MICROLOCAL TECHNIQUES FOR SEMICLASSICAL PROBLEMS IN GEOMETRIC QUANTIZATION

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1. Introduction

Recently the microlocal analysis of Toeplitz operators developed by Boutet de Monvel, Sjöstrand, and Guillemin [10], [9], has been used to obtain various semiclassical results for geometrically quantized compact Kähler manifolds [4], [6], [7]. In this paper we will give an expository account of these methods and results. Part of our approach will be to work out the analogous theory for quantization of $\mathbb{C}^n$. The theory of [9] does not apply directly to this case, because of the non-compactness, but the microlocal structure is the same.[1] And the explicitness with which one can write things down in this case makes the structure of the arguments, and in particular the prominence of stationary phase, more transparent.

The development of this theory for $\mathbb{C}^n$ will also yield an improvement of a result of Coburn [11], a theorem connecting the Poisson bracket of two functions two the semiclassical limit of the commutator of the associated operators. The class of functions for which this was proven in [11] constists of trigonometric polynomials plus functions with compact support. We are able to enlarge this to the class of bounded functions with bounded derivatives (see Section 4).

2. The compact Kähler case

Let X be a compact Kähler manifold, of real dimension $2n$ ($\mathbb{C}P^n$, for example, or a coadjoint orbit of a compact Lie group), with symplectic form ω. Let $L \to X$ be a quantizing line bundle, i.e. a holomorphic hermitian line bundle, the curvature of whose natural connection is ω. We form Hilbert spaces by taking holomorphic sections of tensor powers of L,

$$\mathcal{H}_k := L^2_{hol}(X, L^{\otimes k}),$$

where the inner product is given by integrating the hermitian structure with respect to the volume form $\omega^n/n!$. The curvature of the connection induced on $L^{\otimes k}$ is $k\omega$, so according to the principles of geometric quantization, $k = 1/\hbar$. For k sufficiently large, the dimension of $\mathcal{H}_k$ is given by the Riemann-Roch formula.

For this setup there is a simple way to associate operators to functions, often referred to as the Berezin-Toeplitz quantization, after [1], [2]. Let

$$\Pi_k : L^2(X, L^{\otimes k}) \to \mathcal{H}_k$$

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[1] This is well-known, although to my knowledge the details worked out here do not appear elsewhere in the literature.

be the orthogonal projection. Given $f \in C^\infty(X)$ we define $T_k(f) := \Pi_k \circ M(f)$ acting on $\mathcal{H}_k$, where $M(f)$ is the operator of multiplication by f.

It is not obvious at this point how microlocal analysis is relevant to the situation, since the Π_k's are projections onto finite dimensional spaces and thus have smooth kernels. The idea is to roll all the $\mathcal{H}_k$'s together, using a standard trick of analysis of operators on vector bundles.

Let $P \subset L^*$ be the unit circle bundle. P is a principal S^1 bundle, from which each $L^{\otimes k}$ may be constructed as an associated bundle for the appropriate representation of S^1. Thus $C^\infty(X, L^{\otimes k})$ may be naturally identified with $C^\infty(P)_k$, the set of functions on P satisfying $f(p \cdot e^{i\theta}) = e^{ik\theta} f(p)$. P inherits a connection form α from the connection on L, and the condition on the curvature of L implies that $d\alpha = \pi^*\omega$. Thus α is a contact structure, and the corresponding volume form is $dp = \alpha \wedge (d\alpha)^n/2\pi n!$. The identification of sections of $L^{\otimes k}$ with equivariant functions on P extends to an isomorphism

$$L^2(X, L^{\otimes k}) \cong L^2(P, dp)_k.$$

We can now think of $\mathcal{H}_k \subset L^2(P)$ for all k and so define

$$\mathcal{H} = \bigoplus_{k=0}^{\infty} \mathcal{H}_k.$$

This total Hilbert space has a very nice interpretation: $\mathcal{H}$ is the Hardy space of P, i.e. the L^2 space of boundary values of holomorphic functions on the unit disk bundle of L^*. Another way to phrase this is that $\mathcal{H}$ is the L^2 completion of the kernel of the boundary Cauchy-Riemann operator $\overline{\partial}_b$ on P, which is defined by restriction of $\overline{\partial}$ from the unit disk bundle. The orthogonal projector $\Pi : L^2(P) \to \mathcal{H}$ is called the Szegö projector.

As an operator on $L^2(P)$, the quantization of a function $f \in C^\infty(X)$ is

$$T_k(f) = \Pi_k M(f) \Pi_k,$$

where f is pulled back from X to P. (This is of course equivalent to the quantization map given above, but now we regard all operators as acting on $L^2(P)$.) The full operator $T(f) = \Pi M(f) \Pi$ is an example of a Toeplitz operator. A Toeplitz operator of order m is defined to be an operator on $L^2(P)$ which can be written in the form $\Pi Q \Pi$, where $Q \in \Psi^m(P)$, the space of pseudodifferential operators of order m on P. Denote by $\mathcal{T}^m$ the set of Toeplitz operators of order m.

The work of Boutet de Monvel, Sjöstrand, and Guillemin, [10], [9], provides a detailed picture of the microlocal structure of Π.

Theorem 2.1. [9] *The wave front set of the Schwartz kernel of* Π *is*

$$WF(\Pi) = \{(p, r\alpha_p;\ p, -r\alpha_p) \in T^*P \times T^*P;\ p \in P,\ r > 0\}$$

In fact Π is shown to be a Fourier integral operator of Hermite type and its principal symbol is determined. (Hermite type refers to the fact that the wave front set is isotropic but not Lagrangian; this makes the invariant description of the symbol a good deal more complicated.) For semiclassical purposes, the key point is that the only singularities of Π are large k singularities (since each Π_k was smooth). So analysis of the singularities of Π translates into analysis of the semiclassical behavior of Π_k.

The set of all Toeplitz operators is in fact closed under composition. This highly non-trivial fact is a consequence of the following result.

Theorem 2.2. [9] *Given $A \in \mathcal{T}^m$, there exists $Q \in \Psi^m(P)$ satisfying $[\Pi, Q] = 0$ such that $A = \Pi Q \Pi$.*

Theorem 2.1 indicates that Π projects wave-front sets into the set

$$\Sigma = \{(p, r\alpha_p) \in T^*P;\ r > 0\}.$$

Thus as far as the operator $\Pi Q \Pi$ is concerned, the crucial feature of Q should be the behavior of the total symbol near Σ. In fact, the following result shows that the restriction $\sigma(Q)|_\Sigma$ gives a meaningful principal symbol for the Toeplitz operator $\Pi Q \Pi$.

Theorem 2.3. [9] *If $\Pi Q_1 \Pi = \Pi Q_2 \Pi$ then $\sigma(Q_1)|_\Sigma = \sigma(Q_2)|_\Sigma$. Furthermore, if $Q \in \Psi^m(P)$ and $\sigma(Q)|_\Sigma = 0$, then $\Pi Q \Pi \in \mathcal{T}^{m-1}$.*

In [13], Guillemin pointed out that these results lead to a $*$-product on $C^\infty(X)$, which we'll now describe. Let $D_\theta = -i\frac{\partial}{\partial\theta}$, where $\frac{\partial}{\partial\theta}$ is the generator of the S^1 action on P. Thus $D_\theta \Pi_k = k\Pi_k$. We can find a parametrix D_θ^{-1} such that $D_\theta^{-1}\Pi_k = k^{-1}\Pi_k$ for $k \geq 1$.

Lemma 2.4. *For $Q \in \Psi^m(P)$, there exists a constant C such that for all $k \geq 1$*

$$\|\Pi_k Q \Pi_k\| \leq C k^m.$$

Proof. The operator $D_\theta^{-m} Q$ is of order zero, and hence bounded in $L^2(P)$ because P is compact. Let $C = \|D_\theta^{-m} Q\|$. Since Π_k is an orthogonal projection, $\|\Pi_k D_\theta^{-m} Q \Pi_k\| \leq C$. But $\Pi_k D_\theta^{-m} Q \Pi_k = k^{-m} \Pi_k Q \Pi_k$ for $k \geq 1$. □

The following theorem (a slight variation on a theorem of [13]) allows us to expand general Toeplitz operators in terms of operators of the form $T_k(f)$.

Theorem 2.5. *Given $Q \in \Psi^m(P)$ with $[D_\theta, Q] = 0$, there exists a series of functions $f_i \in C^\infty(X)$ such that*

$$\Pi_k Q \Pi_k \sim \sum_{j=0}^{\infty} k^{m-j} T_k(f_j),$$

meaning that if we approximate $\Pi_k Q \Pi_k$ with terms up to k^{-N} the norm of the remainder is $O(k^{-N-1})$.

Proof. The function $\sigma(Q)|_\Sigma$ is invariant under the S^1 action, of order m in r. Hence it must be of the form $r^m f_m$, where f_m is (the pull-back of) a function on X. Now $\Pi Q \Pi - \Pi D_\theta^m M(f_m)\Pi \in \mathcal{T}^{m-1}$ and we simply iterate the process. Lemma 2.4 gives the bound on the remainder □

The $*$-product on $C^\infty(X)$ is the formal power series

$$f * g := \sum_{j=0}^{\infty} k^{-j} B_j(f, g),$$

where $B_j(f, g) \in C^\infty(X)$ is the j-th function in the expansion of $T(f)T(g)$ according to Theorem 2.5 (where $T(f) := \Pi M(f) \Pi$). Associativity of the $*$-product follows from the associativity of Toeplitz operators, and it is clear that $B_0(f, g) = fg$. Consider $[T(f), T(g)]$. According to Theorem 2.2 there are operators Q_f, Q_g, commuting with Π, which yield $T(f)$ and $T(g)$. Then we have

$$[T(f), T(g)] = \Pi[Q_f, Q_g]\Pi.$$

Since Σ is symplectic, the restriction $\sigma([Q_f, Q_g]) = i\{\sigma(Q_f), \sigma(Q_g)\}$ to Σ is the Poisson bracket $i\{f,g\}_\Sigma$. The symplectic form on Σ is $dr \wedge d\alpha + r d\alpha$. Recalling that $d\alpha = \pi^*\omega$, we have then $\{f,g\}_\Sigma = r^{-1}\{f,g\}_\omega$. This means that

$$B_1(f,g) - B_1(g,f) = i\{f,g\}_\omega,$$

so indeed we have a $*$-product.

Of course, Theorem 2.5 gives more than a purely formal $*$-product, since we have operator norm bounds on remainders. We can in particular show that the map $f \mapsto T_k(f)$ satisfies two important properties in Rieffel's definition of deformation quantization in the the category of $\mathbb{C}^*$-algebras [19]. This was proven in [4].

Corollary 2.6. *For $f, g \in C^\infty(X)$,*

$$\|T_k(f)T_k(g) - T_k(fg)\| = O(k^{-1}),$$

and

$$\|k[T_k(f)T_k(g)] - iT_k(\{f,g\})\| = O(k^{-1}).$$

To close this section, we consider the issue of orderings. Suppose we have a map $R : C^\infty(X) \to \Psi^0(P)$ such that $\sigma(R_f)|_\Sigma = f$. If we then let $T_k'(f) = \Pi_k R_f \Pi_k$, then all of the properties we have just described remain the same. This amounts to choosing a different ordering in our quantization of f. In particular, Tuynman [20] has pointed out that the operator associated to f by geometric quantization can be written $\Pi_k(f - \frac{1}{2}D_\theta^{-1}\Delta f)\Pi_k$. In Section 3 we will show how to obtain the Weyl quantization by this technique. In general, semiclassical results will depend on the choice of ordering, but most formulas can be translated into a different ordering scheme with simple modifications.

3. Quantization of $\mathbb{C}^n$

We now turn to the $\mathbb{C}^n$ version of the setup just described. We'll use the symplectic form

$$\omega = i\sum_{j=1}^n dz_j \wedge d\overline{z}_j.$$

This corresponds to the standard symplectic form on $\mathbb{R}^{2n}$ under the identification $z_j = (q_j + ip_j)/\sqrt{2}$. We need to find a holomorphic hermitian line bundle L, whose curvature is given by ω. Of course, this line bundle will be trivial, so the hermitian structure can be specified by a positive function $e^{-h(z)}$ (the norm of the constant section 1 at the point z). The curvature for the associated connection will be given by $2i\partial\overline{\partial}h$, so we set $h(z) = |z|^2/2$.

The Hilbert spaces of holomorphic sections of $L^{\otimes k}$ naturally isomorphic to weighted Bargmann spaces

$$\mathcal{H}_k = L^2_{hol}(\mathbb{C}^n, e^{-k|z|^2}\, dl_z),$$

where dl_z is Lebesgue measure $= d^n q\, d^n p$.

The principal bundle $P = \mathbb{C}^n \times S^1$ inherits the connection form

$$\alpha = d\theta + \frac{i}{2}\sum_{j=1}^n (z_j d\overline{z}_j - \overline{z}_j dz_j).$$

Thus the contact volume form is $dp = d\theta\, dl/2\pi$. As in Section 2 we rewrite the Hilbert spaces as subspaces of $L^2(P)$:

$$\mathcal{H}_k = \{F(z,\theta) \in L^2(P) : F(z,\theta) = f(z)e^{ik\theta - k|z|^2/2},\ f \text{ is holomorphic}\}.$$

Then we define the total Hilbert space $\mathcal{H} = \oplus_{k=1}^{\infty} \subset L^2(P)$ and orthogonal projector $\Pi : L^2(P) \to \mathcal{H}$. Although $\mathcal{H}$ indeed consists of boundary values of holomorphic functions on the domain $\{(z,\zeta) \in \mathbb{C}^n \times \mathbb{C} : |\zeta|^2 < e^{-|z|^2}\}$, this Π is not a Szegö projector because the domain is not compact.

The boundary Cauchy-Riemann operator for this case is

$$\overline{\partial}_b f = \sum_j \left(\frac{\partial f}{\partial \overline{z}_j} - \frac{i z_j}{2} \frac{\partial f}{\partial \theta} \right) d\overline{z}_j.$$

Note that $\overline{\partial}_b f = 0$ has solutions which are not boundary values of holomorphic functions, e.g. $e^{-ik\theta + k|z|^2/2}$. But all such solutions grow exponentially so they are not in L^2. We still can write $\mathcal{H}$ as the closure of $\ker \overline{\partial}_b \cap L^2(P)$.

A basis for $\mathcal{H}_k \subset L^2(P)$ is given by

$$\phi_{k,\alpha}(z,\theta) = z^\alpha e^{ik\theta - k|z|^2/2},$$

where $\alpha \in \mathbb{N}^n$ is a multi-index: $z^\alpha = z_1^{\alpha_1} \ldots z_n^{\alpha_n}$. Using this basis allows us to write Π_k explicitly:

$$\begin{aligned}\Pi_k(z,\theta;w,\eta) &= \sum_\alpha \frac{\phi_{k,\alpha}(z,\theta)\overline{\phi_{k,\alpha}(w,\eta)}}{\|\phi_{k,\alpha}\|^2} \\ &= \left(\frac{k}{2\pi}\right)^n e^{ik(\theta-\eta)} e^{kz\cdot\overline{w} - k|z|^2/2 - k|w|^2/2}\end{aligned}$$

This is just the reproducing kernel on Bargmann space, with a phase factor. Using this explicit formula, it is a simple matter to check that

$$WF(\Pi) = \{(p, r\alpha_p; p, -r\alpha_p) \in T^*P \times T^*P; r > 0\},$$

in agreement with Theorem 2.1. As before, let $\Sigma = \{(p, r\alpha_p) \in T^*P; r > 0\}$.

Our basic quantization map is $f \mapsto T_k(f)$. Let

$$\rho_j := T_k(z_j), \qquad \rho_j^* = T_k(\overline{z}_j).$$

A simple computation shows that

$$\rho_j^* \phi(z) e^{ik\theta - k|z|^2/2} = \frac{1}{k}\frac{\partial \phi}{\partial z_j} e^{ik\theta - k|z|^2/2}.$$

Thus ρ and ρ^* behave like creation and annihilation operators, respectively, and

$$[\rho_i^*, \rho_j] = \frac{1}{k}\delta_{ij}.$$

The map T_k moves creation operators to the right:

$$T_k(z^\alpha \overline{z}^\beta) = (\rho^*)^\beta \rho^\alpha,$$

an ordering referred to as anti-Wick, since Wick ordering moves creation operators to the left. For f polynomial, we can write

$$T_k(f) = e^{\rho^* \cdot \frac{\partial}{\partial \overline{z}}}\, e^{\rho \cdot \frac{\partial}{\partial z}}\, f\big|_{z=\overline{z}=0}.$$

On the other hand, the Weyl ordering would symmetric:

$$W_k(f) := e^{\rho^* \cdot \frac{\partial}{\partial \overline{z}} + \rho \cdot \frac{\partial}{\partial z}}\, f\big|_{z=\overline{z}=0}.$$

For example, the two quantizations of the harmonic oscillator potential are

$$T_k(|z|^2) = \rho^* \cdot \rho, \qquad W_k(|z|^2) = (\rho^* \cdot \rho + \rho \cdot \rho^*)/2.$$

To translate between the two schemes amounts to a simple calculation:

$$e^{\rho^* \cdot \frac{\partial}{\partial \bar z} + \rho \cdot \frac{\partial}{\partial z}} = e^{\rho^* \cdot \frac{\partial}{\partial \bar z}} e^{\rho \cdot \frac{\partial}{\partial z}} e^{-\frac{1}{2k} \frac{\partial}{\partial z} \cdot \frac{\partial}{\partial \bar z}}.$$

This means that $W_k(f) = \Pi_k(Wf)\Pi_k$ (at least for f polynomial), where

$$W = e^{-\frac{1}{2} D_\theta^{-1} \frac{\partial}{\partial z} \cdot \frac{\partial}{\partial \bar z}}.$$

Note that Wf is a pseudodifferential operator on P, with principal symbol given by f. One can extend W from polynomials to a class of smooth functions, but we will not go into the details here.

4. Deformation estimates

We want the map $f \mapsto T_k(f)$ to satisfy Rieffel's deformation estimates:

$$\lim_{k\to\infty} \|T_k(f)\| = \|f\|_\infty, \tag{4.1}$$

and

$$\lim_{k\to\infty} \|k[T_k(f), T_k(g)] - iT_k(\{f,g\})\| = 0. \tag{4.2}$$

For the $\mathbb{C}^n$ setup we have described, (4.1) was proven in [11] for all continuous bounded functions, and (4.2) was proven for a restricted class of functions as noted in the introduction.

In this section, we will show that various facts from the theory of [9] may be carried over to $\mathbb{C}^n$ without difficulty despite the non-compactness. We will arrive at a proof of property (4.2) for functions in $C_b^{4n+6}(\mathbb{C}^n)$, where the subscript b indicates that the function and its derivatives out to order $4n+6$ are bounded.

Define the symbol class $S_l^m(T^*P)$ to consist of those $a \in C^l(T^*P)$ satisfying

$$|\partial_u^\alpha \partial_\eta^\beta a(u,\eta)| \le C_{\alpha,\beta}(1+|\eta|)^{m-|\beta|} \qquad \text{for } |\alpha|+|\beta| \le l, (u,\eta) \in T^*P. \tag{4.3}$$

Let $\Psi_l^m(P)$ be the class of pseudodifferential operators whose total symbols are contained in $S_l^m(T^*P)$. Since $\dim T^*P = 4n+2$, the Calderon-Vaillancourt Theorem tells us that any $Q \in \Psi_{4n+3}^0(P)$ is bounded on $L^2(P)$.

Motivated by [9] and the method of proof of Corollary 2.6, our strategy is to find, for a given $f \in C_b(X)$, the pseudodifferential operator Q_f for which

$$[Q_f, \Pi] = 0 \qquad \text{and} \qquad \Pi Q_f \Pi = \Pi M(f) \Pi.$$

All other facts may then be derived from this construction.

First we consider the commutation. To make Q_f commute with Π, it is natural to start by solving the easier problem of $[\bar\partial_b, Q] = 0$. We will adopt (z,ξ) as complex coordinates for $T^*\mathbb{C}^n$. That is, $\xi_j = (\xi_j^q + i\xi_j^p)/\sqrt{2}$. Note that the standard phase function for a pseudodifferential operator becomes $2\,\mathrm{Re}\, z \cdot \bar\xi$ in this notation. This means that $\sigma(\frac{\partial}{\partial z_j}) = i\bar\xi_j$ and $\sigma(\frac{\partial}{\partial \bar z_j}) = i\xi_j$. For future reference, we compute the Fourier transform of a Gaussian in these coordinates.

Lemma 4.1.

$$\int e^{2i\,\mathrm{Re}\, z\cdot\bar\xi} z^\alpha e^{-\lambda|z|^2} dl_z = \left(\frac{2\pi}{\lambda}\right)^n \left(\frac{i}{\lambda}\xi\right)^\alpha e^{-|\xi|^2/\lambda}.$$

Now let q be the total symbol of Q. We'll assume that $[\frac{\partial}{\partial\theta}, Q] = 0$, which means that q is independent of θ. By computing the total symbol of $[\overline{\partial}_b, Q]$, we see that

$$[\overline{\partial}_b, Q] = 0 \Longleftrightarrow \frac{\partial}{\partial\overline{z}_j} q + \frac{i\kappa}{2}\frac{\partial}{\partial\overline{\xi}_j} q = 0, \quad j = 1, \dots, n. \tag{4.4}$$

Lemma 4.2. *For $Q \in \Psi^0_{4n+3}(P)$, if $[D_\theta, Q] = 0$ and $[\overline{\partial}_b, Q] = 0$ then we have $Q\Pi = \Pi Q\Pi$.*

Proof. By Calderon-Vaillancourt, Q is continuous on $L^2(P)$. Also, by assumption $\overline{\partial}_b Q u = Q\overline{\partial}_b u = 0$ for $u \in \mathcal{H}$. Thus Q preserves $\mathcal{H}$, which is another way of saying $Q\Pi = \Pi Q\Pi$. □

In our complex notation, Σ is the set $\{(z,\theta;\xi,\kappa) \in T^*P : \xi = i\kappa z/2, \kappa > 0\}$. We therefore expect the condition $\Pi Q\Pi = \Pi M(f)\Pi$ to require that $q(z,\xi,\kappa) = f(\frac{1}{2}z - \frac{i}{\kappa}\xi)$ to leading order in κ. It turns out the full expression is given in terms of a heat kernel. This is very reminiscent to the relation between the Weyl symbol of an operator on $L^2(\mathbb{R}^n)$ and the Wick or anti-Wick symbol defined through the Bargmann transform (see §2.7 of [12]). Indeed, one should be able to derive Q_f from these known results.

As above, let $C^l_b(\mathbb{C}^n)$ be the space of bounded functions with bounded derivatives to order l.

Theorem 4.3. *For $f \in C^l_b(\mathbb{C}^n)$ let Q_f be the pseudodifferential operator on P with total symbol q_f, which for $\kappa \geq 1$ is given by*

$$q_f(z,\xi,\kappa) = (\tfrac{\kappa}{\pi})^n \chi(\kappa) \int e^{-2\kappa|z/2 - i\xi/\kappa - w|^2} f(w)\, dl_w, \tag{4.5}$$

where $\chi \in C^\infty(\mathbb{R})$ satisfies $\chi(\kappa) = 0$ for $\kappa < 1/2$ and $\chi(\kappa) = 1$ for $\kappa \geq 1$. Then $Q_f \in \Psi^0_l(P)$ with

$$[\Pi, Q_f] = 0 \qquad \text{and} \qquad \Pi Q_f \Pi = \Pi M(f)\Pi.$$

Proof. The cutoff χ is needed to make q_f smooth but serves no other purpose. Indeed, since taking $Q_f\Pi_k$ will set $\kappa = k$, and $\Pi = \sum_{k\geq 1}\Pi_k$, the choice of χ has no effect on the result. Henceforth we will simply drop it from the notation.

By a simple change of variables,

$$q_f(z,\xi,\kappa) = (\tfrac{\kappa}{\pi})^n \int e^{-2\kappa|w|^2} f(w + \tfrac{1}{2}z - \tfrac{i}{\kappa}\xi)\, dl_w.$$

This allows us to estimate derivatives of q_f in terms of derivatives of f, so that $f \in C^l_b(\mathbb{C}^n)$ implies $q_f \in S^0_l(T^*P)$. And an application of (4.4) shows that $[D_j, Q_f] = 0$. Thus $Q_f\Pi = \Pi Q_f \Pi$ from Lemma 4.2.

Now we need to show that $\Pi Q_f\Pi = \Pi M(f)\Pi$, which is equivalent to showing $(\phi_{k,\beta}, Q_f\phi_{k,\alpha}) = (\phi_{k,\beta}, f\phi_{k,\alpha})$, for all k, α, β. But note that

$$\Big(\frac{\partial}{\partial z}\Big)^\beta \phi_{k,\alpha}(0) = \beta!\delta_{\alpha,\beta},$$

which implies $(\phi_{k,\beta}, u) = \beta!(\frac{\partial}{\partial z})^\beta u(0)$ for $u \in \mathcal{H}_k$. Since Q_f preserves $\mathcal{H}_k$, it will thus suffice to compare $(\frac{\partial}{\partial z})^\beta Q_f\phi_{k,\alpha}(0)$ to

$$\Big(\frac{\partial}{\partial z}\Big)^\beta \Pi M(f)\phi_{k,\alpha}(0) = \Big(\frac{k}{2\pi}\Big)^n \int (k\overline{z})^\beta f(z) z^\alpha e^{-k|z|^2}\, dl_z \tag{4.6}$$

Using the definition (4.5),

$$\begin{aligned}
\Big(\frac{\partial}{\partial z}\Big)^{\beta} Q_f\phi_{k,\alpha}(0) &= \tfrac{1}{(2\pi)^{2n}}\Big(\frac{\partial}{\partial z}\Big)^{\beta}\int e^{2i\,\mathrm{Re}(z-w)\cdot\bar\xi} q_f(z,\xi,k) w^{\alpha} e^{-k|w|^2/2}\,dl_\xi\,dl_w\Big|_{z=0}\\
&= (\tfrac{k}{4\pi^3})^n\int e^{-2i\,\mathrm{Re}\,w\cdot\bar\xi}(i\bar\xi+\tfrac{\partial}{\partial z})^{\beta} e^{-2k|z/2-i\xi-y|^2} f(y) w^{\alpha} e^{-k|w|^2/2}\,dl_\xi\,dl_w\,dl_y\Big|_{z=0}\\
&= (\tfrac{k}{2\pi^3})^n\int e^{-2i\,\mathrm{Re}\,w\cdot\bar\xi}(k\bar y)^{\beta} e^{-2|\xi-iky|^2/k} f(y) w^{\alpha} e^{-k|w|^2/2}\,dl_\xi\,dl_w\,dl_y
\end{aligned}$$

Performing the w integration using Lemma 4.1 gives

$$\begin{aligned}
\Big(\frac{\partial}{\partial z}\Big)^{\beta} Q_f\phi_{k,\alpha}(0) &= \tfrac{1}{\pi^{2n}}\int (k\bar y)^{\beta} f(y)(-\tfrac{2i}{k}\xi)^{\alpha} e^{-2k|z/2-i\xi-y|^2} e^{-2|\xi|^2/k}\,dl_\xi\,dl_y.\\
&= \tfrac{1}{\pi^{2n}}\int (k\bar y)^{\beta} f(y)(-\tfrac{2i}{k}\xi)^{\alpha} e^{-4|\xi-iky/2|^2/k} e^{-k|y|^2}\,dl_\xi\,dl_y.
\end{aligned}$$

Now change variables $\xi\to\xi-iky/2$,

$$\Big(\frac{\partial}{\partial z}\Big)^{\beta} Q_f\phi_{k,\alpha}(0) = \tfrac{1}{\pi^{2n}}\int (k\bar y)^{\beta} f(y)(y-\tfrac{2i}{k}\xi)^{\alpha} e^{-4|\xi|^2/k} e^{-k|y|^2}\,dl_\xi\,dl_y.$$

If we expand $(y-\frac{2i}{\kappa}\xi)^{\alpha}$ and integrate over ξ only the y^{α} term survives (the others are zero by symmetry). So we finally obtain

$$\Big(\frac{\partial}{\partial z}\Big)^{\beta} Q_f\phi_{k,\alpha}(0) = (\tfrac{k}{2\pi})^n\int (k\bar y)^{\beta} f(y) y^{\alpha} e^{-k|y|^2}\,dl_\xi\,dl_y.$$

Comparing this to (4.6), we have shown that $(\phi_{k,\beta}, Q\phi_{k,\alpha}) = (\phi_{k,\beta}, f\phi_{k,\alpha})$ and thus $\Pi Q_f\Pi = \Pi M(f)\Pi$.

To conclude the proof, a simple calculation shows that $(Q_f)^* = Q_{\bar f}$. This implies

$$\Pi Q_f = (Q_{\bar f}\Pi)^* = (\Pi M(\bar f)\Pi)^* = \Pi M(f)\Pi = Q_f\Pi.$$

□

Lemma 4.4. *The principal symbol of Q_f is*

$$\sigma(Q_f)(z,\xi,\kappa) = f(\tfrac12 z - \tfrac{i}{2\kappa}\xi)$$

More precisely, for $f\in C_b^l(\mathbb{C}^n)$,

$$q_f(z,\xi,\kappa) - f(\tfrac12 z-\tfrac{i}{2\kappa}\xi)\in S^{-1}_{l-2}(T^*P)$$

Proof. The expression,

$$q_f(z,\xi,\kappa) = (\tfrac{\kappa}{\pi})^n\int e^{-2\kappa|w|^2} f(w+\tfrac12 z-\tfrac{i}{\kappa}\xi)\,dl_w,$$

can be estimated for large κ by a variation of the stationary phase argument. To apply this, one needs to cut off the integral for large w, which gives an error term that decays exponentially in κ. Neglecting this correction, we have

$$|q_f(z,\xi,\kappa) - f(\tfrac12 z-\tfrac{i}{\kappa}\xi)| \le C\kappa^{-1}\sum_{|\alpha|=2}\sup(\partial^{\alpha} f)$$

For the derivatives, first differentiate inside the integral and then make a similar estimate. In each case 2 extra derivatives are required for the estimate. □

Theorem 4.5. *For $f, g \in C_b^{4n+5}(\mathbb{C}^n)$,*

$$\|T_k(f)T_k(g) - T_k(fg)\| = O(k^{-1}). \tag{4.7}$$

And for $f, g \in C_b^{4n+6}(\mathbb{C}^n)$,

$$\|k[T_k(f), T_k(g)] - T_k(f,g)\| = O(k^{-1}). \tag{4.8}$$

Proof. To prove (4.7) note that

$$T_k(f)T_k(g) - T_k(fg) = (Q_f Q_g - Q_{fg})\Pi.$$

By Lemma 4.4, $Q_f Q_g$ and Q_{fg} have the same principal symbol, so $Q_f Q_g - Q_{fg} \in \Psi^{-1}_{4n+3}(P)$. This implies $D_\theta(Q_f Q_g - Q_{fg}) \in \Psi^0_{4n+3}(P)$, where $D_\theta = -i\frac{\partial}{\partial\theta}$. By Calderon-Vaillancourt $D_\theta(Q_f Q_g - Q_{fg})$ is bounded. But

$$D_\theta(Q_f Q_g - Q_{fg})\Pi_k = k(T_k(f)T_k(g) - T_k(fg)).$$

Since $\|\Pi_k\| = 1$ we see that

$$k\|T_k(f)T_k(g) - T_k(fg)\| \le C,$$

where C is independent of k, so (4.7) is proven.

The principle is the same for (4.8). With our complex notation for T^*P, we have

$$\begin{aligned}
\sigma([Q_f, Q_g]) &= -i\{f(\tfrac{1}{2}z - \tfrac{i}{\kappa}\xi), g(\tfrac{1}{2}z - \tfrac{i}{\kappa}\xi)\}_{T^*P} \\
&= -2i\sum_j \left(\frac{\partial}{\partial z_j}\frac{\partial}{\partial \bar\eta_j} + \frac{\partial}{\partial \bar z_j}\frac{\partial}{\partial \eta_j} - \frac{\partial}{\partial \xi_j}\frac{\partial}{\partial \bar w_j} - \frac{\partial}{\partial \bar\xi_j}\frac{\partial}{\partial w_j}\right) f(\tfrac{1}{2}z - \tfrac{i}{\kappa}\xi)g(\tfrac{1}{2}w - \tfrac{i}{2\kappa}\eta)\Big|_{\substack{w=z\\ \eta=\xi}} \\
&= \frac{1}{\kappa}\sum_{j=1}^n \left(\frac{\partial f}{\partial z_j}\frac{\partial g}{\partial \bar z_j} - \frac{\partial f}{\partial \bar z_j}\frac{\partial g}{\partial z_j}\right)\Big|_{z/2 - i\xi/2\kappa} \\
&= \frac{i}{\kappa}\{f, g\}\Big|_{z/2 - i\xi/2\kappa}.
\end{aligned}$$

So that $\sigma([Q_f, Q_g]) = \kappa^{-1}\sigma(Q_{\{f,g\}})$. Thus $D_\theta[Q_f, Q_g] - Q_{\{f,g\}} \in \Psi^0_{4n+3}(P)$ and the result follows as above. $\square$

It is natural to try to extend these results to more general complex domains. Indeed, in [5] the deformation estimates were proven for all non-compact Cartan domains. The proof relied heavily on the symmetry of the domains and required functions to have compact support. The methods presented above make no use of symmetry, so one could imagine extending the results to a much broader class of domain. The key difficulty is to prove estimates for q_f which show that Q_f and associated operators are bounded. One cannot localize the problem in the usual manner, because the smooth error terms may be unbounded. Work to overcome this difficulty is now in progress.

5. Semiclassical spectral theory

If we fix a Hamiltonian function $H \in C^\infty(X)$, then we have a corresponding family of operators

$$\hat H_k := \Pi_k M(\pi^* H)\Pi_k \text{ on } L^2(P)$$

Define a set eigenvalues and normalized eigenfunctions:

$$\hat H_k \psi_{k,j} = \lambda_{k,j}\psi_{k,j}, \qquad j \in \mathbb{N}.$$

One of the major goals of semiclassical analysis is to show how the classical theory (i.e. the Hamiltonian flow) is manifested in the large k behavior of $\psi_{k,j}$ and $\lambda_{k,j}$. Let ϕ_τ denote the Hamiltonian flow of H on X.

A standard way to study the semiclassical behavior of the spectrum of $\hat{H}_k$ is to take a trace weighted near a particular energy. Let φ be a test function. Fix an energy E and consider the sums

$$\sum_j \varphi(k(\lambda_{k,j} - E)) \tag{5.1}$$

and

$$\sum_j \varphi(k(\lambda_{k,j} - E))\, \psi_{k,j}(p_1)\, \overline{\psi_{k,j}(p_2)}, \qquad p_1, p_2 \in P. \tag{5.2}$$

The effect of the test function is to restrict to a range of energies near E, whose size is $O(k^{-1})$. The goal is to produce large k asymptotic expansions for these expressions. Expansions of traces of the form (5.1) have now been proven in various settings. The more unusual localized form (5.2) is a Kähler version of expressions whose asymptotic expansions were studied in [18].

For X compact Kähler, large k asymptotic expansions were given for these sums in [7] (with restrictions on the regularity of ϕ_τ and with the Fourier transform of φ compactly supported). The trace (5.1) has an expansion whose terms correspond to fixed point sets in the energy surface $H^{-1}(E)$ of the map ϕ_τ for some value τ. For instance, the $\tau = 0$ fixed point set is the whole energy surface, and the leading contribution is Ck^{n-1} times the volume of $H^{-1}(E)$. (When E is a regular value of H the energy surface inherits a natural volume form from the Liouville form.)

And the localized trace (5.2) decays rapidly unless both points are on the energy surface and connected by a finite time classical path. Each classical path connecting the points gives rise to a term in the expansion.

To illustrate this behavior, we turn the classical example of a harmonic oscillator. Take $X = \mathbb{C}^n$ with the quantization setup as in Section 3, and let $H(z) = |z|^2$. The basis $\{\phi_{k,\alpha}\}$ conveniently provides the set of eigenstates:

$$\hat{H}_k \phi_{k,\alpha} = \tfrac{1}{k}(|\alpha| + n)\phi_{k,\alpha}.$$

(See the discussion of ordering in Section 3.)

For simplicity, take φ supported in $(n - 1/2, n + 1/2)$, with $\varphi(n) = 1$. And assume that E is an integer. Then (5.1) becomes

$$\sum_\alpha \varphi(k^{-1}(\lambda_{k,\alpha} - E)) = \#\{\alpha \in \mathbb{N}^n : |\alpha| = kE\} = \binom{n + kE - 1}{n - 1}$$

Applying Stirling's formula gives the asymptotic formula

$$\sum_\alpha \varphi(k^{-1}(\lambda_{k,\alpha} - E)) = \frac{(kE)^{n-1}}{(n-1)!} + O(k^{n-2})$$

This calculation agrees with the result quoted above, the canonical volume of $H^{-1}(E)$ being $\pi^n E^{n-1}/(n-1)!$.

To compute the localized trace (5.2), we must normalize $\phi_{k,\alpha}$ by a factor of $\sqrt{k^{n+|\alpha|}/(2\pi)^n\alpha!}$. Then we have

$$\begin{aligned}\sum_\alpha \varphi(k^{-1}(\lambda_{k,\alpha}-E))(\tfrac{k}{2\pi})^n \tfrac{k^{|\alpha|}}{\alpha!}\phi_{k,\alpha}(z,0)\overline{\phi_{k,\alpha}(w,0)} \\ = \sum_{|\alpha|=kE} (\tfrac{k}{2\pi})^n \tfrac{k^{|\alpha|}}{\alpha!} z^\alpha \overline{w}^\alpha e^{-k|z|^2/2-k|w|^2/2} \\ = (\tfrac{k}{2\pi})^n k^{kE} \tfrac{1}{(kE)!}(z\cdot\overline{w})^{kE} e^{-k|z|^2/2-k|w|^2/2}.\end{aligned}$$

By Stirling's formula this expression is

$$\sim \frac{k^{n-1/2}}{\pi^n\sqrt{2\pi E}}\left(\frac{z\cdot\overline{w}}{E}\right)^{kE} e^{-k|z|^2/2-k|w|^2/2+kE}$$

for large k. Two conditions have to be met for this expression not to decay exponentially for large k. Namely $|z|^2=|w|^2=E$ (the points must lie on the energy surface) and $z=e^{i\theta}w$ for some θ (the points must be connected by a classical trajectory). When these conditions are satisfied the resulting order is $k^{n-1/2}$.

Note that for both cases the asymptotics resulted from an application of Stirling's formula. When one considers that Stirling's formula can be obtained through stationary phase approximation, the method used here is really quite close to that of [7].

The proofs in [7] are based on the following strategy, which originated in [14]. Suppose you want to know the large k behavior of quantities A_k. Define

$$\Upsilon(\theta)=\sum_k A_k e^{ik\theta}.$$

If Υ turns out to be smooth, then the A_k's must decay rapidly. And if Υ has singularities, one can translate precise knowledge of these singularities into knowledge of the large k behavior of the A_k.

The trick then is to write Υ as a composition of distributions, in such a way that one can trace the singularities. To analyze (5.1) or (5.2), the appropriate Υ is written as a composition of kernels of Hermite Fourier integral operators. The necessary composition theorems were worked out in [9]. The kernels of the operators are written locally as oscillatory integrals of a particular form, and the determination of singularities in the composition basically reduces to a stationary phase approximation.

For the case of the localized trace formula (5.2), Υ can be constructed from the following operators. Let

$$B=e^{-t\Pi D_\theta M\Pi},$$

as an operator from $C^\infty(P\times P)\to C^\infty(\mathbb{R})$. Let $\mathcal{B}$ be the Schwartz kernel of this operator. Having chosen φ, E, p_1, p_2, we define $F: C_0^\infty(\mathbb{R}\times P\times P)\to C^\infty(S^1)$ by

$$F(f)(\beta)=\int e^{i\kappa(\beta+tE-\theta)}\hat{\varphi}(t)f(t,p_1\cdot e^{i\theta},p_2)\,d\kappa\,d\theta\,dt.$$

To see that the desired Υ is $F(\mathcal{B})$, write $\mathcal{B}$ as a sum over the basis $\psi_{k,j}$. (Of course, one needs to check first that F can be extended to distributions such as $\mathcal{B}$.)

To see where singularities may occur in Υ is a simple matter of composing wavefront sets (most of the work comes in computing the leading behavior). F is an

FIO of standard type, with wave front set

$$\Gamma \subset T^*S^1 \times T^*\mathbb{R} \times T^*P \times T^*P$$

given by

$$\Gamma = \{(\theta - tE, J(p_1,\eta_1);\ t, EJ(p_1,\eta_1);\ p_1 \cdot e^{i\theta}, \eta_1; p_2, \eta_2) : \theta \in \mathbb{R}/2\pi\mathbb{Z}, t \in \mathbb{R}, \eta_i \in T^*_{p_i}P\}, \tag{5.3}$$

where $J = \sigma(D_\theta)$. Because α is a connection form, $J(p, r\alpha_p) = r$.

The wave front set of $\mathcal{B}$ is the moment Lagrangian in $T^*\mathbb{R} \times \Sigma \times \Sigma$ (which is isotropic as a subset of $T^*\mathbb{R} \times T^*P \times T^*P$):

$$\tilde{\Sigma} = \{(t, rH(\pi(p));\ \phi_t^h(p) \cdot e^{itH(p)}, r\alpha;\ p, r\alpha) : t \in \mathbb{R}, p \in P, r > 0\} \tag{5.4}$$

where ϕ^h is the horizontal lift of ϕ from X to P.

The wave front set of Υ is contained in the composition $\Gamma \circ \tilde{\Sigma} \subset T^*S^1$, where $\Gamma \circ \tilde{\Sigma}$ is the set of points $a \in T^*S^1$ for which there exists $b \in \tilde{\Sigma}$ such that $(a, b) \in \Gamma$. ¿From (5.3) and (5.4) we see that $\Gamma \circ \tilde{\Sigma}$ consists of points $(\theta, r) \in T^*S^1, r > 0$, such that $H(p_1) = E$ and $p_1 = \phi_\tau(p_2) \cdot e^{i\theta}$ for some τ. So we see that the terms decay rapidly unless these classical conditions are met.

6. Semiclassical states

The WKB method in quantum mechanics gives way of constructing approximate semiclassical solutions of Schrödinger's equation for some Hamiltonian to a Lagrangian submanifold Λ in phase space (see [3] for a nice exposition). The construction requires that Λ satisfy a Bohr-Sommerfeld type condition, reflecting the fact that only certain energies occur quantum mechanically.

For compact Kähler X, quantized as in Section 2, the Bohr-Sommerfeld condition is that Λ should have a horizontal lifting up to P. Or equivalently, Λ should be given as $\pi(\tilde{\Lambda})$ where $\tilde{\Lambda}$ is a Legendrian submanifold of P. (Legendrian means simply $\alpha|_{T\tilde{\Lambda}} = 0$). A simple way to associate states to $\tilde{\Lambda}$ (not using WKB) is to take $\Pi(\delta_\nu)$, where δ_ν is the delta-function distribution associated to some density ν on $\tilde{\Lambda}$. Since Π applied to the delta-function of a point gives a coherent state, this essentially amounts to integrating coherent states over a submanifold of X. The lifting to P amounts to a choice of phase for these states.

The problem addressed in [6] is to relate the inner products of states constructed in this way to the underlying Lagrangian submanifolds, in the semiclassical limit. The results are that the norm of such a state has an asymptotic expansion with leading term given by the volume of Λ with respect to the chosen density. And the inner product of two different states has an expansion with terms coming from intersections of the submanifolds.

We will demonstrate how this comes about in the case of $X = \mathbb{C}$. Our calculations are essentially a repeat of calculations done in [17]. Let $\gamma(t)$ be a smooth regular closed curve in $\mathbb{C}$ with period 2π. Because we are in dimension two, any such γ is a Lagrangian submanifold. To lift to P, we add an angle component $\vartheta(t)$. The pair (γ, ϑ) will be horizontal if $\alpha(\gamma', \vartheta') = 0$, or in other words, if

$$\vartheta' = -\tfrac{i}{2}(\gamma\overline{\gamma}' - \overline{\gamma}\gamma'). \tag{6.1}$$

The Bohr-Sommerfeld condition for such a curve is simply $\vartheta(2\pi) = \vartheta(0)$. From the formula for ϑ' we see that $\vartheta(2\pi)$ is 2 times the area enclosed by γ, counted with signs and multiplicities. Hence this area must be π times an integer.

The coherent state is $\Pi(\delta_{w,\eta})(z,\theta) = \frac{k}{2\pi} e^{kz\overline{w} - k|z|^2/2 - k|w|^2/2}$. For simplicity, we'll take the density on the curve to be the unit of arclength. Therefore to the curve γ, with ϑ determined by (6.1), we associate the state

$$\Phi_{k,\gamma}(z,\theta) = \tfrac{k}{2\pi} e^{-k|z|^2/2} \oint e^{ik\vartheta(t)} e^{kz\overline{\gamma}(t) - k|\gamma(t)|^2/2} |\gamma'(t)|\, dt.$$

Consider first the norm of such a state. Using the reproducing property of the coherent state, the norm squared becomes a double integral.

(6.2)

$$\|\Phi_{k,\gamma}\|^2 = \tfrac{k}{2\pi} \oint \oint e^{ik(\vartheta(t) - \vartheta(s))} e^{k\overline{\gamma}(s)\gamma(t) - k|\gamma(t)|^2/2 - k|\gamma(s)|^2/2}\, |\gamma'(t)|\, |\gamma'(s)|\, ds\, dt.$$

We can apply stationary phase to the s integration. The equation for the critical point is

$$0 = -\vartheta'(s) - i\gamma(t)\overline{\gamma}'(s) + \tfrac{i}{2}(\gamma(s)\overline{\gamma}'(s) + \overline{\gamma}(s)\gamma'(s)).$$

Since the curve was horizontal, by (6.1), this equation reduces to

$$i(\gamma(s) - \gamma(t))\overline{\gamma}'(s) = 0.$$

By the simplicity and regularity of γ, the only critical point occurs when $s = t$, and it is non-degenerate. The second derivative of the phase at the critical point is $i|\gamma'(t)|^2$. Applying stationary phase,

$$\oint e^{ik(\vartheta(t) - \vartheta(s))} e^{k\overline{\gamma}(s)\gamma(t) - k|\gamma(t)|^2/2 - k|\gamma(s)|^2/2}\, |\gamma'(s)|\, ds = \sqrt{\tfrac{2\pi}{k}} + k^{-1} R(t),$$

where $R(t)$ is smooth and bounded independently of k. Returning to the norm (6.2), we see that

$$\begin{aligned} \|\Phi_{k,\gamma}\| &= (\tfrac{k}{2\pi})^{1/2} \oint |\gamma'(t)|\, dt + O(k^0), \\ &= (\tfrac{k}{2\pi})^{1/2} \times \text{length}(\gamma) + O(k^0). \end{aligned}$$

The inner product may be dealt with in a similar way. We can apply stationary phase to the full integration in

$$\begin{aligned} (\Phi_{k,\gamma_1}, \Phi_{k,\gamma_2}) = \tfrac{k}{2\pi} \oint \oint e^{ik(\vartheta_2(t) - \vartheta_1(s))} & e^{k\overline{\gamma}_1(s)\gamma_2(t) - k|\gamma_2(t)|^2/2 - k|\gamma_1(s)|^2/2} \\ & \times |\gamma_2'(t)|\, |\gamma_1'(s)|\, ds\, dt. \end{aligned}$$

Using the condition (6.1) as above, we see that critical points occur when $\gamma_1(s) = \gamma_2(t)$. These will be non-degenerate for transverse intersections. The result is an asymptotic expression for the inner product as a sum over intersection points.

The proofs in [6] follow the same basic lines as the arguments presented here. The theory of [9] is used to write the state (locally) as an oscillatory integral. The main difficulty is to compute the leading term of this integral. Then inner products between states are estimated by stationary phase.

7. Almost-Kähler quantization

The Kähler restriction of Section 2 can in fact be dropped. The Appendix of [9] shows that one can construct a "generalized Toeplitz structure," namely a projector $\Pi : L^2(P) \to \mathcal{H}$ with the microlocal properties we described, for any compact contact manifold P. In particular, if X is a compact symplectic manifold with quantizing circle bundle P, then there is a Toeplitz structure on P. The question is: can we define such an $\mathcal{H}$ in a natural way? This is also the problem with geometric quantization of X. There is no natural choice of polarization unless X is Kähler.

In [8] a quantization scheme was proposed which at least has the virtue of generalizing the the Kähler case directly. For X Kähler one could think of $\mathcal{H}_k$ as the kernel of the Laplacian $\overline{\partial}^*\overline{\partial}$ acting on sections of $L^{\otimes k}$. This is quite not the same as the Laplacian on sections Δ_k coming from the metric and connection. The two are related by $\overline{\partial}^*\overline{\partial} = \Delta_k - nk$.

In the general case $\Delta_k - nk$ will have no kernel. But in [14] it was shown that a finite number of eigenfunctions of this shifted Laplacian stay clustered around 0 as $k \to \infty$. The rest drift to the right like $O(k)$.

In fact if we define $\mathcal{H}_k$ to be this collection of low-lying eigenstates, and set up $\Pi : L^2(P) \to \mathcal{H}$ as in the Kähler case, this Π is a generalized Toeplitz structure. (This was actually the basis of the proof in [14], rather than the main result.) Therefore the proofs of semiclassical results in [4], [6], and [7] apply to this setting with no alterations.

We have no physical motivation to justify this method of quantizing by taking low level eigenstates of the Laplacian, and other schemes are certainly possible. It would be nice to discover some underlying principle. More generally, we would like to develop some basic criteria for quantization theories which will guarantee the kind of good semiclassical behavior we have seen here. Such criteria would presumably be microlocal in nature, but hopefully one could require considerably less structure than the full theory of [9].

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