Decomposition theorems for groups of diffeomorphisms in the sphere.

R. de la Llave
Dept. Math.
U. Texas at Austin
Austin TX 78712
USA
e-mail llave@math.utexas.edu

R. Obaya
Dept. Matemática Aplicada a la Ingeniería
Escuela Superior de Ingenieros Industriales
Univ. de Valladolid
47011 Valladolid
SPAIN
e-mail rafoba@wmatem.eis.uva.es

Abstract

We study the algebraic structure of several groups of differentiable diffeomorphism in $\mathbf{S}^n$. We show that given any sufficiently smooth diffeomorphism can be written as the composition of a finite number of diffeomorphisms which are symmetric under reflection, essentially one-dimensional and about as differentiable as the given one.

Keywords: Decomposition theorems, diffeomorphism groups.

AMS 1991 Classification: 58D05, 57S25, 57S05

1 Introduction

The goal of this paper is to prove several decomposition theorems for diffeomorphism groups.

We recall that decomposition theorems for groups, roughly, state that any element of the group can be can be written as a finite product of elements lying in a smaller subgroup.

In our case, the groups considered will be groups of differentiable diffeomorphisms of the sphere and the subgroups in which we factor will be groups of diffeomorphisms that commute with reflections across a plane and which are essentially one dimensional. The factors into which we can decompose a given map will be slightly less differentiable than the original one.

There are several motivations for the study of theorems of this type. For example, in [5] it is shown that a theorem of this type for the circle can be used to solve the inverse problem for scattering of geodesic fields in surfaces of genus one. Physically, this is the problem of given the transformation to be effected by a lens, find the distribution of the refractive index that produces the desired effect. The problem is such that it it is solved for two diffeomorphisms, it is solved for their composition. Moreover, for symmetric mappings a simple construction works. Hence, the decomposition theorem, shows that it can be solved for all mappings. For the application above, the loss of differentiability incurred in the factorization does not affect essentially the conclusions.

We also note that these theorems are analogues of the usual factorization theorems in Lie algebras and they could be useful in the problem of computing representations of diffeomorphism groups [1].

If we consider these theorems as infinite dimensional versions of factorization theorems for Lie groups, one first difficulty is that for diffeomorphism groups, the corresponding Lie algebra – the Lie algebra of vector fields – is considerably worse behaved than those of finite dimensional Lie groups. For example, the exponential is not surjective in any neighborhood of the identity. Hence, it is not possible to apply the usual implicit function theorem in Banach spaces to obtain results in a small enough neighborhood of the origin using the *"infinitesimal results"* obtained from the Lie algebra. Nevertheless, we will show that it is possible to obtain similar results to those in the finite dimensional case by using appropriate *"hard"* implicit function theorems of the Nash–Moser type. However, one needs to reformulate the problem so that it has some group structure. This requires some assumptions on the structure of the manifold. (Of course, the existence of the reflections is included in the assumptions.) In our proof we proceed to prove the theorem by induction on the dimension. The construction is somewhat delicate since we want to obtain a loss of differentiability that is independent of the dimension, in spite of the fact that the number of inductive steps has to grow with the dimension.

2 Acknowledgments

The work of R.L. has been supported by NSF grants. Also R.L wants to acknowledge the hospitality enjoyed at Valladolid during several visits. The work of R.O. has been partially supported by Junta de Castilla y Leon. Letters from J. Langer and F. Bien discussing the implications of these theorems were very useful to us and provided encouragement.

3 Decomposition theorem for spheres.

3.1 Notation and statement of results

We start this section by recalling some known results which show how to endow certain function spaces on compact sets with differentiable manifold structure, necessarily of infinite dimension.

If l is an integer, and Ω is a compact set in $\mathbb{R}^d$ we will say that $f : \Omega \to \mathbb{R}$ is C^l if it has continuous derivatives of order l. We will denote by $C^l(\Omega)$, the space of C^l functions in Ω endowed with the norm $\| \|_{C^l(\Omega)}$ defined as supremum of the derivatives of order up to l. It is well known that it is a Banach space.

If l is not an integer, we define as usual

$$\|f\|_{C^l(\Omega)} = \max(\sup_{x\neq y} \|D^l f(x) - D^l f(y)\|/|x-y|^{l-[l]}, \|f\|_{C^{[l]}(\Omega)}) \qquad (1)$$

To cover Lipschitz conditions as well, we introduce the notation $l^- = (l-1) + 1^-$. In this notation, 1^- will be bigger than any number in $[0,1)$ but smaller than 1.

These notions of regularity can be lifted to geometric objects that can be expressed in coordinates by saying that an object is C^l if we can find coordinate patches that cover the manifold on which the expression is C^l.

Furthermore, if we fix a set of coordinate patches that cover the manifold we can talk about the C^l distance between two objects by declaring it to be the maximum of the C^l distance between the coordinate expressions. A moment's reflection will show that the statement that a geometric object is C^l is independent of the coordinate patches chosen (that is, if the coordinate expressions in one patch are C^l so are in all others).

Unfortunately, the C^l distances do depend on the patch. Even if it is easy to show that the distances obtained using two patches are equivalent, they do differ and some statements such as that certain operators acting on C^l functions are contractions require specific choices of coordinate patches. When l is an integer, there is a geometrically natural notion of C^l distance in Riemannian geometry based on the notion of jets. (See e.g. [3] Ch. 2) For non-integer l, since the definition of Hölder distance (1) requires comparing the values at different points, there are no geometrically natural notions of C^l distance. For our purposes, the dependence on a coordinate patch is not a shortcoming since many analytical operations that we will need to perform require taking coordinates anyway and we will always take a fixed coordinate system.

We will denote by $\mathrm{Diff}^l(M,N)$, the set of diffeomorphisms of M into N and, when $M = N$ we will simply write $\mathrm{Diff}^l(N)$. When l is finite $l \geq 1$, these sets can be given the structure of Banach manifolds as follows. Using charts, we can define a norm on the space of C^l vector fields on N that makes it a Banach space. Given a diffeomorphism $f \in \mathrm{Diff}^l(M,N)$ and a C^l vector field v, the map $\mathrm{Exp}(f,v) : M \to N$ defined by $\mathrm{Exp}(f,v)(x) = \exp_{f(x)} v(f(x))$ – where exp denotes the Riemannian geometry exponential associated to a C^∞ metric – is a C^l mapping. The implicit function theorem shows that for sufficiently small v, this mapping is also a diffeomorphism and, moreover, any diffeomorphism in a C^l neighborhood of f can be written in this way. If $l < 1$, we cannot apply the implicit function theorem to conclude that $\mathrm{Exp}(f,v)$ is a diffeomorphism, nevertheless, it is possible to show that Exp covers a neighborhood of f in the space of C^l mappings. (A Similar argument shows that $\mathrm{Diff}^\infty(M)$ is a Frechet manifold).

In $\mathrm{Diff}^l(M)$ it is natural to define the group operation of composition. This set is a topological group for integer exponents; this is the reason we formulate the statements in this paper only for this case. Unfortunately, when l is not an integer, this operation will be discontinuous. Moreover, when l is an integer, even if composition is continuous, it will not be differentiable. For properties of the composition operator on Hölder spaces we refer to [7] where there is a very systematic treatment of these properties.

We will also consider families of diffeomorphisms depending of a parameter. We will denote the parameters by subindices. We will say that f_λ is a C^l family of diffeomorphisms when for every fixed $\lambda \in \Lambda$, where $\Lambda \subset \mathbb{R}^m$ is an open set or a compact manifold, the map $x \to f_\lambda(x)$ is a C^l diffeomorphism and, moreover $(\lambda, x) \to f_\lambda(x)$ is also C^l.

The composition of diffeomorphisms can be extended in a natural way to families. We denote by $g_\lambda \circ f_\lambda$ the family $(\lambda, x) \to g(\lambda, f_\lambda(x))$. The notation f_λ is somewhat ambiguous since it denotes at the same time the family and the mapping for a concrete value of λ. The notation is, however, standard use and it does not lead to confusion.

Remark. It is also possible to think of families as mappings from $\Lambda \times M$ to $\Lambda \times N$ such that the first variable remains unaltered. This point of view makes it clear that many results about composition proved for C^l mappings are true for C^l families.

Notice that the map $\Lambda \to C^l(M, N)$ defined by $\lambda \to f_\lambda$ is only C^0. There are many C^0 mappings $\Lambda \to C^l(M, N)$ which are not C^l families.

From now on, the manifold under consideration will be the n-dimensional sphere $\mathbf{S}^n$. We take its standard embedding in the Euclidean space

$$\mathbf{S}^n = \left\{ x = (x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} | \sum_{i=1}^{n+1} x_i^2 = 1 \right\}.$$

If x is a point in $\mathbf{S}^n$, we will denote by Π_x the hyperplane passing through the origin and consisting of vectors orthogonal to x, and by $\mathcal{M}_{x,y}$ the one-dimensional maximum circle going from x to $-x$ and passing through the point $y \in \mathbf{S}^n - \{x, -x\}$. We will refer to those sets as *meridians*. We will also represent by $\mathcal{P}_{x,\gamma}$, $-1 \le \gamma \le 1$, the subsets of $\mathbf{S}^n$ whose points have a constant projection γ along the line joining the origin and the point x. The sets of these form will be referred to as *parallels*. Finally, we denote by δ_x, the reflection across the plane Π_x.

We call attention to the fact that, with our notation, meridians are always one-dimensional, but parallels in $\mathbf{S}^n$ are copies of $\mathbf{S}^{n-1}$. Outside a neighborhood of the poles $\{x,-x\}$, meridians $\mathcal{M}_{x,y}$ and parallels $\mathcal{P}_{x,\gamma}$ constitute two transversal foliations.

We will denote by $\mathrm{Diff}^l_0(\mathbf{S}^n)$ the connected component of the identity of $\mathrm{Diff}^l(\mathbf{S}^n)$ and by Sym^l_x the subset of $\mathrm{Diff}^l(\mathbf{S}^n)$ commuting with δ_x. The symbol $\mathrm{Sym}^l_{x,0}$ will stand for the connected component of Sym^l_x containing the identity.

Next we study the algebraic structure of the group of diffeomorphisms in $\mathbf{S}^n$. The main theorem in this paper is:

Theorem 3.1 *Let $n \in \mathbb{N}$, $n \geq 2$. For any $l \in \mathbb{N} \cup \{\infty\}$, $l \geq 7$, we can find points $y_1, \ldots y_{(n+1)^2}$ in such a way that any $f \in \mathrm{Diff}^l_0(\mathbf{S}^n)$ can be written as $f = f_1 \circ f_2 \circ \cdots \circ f_N$, where each $f_i \in \mathrm{Sym}^{l-2}_{y_j,0}$ for some $j = 1, \ldots, (n+1)^2$. (The number N may depend on f)*

For $n = 1$, there exist points y_1, y_2 such that any $f \in \mathrm{Diff}^l_0(\mathbf{S}^1)$ can be written as $f = f_1 \circ f_2 \circ \cdots \circ f_N$ with each $f_i \in \mathrm{Sym}^{l-2}_{y_j,0}$ for some $j = 1, 2$. This theorem was proved in [5] for $l = \infty$. A different proof valid in the classes of finitely differentiable functions was given in [6] where an analogue result for $\mathbf{T}^n$ was also proved.

We also remark that, as a corollary of the proof of Theorem 3.1, we will obtain that the diffeomorphisms appearing as factors can be identified with families of diffeomorphisms of the circle.

3.2 Proof of Theorem 3.1

First at all, observe that it suffices to prove the result in an arbitrary small neighborhood of the identity in the C^l-topology. In effect, given any open neighborhood $\mathcal{U}$ of the identity, any diffeomorphism in the connected component of the identity can be written as the composition of a finite number of diffeomorphisms in $\mathcal{U}$.

We will establish the local theorem for families of diffeomorphisms by induction on the dimension. We introduce in the inductive argument that the map $\mathcal{F}$, constructed through the proof, which assigns to each family of diffeomorphisms $f_\lambda \in \mathcal{U}$ the factors $f_{\lambda,i}$, $i = 1, \ldots, N$ in a neighborhood of the identity in the C^{l-2} topology, is continuous and that for each value of the parameter $\lambda \in \Lambda$ with $f_\lambda = Id$, one obtain factors $f_{\lambda,i} = Id$ for every

$i = 1, \ldots, N$. This result for the one dimensional case is stated explicitly in [6]

In the statement of the theorem the points $y_1, \ldots, y_{(n+1)^2}$ are left to our choice. We fix a real number ρ and take those $y = (y_k)$ of $\mathbf{S}^n$ whose components are defined by one of the following conditions:

- there is $1 \leq i \leq n+1$ such that $y_i = 1$ and $y_k = 0$ for $k \neq i$, or
- there are $1 \leq i, j \leq n+1$ such that $y_i = \cos\rho$, $y_j = \sin\rho$ and $y_k = 0$ for $k \neq i, j$.

(However, it will be clear that a more careful selection of these points would allow us to reduce their number. We have not bothered with this issue.)

The choices of the angle of separation ρ that are convenient for the method of proof are those satisfying the so-called *Diophantine* conditions,

$$\left|\rho - \frac{p}{q}\right|^{-1} \leq Cq^{2+\delta} \tag{2}$$

for all pair of integers p, q with $q > 0$.

It is well known that for $\delta > 0$, the numbers satisfying this kind of inequalities have full Lebesgue measure. For $\delta = 0$, the set of numbers satisfying these inequalities is called the set of constant type numbers. Moreover relation (2) with $\delta = 0$ is equivalent to ρ having a bounded continued fraction expansion. All irrational numbers that satisfy a quadratic equation with integer coefficients are constant type; in particular, constant type numbers are dense.

Even if all Diophantine ρ's would lead to a theorem of the type we want and, for δ small, with the conditions of differentiability required in the statement, constant type numbers will lead to the sharpest differentiability conclusions. From now on we will assume we have chosen a constant type number. The actual choice will affect the sizes of the neighborhoods where the local theorem holds, but not the differentiability properties.

In order to apply the induction process, we will decompose diffeomorphisms of $\mathbf{S}^n$ in a neighborhood of Id into factors with the same order of differentiability that preserve lower dimensional spheres. We will start by proving a parameterized version of the well known *fragmentation lemma.*

Lemma 3.2 *Let M be a compact manifold, and $\{K_i\}_{i=1}^n$ be a collection of compact subsets of M such that*

$$\bigcap_{i=1}^{n} K_i = \emptyset \,.$$

Then, for each $l \in \mathbb{N} \cup \{\infty\}$ we can find a neighborhood $\mathcal{U}$ of the identity in the space of C^l families of diffeomorphisms, and a map $\mathcal{F}_0$ that, to each family f_λ in $\mathcal{U}$, associates C^l families $f_\lambda^1, \cdots, f_\lambda^n$, satisfying

i) $f_\lambda = f_\lambda^1 \circ \cdots \circ f_\lambda^n$.

ii) $f_\lambda^i|_{K_i} = Id$.

iii) if for some $x \in M$ and $\lambda \in \Lambda$ one has $f_\lambda(x) = x$ then $f_\lambda^i(x) = x$ for every $i = 1, \ldots, n$.

The mapping $\mathcal{F}_0$ is continuous when we give its domain and its range the topology of C^l families.

Proof. We will assume that $\mathcal{U}$ is small enough so that $f_\lambda(x)$ is within the injectivity radius of the exponential. We will also need that some intermediate steps in our construction lie in this domain. Since these are a finite number of conditions, it will be easy to check that they are satisfied for a non-trivial neighborhood.

Suppose first the case $n = 2$; then we can find a C^∞-function ϕ such that

$$\phi|_{K_1} \equiv 1 \,, \quad \phi|_{K_2} \equiv 0 \,.$$

If f_λ can be represented as the exponential of the vector field $F_\lambda = E^{-1}(f_\lambda)$, then $f_\lambda^2 = E(F_\lambda \tau)$ is the identity restricted to K_2. Moreover, since $f_\lambda(x) = x \Leftrightarrow E^{-1}(f_\lambda)(x) = 0$, the set of fixed points of f_λ^2 is bigger than those of f_λ. Provided that f_λ is sufficiently close to the identity –which depends only on the K's and ϕ–, we have $f_\lambda^2|_{K_1} = f_\lambda$ and $f_\lambda^2|_{K_2} = Id$.

Hence, setting $f_\lambda^1 = f_\lambda \circ (f_\lambda^2)^{-1}$, the claims of the lemma are satisfied. Since the choice of ϕ depends only on the compacts K_1 and K_2 the continuity follows. Notice that the previous argument works even in the case that one of the compacts is the empty set.

To prove the general case, we will proceed by induction in n, the number of compact sets, starting from teh aleady verified case $n = 2$. We will assume that the conclusion is true for $n \leq n_0$, $n_0 \geq 2$, and will check it for $n = n_0+1$.

If $K_1, \ldots, K_{n_0+1}$ are compact sets with empty intersection, we can find open sets $V_1, \ldots, V_{n_0+1}$ such that $K_i \subset V_i$, and $\cap_{i=1}^{n_0+1} \overline{V}_i = \emptyset$.

Set $L_0 = \cap_{i=1}^{n_0} \overline{V}_i$ and $L_{n_0+1} = K_{n_0+1}$. Since $L_0 \cap L_{n_0+1} = \emptyset$ we apply the previous result with $n_0 = 2$ to obtain families f_λ^0, $f_\lambda^{n_0+1}$ which are the identity on L_0, L_{n_0+1} respectively, and with not less fixed points than f_λ such that $f_\lambda = f_\lambda^0 \circ f_\lambda^{n_0+1}$.

We can now consider the compact sets $L_i = K_i - \cap_{i=1}^{n_0} V_i$. They have empty intersection because $\cap_{i=1}^{n_0} K_i \subset \cap_{i=1}^{n_0} V_i$, so we can apply the induction hypothesis to f_λ^0 to write $f_\lambda^0 = f_\lambda^1 \circ \cdots \circ f_\lambda^{n_0}$. Each one of the f_λ^i will be the identity on a neighborhood of K_{n_0+1}. This gives us the representation $f_\lambda = f_\lambda^1 \circ \cdots \circ f_\lambda^{n_0+1}$ satisfying all the conditions of the statement.

The continuity of $\mathcal{F}_0$ is also a consequence of the induction process.

We emphasize that, since, properties i)-iii) do not determine $\mathcal{F}_0$ uniquely the claim of continuity refers only to the one constructed above.

$\blacksquare$

In the next lemma, we show how diffeomorphisms can be factored into simpler maps that conserve the parallels and the meridians, which are in essence diffeomorphisms of $\mathbf{S}^{n-1}$ and $\mathbf{S}^1$ respectively. We will henceforth refer to diffeomorphisms and families that preserve the parallels as *horizontal* and those that preserve the meridians as *vertical*.

Lemma 3.3 *Let x_0 be a point in $\mathbf{S}^n$, $-x_0$ the antipodal point and U, U' two open neighborhoods around x_0, $-x_0$ respectively. Then, for every $l \in \mathbb{N} \cup \{\infty\}$, we can find a neighborhood $\mathcal{U}$ of the identity in the space of C^l families of diffeomorphisms in such a way that, for any family f_λ in $\mathcal{U}$, such that $f_\lambda|_U = Id$, $f_\lambda|_{U'} = Id$, there exist C^l families f_λ^h, f_λ^v satisfying:*

i) $f_\lambda = f_\lambda^h \circ f_\lambda^v$,

ii) $f_\lambda^v \mathcal{M}_{x_0,x} = \mathcal{M}_{x_0,x}$ *for all* $x \in \mathbf{S}^n - \{x_0, -x_0\}$,

iii) $f_\lambda^h \mathcal{P}_{x,\gamma} = \mathcal{P}_{x,\gamma}$ *for all* $\gamma \in [-1, 1]$, *iv) if for some $x \in \mathbf{S}^n$ and $\lambda \in \Lambda$ one has $f_\lambda(x) = x$ then $f_\lambda^h(x) = f_\lambda^v(x) = x$. In particular, $f_\lambda^h|_{U\cup U'} = f_\lambda^v|_{U\cup U'} = Id$.*

Moreover, if we restrict f_λ^h, f_λ^v to be in a neighborhood of the identity, they are unique, and the mapping $\mathcal{F}_1$ that assigns to each f_λ the pair f_λ^h, f_λ^v is continuous when we give its domain and its range the topology of C^l families.

Proof. For any $x \in \mathbf{S}^n - \{x_0, -x_0\}$, we represent by $\gamma(x)$ the projection of x along the line joining the origin and the point x_0. Obviously, we should set $f_\lambda^v(x)$ to be the point in the meridian of x and on the parallel of $f_\lambda(x)$. Then, f_λ^h would move along the parallels so as to get to $f_\lambda(x)$.

More formally, using the notation for parallels and meridians introduced in the previous section, we have:

$$\begin{aligned} f_\lambda^v(x) &= \mathcal{M}_{x_0,x} \cap \mathcal{P}_{x_0,\gamma(f_\lambda(x))} \\ f_\lambda^h(x) &= \mathcal{M}_{x_0,f_\lambda((f_\lambda^v)^{-1}(x))} \cap \mathcal{P}_{x_0,\gamma(x)}. \end{aligned}$$

Using the uniform transversality of the meridian and parallel foliations outside a neighborhood of the poles, the usual implicit function theorem, establishes that $f_\lambda^v(x)$ $f_\lambda^h(x)$ depend jointly C^l on x and on the parameter λ, and consequently, they are C^l families of diffeomorphisms.

■

The induction on the dimension to prove Theorem 3.1 will proceed as follows. For $0 < \varepsilon < 1$, we will consider the sets $K_i^\varepsilon = \{x \in \mathbf{S}^n \,/\, |x_i| \le \varepsilon\}$. It is obvious that $\cap_{i=1}^n K_i^\varepsilon = \emptyset$ when ε is sufficiently small.

Lemma 3.2 will allows us to write any diffeomorphism f as composition of diffeomorphisms f_i, $1 \le i \le n+1$, each of which is the identity restricted to K_i^ε. From now on we consider the case $i = 1$; it is clear that the same arguments could be applied to the rest of diffeomorphisms f_i, $i = 2, \cdots, n+1$.

Furthermore, for f_1 we will pick a polar axis going through the points $x_\pm = (0, \ldots, \pm 1)$ and using Lemma 3.3 we will write $f_1 = f^h \circ f^v$, as a composition of an horizontal and a vertical diffeomorphisms. If f_1 was sufficiently close to the identity then f^h, f^v would be the identity in K_1^ε. We will apply the induction hypothesis to f^h and will show how to factor f^v.

To apply the induction hypothesis, we will think of $f = f^h$ as a family of maps sending $\mathbf{S}^{n-1}$ into $\mathbf{S}^{n-1}$, the parameter being the height of the parallel. That is the map will be $f^h(p,\gamma) = (\tilde{f}_\gamma(p), \gamma)$, where $\tilde{f}_\gamma$ is a diffeomorphism of $\mathbf{S}^{n-1}$ for every $\gamma \in [-1,1]$. Moreover we have $\tilde{f}_\gamma = Id$ on $\mathbf{S}^{n-1}$ when γ is close enough to -1 or 1. Notice how we are lead to consider families of diffeomorphisms even if we start with a single one. With hindsight, this is the reason why we have formulated the inductive hypothesis and the statements of the paper in terms of families.

We obtain $\tilde{f}_\gamma = \tilde{f}_{\gamma,1} \circ \cdots \circ \tilde{f}_{\gamma,N'}$ as composition of C^{l-2} families of diffeomorphisms of $\mathbf{S}^{n-1}$. For each $i = 1, \ldots, N'$ there exists $j \in \{1, \ldots, n^2\}$ (or $j \in \{1,2\}$ when $n = 2$) such that $\tilde{f}_{\gamma,i} \circ \delta_{y_j} = \delta_{y_j} \circ \tilde{f}_{\gamma,i}$ on $\mathbf{S}^{n-1}$ for all the values of γ. Since the induction includes in its conclusions that the factors $\tilde{f}_{\gamma,i}$ depend smoothly on the height, and preserve the identity in a neighborhood of the poles, we conclude that these factors are really diffeomorphisms in $\mathbf{S}^n$ commuting with reflections. Clearly, the above arguments apply without any change to families of horizontal diffeomorphisms.

We now proceed to establish the factorization of f^v into symmetric diffeomorphisms. According with the above remarks we also can take the height as a periodic variable and consider the parameters in a set $\Lambda = \mathbf{T}^{m-2}$.

Proposition 3.4 *For an $l \in \mathbb{N} \cup \{\infty\}$, $l \geq 7$, we can find two points y_1, y_2 in the sphere and a neighborhood $\mathcal{U}$ of the identity in the space of C^l families of diffeomorphisms such that, for every family $f^v_\lambda \in \mathcal{U}$ with*

(h.1) $f^v_\lambda \mathcal{M}_{x_+,x} = \mathcal{M}_{x_+,x} \qquad \forall\, x \in \mathbf{S}^n - \{x_\pm\}$,

(h.2) $f^v_\lambda|_{K_1^\varepsilon} = Id$,

there exist two C^{l-2} families $f^{v,1}_\lambda$, $f^{v,2}_\lambda$ in a neighborhood of the identity, satisfying

i) $f^{v,1}_\lambda \mathcal{M}_{x_+,x} = f^{v,2}_\lambda \mathcal{M}_{x_+,x} = \mathcal{M}_{x_+,x} \qquad \forall\, x \in \mathbf{S}^n - \{x_\pm\}$,

ii) $f^{v,1}_\lambda \circ \delta_{y_1} = \delta_{y_1} \circ f^{v,1}_\lambda$,

$f^{v,2}_\lambda \circ \delta_{y_2} = \delta_{y_2} \circ f^{v,2}_\lambda$,

iii) $f^v_\lambda = f^{v,1}_\lambda \circ f^{v,2}_\lambda$, *iv) if for some λ one has $f_\lambda^v = Id$ then $f_\lambda^{v,1} = f_\lambda^{v,2} = Id$.*

Besides, the map $\mathcal{F}_2$ which assigns to f_λ^v the factors $f_\lambda^{v,1}$ and $f_\lambda^{v,2}$ is continuous for the C^l-topology in its domain and the C^{l-2}-topology in its range.

Proof. We remark again that, since the pair $f_\lambda^{v,1}, f_\lambda^{v,2}$ is not uniquely determined (even when y_1, y_2 are fixed), the conclusions of continuity in the statement are valid only for the map $\mathcal{F}_2$ constructed below.

We will take polar coordinates in $\mathbf{S}^n - K_1^\varepsilon$ defined by:

$$
\begin{array}{rcl}
x_{n+1} & = & \cos \pi\varphi \\
x_n & = & \sin \pi\varphi \ \cos \pi\omega_1 \\
x_{n-1} & = & \sin \pi\varphi \ \sin \pi\omega_1 \ \cos \pi\omega_2 \\
\vdots & & \quad \vdots \\
x_3 & = & \sin \pi\varphi \ \sin \pi\omega_1 \cdots \cos \pi\omega_{n-2} \\
x_2 & = & \sin \pi\varphi \ \sin \pi\omega_1 \cdots \sin \pi\omega_{n-2} \ \cos 2\pi\theta \\
x_1 & = & \sin \pi\varphi \ \sin \pi\omega_1 \cdots \sin \pi\omega_{n-2} \ \sin 2\pi\theta
\end{array}
$$

for φ, $\omega_1, \ldots, \omega_{n-2}$, $\theta \in (0, 1)$.

Notice that if $|x_1| \geq \varepsilon$, then $|x_{n+1}|^2 \leq 1-\varepsilon^2$ so that φ is defined univocally, and, since $|x_n|^2 + |x_{n+1}|^2 \leq 1 - \varepsilon^2$, ω_1 is likewise outside of the critical values of sin; this same argument is applicable to $\omega_2 \ldots \omega_{n-2}$. In others words, this polar coordinate system defines a smooth transformation Ψ which identifies $\mathbf{S}^n - K_1^\varepsilon$ with an open subset V of $\mathbf{T}^n$. Let δ be a small real positive number and $V_\delta = (\delta, 1-\delta)^{n-1} \times [0, 1] \subset \mathbf{T}^n$. Then $U_\delta = \Psi^{-1}(V_\delta)$ is an open subset of $\mathbf{S}^n$ which contains $\mathbf{S}^n - K_\varepsilon$, where the above coordinate system still works. In particular the images by the map Ψ^{-1} of the points of the plane $\theta = 0$ lie in K_1^ε. It is clear that every diffeomorphism f of $\mathbf{S}^n$ with $f|K_1^\varepsilon = Id$ can be extended by the identity to a diffeomorphism of $\mathbf{T}^n$.

We observe that when we express the family f_λ^v of vertical diffeomorphisms of $\mathbf{S}^n$ in these coordinates, the value of φ changes but the others coordinates remain unaltered. Therefore, a reflection across the plane Π_{y_1} with $y_1 = (1, \ldots, 0)$ is expressed as a change in sign for the θ coordinate while all the others remain fixed. More generally, a reflection across Π_{y_2} with $y_2 = (\cos \rho, \sin \rho, \ldots, 0)$ can be expressed in the coordinates (3.2) by sending θ into $2\rho - \theta$ and leaving all the other variables unchanged.

If we take coordinates $(\varphi, \omega, \theta)$, the family f^v_λ will be written as

$$(\varphi, \omega, \theta) \to (\varphi + v_\lambda(\varphi, \omega, \theta), \omega, \theta),$$

and we will look for $f^{v,1}_\lambda$ $f^{v,2}_\lambda$ that can be written as

$$(\varphi, \omega, \theta) \to (\varphi + v^i_\lambda(\varphi, \omega, \theta), \omega, \theta)$$

with

$$\begin{aligned} v^1_\lambda(\varphi, \omega, \theta) &= v^1_\lambda(\varphi, \omega, -\theta), \\ v^2_\lambda(\varphi, \omega, \theta) &= v^2_\lambda(\varphi, \omega, -\theta + 2\rho). \end{aligned} \tag{3}$$

The equation that v^1_λ v^2_λ have to solve is, as an straightforward calculation shows

$$v_\lambda(\varphi, \omega, \theta) = v^1_\lambda(\varphi + v^2_\lambda(\varphi, \omega, \theta), \omega, \theta) + v^2_\lambda(\varphi, \omega, \theta). \tag{4}$$

Notice that there is an open set of values of w for which all the points corresponding to $(\varphi, \omega, \theta)$ are in K^ε_1, hence $v_\lambda(\varphi, \omega, \theta) = 0$. For these values of w, equation (4) admits the trivial solution $v^1_\lambda(\varphi, \omega, \theta) = v^2_\lambda(\varphi, \omega, \theta) = 0$. Once established Proposition 3.4, its assertion $iv)$ would imply that $f^{v,1}_\lambda = Id$, $f^{v,2}_\lambda = Id$ for the points in the sphere satisfying this condition. (We will show a similar behavior in the variable φ).

Clearly, the solutions of (4) can be extended by the identity to all the points of the sphere in K^ε_1 even for those for which our coordinate system does not work. Under conditions of proximity to the identity of f^v_λ, $f^{v,1}_\lambda$, $f^{v,2}_\lambda$, these will be solutions of our original problem.

Since, except for these considerations, ω does not enter in either (4) or in (3), we can consider these variables as parameters in the problem in the same way as λ. We will hence assume that λ refers both to the parameters in the family and to ω, and omit ω from the notation.

We will carry out the proof of the proposition considering the family (f^v_λ) as a vertical diffeomorphism in $\mathbf{T}^m$ where we will solve the functional equation (4). Finally, we will show that the factors obtained are in fact diffeomorphism of $\mathbf{S}^n$.

Proceeding heuristically for the moment, the importance of the Diophantine condition already imposed on ρ becomes apparent if we consider the *linearized equations* obtained by expanding formally (4) in the unknowns and keeping only the linear terms. We obtain

$$v_\lambda(\varphi,\theta) = v_\lambda^1(\varphi,\theta) + v_\lambda^2(\varphi,\theta) \tag{5}$$

By expanding v_λ, v_λ^1, and v_λ^2 in Fourier coefficients in the variable θ, $v_\lambda(\varphi,\theta) = \sum_k \hat{v}_{\lambda,k}(\varphi)e^{2\pi i k\theta}$ and $v_\lambda^j(\varphi,\theta) = \sum_k \hat{v}_{\lambda,k}^j(\varphi)e^{2\pi i k\theta}$ for $j = 1,2$, we see that (5) is equivalent to

$$\hat{v}_{\lambda,k}(\varphi) = \hat{v}_{\lambda,k}^1(\varphi) + \hat{v}_{\lambda,k}^2(\varphi). \tag{6}$$

Similarly, the symmetry conditions for v_λ^1, v_λ^2 can be expressed as

$$\begin{aligned} \hat{v}_{\lambda,-k}^1(\varphi) &= \hat{v}_{\lambda,k}^1(\varphi), \\ \hat{v}_{\lambda,-k}^2(\varphi)e^{4\pi i\rho} &= \hat{v}_{\lambda,k}^2(\varphi). \end{aligned} \tag{7}$$

When $k \neq 0$, we can group the equations for k, $-k$ in (6) and, using (7) we obtain

$$\begin{aligned} \hat{v}_{\lambda,k}(\varphi) &= \hat{v}_{\lambda,k}^1(\varphi) + \hat{v}_{\lambda,k}^2(\varphi), \\ \hat{v}_{\lambda,-k}(\varphi) &= \hat{v}_{\lambda,-k}^1(\varphi) + \hat{v}_{\lambda,k}^2(\varphi)e^{-4\pi i\rho}. \end{aligned}$$

These equations admit the solutions

$$\begin{aligned} \hat{v}_{\lambda,k}^1(\varphi) &= (\hat{v}_{\lambda,k}(\varphi)e^{-4\pi i\rho k} - \hat{v}_{\lambda,-k}(\varphi))/(e^{-4\pi i\rho k} - 1), \\ \hat{v}_{\lambda,k}^2(\varphi) &= (\hat{v}_{\lambda,k}(\varphi) - \hat{v}_{\lambda,-k}(\varphi))/(e^{-4\pi i\rho k} - 1). \end{aligned} \tag{8}$$

When $k = 0$, there are many solutions. Since $v_\lambda(\varphi, 0) = 0$ for every $\varphi \in \mathbf{T}$ we choose

$$\begin{aligned} \hat{v}_{\lambda,0}^1(\varphi) &= -\textstyle\sum_{k\neq 0} \hat{v}_{\lambda,k}^1(\varphi), \\ \hat{v}_{\lambda,0}^2(\varphi) &= -\textstyle\sum_{k\neq 0} \hat{v}_{\lambda,k}^2(\varphi), \end{aligned} \tag{9}$$

which provides $v^1(\varphi, 0) = v^2(\varphi, 0) = 0$ for every $\varphi \in \mathbf{T}$.

In addition, if $\varphi \in \mathbf{T}$ with $|\sin(\pi\varphi)| < \varepsilon$, then $v_\lambda(\varphi,\theta) = 0$ for every $\theta \in \mathbf{T}$, hence their Fourier coefficients are $\hat{v}_{\lambda,k}(\varphi) = 0$ for every k. From (8) and (9) we deduce that both functions $v_\lambda^1(\varphi,\theta) = v_\lambda^2(\varphi,\theta) = 0$ for all θ.

Notice that, if ρ were rational, the denominators in (8) would become zero; that is, there would be no solution for the equation. Even if ρ is irrational the denominators would become arbitrarily close to zero. Nevertheless, if ρ is a number of constant type, we have the bounds

$$|(e^{-2\pi ik\rho} - 1)^{-1}| \leq C'|k|, \tag{10}$$

that allow us to control the new Fourier expansions.

If we think of (4) as an equation in spaces of functions, expression (5) would be the derivative –in some sense that will have to be specified– at zero. The calculation shows that the derivative, even if invertible, is not bounded; hence, the usual implicit function theorem in Banach spaces does not apply.

We also emphasize that the above calculation shows only existence of the inverse of the derivative at zero. If we had started making perturbations around a non-zero, v^1_λ, v^2_λ we would have been lead to an equation that cannot be readily analyzed in terms of Fourier components.

The above heuristic discussion suggests that, among the several *hard* implicit function theorems which can cope with unbounded inverses, the one that will be useful for this problem is that of Zehnder [10], which requires existence of inverse only at one point, but which also requires a *group* structure for the equation. We continue the proof by recalling Zehnder's theorem, which we have taken from [10] in the version we are going to apply.

Definition 3.5 We say that $(X_\alpha)_{\alpha\geq 0}$ is a *family of regular spaces* when X_α are real Banach spaces with norm $\| \quad \|_\alpha$ and, whenever for $\alpha' \leq \alpha$ we have

$$\begin{aligned} &X_0 \supset X_{\alpha'} \supset X_\alpha \supseteq X_\infty = \bigcap_{\alpha\geq 0} X_\alpha, \\ &\|x\|_{\alpha'} \leq \|x\|_\alpha \quad \forall x \in X_\alpha. \end{aligned}$$

Furthermore, it is possible to define on them a C^∞-regularization, that is, a family $(S_t)_{t>0}$ of linear mappings $S_t : X_0 \to X_\infty$ and constants $C(\alpha, \alpha')$ satisfying:

$$\begin{aligned} &i) \quad \lim_{t\to\infty} \|S_t x - x\|_0 = 0 \quad \forall x \in X_0, \\ &ii) \quad \|S_t x\|_\alpha \leq t^{\alpha-\alpha'} C(\alpha,\alpha')\|x\|_{\alpha'} \quad \forall x \in X_{\alpha'} \quad 0 \leq \alpha' \leq \alpha, \\ &iii) \quad \|S_t x - x\|_{\alpha'} \leq t^{\alpha'-\alpha} C(\alpha,\alpha')\|x\|_\alpha \quad \forall x \in X_\alpha \quad 0 \leq \alpha' \leq \alpha. \end{aligned}$$

Notice that X_∞ is dense in X_0 but not necessarily in X_α for $0 < \alpha$; besides, the closure of X_∞ in X_α contains $X_{\alpha'}$ for every $\alpha < \alpha'$. We now consider several families of regular spaces and denote by the same symbol their norms which are probably different. However, this notation is standard and clear.

Theorem 3.6 *Let $(X_\alpha)_{\alpha\geq 0}$, $(Y_\alpha)_{\alpha\geq 0}$, $(Z_\alpha)_{\alpha\geq 0}$ be families of regular spaces and $(x_0,y_0)\in X_\infty\times Y_\infty$.*

Take $F\,:\,X_0\times Y_0\to Z_0$ with $F(x_0,y_0)=0$ and continuous restriction on the set B_0. (For any α, we will write $B_\alpha=\{(x,y)\in X_\alpha\times Y_\alpha\ /\ \|x-x_0\|_\alpha<1/2\,,\ \|y-y_0\|_\alpha<1/2\}$.)

Assume:

(h.1) *$F(x,\cdot)\,:\,Y_0\to Z_0$ is twice differentiable and*

$$\|d_2F(x,y)\|\leq M_0\,,\quad \|d_2^2F(x,y)\|\leq M_0\quad \forall\, x,y\in B_0\,.$$

(h.2) *F is uniformly Lipschitz in X_0. That is, for all $(x,y)\in B_0$, $(x',y)\in B_0$*

$$\|F(x,y)-F(x',y)\|_0\leq M_0\|x-x'\|_0\,.$$

(h.3) *The triple (F,x_0,y_0) is of order ∞. That is*

- (h.3.1) *$F(B_0\cap X_\alpha\times Y_\alpha)\subset Z_\alpha\quad 0\leq\alpha$.*
- (h.3.2) *there exist constants M_α, $1\leq\alpha$ such that if $(x,y)\in X_\alpha\times Y_\alpha\cap B_1$ satisfies $\|x-x_0\|_\alpha\leq K$, $\|y-y_0\|_\alpha\leq K$, then $\|F(x,y)\|_\alpha\leq K\,M_\alpha$.*

(h.4) *There exist an approximate right-inverse of loss σ, $1\leq\sigma$. That is, for every $\alpha\geq\sigma$ and $(x,y)\in B_\sigma\cap(X_\alpha\times Y_\alpha)$ there exist a linear mapping $\eta(x,y)\in\mathcal{L}(Z_\alpha,Y_{\alpha-\sigma})$ such that:*

- (h.4.1) *$\|\eta(x,y)z\|_0\leq M_0\|z\|_\sigma$ for all $z\in Z_\sigma$.*
- (h.4.2) *$\|d_2F(x,y)\circ\eta(x,y)-1)z\|_0\leq M_0\|F(x,y)\|_\sigma\,\|z\|_\sigma$ for all $z\in Z_\sigma$.*
- (h.4.3) *if $\|x-x_0\|_\alpha\leq K$, $\|y-y_0\|_\alpha\leq K$, then $\|\eta(x,y)(F(x,y))\|_{\alpha-\sigma}\leq K\,M_\alpha$.*

Then, if ε is a small positive number and $\lambda=2\sigma+\varepsilon$, we can find an open neighborhood $D_\lambda=\{x\in X_\lambda\ /\ \|x-x_0\|_\lambda<C_\lambda\}$ and a mapping $\psi\,:\,D_\lambda\to Y_\sigma$ such that

- *i)* $F(x,\psi(x))=0$.
- *ii)* $\|\psi(x)-y_0\|_\sigma\leq C_\lambda^{-1}\|x-x_0\|_\lambda$.

iii) there exist $0 < \beta < 1$ *and constants* C_τ *for* $\lambda < \tau$, *such that for every* $x \in D_\lambda \cap X_\tau$ *one has* $\psi(x) \in Y_{\beta(\tau-1)}$ *and* $\|\psi(x) - y_0\|_{\beta(\tau-1)} \leq C_\tau \|x - x_0\|_\tau^\beta$.

In particular, $\psi(D_\lambda \cap X_\infty) \subset Y_\infty$. *Moreover, if* $\eta(x,y)$ *depends continuously on* (x,y), *then the mapping* $\psi : D_\lambda \to Y_\sigma$ *is continuous.*

We next precise the spaces and functional to which the implicit function theorem will be applied to yield the main result. For each $\mu \in \mathbf{T}$, we denote by $C^r_\mu(\mathbf{T}^m)$ the set of functions v of $C^r(\mathbf{T}^m)$ satisfying

$$v_\lambda(\varphi, \theta) = v_\lambda(\varphi, -\theta + 2\mu), \quad \forall\, (\lambda, \varphi, \theta) \in \mathbf{T}^m\,.$$

Let us fix $l \in \mathbb{N}$, $l \geq 7$, we take

$$X_\alpha = Z_\alpha = \Big\{ v \in C^{l-7+\alpha}(\mathbf{T}^m) \,/\, v_\lambda(\varphi, 0) = 0 \quad \forall\, (\lambda, \varphi) \in \mathbf{T}^{m-1} \Big\}$$

and

$$\begin{aligned} Y_\alpha = \Big\{ (v^1, v^2) \in C_0^{l-5+\alpha}(\mathbf{T}^m) \times C_\rho^{l-5+\alpha}(\mathbf{T}^m) \,/ \\ v^i_\lambda(\varphi, 0)) = 0,\ \forall\, (\lambda, \varphi) \in \mathbf{T}^{m-1},\ i = 1, 2 \Big\} \end{aligned}$$

for any $\alpha \geq 0$, with the corresponding Hölder norms and define the operators

$$\begin{array}{rll} F \ : & X_\alpha \times Y_\alpha & \longrightarrow Z_\alpha \\ & (v, v^1, v^2) & \longrightarrow v^1_\lambda(\varphi + v^2_\lambda(\varphi, \theta), \theta) + v^2_\lambda(\varphi, \theta) - v_\lambda(\varphi, \theta). \end{array}$$

¿From Lemma 2.5 of [10] it is immediate to show that $(X_\alpha)_{\alpha \geq 0}$, $(Y_\alpha)_{\alpha \geq 0}$ and $(Z_\alpha)_{\alpha \geq 0}$ are families of regular spaces. Observe that we only use those diffeomorphisms in $\mathbf{T}^m$ which are the identity on the plane $\theta = 0$ in order to guarantee the uniqueness of the decomposition into symmetric factors.

Remark. Consider a C^r vertical diffeomorphism of $\mathbf{T}^m$ in a small neighborhood of the identity, $f_\lambda(\varphi, \theta) = (\varphi + v_\lambda(\varphi, \theta), \theta)$ with $r \geq 1$. Then f_λ^{-1} is also a C^r vertical diffeomorphism $f_\lambda^{-1}(\varphi, \theta) = (\varphi + u_\lambda(\varphi, \theta), \theta)$. Besides, taking $B_r = \{ v \in C^r / ||v||_{C^r} < 1/2 \}$ the map $B_r \subset C^r(\mathbf{T}^m) \to C^{r'}(\mathbf{T}^m)$, $v \to u$ is continuous for every $r' < r$. In the case that $v \in C^r_\mu(\mathbf{T}^m)$ and $v_\lambda(\varphi, 0) = 0$ for every $(\lambda, \varphi) \in \mathbf{T}^{m-1}$ it is obvious that $u \in C^r_\mu(\mathbf{T}^m)$ and $u_\lambda(\varphi, 0) = 0$ for every $(\lambda, \varphi) \in \mathbf{T}^{m-1}$.

We will show that the above F satisfy all the conditions of the previous statement around $(0,0,0)$. Hence, we will obtain the result by applying Zehnder's theorem. Note that the application Zehnder's theorem also provides the proof of Proposition 3.4 for C^∞ families of diffeomorphisms in the sphere.

The conditions (h.1) (h.2) (h.3) are differentiability properties of the composition. They follow from Theorems 4.3 and 6.11 in [7]. The most crucial hypothesis is that about the existence of an approximate right-inverse of the linearized operator, which we now discuss.

We will start the construction of the exact inverse at $(0,0,0)$. To study its differentiability properties, we will use the following result proved in Rüssmann [9].

Lemma 3.7 *Let ρ be a constant type number and*

$$v(z) = \sum_{k\in\mathbb{Z}} \hat{v}_k e^{ikz}$$

an analytic complex function in the strip $U_\tau = \{z \in \mathbf{C} / \ |\mathrm{Im} z| < \tau\}$ with $|v(z)| \leq M \ \ \forall\, z \in U_\tau$. The function with mean value zero

$$h(z) = \sum_{k\in\mathbb{Z}-\{0\}} \frac{\hat{v}_k}{e^{2\pi ik\rho}-1} e^{ikz}$$

is analytic in $U_{\tau'}$ for every $\tau' < \tau$ and, there is a constant $\kappa > 0$, depending only on ρ, such that

$$|h(z)| \leq \frac{\kappa\, M}{\tau - \tau'} \qquad \text{for} \quad |\mathrm{Im} z| < \tau'. \tag{11}$$

We take a periodic real analytic function $v_\lambda(\varphi,\theta)$ defined in the strip U_τ^m with $v_\lambda(\varphi,0) = 0$ for every $(\lambda,\varphi) \in \mathbf{T}^{m-1}$ and consider the pair h_λ^1, h_λ^2 whose Fourier coefficients in the variable θ ,$(\hat{h}_{\lambda,k}(\varphi))_{k\in Z}$, are given by the relations (8) when $k \neq 0$ and by $\hat{h}_{\lambda,0}(\varphi) = 0$ for every λ, φ. Define

$$v_\lambda^1(\varphi,\theta) = h_\lambda^1(\varphi,\theta) - h_\lambda^1(\varphi,0)$$

$$v_\lambda^2(\varphi,\theta) = h_\lambda^2(\varphi,\theta) - h_\lambda^2(\varphi,0).$$

Observe that the Fourier coefficients in the variable θ of v^1_λ, v^2_λ are given by the equations (8) and (9). The functions $h^1_\lambda, h^2_\lambda, v^1_\lambda, v^2_\lambda$ are analytic in the strip $U^m_{\tau'}$ for every $\tau' < \tau$ and satisfy a inequality similar to (11) with maybe some other constants. (Notice that if ρ is a constant type number, so is -2ρ, $e^{-4\pi i\rho k}/(e^{-4\pi i\rho k}-1) = 1 + 1/(e^{-4\pi i\rho k}-1)$ and that $\sup_{\Im z\leq\tau} |\sum_k \hat{v}_k e^{2\pi ikz}| = \sup_{\Im z\leq\tau} |\sum_k \hat{v}_{-k} e^{2\pi ikz}|$). Besides $v_\lambda(\varphi,\theta) = v^1_\lambda(\varphi,\theta) + v^2_\lambda(\varphi,\theta)$.

¿From Lemma 3.7 it is also possible to deduce regularity properties of the above decomposition in the classes of finitely differentiable functions. The key is that the differentiability properties of functions can be read of from the size of analytic approximations on thin strips –see [8]–. The natural spaces on which to consider such characterizations are the Λ_r spaces endowed with the Zygmund norms. We just remark that, when $r \notin \mathbb{N}$, $\Lambda_r = C^r$, whereas $r \in \mathbb{N}$, $C^r \subset \Lambda_r$ and the inclusion is strict. Notice, however that the definition of C^r functions makes sense on Banach spaces (for $0 \leq r \leq 1^-$ it makes sense in metric spaces) whereas the characterization by analytic approximations is useful only for functions on $\mathbb{R}^n$.

A detailed argument along these lines can be found in [8] p. 528, [10], [4] which allows us to state the following:

Lemma 3.8 *Let $v \in C^r(\mathbf{T}^m)$, $r \notin \mathbb{N}$ $r > 1$, with $v_\lambda(\varphi, 0) = 0$ for every $(\lambda,\varphi) \in \mathbf{T}^{m-1}$. There are functions $v^1 \in C^{r-1}_0(\mathbf{T}^m)$, $v^2 \in C^{r-1}_\rho(\mathbf{T}^m)$ and a constant κ depending only on ρ and r such that*

i) $v = v^1 + v^2$.

ii) $v^1_\lambda(\varphi,0) = v^2_\lambda(\varphi,0) = 0$ *for all* $(\lambda,\varphi) \in \mathbf{T}^{m-1}$.

iii) $||v^1||_{C^{r-1}}, ||v^2||_{C^{r-1}} \leq \kappa||v||_{C^r}$.

Notice that $F : X_\alpha \times Y_\alpha \to Z_\alpha$ is differentiable with respect to the $Y's$ components and $d_2F(0,0,0)(v^1,v^2) = v^1 + v^2$ in all the cases. Consequently, the conclusions of Lemma (3.8) provides a direct estimation of the exact inverse at $(0,0,0)$ of d_2F. We obtain

Lemma 3.9 *Let $\epsilon > 0$ be a small parameter and $\sigma = 3+\epsilon$. For every $\alpha \geq \sigma$ and $v \in Z_\alpha$, there is an unique element $(v^1,v^2) \in Y_{\alpha-\sigma}$ such that $v = v^1 + v^2$. Moreover, the map*

$$\begin{array}{rccc} \eta : & Z_\alpha & \longrightarrow & Y_{\alpha-\sigma} \\ & v & \longrightarrow & (v^1,v^2) \end{array}$$

is continuous.

(More precisely, the map $\eta\ : Z_{\alpha-\epsilon/2} \to Y_{\alpha-\sigma}$ is also continuous). Now an approximate right-inverse in a complete neighborhood of $(0,0,0)$ is required. An important observation is that one exact right-inverse can be obtained thanks to the group action. See Hamilton [2], pg 198 and Zehnder [10], pg. 133.

Let us consider $F_2 : Y_\alpha \to Z_\alpha,\ (v^1,v^2) \to F(0,v^1,v^2)$. It is obvious that $d_2F(v',v^1,v^2) = dF_2(v^1,v^2)$. We write $B^2_\alpha = \{(v^1,v^2) \in Y_\alpha\ /\ ||v^1||_\alpha < 1/2, ||v^2||_\alpha < 1/2\}$.

Lemma 3.10 *For each $\alpha \geq \sigma$, there exists a continuous map $\eta :\ B^2_\sigma \cap Y_\alpha \to \mathcal{L}(Z_\alpha, Y_{\alpha-\sigma})$ such that*

$$dF_2(v^1,v^2) \circ \eta(v^1,v^2)(v) = v$$

for every $(v^1,v^2) \in B^2_\sigma \cap Y_\alpha$ and $v \in Z_\alpha$.

Proof. First of all we solve this question in the spaces of infinitely differentiable functions. The sets $X_\infty, Y_\infty, Z_\infty$ can be understood as tangent spaces at the identity of groups of C^∞ diffeomorphisms in $\mathbf{T}^m$.

Let us fix $(v^1,v^2) \in B^2_0 \cap Y_\infty$ (that is $||v^i||_{C^{l-5}} < 1/2, i = 1,2$), with which we associate the following differentiable functions

$$\begin{array}{llll} \Phi_{(v^1,v^2)} : & Z_\infty & \longrightarrow & Z_\infty \\ & w & \longrightarrow & v^1_\lambda(\varphi + w_\lambda(\varphi + v^2_\lambda(\varphi,\theta),\theta) + v^2_\lambda(\varphi,\theta),\theta) \\ & & & \quad +w_\lambda(\varphi + v^2_\lambda(\varphi,\theta),\theta) + v^2_\lambda(\varphi,\theta) \end{array}$$

and

$$\begin{array}{llll} \mathcal{C}_{(v^1,v^2)} : & Y_\infty & \longrightarrow & Y_\infty \\ & (\omega^1,\omega^2) & \longrightarrow & (v^1_\lambda(\varphi + \omega^1_\lambda(\varphi,\theta),\theta) + \omega^1_\lambda(\varphi,\theta), \\ & & & \quad \omega^2_\lambda(\varphi + v^2_\lambda(\varphi,\theta),\theta) + v^2_\lambda(\varphi,\theta)). \end{array}$$

Take the vertical diffeomorphisms $f^i_\lambda(\varphi,\theta) = (\varphi + v^i_\lambda(\varphi,\theta)),\theta)$ with inverses $(f^i_\lambda)^{-1}(\varphi,\theta) = (\varphi + u^i_\lambda(\varphi,\theta)),\theta),\ i = 1,2$. Notice that $F_2(v^1,v^2) = \Phi_(v^1,v^2)(0)$; besides we find the following relations

$$\Phi_{(v^1,v^2)} \circ \Phi_{((u^1),(u^2))} = Id. \tag{12}$$

$$\Phi_{(v^1,v^2)} \circ F_2 = F_2 \circ \mathcal{C}_{(v^1,v^2)}. \tag{13}$$

By differentiating (13) we obtain

$$d\Phi_{(v^1,v^2)}(0) \circ dF_2(0,0) = dF_2(v^1,v^2) \circ d\mathcal{C}_{(v^1,v^2)}(0,0);$$

therefore, from relation (12) and Lemma (3.9), we can deduce that $\eta(v^1,v^2) = d\mathcal{C}_{(v^1,v^2)}(0,0) \circ \eta \circ d\Phi_{(u^1,u^2)}(F_2(v^1,v^2))$ is a linear continuous map from Z_∞ to Y_∞ with

$$dF_2(v^1,v^2) \circ \eta(v^1,v^2)(v) = v \tag{14}$$

for every $v \in Z_\infty$.

We now focus our attention to the classes of finitely differentiable functions where we consider defined $\Phi_{(v^1,v^2)}$ and $\mathcal{C}_{(v^1,v^2)}$. It follows from Propositions 6.1, 6.2 and Theorem 6.10 in [7] that if $(v^1,v^2) \in B^2_\sigma \cap Y_\alpha$ then $\eta(v^1,v^2) \in \mathcal{L}(Z_\alpha, Y_{\alpha-\sigma})$ and besides $\eta:\ B^2_\sigma \cap Y_\alpha \to \mathcal{L}(Z_\alpha, Y_{\alpha-\sigma})$, $(v^1,v^2) \to \eta(v^1,v^2)$ is continuous.

Actually, a more precise analysis of the continuity of η allows us to state that, if $(v^1,v^2) \in B^2_0 \cap Y_{\alpha-\epsilon/2}$, then $\eta(v^1,v^2) \in \mathcal{L}(Z_{\alpha-\epsilon/2}, Y_{\alpha-\sigma})$ and the map $\eta:\ B^2_0 \cap Y_{\alpha-\epsilon/2} \to \mathcal{L}(Z_{\alpha-\epsilon/2}, Y_{\alpha-\sigma})$ still remains continuous.

Taking into account that the closure of $B^2_0 \cap Y_\infty$ and Z_∞ in the natural topologies of $Y_{\alpha-\epsilon/2}$ and $Z_{\alpha-\epsilon/2}$ contain $B^2_\sigma \cap Y_\alpha$ and Z_α respectively, we deduce from (14) that

$$dF_2(v^1,v^2) \circ \eta(v^1,v^2)(v) = v$$

for every $(v^1,v^2) \in B^2_\sigma \cap Y_\alpha$ and $v \in Z_\alpha$. This proves that η is an exact right-inverse of loss σ of d_2F.

$\blacksquare$

We have verified all the hypotheses of Theorem (3.6). Its conclusions provide the factorization of every C^l vertical diffeomorphism of $\mathbf{T}^m$ in a neighborhood of Id into C^{l-2} factors commuting with reflections across the planes $\theta = 0$ and $\theta = \rho$ respectively. In these conditions we can return to our original problem in the sphere.

Let us assume that one of the above diffeomorphisms on the torus becomes from f^v_λ, a C^l family of vertical diffeomorphisms of $\mathbf{S}^n$ with $f^v_\lambda|_{K^1_\varepsilon} = Id$, and consider its decomposition $f^v_\lambda = f^{v,1}_\lambda \circ f^{v,2}_\lambda$ according to the thesis of Theorem (3.6). We denote, as usual, $f^v_\lambda(\varphi,\theta) = (\varphi + v_\lambda(\varphi,\theta), \theta)$, the factors $f^{v,i}_\lambda(\varphi,\theta) =$

$(\varphi + v^i_\lambda(\varphi,\theta),\theta)$ and their inverses $(f^{v,i}_\lambda)^{-1}(\varphi,\theta) = (\varphi + u^i_\lambda(\varphi,\theta),\theta)$, $i = 1,2$. Let us fix $\lambda \in \mathbf{T}^{m-2}$ and take $\varphi \in \mathbf{T}$ with $|\sin(\pi\varphi)| \leq \varepsilon$. Then $v_\lambda(\varphi,\theta) = 0$, which implies that $(f^{v,1}_\lambda)^{-1}(\varphi,\theta) = (f^{v,2}_\lambda)(\varphi,\theta)$ and hence $u^1_\lambda(\varphi,\theta) = v^2_\lambda(\varphi,\theta)$ for every $\theta \in \mathbf{T}$. An analysis of this equality based on the Fourier coefficients, given in (7) and (9), shows that $u^1_\lambda(\varphi,\theta) = v^2_\lambda(\varphi,\theta) = 0$ and hence $f^{v,1}_\lambda(\varphi,\theta) = f^{v,2}_\lambda(\varphi,\theta) = (\varphi,\theta)$ for every $\theta \in \mathbf{T}$. This same argument proves that if for some λ one has $f^v_\lambda = Id$ then also $f^{v,1}_\lambda = f^{v,2}_\lambda = Id$.

Thus, it is clear that $f^{v,1}_\lambda, f^{v,2}_\lambda$ can be extended by the identity on the open subset of $\mathbf{S}^n$, where our coordinate system does not work, and define C^{l-2} families of vertical diffeomorphisms of $\mathbf{S}^n$ satisfying all the conditions $(i)-(iv)$ of Proposition (3.4). This finishes the induction process and completes the proof of Theorem (3.1).

$\blacksquare$

References

[1] F. Bien *Global representations of the diffeomorphism group of the circleInfinite dimensional Lie Algebras and groups, Marseille 1988* World Scientific (1989)

[2] R. Hamilton *The inverse function theorem of Nash and Moser Bull. A.M.S.* **7** 65-222 (1982)

[3] M. Hirsch *Differential Topology* Springer Verlag (1976)

[4] S. Krantz *Lipschitz spaces, smoothness of functions and approximation theory Expo. Mat.* **3** 193–260 (1983)

[5] J. Langer, D.A. Singer *Diffeomorphisms of the circle and geodesic fields on Riemann surfaces of genus one Inv. Mat.* **69** 229-242 (1982)

[6] R. de la Llave *Remarks on J. Langer and D.A. Singer decomposition theorem for diffeomorphisms of the circle Comm. Math. Phys.* **104** 387-401 (1986)

[7] R. de la Llave, R. Obaya *Regularity of the composition operator in spaces of Hölder functions* Preprint

[8] J. Moser *A rapidly convergent iteration method and non-linear differential equations II* Ann. Sc. Norm.Pisa **20** 499-535 (1966)

[9] H. Rüssmann *On optimal estimates for the solutions of linear difference equations on the circle* Cel. Mech. **14** 33–37 (1976)

[10] E. Zehnder *Generalized implicit function theorems with applications to some small divisor problems I* Comm. Pure and Appl. Math. **28** 91–140 (1975)

[11] E. Zehnder *Generalized implicit function theorems with applications to some small divisor problems II* Comm. Pure and Appl. Math. **29** 49–111 (1976)