, $\bar v>v$, with $a_{\bar l}=1$ and for any $l\in T_{\hat v}$, $l\neq\bar l$, $\bar v\ge\hat v>v$ $a_l=0$. If v is such that $|P_v|=2$ and the Kronecker δ of (2.26) is verified then $\hat\psi(P_v)$ contains a field $\partial^2\psi_{\mathbf{x}_{ij}}$ or $\Delta\psi$ and there are two (possibly coinciding) lines $l_1\in T_{\bar v_1}$, $l_2\in T_{\bar v_2}$ with $\bar v_1>v$, $\bar v_2>v$ with $a_l=1$ and for any $l\in T_{\hat v}$, $l\neq l_1,l_2$ $\bar v_1\ge\hat v>v$, $\bar v_2\ge\hat v>v$ one has $a_l=0$.

4)$\{\int dt\}$ is a product of integral over the interpolation parameters.

5) V_1 is the set of the end points of the tree, and V_2 is the set of end points *not* associated to running coupling constants. Moreover $\gamma_a=\pm1$ in correspondence of end-points of kind τ or ι and zero otherwise, and $\delta_a=1$ in correspondence of end-points of kind ν and zero otherwise.

Summing (3.22) over the index n_i *i.e.* summing over all the trees differing only for the choices of the addend $\mathcal{R}V^0$ associated to a given end-point(*i.e.* summing over trees with the same labels and differing only for the choices of n_i), we find

$$\int d\mathbf{x}^{v_0}\sum_{P_{v0}}\sqrt{Z_{k-1}}^{|P_{v_0}|}\tilde\psi^{\le k}(P_{v_0})\sum_{\{P_v\}}\{\int dt\}[\prod_{i\in V_2}\sum_{n_i}]\prod_{v\,not\,e.p.}|\frac{Z_{h_v-1}}{Z_{h_v-2}}|^{|P_v|/2}\frac{1}{s_v!}\sum_{\{q\}}^{*}$$

$$\tilde{\mathcal{E}}^T_{h_v}(\hat\psi(P_{v^1}/Q_{v^1}),...,\hat\psi(P_{v^{s_v}}/Q_{v^{s_v}})[\prod_{i\in V_1}(\vec{v}_{h_i})_a e^{2ip_F x_i\gamma_a}\gamma^{h_i\delta_a}][\prod_{i\in V_2}\hat\varphi_{n_i}e^{in_i p\mathbf{x}_i}] \qquad (3.23)$$

We can bound $detG^T$ by the Grahm-Hadamard inequality ([Le],[BGPS]). Once that we have bounded the determinant, we have to make the integration over the coordinates and over the interpolation variables. It is convenient to change variables from $\{\mathbf{x}\}$ to $\{\mathbf{r}\}$, where $\{\mathbf{r}\}$ is the collection of the difference $\mathbf{x}_l-\mathbf{y}_l$ appearing in the factors $\prod_l \partial^{q_l}_{\mathbf{x}_l}\partial^{q'_l}_{\mathbf{y}_l}(\mathbf{x}_l-\mathbf{y}_l)^{a_l}g^h(\mathbf{x}_l-\mathbf{y}_l)$. Note that $\mathbf{x}_l\equiv x_{ij}(t)$ so that the determinant of the Jacobian of this trasformation is a function of t, and one can worry about its integrability. However it is possible to show (see [BM1], App. 3) that such determinant is exactly 1. So we obtain a bound for W^k (3.13), estimating the propagators by lemma 2.8, using lemma 2.6 and (3.10) and summing over N_v istead over n_i:

$$\Lambda\gamma^{-kD(P_{v_0})}\sum_{\{P_v\}}[\sum_{\{N_v\geq C[\gamma^{\frac{-h_v}{\tau}}-|P_v|]\}}\prod_{v not e.p.}\delta_{N_v,N_{v^1}+...+N_{v^{s_v}}}][\prod_{v not e.p.}|\frac{Z_{h_v-1}}{Z_{h_v-2}}|^{\frac{|P_v|}{2}}]$$

$$C^{|Q_v|-|P_v|}J(\tau,P_{v_0},x_{v_0})\gamma^{-[D(P_v)+z_v(N_v,P_v)](h_v-h_{v'})}[\prod_{i\in V_1}(\vec{v}_{h_i})_a][\prod_{i\in V_2}|\hat\varphi_{n_i}|]$$

where

$$\int d\{\mathbf{r}_l\}J(\tau,P_{v_0},x_{v_0})=\int d\{\mathbf{r}_l\}\prod_{v not e.p.}\frac{1}{s_v!}\sum_{T_v}\int dt\prod_{l\in T}\partial^{q_l+q'_l}_{\mathbf{r}_l}[(\mathbf{r})^a_l\bar{g}^{h_v}(\vec{r}_l)]$$

and $D(P_v)=-2+\sum_{f\in P_v}(1/2+m_f)$, with m_f being the order of derivatives applied to the field of label f, v' is the vertex preceding v on the tree (so that $h_{v'}=h_v-1$). The presence $z_v(N_v,P_v)$ is due to the renormalization procedure and it is defined as

1) $z_v(N_v,P_v)=1$ if $|P_v|=4$, $\sum_{f\in P_v}m_f=0$ and $\sum_i\varepsilon_i\omega_i p_F+2N_v p=0$

2) $z_v(N_v,P_v)=1$ if $|P_v|=2$, $\sum_{f\in P_v}m_f=1$ and $\sum_i\varepsilon_i\omega_i p_F+2N_v p=0$

3) $z_v(N_v,P_v)=2$ if $|P_v|=2$, $\sum_{f\in P_v}m_f=0$ and $\sum_i\varepsilon_i\omega_i p_F+2N_v p=0$

3.6 We can write then

$$\prod_{v not e.p.}\gamma^{-[D(P_v)+z_v(N_v,P_v)](h_v-h_{v'})}\leq$$

$$\prod_{v not e.p.}\gamma^{\frac{-|P_v|}{6}}[\prod_{v\in T_4}\gamma^{h_v-h_{v'}}][\prod_{v\in T_2}\gamma^{h_v-h_{v'}}][\prod_{v\in T_3}\gamma^{2(h_v-h_{v'})}]$$

where

1) T_4 is the set of clusters with $|P_v|=4$, $\sum_{f\in P_v}m_f=0$ and $\sum_i\varepsilon_i\omega_i p_F+2N_v p\neq 0$

2) T_2 is the set of clusters with $|P_v|=2$, $\sum_{f\in P_v}m_f=1$ and $\sum_i\varepsilon_i\omega_i p_F+2N_v p\neq 0$

3) T_3 is the set of clusters with $|P_v|=2$, $\sum_{f\in P_v}m_f=0$ and $\sum_i\varepsilon_i\omega_i p_F+2N_v p\neq 0$

Let we call *hard vertices* $\bar v$ the vertices such that $|P_{\bar v'}|\neq|P_{\bar v}|$; the set of such vertices is called $\mathcal{T}$. Given $\tau\in\tau_{n,k}$ the number of hard vertices is bounded dy Cn, if C is a suitable constant. This can be easily checked remembering the graph rapresentation of the above series, see for instance [BGPS]; in each graph associated to the tree the clusters associated to hard vertices must differ at least for a line, and the number of lines is of course $O(n)$. To an hard vertex $\bar{v}$ we associate a *depth*, defined in the following way: if $\bar{v}$ is the first hard vertex preceding an end point on the tree then $D_{\bar{v}} = 1$, otherwise $D_{\bar{v}} = 1 + max_{\bar{v}''}\{D_{\bar{v}''}\}$, where $\bar{v}''$ are hard vertices following $\bar{v}$ on the tree and such that there are no other hard vertices between $\bar{v}$ and anyone of the $\bar{v}''$. Note also that $D_{\bar{v}} \le -h_{\bar{v}} + 2$.

We can write

$$\prod_{v note.p.} \gamma^{-[D(P_v)+z_v(N_v,P_v)](h_v-h_{v'})} \le \prod_{v note.p.} \gamma^{\frac{-|P_v|}{6}}[\prod_{\bar{v}\in\mathcal{T}_4}\gamma^{-h_{\bar{v}'}}][\prod_{\bar{v}\in\mathcal{T}_2}\gamma^{-h_{\bar{v}'}}][\prod_{\bar{v}\in\mathcal{T}_3}\gamma^{-2h_{\bar{v}'}}]$$

where $\bar{v}'$ is a the vertex preceding $\bar{v}$ and $\mathcal{T}_i$ is the intersection between $\mathcal{T}$ and T_i . We write $\mathcal{T}_i = \mathcal{T}_i' \bigcup \mathcal{T}_i''$ with $\mathcal{T}_i'$ such that $N_v \neq 0$ and T_i'' such that $N_v = 0$. By Lemma 2.7 it follows that

$$\prod_{\bar{v}\in\mathcal{T}_i''} \gamma^{-h_{\bar{v}'}} \le C^n$$

On the other hand it is easy to show that

$$\prod_i |\hat{\varphi}_{n_i}| \le e^{\frac{\xi n}{2}} \prod_i e^{-\xi|n_i|/2} \prod_{\bar{v}\in\mathcal{T}_2'\bigcup\mathcal{T}_3'\bigcup\mathcal{T}_4'} e^{\frac{-\xi|N_{\bar{v}}|}{2^{D_{\bar{v}}+1}}} \tag{3.24}$$

In fact let us consider a hard vertex $\bar{v}_1$ not followed by any other hard vertex. Then we can write

$$\prod_{i\in L_{\bar{v}_1}} e^{\frac{-\xi|n_i|}{2}} \le e^{\frac{-\xi|N_{\bar{v}_1}|}{4}} e^{\frac{-\xi|N_{\bar{v}_1}|}{4}}$$

where the product is over the end-points contained in the cluster $\bar{v}_1$. So for the hard vertex $\bar{v}_1$ we have found the factor $e^{\frac{-\xi|N_{\bar{v}_1}|}{2^{D_{\bar{v}_1}+1}}}$ and we have an extra factor $e^{\frac{-\xi|N_{\bar{v}_1}|}{4}}$.

Let us consider now a vertex $\bar{v}_2$ with depth 2 *i.e.* followed only by end points and hard vertex $\bar{v}_i$ with depth 1; we can write

$$\prod_{i\in L_{\bar{v}_2}} e^{\frac{-\xi|n_i|}{2}} \prod_{\bar{v}_1:L_{\bar{v}_1}\subset L_{\bar{v}_2}} e^{\frac{-\xi|N_{\bar{v}_1}|}{4}} \le e^{\frac{-\xi|N_{\bar{v}_1}|}{8}} e^{\frac{-\xi|N_{\bar{v}_1}|}{8}}$$

so again the the factor $e^{\frac{-\xi|N_{\bar{v}_1}|}{2^{D_{\bar{v}_1}+1}}}$ and an extra factor $e^{\frac{-\xi|N_{\bar{v}_1}|}{8}}$. Proceding in this way we have (3.24), noting that to the end-points of kind ι,τ it is not associated any factor $e^{-\xi}$ (this explains the factor $e^{\frac{\xi n}{2}}$ in front of (3.24)).

Using Lemma 2.6 and the fact that $D_{\bar{v}} \le -h_{\bar{v}'} + 2$ we have

$$\prod_i |\hat{\varphi}_{n_i}| \le \prod_{\bar{v}\in\mathcal{T}_2'\cup\mathcal{T}_3'\cup\mathcal{T}_4'} e^{\frac{-\xi C_2 \gamma^{\frac{-h_{\bar{v}'}}{\tau}}}{2^{-h_{\bar{v}'}+3}}} \tag{3.25}$$

At the end, summing over N_v, we can write

$$|\hat{V}^k(\hat{\tau}, \sqrt{Z_{k-1}}\psi^{\le k})| \le C^n \varepsilon^n \gamma^{-kD(P_{v_0})} \sum_{\{P_v\}} \prod_{v note.p.} \gamma^{\frac{-|P_v|}{8}}$$

$$\prod_{\bar{v}\in\mathcal{T}_1'} \gamma^{-h_{\bar{v}'}} e^{\frac{-\xi C_2 \gamma^{\frac{-h_{\bar{v}'}}{\tau}}}{2^{-h_{\bar{v}'}+3}}} \prod_{\bar{v}\in\mathcal{T}_2'} \gamma^{-h_{\bar{v}'}} e^{\frac{-\xi C_2 \gamma^{\frac{-h_{\bar{v}'}}{\tau}}}{2^{-h_{\bar{v}'}+3}}} \prod_{\bar{v}\in\mathcal{T}_3'} \gamma^{-2h_{\bar{v}'}} e^{\frac{-\xi C_2 \gamma^{\frac{-h_{\bar{v}'}}{\tau}}}{2^{-h_{\bar{v}'}+3}}}$$

Choosing γ so that $\gamma^{\frac{1}{\tau}}/2 > 1$ and remembering that the hard vertices are $\leq C_1 n$ we have that

$$\prod_{\bar{v}\in\mathcal{T}_i'} \gamma^{-2h_{\bar{v}'}} e^{\frac{-\xi C_2 \gamma^{\frac{-h_{\bar{v}'}}{\tau}}}{2^{-h_{\bar{v}'}+3}}} \leq C_2^n$$

By a standard calculation

$$\sum_{\tau\in\mathcal{T}_{k,n}} \sum_{\{P_v\}} \prod_{v note.p.} \gamma^{\frac{-|P_v|}{8}} \leq C_3^n$$

and this completes the proof of Theorem 3.3 (the last factor in the bound comes from (3.25)).

Remark: Formula (3.25) is the analogue of *Brjuno lemma* for this problem; it ensures that the small denominator problem arising in the series for the uncommensurability of ϕ_x, can be controlled taking into account the Diophantine condition. The same role is plaied by the original Brjuno lemma in the proof of the convergence of the Lindstedt series.

We have now to perform the last integration (2.69).

3.7THEOREM *If there exists constants κ, C_2, ε such that $\frac{\sigma_{h^*-1}}{\kappa} \geq \kappa$ and $|\vec{v}_{h^*}| \leq \varepsilon$, $|\frac{Z_{h^*}}{Z_{h^*-1}}| \leq e^{C_2\varepsilon^2}$ than*

$$|\int \tilde{P}_{Z_{h^*-1}}(d\psi^{(\leq h^*)}) e^{-\tilde{\mathcal{V}}^{h^*}(\sqrt{Z_{h^*}}\psi^{(\leq h^*)})}| \leq C_3|\Lambda|\gamma^{2h^*}$$

where C_3 is a suitable constant

The proof of this theorem is rather straigthforward. We can write

$$\int \tilde{P}_{Z_{h^*-1}}(d\psi^{\leq h^*}) e^{-\tilde{\mathcal{V}}^{h^*}(\sqrt{Z_{h^*-1}}\psi^{\leq h^*})} = \sum_n \frac{(-1)^n}{n!} E^T_{\leq h^*}(\hat{\mathcal{V}}^{h^*}, ..., \hat{\mathcal{V}}^{h^*}) =$$

$$= \sum_n \sum_{i_1,\ldots,i_n} \frac{(-1)^n}{n!} E^T_{\leq h^*}(\hat{\mathcal{V}}^{h^*}_{i_1}, ..., \hat{\mathcal{V}}^{h^*}_{i_1})$$

and $\hat{V}^{h^*}_i$ are terms of the form

$$\int \prod_{j=1}^{i} d\mathbf{k}_i \prod_{j=1}^{i} \psi^{\varepsilon_i,\leq h^*}_{\mathbf{k}_i+\omega_i p_F,\omega_i} W^{h^*}_i(\mathbf{k}_1,..,\mathbf{k}_i)\delta(\sum_{j=1}^{i}\varepsilon_j(\mathbf{k}_j+\omega_j p_F)+2np) \tag{3.26}$$

It is easy to check that each addend in the above sum will be bounded by

$$\gamma^{2h^*}\Lambda C^n \prod_i \gamma^{h^*(i/2-2)m_i} \gamma^{h^* m_2^1} e^{\frac{-\xi c_2 \gamma^{-h^*}|m_2^2|}{2^{-h^*+3}}}$$

if m_i is the number of terms of the form (3.26), m_2^1 the number of such terms with $i=2$, $n=0$ and m_2^2 the number of terms with $i=2$, $n\neq 0$. We have used that the number of loops is $\sum_i(\frac{i}{2}-1)m_i+1$, the number of g^{h^*} is $\sum_i \frac{i m_i}{2}$.

4. The flow of the Renormalization group

4.1 The convergence of the expansion for the partition function is proved by theorems 3.4,3.7 under suitable assumptions on the running coupling constants *i.e.* that there exists a finite

h^* (2.68) such that, given a constant ε, then $\max_{k\geq h^*}|\vec{v}_k| le\varepsilon$ and $\max_{k\geq h^*}|\frac{Z_k}{Z_{k-1}}|\leq e^{c_2\varepsilon^2}$. In this section we prove that it is possible to choose $|\lambda|, u$ so small and a proper ν_0 so that the above conditions are indeed verified.

Let we say first that the property $\gamma>1$ can be used to show that there is a ν_0 such that $|\nu_h|\leq\varepsilon$ and $\nu_h=O(\gamma^h)$. In fact ν_h obeys to the equation $\nu_{h-1}=\gamma\nu_h+\beta^h_\nu$, with $|\beta^h_\nu|<K\varepsilon^2_h$, if K is a constant. One obtains that $\nu_h=\gamma^{-h}(\nu_0+\sum_{j=h+1}^0\gamma^{j-1}\beta^j_\nu)$ so that, choosing ν such that ν_0 satisfies $\nu_0+\sum_{j=h^*}^0\gamma^{j-1}\beta^j_\nu=0$ of course $\nu_{h^*}=0$ and $|\nu_h|\leq\varepsilon$ for any $h\geq h^*$. The value of ν_0 so that $|\nu_h|\leq\varepsilon$ for any $h\geq h^*$ is not unique; from the value of h^* computed in (4.4) it is clear that, if $\bar\nu_0$ is the value such that $\nu_{h^*}=0$, any $\nu_0=\bar\nu_0+O(u^2)$ has the effect that $|\nu_h|\leq\varepsilon$ for any $h\geq h^*$. This is expected as we can fix the chemical potential as we like inside the gap without changing the physical properties of the system.

Eliminating on the Beta function the ratios $\frac{Z_h}{Z_{h'}}$ and the dependence on $\nu_{h+1},...,\nu_0$ (see [BGPS], app. 4 for the detail) we can write, for $h\geq h^*$

$$\begin{aligned}
\lambda_{h-1}&=g_h+G^{1,h}_\lambda+G^{2,h}_\lambda+\gamma^hR^h_\lambda\\
\sigma_{h-1}&=\sigma_h+G^{1,h}_\sigma+\gamma^hR^h_\sigma\\
\delta_{h-1}&=\delta_h+G^{1,h}_\delta+G^{2,h}_\delta+\gamma^hR^h_\delta\\
\tau_{h-1}&=\tau_h+G^{2,h}_\tau+\gamma^hR^h_\tau\\
\iota_{h-1}&=\vartheta_h+G^{2,h}_\iota+\gamma^hR^h_\iota\\
\frac{Z_{h-1}}{Z_h}&=1+G^{1,h}_z+G^{2,h}_z+\gamma^hR^h_z
\end{aligned}\tag{4.1}$$

where:

1) $G^{1,h}_\lambda\equiv G^{1,h}_\lambda(\lambda_h,\delta_h;...;\lambda_0,\delta_0)$, $G^{1,h}_\delta\equiv G^{1,h}_\delta(\lambda_h,\delta_h;...;\lambda_0,\delta_0)$ and $G^{1,h}_z\equiv G^{1,h}_z(\lambda_h,\delta_h;...;\lambda_0,\delta_0)$ are given by series of terms involving only the Luttinger model part of the propagator $g^k_{\omega,L}(x-y)$, $k\geq h$, see lemma 2.4.

2) $G^{2,h}_\lambda, G^{2,h}_\delta, G^{2,h}_z, G^{2,h}_\tau, G^{2,h}_\iota$ depend on all the running coupling constants and are given by a series of terms involving at least a propagator $C^{(k)}_{2,\omega}(\mathbf{x}-\mathbf{y})$ or $g^{(k)}_{\omega,-\omega}(\mathbf{x}-\mathbf{y})$, $k\geq h$. By lemma 2.4:

$$|G^{2,h}_\lambda|,|G^{2,h}_\delta|,|G^{2,h}_z|,|G^{2,h}_\tau|,|G^{2,h}_\iota|\leq\varepsilon^2_h|\frac{\sigma_h}{\gamma^h}|$$

3) By a second order computation one obtains

$$G^{1,h}_\sigma=\sigma_h[-\beta_2g_h+\bar G^{1,h}_\sigma]\quad G^{1,h}_z=g^2_h[\beta_1+\bar G^{1,h}_z]$$

with β_1,β_2 non vanishing positive contants and $|\bar G^{1,h}_\sigma|\leq\varepsilon_h$ and $|\bar G^{1,h}_z|\leq\varepsilon_h$. Moreover $G^{1,h}_\lambda, G^{1,h}_\delta$ are vanishing at the second order.

4)R^h_i, $i=\lambda,z,\sigma,\delta,\tau,\vartheta$ depend on all the running coupling constants and are given by a series of terms involving at least a propagator $C^{(k)}_{1,\omega}(x-y)$, $k\geq h$ so that,from lemma 2.4, $|R^h_i|\leq\varepsilon^2_h\gamma^h$.

The fact that $G^{1,h}_\lambda, G^{1,h}_\delta$ are vanishing at the second order could generate a problem as one cannot exclude *a priori* there there is some non-vanishing term at some large order, and the evolution of the running coupling constants will depend critically on this unknown term. Fortunately it is possible to prove that:

$$\lim_{h\to-\infty} G_{\lambda}^{1,h}(g,\delta;\ldots;g,\delta)=0 \tag{4.2}$$

$$\lim_{h\to-\infty} G_{\delta}^{1,h}(g,\delta;\ldots;g,\delta)=0$$

i.e. it is vanishing at *all* orders. In fact $G_{\lambda}^{1,h}$, $G_{\delta}^{1,h}$ coincide, up to terms vanishing as $h\to-\infty$ as $O(\gamma^h)$, with the correspondent quantities of the Luttinger model , and, by using the properties of the exact solution [ML],[M] one can prove that it is vanishing. This non perturbative result was proved in [BGPS] using some properties of the exact solution obtained in [BeGM],[BM1].

As a straigthforward consequences of the above facts one can prove.

4.2LEMMA: There exist positive constants c_1,c_2,c_3,c_4 such that, if λ,u are small enough:

$$\begin{gathered}
|g_{h-1}-g_0|<c_1\lambda^2\\
\lambda\beta_1 c_3 h<\log\left(\frac{\sigma_{h-1}}{\sigma_0}\right)<\lambda\beta_1 c_4 \\
-\beta_3 c_1\lambda^2 h<\log(|Z_{h-1}|)<-\beta_3 c_2\lambda^2 h \qquad |\tau_{h-1}-\tau_0|<c_1|\lambda|\\
|\vartheta_{h-1}-\vartheta_0|<c_1|\lambda| \quad |\delta_{h-1}-\delta_0|<c_1|\lambda
\end{gathered} \tag{4.3}$$

PROOF We proceed by induction. Assume that the above inequalities hold for any $k\geq h$ and we prove them for $h-1$. By (4.3) and the analiticity of the Beta function we obtain

$$\sigma_h(1-\beta_2 g_h-c_a\varepsilon_h^2)\leq\sigma_{h-1}\leq\sigma_h(1-\beta_2 g_h+c_b\varepsilon_h^2)$$

with $c_b,c_a>0$ are suitable constants. Then, if $\lambda>0$ (for fixing ideas) there exists a $c_4<1$ such that

$$\sigma_{h-1}\leq u\gamma^{c_4\lambda\beta_2(h-1)}\frac{(1-\beta_2\lambda+c_b\varepsilon_h^2)}{\gamma^{-\lambda c_4\beta_2}}\leq u\gamma^{\lambda c_4\beta_2(h-1)}$$

and a $c_3>1$ such that

$$\sigma_{h-1}\geq u\gamma^{c_3\lambda\beta_2(h-1)}\frac{(1-\beta_2\lambda-c_a\varepsilon_h^2)}{\gamma^{-\lambda c_3\beta_2}}\geq u\gamma^{c_4\beta_2(h-1)}$$

We proceed in the same way for Z_h. Moreover :

$$\lambda_h\leq\lambda^2\sum_{h=h^*}^{0}\frac{\tilde{\sigma_h}}{\gamma^h}\leq\lambda^2\sigma_{h^*}\gamma^{-h^*}\sum_{h=h^*}^{0}\gamma^{-h+h^*}\frac{\sigma_h}{\sigma_{h^*}}\leq K$$

if K is a constant. For the other constants we proceed in a similar way.■

As a consequence of the above estimates on the running coupling constants fi find that h^* is finite:

$$\frac{\log_\gamma(u)}{1-\lambda\beta_2 c_\alpha}\leq h^*\leq\frac{\log_\gamma(u)+1}{1-\lambda\beta_2 c_\beta} \tag{4.4}$$

Noting that the above recursive relation of course implyes that $K_1\sigma_h\leq\sigma_h(k)\leq K_2\sigma_h$ the proof of the convergence for the exapnsion for the partition function is completed.

4.3 From the proof of the convergence of the partition function it is possible, repeating essentially word by word the analogous proof in [BGPS], to deduce the convergence of the Schwinger function. We do not repeat such computations here. It holds that

$$S(x,y) = S^{u.v.}(x,y) + \sum_{h=h^*}^{0} \sum_{\omega_1,\omega_2} e^{i\pi(\omega_1 \mathbf{x} - \omega_2 \mathbf{y})} \cdot \Big(\frac{g^{(h)}_{\omega_1,\omega_2}(x-y)}{Z_h} + \frac{\bar{S}^{(h)}_{\omega_1,\omega_2}(x,y)}{Z_h}\Big) \tag{4.5}$$

where we call $g^{(\leq h^*)}(x-y)$ simply $g^{(h^*)}(x-y)$ and the first addend is obtained by the integration of $\psi^{(1)}$ (and it is bounded by $\frac{C_N}{1+|\mathbf{x}-\mathbf{y}|^N}$; $\bar{S}^{(h)}_{\omega_1,\omega_2}(x,y)$ is given by a sum of Feynmann graphs similar to the ones contributing to the effective potential, see sec.3, with the difference that:

1) they have two external lines, to which the propagators $g^{(h_1)}_{\omega_1,\omega'}(x-x')$ or $g^{(h_2)}_{\omega'',\omega_2}(y-y')$ are associated, with $h_1, h_2 \geq h$ and if x', y' are respectively the coordinates of the vertex the external lines is entering in or coming out.

2) h is the smallest scale of the propagators contributing to $\bar{S}^{(h)}_{\omega_1,\omega_2}(x-y)$ (and not, as for the graphs for the effective potential, the scale of the external lines). No $\mathcal{R}$ acts on clusters containing the external lines.

One can check, by repeating the computations in [BGPS], sec.6, that $|\bar{S}^{(h)}_{\omega,\omega}(x,y)| \leq A\tilde{\varepsilon}\gamma^h \frac{C_N}{1+(\gamma^h|x-y|)^N}$ if $\tilde{\varepsilon} = \max(u, u^{1+\eta_1}, |\lambda|)$.

We are now ready to prove Theorem 1. First of all, we note that the r.h.s. of (4.5) has a meaning also if we take the formal limit $i \to \infty$, $\beta \to \infty$, (recall that $L = L_i$), by doing the substitution

$$\frac{1}{L_i \beta} \sum_{\mathbf{k} \in \mathcal{D}_{L_i,\beta}} \to \int_{T^1 \times R} \frac{d\mathbf{k}}{(2\pi)^2} \ . \tag{4.6}$$

The substitution (4.6), applied to (4.5), allows to define $g^{(h)}(\mathbf{x};\mathbf{y})$ in the limit $i \to \infty$, $\beta \to \infty$, at least for $h \leq 0$. For $h = 1$, one has to be careful, since the integral over k_0 is not absolutely convergent. However it is easy to prove, by using standard well known arguments, that the limit as $i \to \infty$ and $\beta \to \infty$ of $g^{(1)}(\mathbf{x};\mathbf{y})$ is well defined for $x_0 \neq y_0$ and has the same discontinuity in $x_0 = y_0$ of the same limit taken on the free propagator (2.1). The previous considerations says that, for λ real and small enough, there exists the limit

$$S(\mathbf{x};\mathbf{y}) = \lim_{\substack{\beta \to \infty \\ i \to \infty}} S^{L_i,\beta}(\mathbf{x};\mathbf{y}) \tag{4.7}$$

and that this limit is obtained by doing the substitution (4.6) in all quantities appearing in the r.h.s. of (4.5). In fact let us consider one of the finite $L = L_i$ and β quantities appearing in the r.h.s. of (4.5); if we interpret it as a Riemann sum of the corresponding $L_i = \beta = \infty$ quantity it is easy to prove that the difference between the two quantities can be bounded by a constant times $(1/L_i + 1/\beta)$. In fact one gets essentially the same bounds up to a factor $\gamma^{-h}(1/L_i + 1/\beta)$, coming from the comparison of the integral and the corresponding Riemann sum; we shall not give the details, which are completely straightforward.

4.3 Let us define

$$\eta_1 = -\frac{\log(\bar{Z}_{h^*})}{\log u} \qquad 1 + \eta_2 = \frac{\log(\sigma_{h^*})}{\log u}$$

which are respectively $\beta_1 \lambda^2 + O(\lambda^3)$ and $\beta_2 \lambda + O(\lambda^2)$. The bounds (1.5) can be obtained by (4.5). In fact if $1 \leq |\mathbf{x} - \mathbf{y}| \leq \hat{u}(p_F)^{-1}$ it holds, for $\gamma^{-h_x - 1} < |x-y| \leq \gamma^{-h_x}$, $h_x > h^*$,

$$\sum_{h=0}^{h^*} |\frac{g^{(h)}_{\omega_1,\omega_2}(x-y)}{Z_h} + \frac{\bar{S}^{(h)}_{\omega_1,\omega_2}(x,y)}{Z_h})| \leq A_1 \left[\sum_{h=h^*}^{h_x - 1} \frac{\gamma^h}{Z_h} + \sum_{h=h_x}^{0} \frac{\gamma^h}{Z_h} \frac{C_N}{\gamma^{Nh}|x-y|^N} \right] \leq A_2 \gamma^{h_x(1-\eta_3)}$$

On the other hand if $|\mathbf{x}-\mathbf{y}|\geq \hat u(p_F)^{-1}$:

$$\sum_{h=0}^{h^*}|\frac{g^{(h)}_{\omega_1,\omega_2}(x-y)}{Z_h}+\frac{\bar S^{(h)}_{\omega_1,\omega_2}(x,y)}{Z_h})|\leq \frac{C_N}{|x-y|^N}\sum_{h=h^*+1}^{1}\frac{\gamma^{-(N-1)h}}{Z_h}A_2\frac{\gamma^{h^*}}{Z_{h^*}}\leq$$

$$\frac{C_N}{(\gamma^{h^*}|x-y|)^N}$$

¿From the above bounds and lemma 2.6 it follows that

$$\sum_{h=0}^{h^*}\sum_{\omega_1,\omega_2}e^{ip_F(\omega_1\mathbf{x}-\omega_2\mathbf{y})}\frac{g^{(h)}_{\omega_1,\omega_2}(x-y)}{Z_h}-\sum_{\omega}e^{ip_F(\omega(x-y)}\frac{1}{(L\beta)}\sum_{\mathbf{k}'\in\mathcal{D}_{L,\beta}}\frac{f_h(\mathbf{k}')}{-ik_0+\omega k'}\frac{1}{|\mathbf{k}'|^{\eta_3}}\leq$$

$$C\sum_{h=0}^{h^*}[\frac{\sigma_h}{Z_h}]\leq C_1\frac{\tilde\varepsilon}{\hat Z(p_F)}\log\hat u(P_F)$$

so proving (1.10).

4.3 Let us prove the decomposition (1.8). By diagonalizing the quadratic form $\hat g^{(h)}(x,y)=\sum_{\omega_1,\omega_2}e^{ip_F(\omega_1\mathbf{x}-\omega_2\mathbf{y})}g^{(h)}_{\omega_1,\omega_2}(x-y)$ it is possible to see that:

$$\hat g^{(h)}(x,y)=\frac{1}{(L\beta)}\sum_{\mathbf{k}_1'\in\mathcal{D}_{L,\beta}}e^{ik'(x-y)}\frac{\tilde f_h(k')}{Z_h}[\frac{F_{xy}(k',\sigma_h)}{A+B}+\frac{F_{xy}(-k',-\sigma_h)}{A-B}]$$

where

$$F_{xy}(k',\sigma_h)=\hat\phi(k',\mathbf{x},\sigma_h)\hat\phi(k',-\mathbf{y},\sigma_h)$$

$$\hat\phi(k',\mathbf{x},\sigma_h)=\frac{1}{\sqrt{2B}}[\sqrt{B-C}e^{ip_F\mathbf{x}}-\frac{e^{-ip_F\mathbf{x}}}{\sqrt{B-C}}]$$

and $A=-ik_0+\mathbf{k}'^2$, $B=\sqrt{(C^2+\sigma_h^2)}$ and $C=2\pi\mathbf{k}'$. We can rewrite the above integral in terms of the k variable. Recall that, if $\omega=\mathrm{sign}(\mathbf{k})$ and $f_h(k')\neq 0$, then $k=\omega p_F+\mathbf{k}'$. Hence

$$\hat g^h(x-y)=\sum_{\omega_1,\omega_2}e^{ip_F(\omega_1\mathbf{x}-\omega_2\mathbf{y})}g^h_{\omega_1,\omega_2}(x-y)=$$

$$=\frac{1}{(L\beta)}\sum_{\mathbf{k}\mathcal{D}_{L,\beta}}\frac{\tilde f_h(k)}{Z_h(k)}\frac{\hat\phi(\mathbf{k},\mathbf{x},\sigma_h))\hat\phi(\mathbf{k},-\mathbf{y},\sigma_h)e^{ik_0(x_0-y_0)}}{-ik_0-(\hat\varepsilon(k,\sigma_h)-p_F^2)}$$

where $\hat\varepsilon(k,\sigma_h),u(\mathbf{k},\mathbf{x},\sigma_h)$ are defined in the theorem 1.

4.4 Finally the lower bound on the spectral gap can be obtained showing that we can prove the boundedness on $S(x,y;k_0)$ for k_0 complex, for $|\mathrm{Im}\,(k_0)|\leq\frac{\sigma_{h^*}}{2}$. The expansion disussed before is not suitable for this, as the functions $\hat f_h(\mathbf{k})$ are not analytic in k_0. However one can repeat the above analysis using a decomposition $\hat f_h(k)$ *i.e.* not depending on k_0 (see [BGM] app.4 in which are discussed the (obvious) modifications in the bounds changing the cut-off functions). So in the support of $\hat f_h(k)$, for $h>h^*$ we have that, if $|\eta|\leq\frac{\sigma_{h^*}}{2}$

$$|\mathrm{Re}\,[A_h(k',k_0+i\eta]|\geq a_0\gamma^{2h}+\sigma_h^2-\eta^2\geq\frac{\gamma^{2h}}{4}$$

as, by (2.68), $\gamma^h>\gamma^{h^*}\geq\frac{4\sigma_{h^*}}{a_0}$. On the other hand if $h\leq h^*$

$$|\mathrm{Re}\,[A_h(k',k_0+i\eta]|\geq\frac{c_1}{\gamma^{h^*}}$$

for some constant c_1, so everothng is essentially unchanged.

References

[A] S. Aubry in Polarons and Bipolarons in *Hich T_c superconductors and related materials*, editors Salye E, Alexandrov A., Liang W.

[AAR] S. Aubry, G. Abramovici, J. Raimbaut: Chaotic polaronic and bipolaronic states in the adiabatic Holstein model, *J. Stat. Phys.* **67**, 675–780 (1992).

[B] A.D. Bryuno: The analytic form of differential equations I, *Trans. Moscow Math. Soc.* **25**, 131–288 (1971); II, *Trans. Moscow Math. Soc.* **26**, 199–239 (1972).

[BG] G.Benfatto, G.Gallavotti: Perturbation theory of the Fermi surface in a quantum liquid. A general quasiparticle formalism and one dimensional systems, *J. Stat. Phys.* **59**, 541–664 (1990).

[BGL] G.Benfatto, G.Gallavotti, J. Lebowitz: Disorder in the 1D Holstein model, *Hel.Phys. Acta* (1992).

[BGM1] G.Benfatto,G.Gentile,V.Mastropietro: to appear on *J. Stat. Phys.* (1997).

[BGM2] G.Benfatto,G.Gentile,V.Mastropietro: preprint (1997).

[BGPS] G.Benfatto, G.Gallavotti, A.Procacci, B.Scoppola: Beta functions and Schwinger functions for a many fermions system in one dmension, *Comm.Math.Phys.* **160**, 93–171 (1994).

[BM1] F.Bonetto,V.Mastropietro: Beta function and anomaly of the Fermi surface for a d=1 system of interacting fermions in a periodic potential, *Comm.Math.Phys.* **172**, 57-93 (1995).

[BM2] F.Bonetto,V.Mastropietro: Filled band Fermi systems *Mat.Phys.Elect.Jour* **2**, 1–43 (1996).

[BM3] F.Bonetto,V.Mastropietro: Critical indices in a d=1 filled band Fermi system, *Phys. Rev. B* **56**,3,1296-1308 (1997).

[BM4] F.Bonetto,V.Mastropietro: Critical indices for the Yukawa2 quantum field theory, *Nucl.Phys.B* **497**, 541-554 (1997).

[BeGM] G.Benfatto, G.Gallavotti,V.Mastropietro: Renormalization group and the Fermi surface in the Luttinger model, *Phys. Rev. B* **45**, 5468-5480, (1992).

[BLT] J. Belissard, R. Lima, D. Testard: A metal-insulator transition for almost Mathieu model, *Comm. Math. Phys.* **88**, 207–234 (1983).

[CF] L. Chierchia, C. Falcolini: A direct proof of a theorem by Kolmogorov in hamiltonian systems, *Ann. Scuola Norm. Super. Pisa Cl. Sci.* **21**, No. 4, 541–593 (1995).

[D] H. Davenport: *The Higher Arithmetic*, Dover, New York, 1983.

[DS] E.I. Dinaburg, Ya.G. Sinai: On the one dimensional Schroedinger equation with a quasiperiodic potential, *Funct. Anal. and its Appl.* **9**, 279–289 (1975).

[E] L.H. Eliasson: Floquet solutions for the one dimensional quasi periodic Schroedinger equation, *Comm. Math. Phys.* **146**, 447–482 (1992).

[E1] L.H. Eliasson: Absolutely convergent series expansions for quasi-periodic motions, Report 2–88, 1–31, Department of Mathematics, University of Stockholm (1988) and *Mat.Phys.Elect.Jour* **2** .

[FMRS] J. Feldman, J.Magnen, V.Rivasseau, E.Trubowitz: An infinite volume expansion for many fermions Freen functions, *Helv.Phys.Acta.* **65** 679-721,(1992)

[G1] G. Gallavotti: Twistless KAM tori, *Comm. Math. Phys.* **164**, 145–156 (1994).

[G2] G. Gallavotti: Renormalization group and ultraviolet stability for scalar fields via renormalization group methods *Rev. Mod. Phys.* **57** 471-562 (1985)

[GM] G. Gentile, V. Mastropietro: Methods for the analysis of the Lindstedt series for KAM tori and renormalizability in classical mechanics. A review with some applications, *Rev. Math. Phys.* **8**, 393–444 (1996).

[H] T. Holstein: Studies of polaron motion, part 1. The molecular-crystal model. *Ann. Phys.* **8**, 325–342, (1959).

[JM] R.A. Johnson, J. Moser: The rotation number for almost periodic potentials, *Commun. Math. Phys.* **84**, 403–438 (1982).

[KL] T. Kennedy, E.H. Lieb: An itinerant electron model with crystalline or magnetic long range order, *Physica A* **138**, 320–358 (1986).

[LAMR] Lee P.A., Rice P.M., Anderson P.W., Solid State Comm. 14,703,1974.

[L] Lee P.A. Nature 291,11-12,1981.

[ML] V.Mastropietro: Interacting soluble Fermi systems in one dimension, *Nuovo Cim.B* **1**, 304-312, (1994).

[ML] D.Mattis, E.Lieb: Exact solution of a many fermion system and its associated boson field, *Journ.Math.Phys.* **6**, 304-312, (1965).

[MP] J. Moser, J. Pöschel: An extension of a result by Dinaburg and Sinai on quasi periodic potentials, *Comment. Math. Helv.* **59**, 39–85 (1984).

[NO] J.W. Negele, H. Orland: *Quantum many-particle systems*, Addison-Wesley, New York, 1988.

[PF] L. Pastur, A. Figotin: *Spectra of random and almost periodic operators*, Springer, Berlin, 1991.

[P] R.E. Peierls: *Quantum theory of solids*, Clarendon, Oxford, 1955.