On the Phase Diagram of the Random Field Ising Model on the Bethe Lattice

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Abstract: The ferromagnetic Ising model on the Bethe lattice of degree k is considered in the presence of a dichotomous external random field $\xi_x = \pm\alpha$ and the temperature $T \geq 0$. We give a description of a part of the phase diagram of this model in the T–α plane, where we are able to construct limiting Gibbs states and ground states. By comparison with the model with a constant external field we show that for *all realizations* $\xi = \{\xi_x = \pm\alpha\}$ of the external random field :

(i) the Gibbs state is unique for $T > T_c$ $(k \geq 2)$ and for $\alpha > 3$ $(k = 2)$;

(ii) the $\pm$-phases coexist in the domain $\{T < T_c,\ \alpha \leq H^{\mathrm{F}}(T)\}$,

where T_c is the critical temperature and $H^{\mathrm{F}}(T)$ is the critical external field in the ferromagnetic Ising model on the Bethe lattice with a constant external field.

Then we prove that for *almost all* ξ:

(iii) the $\pm$-phases coexist in a larger domain $\{T < T_c,\ \alpha \leq H^{\mathrm{F}}(T) + \varepsilon(T)\}$ where $\varepsilon(T) > 0$; and

(iv) the Gibbs state is unique for $3 \geq \alpha \geq 2$.

We show that the residual entropy at $T = 0$ is positive for $3 \geq \alpha \geq 2$, and we give a constructive description of ground states, by so-called dipole configurations.

Key words: Random external field, Ising model, Gibbs states, ground states, Bethe lattice, residual entropy, dipole configurations, Griffiths singularities

Pacs: 05.50.+q; 75.10Nr.

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1 Introduction and Definitions

It is known that the Ising model on the Bethe lattice τ_k of degree $k \geq 2$, exhibits rather nontrivial behavior from the point of view of the structure of Gibbs states [BG, BRZa]. The problem gets obviously more complicated when the system is embedded into an external random field. In this paper we will be interested in the *random field Ising model* (RFIM) in the dichotomous external random field taking values $\xi_x = \pm\alpha$, $x \in V(\tau_k)$, where $V(\tau_k)$ is the set of vertices of the Bethe lattice τ_k.

The case $k = 1$ corresponds to the one–dimensional RFIM. In this case there is no phase transition for the inverse temperature $\beta = T^{-1} < \infty$ (i.e. the Gibbs state $\mu_{\beta,\xi}$ is unique for all $\beta < \infty$ and all configurations $\xi = \{\xi_x, x \in V(\tau_k)\}$). The structure of ground states $\mu_{\infty,\xi} = \lim_{\beta\to\infty} \mu_{\beta,\xi}$ is described in our previous paper [BRZb]. The exact formula for the residual entropy $S_\infty > 0$ is derived by Derrida *et al.* [DVP] (see also [BPZ],[V],[PF],[KM]). Some partial results for $\beta < \infty$ are obtained by Bruinsma and Aeppli (see [AB, BA]). For the case $k = 2$ Bruinsma [Br] proposes some clever theoretical arguments, to describe the structure of ground states for RFIM on the Bethe lattice, and to estimate the residual entropy at $\beta \to \infty$.

The aim of the present paper is to give a rigorous study of the phase diagram of the RFIM on the Bethe lattice for the dichotomous random external field. The rest of this section contains main notations and definitions. Section 2 is devoted to formulation of our main results. They are summarized in Fig. 1.1 and they can be formulated as follows:

- The Gibbs state is unique for all α and for *all* realizations $\xi = \{\xi_x,\ x \in V(\tau_k)\}$ of the external field, if $T > T_c$, where T_c is the critical temperature of the model in the absence of external field. (*This result is valid for any degree* $k \geq 2$).
- If $k = 2$ and $\alpha > 3$, then the Gibbs state $\mu_{\beta,\xi}$ is unique for all $T > 0$ and *all* ξ. In this case the ground state $\mu_{\infty,\xi} = \lim_{\beta\to\infty} \mu_{\beta,\xi}$ exists, and it is concentrated on the spin configuration that follows the sign of the external field.
- If $k = 2$ and $2 \leq \alpha \leq 3$, then the Gibbs state is unique for all $T > 0$ and for *almost all* realizations ξ of the external field. We show that

in this case the ground state exists, and it is concentrated on the set of *dipole ground state configurations* described in details in Section 4. In Section 5 we derive an exact formula for the corresponding residual entropy. We show that the residual entropy is constant for $2 < \alpha < 3$ and it has spikes at the endpoints $\alpha = 2$ and $\alpha = 3$ (cf. [Br]).

- In the low temperature domain, $T < T_c$, we show that for all $\alpha \leq H^{\mathrm{F}}(T)$, where $H^{\mathrm{F}}(T) \leq 1$ is the critical constant external field, and for *all* ξ, there are two different extreme Gibbs states, which are limiting Gibbs states $\mu^{\pm}_{\beta,\xi}$ obtained with (+)- and (−)-boundary conditions. For $\alpha < 1$ the corresponding ground states $\mu^{\pm}_{\infty,\xi} = \lim_{\beta\to\infty} \mu^{\pm}_{\beta,\xi}$ are concentrated on (+)- and (−)-spin configurations, respectively.

- We extend the above domain where the (+)- and (−)-Gibbs state are different, using an intermittency of the effective external field for small values of $\varepsilon = \alpha - H^{\mathrm{F}}(T) > 0$ (see Section 6). Namely, we show that there exists $\varepsilon(T)$ such that in the domain $\{0 < T < T_c, H^{\mathrm{F}}(T) < \alpha < H^{\mathrm{F}}(T) + \varepsilon(T)\}$, the two limiting Gibbs states with (+)- and (−)-boundary conditions are different for *almost all* realizations ξ.

  The line $\alpha = H^{\mathrm{F}}(T)$ is the line of the *Griffiths singularities*. On this line there is a discontinuous change of support of the probability distribution of the effective external field. The discontinuous change of support leads to a non-analyticity of some thermodynamic averages as a function of α at $\alpha = H^{\mathrm{F}}(T)$. Since these averages are C^{∞}-functions of α, the line $\alpha = H^{\mathrm{F}}(T)$ is the line of the Griffiths singularities (cf. [Br]).

- We conjecture that in domain $\{0 < T < T_c, H^{\mathrm{AF}}(T) - \eta(T) < \alpha < H^{\mathrm{AF}}(T)\}$ the Gibbs state $\mu_{\beta,\xi}$ is unique for *almost all* ξ. The line $H^{\mathrm{AF}}(T)$ corresponds to the critical constant external field for the antiferromagnetic Ising model, see (2.1).We guess that $H^{\mathrm{AF}}(T)$ is the line of the Griffiths singularities and that the Gibbs state is unique for *all* ξ in domain $\{0 < T < T_c, H^{\mathrm{AF}}(T) < \alpha\}$.

Let us introduce main definitions. The Hamiltonian of the ferromagnetic random field Ising model is given by

$$H = -\sum_{x,y} J_{xy}\, \sigma_x \sigma_y - \sum_x \xi_x\, \sigma_x \tag{1.1}$$

Here σ_x are spin variables taking values ± 1, ξ_x stands for the random external field, and $J_{xy} > 0$ are the coupling constants. We shall consider uniform interactions $J_{xy} = 1$, and a *dichotomous field*, i.e., the ξ_x are real independent random variables taking values $\pm\alpha$ with probability $1/2$. These variables are defined for each site of a lattice and we shall be concerned by the *Bethe lattice* τ_k of degree $k \geq 2$, i.e., τ_k is a tree with exactly $k+1$ vertices coming out from each vertex x. We use V and L to denote respectively the set of vertices and edges of τ_k. There is a distance $d(x, y)$ on V which is the length of the unique path from x to y, assuming that the length of an edge is 1. Let us fix a vertex x_0 as the origin. We define

$$W_n = \{x \in V \ : \ d(x_0, x) = n\}$$

the sphere of radius n and

$$V_n = \{x \in V \ : \ d(x_0, x) \leq n\} = \cup_{m=0}^{n} W_m$$

the ball of radius n with the center at x_0. We let $L_n = \{< x, y >: \ x, y \in V_n, \ d(x, y) = 1\}$ be the set of edges with endpoints in V_n, and for $x \in W_n$, $n = 0, 1, \ldots$, denote by $S(x) = \{y \in W_{n+1} \ : \ d(x, y) = 1\}$ the set of *direct successors* of x. Given a realization $\xi = \{\xi_x\}_{x \in V}$ of the external field, the finite-volume Gibbs measures, on the σ–algebras $\Sigma(V_n) = \{\sigma_n = \{\sigma_x = \pm 1, x \in V_n\}\}$, at inverse temperature $\beta = T^{-1}$, and boundary condition $\bar{\sigma}$ (a configuration on $V \setminus V_n$) are defined by

$$\mu_n(\sigma_n \mid \bar{\sigma}) = Z_n^{-1}(\bar{\sigma}) \exp\left\{\beta \sum_{<x,y> \in L_n} \sigma_x \sigma_y + \beta \sum_{x \in V_n} \xi_x \sigma_x + \beta \sum_{\substack{x \in W_n \\ y \in S(x)}} \sigma_x \bar{\sigma}_y \right\} \quad (1.2)$$

where $Z_n(\bar{\sigma})$ is the partition function.

Below it will be useful to consider a more general setup of the problem corresponding to real-valued boundary conditions. It is related to construction of (nonhomogeneous) *Markov chains* on the Bethe lattice τ_k. Let $h = \{h_x, x \in V\}$ be a set of real numbers. We define for each n, the probability distributions

$$\mu_n(\sigma_n \mid h) = Z_n^{-1} \exp\left\{\beta \sum_{<x,y> \in L_n} \sigma_x \sigma_y + \beta \sum_{x \in V_n} \xi_x \sigma_x + \beta \sum_{x \in W_n} h_x \sigma_x \right\} \quad (1.3)$$

where Z_n is the normalizing factor. These probability distributions are said to be *compatible* if for all $n > 1$,

$$\sum_{\sigma_x = \pm 1, x \in W_n} \mu_n(\sigma_n \mid h) = \mu_{n-1}(\sigma_{n-1} \mid h) \quad (1.4)$$

It is easily verified that the probability distributions (1.3) are compatible if and only if for any $x \in V$ the following equation hold:

$$h_x = \sum_{y \in S(x)} f_\beta(h_y + \xi_y) \tag{1.5}$$

where

$$f_\beta(x) = \frac{1}{2\beta} \log \frac{\cosh \beta(x+1)}{\cosh \beta(x-1)} = \frac{1}{\beta} \operatorname{artanh}[(\tanh \beta)(\tanh \beta x)] \tag{1.6}$$

Indeed, one has

$$\begin{aligned} \sum_{\sigma_y = \pm 1} e^{\beta(\sigma_x + h_y + \xi_y)\sigma_y} &= 2 \cosh \beta(\sigma_x + h_y + \xi_y) \\ &= \exp\{\beta f_\beta(h_y + \xi_y)\sigma_x + \beta d_\beta(h_y + \xi_y)\} \end{aligned} \tag{1.7}$$

for all $y \in W_n$. The second equality is satisfied for $\sigma_x = 1$ and $\sigma_x = -1$. This is done by taking $f_\beta(x)$ defined as above and

$$d_\beta(x) = (2\beta)^{-1} \log[4 \cosh \beta(x+1) \cosh \beta(x-1)]$$

If the probability distributions $\mu_n(\sigma_n \mid h)$ are compatible, then by known theorems they are projections on V_n of an infinite-volume Gibbs measure $\mu(\sigma \mid h)$, and in the opposite direction, any limiting (in particular, any extreme) infinite-volume Gibbs measure $\mu(\sigma \mid h)$ has finite-volume compartible projections of the form (1.3) (cf. [G]). We shall call by a Gibbs state, any infinite-volume Gibbs measure $\mu(\sigma)$ on V.

By the Dobrushin–Lanford–Ruelle theorem, if $\mu(\sigma)$ is a Gibbs state then for every n, the conditional distribution of $\sigma_n = \sigma|_{V_n}$, under the condition that outside of V_n, σ coincides with some fixed configuration $\bar{\sigma}$, is given by (1.2). The conditional distributions (1.2) are called *specifications* of $\mu(\sigma)$, see e.g. [G].

Let us recall that in the case of *homogeneous* Markov chains defined by equations (1.3) when for all x, $\xi_x = H$ (constant field) and $h_x = h_*$, the equations (1.5) read

$$h_* = k f_\beta(h_* + H) \tag{1.8}$$

Then there exists $H^{\mathrm{F}}(T) \leq k-1$, given by the equation

$$H^{\mathrm{F}}(T) = \beta^{-1}\left[k \operatorname{artanh}\left(\frac{k\theta - 1}{k/\theta - 1}\right)^{1/2} - \operatorname{artanh}\left(\frac{k - 1/\theta}{k - \theta}\right)^{1/2}\right] \tag{1.9}$$

where $\theta = \tanh\beta$, such that the equation (1.8) has

(i) a unique solution for $T \geq T_c = 1/\operatorname{artanh}(1/k)$ or for $T < T_c$ and $|H| > H^{\mathrm{F}}(T)$

(ii) two distinct solutions if $T < T_c$ and $|H| = H^{\mathrm{F}}(T)$

(iii) three distinct solutions if $T < T_c$ and $|H| < H^{\mathrm{F}}(T)$

2 Main Results

By $\mu_{\beta,\xi}$, where $0 < \beta < \infty$ and $\xi = \{\xi_x = \pm\alpha, x \in V\}$, we denote a Gibbs state corresponding to specifications (1.2). The existence of $\mu_{\beta,\xi}$ for all β and configurations ξ follows from the weak compactness of the space of probability measures on $\Sigma(V)$, [G].

Theorem 2.1 *If $T > T_c$, then $\mu_{\beta,\xi}$ is unique for all ξ.*

Theorem 2.2 *Let $k = 2$. If $\alpha > 3$, then*

(a) *$\mu_{\beta,\xi}$ is unique for all $0 < \beta < \infty$ and all ξ*

(b) *the limit (ground state) $\mu_{\infty,\xi} \equiv \lim_{\beta\to\infty} \mu_{\beta,\xi}$ exists and it is concentrated on a configuration such that $\sigma_x = \operatorname{sign}\xi_x$ for all $x \in V$.*

Remark. The domain of "the uniqueness for all ξ" in Theorems 2.1 and 2.2 can be probably extended as follows. Consider the critical constant external field of the *antiferromagnetic* Ising model on the Bethe lattice :

$$H^{\mathrm{AF}}(T) = \beta^{-1}\left[k \operatorname{artanh}\left(\frac{k\theta - 1}{k/\theta - 1}\right)^{1/2} + \operatorname{artanh}\left(\frac{k - 1/\theta}{k - \theta}\right)^{1/2}\right] \leq k+1 \quad (2.1)$$

where $\theta = \tanh\beta$ (see Fig. 1.1). We conjecture that if $T < T_c$ and $\alpha > H^{\mathrm{AF}}(T)$, then the Gibbs state $\mu_{\beta,\xi}$ is unique for *all* ξ. It is also plausible that there exists a continuous function $\eta(T) > 0$ on $0 < T < T_c$ such that if $T < T_c$ and $H^{\mathrm{AF}}(T) > \alpha > H^{\mathrm{AF}}(T) - \eta(T)$, then the Gibbs state $\mu_{\beta,\xi}$ is unique for *almost all* ξ. The critical line $\alpha = H^{\mathrm{AF}}(T)$ is probably the line of the Griffiths singularities (cf. [Br]), like the ferromagnetic critical line $\alpha = H^{\mathrm{F}}(T)$. In other words, we expect that some of thermodynamic averages are C^∞ non–analytic functions of α at $\alpha = H^{\mathrm{AF}}(T)$.

Theorem 2.3 *Assume that* $0 < T < T_c$ *and* $\alpha \le H^{\mathrm{F}}(T)$, *see* (1.9). *Then for any* $k \ge 2$ *and all realizations* ξ *of the external field,*

(a) *there exist two different extreme Gibbs states* $\mu^+_{\beta,\xi}$ *and* $\mu^-_{\beta,\xi}$ *which are limiting Gibbs states with* $+$ *and* $-$ *boundary conditions (i.e., for* $+$ *and* $-$ *specifications* (1.2)*)*

(b) *if* $\alpha < 1$ *then the limits (ground states)* $\mu^\pm_{\infty,\xi} \equiv \lim_{\beta\to\infty} \mu^\pm_{\beta,\xi}$ *exist and they are concentrated on configurations* $\{\sigma_x = 1,\ x \in V\}$ *and* $\{\sigma_x = -1,\ x \in V\}$, *respectively.*

Theorem 2.4 *Let* $k = 2$ *and assume that* $2 \le \alpha \le 3$. *Then*

(a) *for all* $\beta < \infty$ *and for almost all realizations* ξ *of the external field, there exists a unique Gibbs state* $\mu_{\beta,\xi}$

(b) *for almost all* ξ, *there exists a ground state* $\mu_{\infty,\xi} \equiv \lim_{\beta\to\infty} \mu_{\beta,\xi}$ *and it is a probability measure concentrated on the set of dipole ground state configurations which are described in Section* 4 *below.*

The residual entropy S_∞ of the ground state $\mu_{\infty,\xi}$ is calculated in Section 5.

Theorem 2.5 *Let* $k = 2$. *Then there exists a positive continuous function* $\varepsilon(T)$ *on* $0 \le T \le T_c$ *(*$\varepsilon(0) = \varepsilon(T_c) = 0$*) such that if* $0 < T < T_c$ *and* $H^{\mathrm{F}}(T) < \alpha < H^{\mathrm{F}}(T) + \varepsilon(T)$, *then for almost all* ξ, $\mu^+_{\beta,\xi} \ne \mu^-_{\beta,\xi}$.

Theorem 2.5 is derived from the following

Theorem 2.6 *Under the assumptions of Theorem* 2.5, *Let* $0 < T < T_c$ *and* $H^{\mathrm{F}}(T) < \alpha < H^{\mathrm{F}}(T) + \varepsilon(T)$. *Then under the assumptions of Theorem* 2.5 *one has :*

$$\mathbb{E}_\xi \int \sigma_x \, \mu^+_{\beta,\xi}(\sigma) \equiv \mathbb{E}_\xi \langle \sigma_x \rangle^+_{\beta\xi} > 0$$

Before to pass to proofs we would like to make few remarks about our results.

Theorems 2.1, 2.2, and 2.3 are valid for *all* realizations ξ of the random external field and these theorems are relatively easy. To prove them we use some contraction estimates and F.K.G. correlation inequalities to show that the ferromagnetic Ising model on the Bethe lattice with a *dichotomous random* external field is majorized, in an appropriate sense, by the model with a *constant* external field of the same strength. In fact, Theorems 2.1 and 2.2 give an example of a *strong uniqueness* of Gibbs measures introduced in [CvE],[vE].

On the contrary, Theorems 2.4 and 2.5 are valid only for *almost all* realizations of ξ and their proof is much more difficult. It is worth to notice that in these theorems the condition "for almost all realizations ξ" cannot be replaced by the one "for all realizations ξ". For instance, in Theorem 2.4 one can take $2 \leq \alpha < H^{\mathrm{AF}}(T)$ and a *chess-board* realization ξ (a realization with alternating pluses and minuses). Then the Gibbs state $\mu_{\beta,\xi}$ is *not* unique for this ξ, although by Theorem 2.4 it is *unique* for *almost all* ξ. Similarly, in Theorem 2.5 $\mu^+_{\beta,\xi} = \mu^-_{\beta,\xi}$ if one takes $\xi_x \equiv \alpha > H^{\mathrm{F}}(T)$, a *constant realization*, while $\mu^+_{\beta,\xi} \neq \mu^-_{\beta,\xi}$ for *almost all* ξ.

An interesting feature of the Ising model with the dichotomous random external field is that the residual entropy S_∞ at $T = 0$ is *positive* for $2 \leq \alpha \leq 3$. We conjecture that it is *positive* in the interval $1 \leq \alpha < 2$ as well but we cannot prove it. Convincing heuristic arguments in the favor of this conjecture are given by Bruinsma [Br]. We calculate S_∞ in Section 5 below, and it turns out that $S_\infty(\alpha) = const$ in the interval $2 < \alpha < 3$ while at $\alpha = 2$ and $\alpha = 3$ the residual entropy has two it spikes. This behavior of the residual entropy is easily explained by the structure of the Gibbs measure in the limit $T \to 0$. Namely, for all $2 < \alpha < 3$ the limiting Gibbs measure $\mu_{\infty,\xi}$ is *independent* of α, while for $\alpha = 2$ and $\alpha = 3$ the corresponding

Gibbs measures are concentrated on much bigger sets of configurations than for $2 < \alpha < 3$ (see Section 4). Bruinsma [Br] derives a good approximate formula for $S_\infty(\alpha)$ and he shows that this approximate formula predicts that $S_\infty(\alpha)$ is constant in every interval $1 + \frac{2}{n+1} < \alpha < 1 + \frac{2}{n}$, $n = 1, 2, \ldots$, with spikes at $\alpha = 1 + \frac{2}{n}$.

The central point in the proof of Theorem 2.5 in Section 6 is to show that the limiting probability distribution $\nu(dh_x)$ of the effective external field h_x is not symmetric under plus boundary conditions. To prove the asymmetry of $\nu(dh_x)$ we use the intermittency of the iterations of h_x for a small difference $\alpha - H^{\mathrm{F}}(T) > 0$, and we show that the main mass of $\nu(dh_x)$ is concentrated on the positive half-axis, which gives Theorem 2.6. Then we derive Theorem 2.5 from Theorem 2.6 using some soft ergodic arguments.

3 Proof of Theorems 2.1–2.3 and 2.4a

We introduce the variables $g_x = f_\beta(\xi_x + h_x)$. Then the recursive equation (1.5) reads

$$h_x = \sum_{y \in S(x)} g_y \tag{3.1}$$

This implies that g_x satisfies

$$g_x = f_\beta\Big(\xi_x + \sum_{y \in S(x)} g_y\Big) \tag{3.2}$$

By (1.3) and (3.1), the probability distribution $\mu(\sigma_n \mid g)$, $\sigma_n \equiv \sigma_{|V_n}$, can be written as

$$\mu(\sigma_n \mid g) = Z_n^{-1} \exp\Big\{\beta \sum_{<x,y> \in L_n} \sigma_x \sigma_y + \beta \sum_{x \in V_n} \xi_x \sigma_x + \beta \sum_{\substack{x \in W_n \\ y \in S(x)}} \sigma_x g_y\Big\} \tag{3.3}$$

We freely call $g = \{g_x\}$ the *effective* external field, along with $h = \{h_x\}$. We recall that by F.K.G. inequalities [FKG] one has the following proposition (see [LM]).

Proposition 3.1 *The Gibbs states $\mu^+_{\beta,\xi}$ and $\mu^-_{\beta,\xi}$ exist and they are extreme for all ξ. If $\mu^+_{\beta,\xi} = \mu^-_{\beta,\xi}$, for a given ξ, then the Gibbs state is unique for this ξ.*

We denote by $g^\pm = \{g_x^\pm,\ x \in V\}$ the configurations that correspond to the Gibbs states $\mu^\pm_{\beta,\xi}$.

We use below some properties of the function (1.6). They are summarized in the next

Proposition 3.2

$$
\begin{aligned}
f_\beta(-x) &= -f_\beta(x), \qquad f_\beta(\infty) = 1 \\
0 < \frac{d}{dx} f_\beta(x) &< \tanh\beta \quad \forall x \neq 0, \qquad \frac{d}{dx} f_\beta(0) = \tanh\beta &\quad (3.4)\\
\frac{d}{dx} f_\beta(x) &< \frac{1}{2} \quad \text{if} \quad x \geq 1 &\quad (3.5)\\
\frac{d}{dx} f_\beta(x) &< \frac{1}{2}(1-\tanh\beta) \quad \text{if} \quad x \geq 2 &\quad (3.6)\\
\frac{d^2}{dx^2} f_\beta(x) &< 0 \quad \forall x > 0, \qquad \frac{d^2}{dx^2} f_\beta(0) = 0 &\quad (3.7)
\end{aligned}
$$

Proof. All the formulae (3.4)–(3.7) result from the following two equations:

$$
\begin{aligned}
\frac{d}{dx} f_\beta(x) &= \frac{1}{2}[\,\tanh\beta(x+1) - \tanh\beta(x-1)\,] &\quad (3.8)\\
\frac{d^2}{dx^2} f_\beta(x) &= \frac{\beta}{2}\left[\frac{1}{\cosh^2\beta(x+1)} - \frac{1}{\cosh^2\beta(x-1)}\right]
\end{aligned}
$$

QED

Now we turn to the proof of Theorems 2.1–2.3 and 2.4a.

Proof of Theorems 2.1 *and* 2.2. Consider the set of recursive equations (3.2). We have

$$g_x^+ - g_x^- = f_\beta'(c) \sum_{y \in S(x)} (g_y^+ - g_y^-)$$

for some $c \in \left[\xi_x + \sum_{y\in S(x)} g_y^-, \xi_x + \sum_{y\in S(x)} g_y^+\right]$ so that

$$|g_x^+ - g_x^-| \leq k|f_\beta'(c)| \sup_{y \in S(x)} |g_y^+ - g_y^-| \tag{3.9}$$

When $T > T_c$, one has $\tanh\beta < 1/k$, so that by (3.4), $f'_\beta(c) < 1/k$. By applying recursively the inequality (3.9), we get that $g^+_x = g^-_x$. Hence $\mu^+_{\beta,\xi} = \mu^-_{\beta,\xi}$. In virtue of Proposition 3.1 this proves Theorem 2.1.

To prove the Theorem 2.2, we first remark that $|g_y| \le 1$ for all y . When $\alpha \ge 3$, one gets $|\xi_x| \ge 3$ for all x , so that for $k = 2$ we have $|c| \ge 1$ in (3.9). Thus $f'_\beta(c) < (1/2)$ by (3.5) and iterating (3.9) we conclude that $g^+_x = g^-_x$. Hence, the proof of the Statement (a) of Theorem 2.2 proceeds as above by a simple refence to Proposition 3.1.

For the Statement (b), we observe that for $n = 1$ one gets by (3.3) that

$$\mu(\sigma_x \mid g) = Z^{-1} \exp\Big\{\beta\Big[\xi_x + \sum_{y:d(x,y)=1} g_y\Big]\sigma_x\Big\}$$

When $|\xi_x| > 3$ then $|\xi_x + \sum_{y:d(x,y)=1} g_y| > 0$ and $\mathrm{sign}(\xi_x + \sum_{y:d(x,y)=1} g_y) = \mathrm{sign}\,\xi_x$. Thus we finish the proof by taking the limit $\beta \to \infty$. *QED*

Proof of Theorem 2.3. Let $\langle\ \cdot\ \rangle^{\pm}_{\beta\xi}$ denote the expectation with respect to the measure $\mu^{\pm}_{\beta,\xi}$. Then by conditions of the Theorem and by the F.K.G. inequality one get (cf. the the one-point measure above) that

$$\langle\sigma_x\rangle^+_{\beta\xi} \ge \langle\sigma_x\rangle^+_{\beta\{-\alpha\}} > 0$$

and

$$\langle\sigma_x\rangle^-_{\beta\xi} \le \langle\sigma_x\rangle^-_{\beta\{\alpha\}} < 0$$

This proves that $\mu^+_{\beta,\xi} \ne \mu^-_{\beta,\xi}$.Their extremality follows from Proposition 3.1. Since for $\alpha < 1$ $\lim_{\beta\to\infty}\langle\sigma_x\rangle^{\pm}_{\beta\{\mp\alpha\}} = \pm 1$, we obtain that for all realizations ξ $\sigma_x = 1$ a.e. with respect to $\mu^+_{\beta,\xi}$ and $\sigma_x = -1$ a.e. with respect to $\mu^-_{\beta,\xi}$. *QED*

Proof of Statement (a) *Theorem* 2.4. For any x, we denote by y and z, its two direct successors. The recursive equation (3.2) reads

$$g_x = f_\beta(\xi_x + g_y + g_z) \tag{3.10}$$

For a given ξ, let g^+_x and g^-_x be the g_x corresponding respectively to the states $\mu^+_{\beta,\xi}$ and $\mu^-_{\beta,\xi}$. We shall estimate recursively the expectation $\mathbb{E}_\xi\,|g^+_x - g^-_x|$. We have

$$\begin{aligned} g^+_x - g^-_x &= f_\beta(\xi_x + g^+_y + g^+_z) - f_\beta(\xi_x + g^-_y + g^-_z) \\ &= f'_\beta(c)\,[g^+_y - g^-_y + g^+_z - g^-_z] \end{aligned} \tag{3.11}$$

where $c \in [\xi_x + g_y^- + g_z^-, \xi_x + g_y^+ + g_z^+]$. Let us estimate $f'_\beta(c)$. Assume that $\xi_x = \alpha > 0$. Consider different cases for ξ_y and ξ_z.

Case (i) $\quad \xi_y = \xi_z = -\alpha$. Then we use the estimate

$$f'_\beta(c) \leq \tanh\beta \tag{3.12}$$

which is valid for all c (see (3.4))

Case (ii) $\quad \xi_y + \xi_z = 0$. Let for instance $\xi_y = \alpha$ and $\xi_z = -\alpha$. Then

$$g_y^\pm > 0, \quad g_z^\pm > -1\,,$$

hence

$$\xi_x + g_y^\pm + g_z^\pm > 1, \quad c > 1\,.$$

In this case

$$f'_\beta(c) < \frac{1}{2} \tag{3.13}$$

(see (3.5)).

Case (iii) $\quad \xi_y = \xi_z = \alpha$. Then $\xi_x + g_y^\pm + g_z^\pm > 2$, hence $c > 2$ and

$$f'_\beta(c) < \frac{1}{2}\,(1 - \tanh\beta) \tag{3.14}$$

(see (3.6)).

Notice that probabilities of the cases (i),(ii) and (iii) are respectively 1/4, 1/2 and 1/4. Thus by (3.11)–(3.14)one gets for $x \in W_n$ that

$$\mathbb{E}_\xi\,|g_x^+ - g_x^-| < \left[\frac{1}{4}\tanh\beta + \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{4}\cdot\frac{1}{2}\,(1-\tanh\beta)\right]\dot{2}E_{n+1}$$

where

$$E_{n+1} = \max_{t \in W_{n+1}} \mathbb{E}_\xi\,|g_t^+ - g_t^-|$$

This gives that

$$E_n < \frac{3 + \tanh\beta}{4} E_{n+1}$$

Since $(3 + \tanh\beta)/4 < 1$ and $E_n < 2$ for all n , this implies by iterations that $E_n = 0$ for all n. Hence for all x, $g_x^+ = g_x^-$, for almost all configurations ξ, which implies uniqueness by Proposition 3.1. $\quad$ *QED*

4 Dipole Ground States

Assume that $k=2$ and $\xi_x=\pm\alpha$ with $2<\alpha<3$. We discuss the ground state, $\mu_{\infty,\xi}(\sigma)=\lim_{\beta\to\infty}\mu_{\beta,\xi}(\sigma)$. Let $x\in V$. Then the one-site projection of the Gibbs measure (cf. Section 3) can be presented as

$$\mu_{\beta,\xi}(\sigma_x)=Z^{-1}\exp\left[\beta\sigma_x(\xi_x+g_{yx}+g_{zx}+g_{tx})\right] \tag{4.1}$$

where y,z,t are the nearest neighbors of x. In this section we don't fix the origin x_0 in V, and it is more convenient for us to consider the effective field as a function on edges, $g=\{g_{xy}\}$, rather than a function on vertices $g=\{g_x\}$. In this context, what was before g_x, $x\neq x_0$ is now denoted by g_{xy} where the edge $<x,y>$ goes from x in the direction of the origin x_0. The numbers g_{xy} satisfy the recursive equation

$$g_{xy}=f_\beta(\xi_x+g_{zx}+g_{tx}) \tag{4.2}$$

At $\beta=\infty$

$$g_{xy}=f_\infty(\xi_x+g_{zx}+g_{tx}) \tag{4.3}$$

where $f_\infty=\lim_{\beta\to\infty}f_\beta$ is the piecewise linear function:

$$f_\infty(x)=\begin{cases}-1 \text{ if } & x\leq -1\\ \ \ x \text{ if } & -1\leq x\leq 1\\ \ \ 1 \text{ if } & x\geq 1\end{cases} \tag{4.4}$$

For $2<\alpha<3$ and almost all ξ, the equation (4.3) has a unique solution $g_\xi=\{g_{xy}(\xi)\}$ with

$$g_{xy}(\xi)\in\{-1,-\varepsilon,\varepsilon,1\} \tag{4.5}$$

where

$$\alpha=2+\varepsilon,\quad 0<\varepsilon<1 \tag{4.6}$$

Consider the partition of V in three subsets, $\Lambda=\Lambda_+\cup\Lambda_-\cup\Lambda^0$ with

$$\begin{aligned}\Lambda_\pm&=\{x\in V\,:\,\pm(\xi_x+g_{yx}+g_{zx}+g_{tx})>0\}\\ \Lambda^0&=\{x\in V\,:\,\xi_x+g_{yx}+g_{zx}+g_{tx}=0\}\end{aligned}$$

where $g=\{g_{xy}\}$ is the unique solution of (4.3), $g=g(\xi)$. The sets $\Lambda_\pm$, Λ^0 depend on ξ. We show below that for almost all ξ the equation (4.2) has a unique solution $g = g(\xi, \beta)$ and the limit $\lim_{\beta\to\infty} g(\xi,\beta) = g(\xi,\infty)$, exists. Then $g(\xi,\infty)$ is the unique solution to (4.3). ¿From (4.1) it is clear that

$$\mu_{\infty,\xi}(\sigma_x = 1) = \begin{cases} 1 & \text{if} \quad x \in \Lambda_+ \\ 0 & \text{if} \quad x \in \Lambda_- \end{cases} \tag{4.7}$$

In other words, at $\beta = \infty$, $\sigma_x = 1$ on Λ_+ and $\sigma_x = -1$ on Λ_-.

Let us describe σ_x on Λ^0. Let $x \in \Lambda^0$. Assume, for the sake of definiteness, that $\xi_x = 2 + \varepsilon$. Then

$$2 + \varepsilon + g_{yx} + g_{zx} + g_{tx} = 0$$

and all g's are from the set $\{\pm 1, \pm\varepsilon\}$. Hence two of them, say g_{yx} and g_{zx}, are -1 and $g_{tx} = -\varepsilon$ (see Fig. 4.1).

Then by (4.3),

$$-\varepsilon = g_{tx} = \xi_t + g_{ut} + g_{vt}$$

which implies that

$$\xi_t = -2 - \varepsilon, \quad g_{ut} + g_{vt} = 2 \tag{4.8}$$

By (4.3),

$$\begin{aligned} g_{xt} &= \xi_x + g_{yx} + g_{zx} = 2 + \varepsilon - 1 - 1 = \varepsilon \\ g_{tu} &= \xi_t + g_{xt} + g_{vt} = -2 - \varepsilon + \varepsilon - 1 = -1 \\ g_{tv} &= \xi_t + g_{xt} + g_{ut} = -1 \end{aligned} \tag{4.9}$$

Thus,

$$\xi_t + g_{xt} + g_{ut} + g_{vt} = -2 - \varepsilon + \varepsilon + 1 + 1 = 0$$

so that $t \in \Lambda^0$.

This proves that if $x \in \Lambda_0$ and $\xi_x = 2 + \varepsilon$ then there is a neighboring vertex t such that $t \in \Lambda^0$, $\xi_t = -2 - \varepsilon$, and $g_{tx} = -g_{xt} = \varepsilon$. This motivates the following

Definition 4.1 Two neighboring vertices x,t are called a *dipole* if

$$x, t \in \Lambda^0, \quad \xi_x + \xi_t = 0, \quad |g_{xt}| = |g_{tx}| = \varepsilon \tag{4.10}$$

We call ξ_x a *charge* at x and g_{xt} a *current* from x to t.

Then we can say that in any dipole $\{x,t\}$, the charges ξ_x and ξ_t have opposite signs and the currents are

$$-g_{px}=g_{xp}=\begin{cases}\operatorname{sign}(\xi_x\cdot\varepsilon) & \text{if} \quad p=t\\ \operatorname{sign}\xi_x & \text{if} \quad p\neq t\end{cases} \tag{4.11}$$

In addition

$$\xi_y=\xi_z=\xi_t, \quad \xi_u=\xi_v=\xi_x \tag{4.12}$$

so that the charges in the dipole attract from the outside the charges of the opposite sign (see Fig. 4.2). Indeed, consider, for instance, ξ_u. ¿From (4.8) and (4.3), $1=g_{ut}=\xi_u+g_{pu}+g_{qu}$, where p and q are nearest neighbors of u, which implies that ξ_u cannot be $-2-\varepsilon$, hence $\xi_u=2+\varepsilon$. This proves (4.12).

Dipoles can form *dipole polymers* as shown in Fig. 4.2. In Fig. 4.2 the dipole bonds are shown by dashed lines, and the bonds connecting dipoles between themselves and with the environment are shown by solid lines. Observe that $g_{xy}=-g_{yx}=\pm\varepsilon$ on wave lines and $g_{xy}=-g_{yx}=\pm 1$ on solid lines. The sign of g_{xy} is determined by the rule that positive current goes from $+$ to $-$.

In any dipole polymer the charges are alternating. This implies that for almost all ξ there is no infinite polymer. On the other hand, there is a positive probability of appearing a given polymer at a given place in V. Hence Λ^0 consists of an infinite number of finite dipole polymers, $\Lambda^0=\cup_{k=1}^{\infty}V_k$, and the dipole polymers V_k have a positive density on V.

Consider a polymer V_k. Assume that $x\in V_k$ and $y\notin V_k$ with $d(x,y)=1$. Then

$$y\in V_\tau, \quad \tau=\operatorname{sign}\xi_y \tag{4.13}$$

Indeed, let for the sake of definiteness $\xi_y=2+\varepsilon$. Then

$$1=g_{yx}=f_\infty(\xi_y+g_{py}+g_{qy})=f_\infty(2+\varepsilon+g_{py}+g_{qy})$$

which shows that $g_{py}+g_{qy}\geq 1+\varepsilon$. Hence

$$\xi_y+g_{xy}+g_{py}+g_{qy}\geq 2+\varepsilon-1-1-\varepsilon=0$$

so that y cannot be from V_-. Since $y\notin V_0$, this implies that $y\in V_+$. (4.13) is proved. Thus on the boundary of a polymer V_k the charge of any boundary point y determines the component $V_\pm$ to which y belongs.

Let $\{x, t\}$ be a dipole. Let us determine the possible values σ_x, σ_t of a ground state. We have that

$$\mu_{\beta,\xi}(\sigma_x, \sigma_t) = Z^{-1} \exp\left\{\beta[\sigma_x\sigma_t + (\xi_x + g_{yx} + g_{zx})\sigma_x + (\xi_t + g_{ut} + g_{vt})\sigma_t]\right\}$$

Since $x, t \in V_0$,

$$\xi_x + g_{yx} + g_{zx} = -g_{tx}, \quad \xi_t + g_{ut} + g_{vt} = -g_{xt}$$

and

$$\mu_{\beta,\xi}(\sigma_x, \sigma_t) = Z^{-1} \exp\left\{\beta\left(\sigma_x\sigma_t + g_{tx}\sigma_x + g_{xt}\sigma_t\right)\right\} \tag{4.14}$$

Assume, for the sake of definiteness, that $\xi_x = 2 + \varepsilon$, $\xi_t = -2 - \varepsilon$. Then $g_{tx} = -\varepsilon$, $g_{xt} = \varepsilon$ and

$$\mu_{\beta,\xi}(\sigma_x, \sigma_t) = Z^{-1}\{\exp\beta\left(\sigma_x\sigma_t + \varepsilon\sigma_x - \varepsilon\sigma_t\right)\} \tag{4.15}$$

Since $0 < \varepsilon < 1$, the ground states ($\beta \to \infty$) correspond to $\sigma_x\sigma_t = 1$, i.e. $\sigma_x = \sigma_t = 1$, $\sigma_x = \sigma_t = -1$. In other words, on any polymer we have either $(+)$–sate or $(-)$–state.

Let $x, t \in V_k$ be nearest neighbors belonging to two different dipoles. Assume that $\xi_x = 2 + \varepsilon$, $\xi_t = -2 - \varepsilon$. Then $g_{tx} = -1$, $g_{xt} = 1$ and (4.14) reduces to

$$\mu_{\beta,\xi}(\sigma_x, \sigma_t) = Z^{-1} \exp\beta\left(\sigma_x\sigma_t + \sigma_x - \sigma_t\right) \tag{4.16}$$

and the ground states are

$$(\sigma_x = 1, \sigma_t = 1), (\sigma_x = -1, \sigma_t = -1), (\sigma_x = 1, \sigma_t = -1) \tag{4.17}$$

In other words, between two dipoles we can change sign of σ_x from $\operatorname{sign}\xi_x$ to $\operatorname{sign}\xi_t$.

Definition 4.2 Assume that $2 < \alpha < 3$. Let $V_k = V_k(\xi)$ a dipole polymer. A configuration $\sigma = \{\sigma_x, x \in V_k\}$ on V_k is called a *dipole ground state configuration* if

(i) $\sigma_x\sigma_t = 1$ for every dipole $\{x, t\}$ in V_k,

(ii) either $\sigma_x\sigma_t = 1$ or $\{\sigma_x = \operatorname{sign}\xi_x, \sigma_t = \operatorname{sign}\xi_t\}$ for every pair $\{x, t\}$ connecting two dipoles in V_k.

We denote by $M_k = M_k(\xi)$ the set of dipole ground state configurations on $V_k(\xi)$.

Definition 4.2 describes a dipole ground state configuration for the case when $2 < \alpha < 3$. For the cases $\alpha = 3$ and $\alpha = 2$ it should be modified as follows. Observe that for $\varepsilon = 1$ ($\alpha = 3$) the formula (4.15) coincides with (4.16), hence the ground states on a dipole are (4.17). This leads to the following

Definition 4.3 for $\alpha = 3$. Let $\Lambda_k = \Lambda_k(\xi)$ be a dipole polymer. A configuration $\sigma = \{\sigma_x, x \in \Lambda_k\}$ on Λ_k is called a dipole ground state configuration if for every neighboring $x, t \in \Lambda_k$ either $\sigma_x \sigma_t = 1$ or $\{\sigma_x = \operatorname{sign} \xi_x, \sigma_t = \operatorname{sign} \xi_t\}$.

Notice that the difference between a ground state configuration for $2 < \alpha < 3$ and for $\alpha = 3$ is that for $2 < \alpha < 3$, $\sigma_x \sigma_t = 1$ on any dipole while for $\alpha = 3$ either $\sigma_x \sigma_t = 1$ or $\{\sigma_x = \operatorname{sign} \xi_x, \sigma_t = \operatorname{sign} \sigma_t\}$. Therefore the number of the ground state configurations for $\alpha = 3$ is bigger than the number of those for $2 < \alpha < 3$. This is reflected in the behavior of residual entropy, which is higher at $\alpha = 3$ (at the spike) than at $2 < \alpha < 3$ (on the plateau). We evaluate the residual entropy in the next section.

When $\alpha = 2$ we have to change the definition of dipole.

Namely, in the case $\alpha = 2$, if $x \in \Lambda_0$ and, and, say, $\xi_x = 2$, $g_{yx} = -1$, $g_{zx} = -1$, $g_{tx} = 0$, then $t \in \Lambda_0$ but ξ_t can be both 2 and -2. Thus we arrive at the following

Definition 4.4 for $\alpha = 2$. Two neighboring vertices x, t are a dipole if

$$x, t \in \Lambda_0, \qquad g_{xt} = g_{tx} = 0\,.$$

Thus, we don't have the restriction $\xi_x + \xi_t = 0$ as in (4.10). Strictly speaking, (ξ_x, ξ_t) is not a dipole anymore, since it is not necessary neutral. We will call (x, t) a dipole to facilitate a unique formulation of a ground state both for $2 < \alpha \leq 3$ and for $\alpha = 2$. The definition of a dipole ground state configuration for $\alpha = 2$ remains the same as in Definition 4.2. Observe that for $\alpha = 2$ we have more dipoles and, consequently, more dipole polymers than for $2 < \alpha < 3$. This produces a jump of the residual entropy for $\alpha = 2$ (see Section 5).

Theorem 4.1 *Let $k = 2$ and $2 \le \alpha \le 3$. Then for almost all ξ there exist a limit, $\mu_{\infty,\xi}(\sigma) = \lim_{\beta\to\infty} \mu_{\beta,\xi}(\sigma)$ and*

$$\mu_{\infty,\xi}(\sigma) = \mu^{+}_{\infty,\xi}(\sigma_{V_+})\mu^{-}_{\infty,\xi}(\sigma_{V_-}) \prod_{k=1}^{\infty} \mu^{k}_{\infty,\xi}(\sigma_{V_k})$$

where $\mu^{\pm}_{\infty,\xi}(\sigma_{V_\pm})$ is a degenerate measure concentrated on $\{\sigma_x = +1\}$ or $\{\sigma_x = -1\}$, respectively, and $\mu^{k}_{\infty,\xi}(\sigma_{V_k})$ is a measure concentrated on the set M_k of dipole ground state configurations such that $\mu^{k}_{\infty,\xi}(\sigma_{V_k}) = \frac{1}{|M_k|}$ for all dipole ground state configurations Σ_{V_k}.

Proof of Theorem 4.1. Let $\Lambda \subset V$, we have

$$\mu_{\beta,\xi}(\sigma_\Lambda) = Z^{-1} \exp \beta \left(\sum_{\substack{<x,y> \\ x,y\in\Lambda}} \sigma_x\sigma_y + \sum_{x\in\Lambda} \xi_x\sigma_x + \sum_{\substack{<x,y> \\ x\in\Lambda, y\in\Lambda^c}} g_{yx}\sigma_x \right) \tag{4.18}$$

Lemma 4.1 *For every $N > 1$ and almost all ξ, as $\beta \to \infty$,*

$$g_{xy}(\beta,\xi) = g_{xy}(\infty,\xi) + \beta^{-1}c_{xy}(\xi) + O(\beta^{-N}) \tag{4.19}$$

where $g_{xy}(\infty,\xi) \in \{\pm 1, \pm\varepsilon\}$, $\varepsilon = \alpha - 2$, and

$$c_{xy}(\xi) \in M = \{t : t = \frac{1}{2}\log\frac{m}{n}, m, n \in \mathbb{N}\} \tag{4.20}$$

We prove Lemma 4.1 in several steps. First we prove some auxiliary results.

Lemma 4.2 *If $\alpha \ge 2$, then for almost all ξ the equation*

$$g_{xt} = f_\infty(\xi_x + g_{yx} + g_{zx}), \qquad \forall x \in V, \tag{4.21}$$

has a unique solution $g = \{g_{xy}\}$.

Proof. We have the following properties

(1) If $\xi_x = \alpha$, then $g_{xt} \geq 0$;

(2) If $\xi_x = \alpha$ and $\max\{g_{yx}, g_{zx}\} \geq 0$ then $g_{xt} = 1$;

(3) If $\xi_x = \alpha$ and $\max\{\xi_y, \xi_z\} = \alpha$ then $g_{xt} = 1$;

The properties (1) and (2) are obvious from (4.21) and (3) follows from (1) and (2).

Denote by $V_{x,t}$ the half–tree with the root at x, which grows in the direction opposite to t. For a given configuration $\xi = \{\xi_x, x \in V\}$ consider the sets

$$\begin{aligned} \Lambda_+(\xi) &= \Big\{ v \in V_{xt} : \xi_v = \alpha,\ \max\{\xi_y, \xi_z\} = \alpha,\ S(v) = \{y, z\} \Big\} \\ \Lambda_-(\xi) &= \Lambda_+(-\xi) \\ \Lambda(\xi) &= \Lambda_+(\xi) \cup \Lambda_-(\xi) \end{aligned} \tag{4.22}$$

For a given ξ, we say that there is no percolation by $V_{xt} \setminus \Lambda(\xi)$ if every path from x to ∞ contains a point from $\Lambda(\xi)$.

Lemma 4.3 *For almost all ξ there is no percolation by $V_{xt} \setminus \Lambda(\xi)$*

Proof. Let $p_n^{\pm}$ be the probability of ξ's for which there is a percolation from x to W_n under the condition that $\xi_x = \pm\alpha$, respectively. Then if we consider different possibilities for the field ξ_v at $v \in W_0 = \{x\}$ and $v \in W_1$, we obtain the recursive equations

$$\begin{cases} p_{n+1}^{+} = \dfrac{1}{2}\, p_n^{-} - \dfrac{1}{4}\, (p_n^{-})^2 \\[2ex] p_{n+1}^{-} = \dfrac{1}{2}\, p_n^{+} - \dfrac{1}{4}\, (p_n^{+})^2 \end{cases} \tag{4.23}$$

By the symmetry $p_n^+ = p_n^-$, hence

$$p_{n+1}^{+} \leq \frac{1}{2}\, p_n^{+}$$

which shows that $\lim_{n\to\infty} p_n^+ = 0$, so that with probability 1 there is no percolation from x to ∞. Lemma 4.3 is proved. *QED*

End of the proof of Lemma 4.2. For a given $\xi = \{\xi_x, x \in V\}$, define the set of points blocking the percolation, as

$$B(\xi) = \{v \in V_{xt} : v \in \Lambda(\xi) \text{ and } w \notin \Lambda(\xi)\ \forall\ w \in \pi(x,v), w \neq v\} \qquad (4.24)$$

where $\pi(x,v)$ is the path connecting x with v. By the property (3) above, $g_{vw}(\xi) = \pm 1$ if

$$v \in \Lambda_\pm(\xi), v \in S(w), w \in \pi(x,v), d(v,w) = 1$$

Hence the value $g_{vw}(\xi)$ is uniquely determined for almost all ξ the blocking set $B(\xi)$ separate x from ∞, the value $g_{xt}(\xi)$ is uniquely determined by the value of $g_{vw}(\xi)$ for $v \in B(\xi)$ by virtue of the equation (4.19). Lemma 4.2 is proved.

Assume that $\alpha = 2+\varepsilon > 2$. Then the properties (1)–(3) can be strengthened as follows. Let $\beta = \infty$. Then

(1) If $\xi_x = \alpha$, then $g_{xt} \geq \varepsilon$

(2) If $\xi_x = \alpha$ and $\max\{g_{yx}, g_{zx}\} \geq \varepsilon$ then $(\xi_x + g_{yx} + g_{zx}) \geq 1 + 2\varepsilon$

(3) If $\xi_x = \alpha$ and $\max\{\xi_y, \xi_z\} = \alpha$ then $(\xi_x + g_{yx} + g_{zx}) \geq 1 + 2\varepsilon$

This allows us to prove that if $v \in \Lambda_\pm(\xi)$ then as $\beta \to \infty$,

$$g_{vw} = 1 + O(\beta^{-N}) \qquad (4.25)$$

The proof is based on an asymptotic of the function $f_\beta(t)$ as $\beta \to \infty$.

Lemma 4.4 *As* $\beta \to \infty$

$$f_\beta(t) = \begin{cases} t + O(\beta^{-N}) & \text{if} \quad 0 \leq t < 1, \\ 1 + O(\beta^{-N}) & \text{if} \quad t > 1, \end{cases} \qquad (4.26)$$

and

$$f_\beta\left(1 + \frac{s}{\beta}\right) = 1 + \frac{\log(1 + e^{-2s})}{2\beta} + O(\beta^{-N}) \qquad (4.27)$$

We refer the reader to [BRZb] for the proof of Lemma 4.4. Lemma 4.4 shows that for $t \geq 1+\varepsilon$ the function $f_\beta(t)$ is close to 1. We will call the region $t \geq 1+\varepsilon$, the *plateau*.

Proof of Lemma 4.1. The property (3) above implies that if $\xi_v = \alpha$ and $\max\{\xi_y, \xi_t\} = \alpha$ then for sufficiently large β the value $(\xi_v + g_{yv} + g_{zv})$ is on the plateau, and, by virtue of Lemma 4.4, (4.25) holds. This proves (4.19) for $v \in \Lambda(\xi)$. Since the blocking set $B(\xi) \subset \Lambda(\xi)$, the formula (4.25) holds for $v \in B(\xi)$. For almost all ξ the set $B(\xi)$ separates x from ∞. Now, we can prove (4.19) for all v below $B(\xi)$ by induction moving down from $B(\xi)$ to x. So we assume that

$$\begin{cases} g_{yv}(\beta,\xi) = g_{yv}(\infty,\xi) + \beta^{-1}c_{yv}(\xi) + O(\beta^{-N}) \\ \\ g_{yv}(\beta,\xi) = g_{zv}(\infty,\xi) + \beta^{-1}c_{zv}(\xi) + O(\beta^{-N}) \end{cases} \tag{4.28}$$

where g_{yv}, $g_{zv} \in \{\pm 1, \pm\varepsilon\}$ and c_{yv}, $c_{zv} \in M$. Then

$$g_{vt}(\beta,\xi) = f_\beta(\xi_v + g_{yv}(\beta,\xi) + g_{zv}(\beta,\xi) \tag{4.29}$$

Consider three cases

(1) $$|\xi_v + g_{yv}(\infty,\xi) + g_{zv}(\infty,\xi)| \geq 1$$

(2) $$|\xi_v + g_{yv}(\infty,\xi) + g_{zv}(\infty,\xi)| \leq 1$$

(3) $$|\xi_v + g_{yv}(\infty,\xi) + g_{zv}(\infty,\xi)| = 1$$

Then in the case (1), Lemma 4.4 and (4.28) imply that

$$g_{vt} = \pm 1 + O(\beta^{-N})$$

In the case (2) we obtain that

$$\begin{aligned} g_{vt}(\beta,\xi) &= \xi_v + g_{yv}(\infty,\xi) + g_{zv}(\infty,\xi) \\ &\quad + \beta^{-1}c_{yz}(\xi) + \beta^{-1}c_{zv}(\xi) + O(\beta^{-N}) \\ &= g_{vt}(\infty,\xi) + \beta^{-1}\Big(c_{yv}(\xi) + c_{zv}(\xi)\Big) + O(\beta^{-N}) \end{aligned}$$

This gives the asymptotic (4.19) with

$$c_{vt} = c_{yv} + c_{zv} \in M$$

(Observe that the set M in (4.20) is closed with respect to summation).
In the case (3) we similarly obtain that

$$g_{vt}(\beta,\xi) = g_{vt}(\infty,\xi) + c_{vt}(\xi) + O(\beta^{-N})$$

with

$$c_{vt} = \frac{1}{2}\log\left(1 + e^{-2\left(c_{yv}(\xi)+c_{zv}(\xi)\right)}\right)$$

In this case again $c_{vt}(\xi) \in M$, provided that $c_{yv}(\xi)$, $c_{zv}(\xi) \in M$. This induction proves that Lemma 4.2 for $\alpha = 2+\varepsilon > 2$.

In the case $\alpha = 2$ we have to use the following property (4) which follows from the ones (1)–(3).

Property (4) : If $\xi_x = \alpha$ and either

(a) $\quad \xi_y = \xi_z = \alpha$,

or

(b) $\quad \xi_y = \alpha, \qquad \xi_z = -\alpha$ and $\max\{\xi_u,\xi_v\} = \alpha$ where $\{u,v\} = S(y)$,

or

(c) $\quad \xi_y = -\alpha, \qquad \xi_z = \alpha$ and $\max\{\xi_u,\xi_v\} = \alpha$ where $\{u,v\} = S(z)$,

Then $\xi_x + g_{yx} + g_{zx} \geq 2$ (see Fig. 4.3) Indeed, in the case (a), $g_{xy}, g_{zx} \geq 0$ hence $\xi_x + g_{yx} + g_{zx} \geq 2$. In the case (b), $g_{yx} \geq 1$ by property (3) above, hence $\xi_x + g_{yx} + g_{zx} \geq 2+1-1 \geq 2$; the same arguments works for the case (c).

For a given ξ, we define the sets $\Lambda^0_\pm(\xi)$ of vertices $v \in V_{xt}$ for which the assumptions of Property (4) hold with respect to configuration $\pm\xi$, respectively. Let $\Lambda^0(\xi) = \Lambda^0_+(\xi) \cup \Lambda^0_-(\xi)$.

We use the following lemma, which is a replacement of Lemma 4.3.

Lemma 4.5 *For almost all ξ there is no percolation by $V_{xt} \setminus \Lambda^0(\xi)$.*

Proof. Let $p_n^{\pm}$ be probabilities of percolation from the root x to W_n under the condition that $\xi_x = \pm 2$. Then $p_n^+ = p_n^- = p_n$. Considering the different possibilities for ξ_y, ξ_z, ξ_u, and ξ_v, we obtain that

$$p_{n+1} \leq \frac{1}{2}\, p_n + \frac{1}{8}\,(p_n + 2p_{n-1}) = \frac{5}{8}\, p_n + \frac{1}{4}\, p_{n-1}$$

This is majorized by a sequence $\{b_n\}$ satisfying

$$b_{n+1} = \frac{5}{8}\, p_n + \frac{1}{4}\, p_{n-1}, \qquad b_0 = p_0,\ b_1 = p_1$$

Two fundamental solutions for the last equation are $b_n = \lambda_{1,2}^n$ where $\lambda_{1,2}^n$ are to be found from the quadratic equation

$$\lambda^2 = \frac{5}{8}\lambda + \frac{1}{4}\lambda, \qquad \text{or} \qquad 8\lambda^2 - 5\lambda - 2 = 0$$

This gives

$$\lambda_{1,2} = \frac{5 \pm \sqrt{89}}{16}$$

Since $|\lambda_{1,2}| < 1$, this implies that $\lim_{n\to\infty} b_n = 0$, hence $\lim_{n\to\infty} p_n = 0$. Lemma 4.5 is proved *QED*

The rest of the proof of Lemma 4.1 for $\alpha = 2$ is similar to the proof for $\alpha = 2 + \varepsilon > 2$ and we omit it.

Completion of the proof of Theorem 4.1. Let us substitute the asymptotic (4.19) into the formula for $\mu_{\beta,\xi}(\sigma_\Lambda)$:

$$\mu_{\beta,\xi}(\sigma_\Lambda) = Z^{-1} \exp\left[\beta H_0(\sigma_\Lambda) + H_1(\sigma_\Lambda) + O(\beta^{-N})\right] \tag{4.30}$$

where $\Lambda \subset V$, is a finite connected set,

$$H_0(\sigma_\Lambda) = \sum_{\substack{<x,y> \\ x,y\in\Lambda}} \sigma_x\sigma_y + \sum_{x\in\Lambda} \xi_x\sigma_x + \sum_{\substack{<x,y> \\ x\in\Lambda, y\in\Lambda^c}} g_{yx}(\infty,\xi)\sigma_x \tag{4.31}$$

and

$$H_1(\sigma_\Lambda) = + \sum_{\substack{<x,y> \\ x\in\Lambda, y\in\Lambda^c}} c_{xy}\sigma_x \tag{4.32}$$

Denote by $M_\xi(\Lambda) = \{\sigma_\Lambda^{(j)}\}$ the set of ground state configurations of the Hamiltonian $H_0(\sigma_\Lambda)$, i.e.,

$$\min_{\sigma_\Lambda} H_0(\sigma_\Lambda) = H_0(\sigma_\Lambda^{(j)}), \qquad \forall \sigma_\Lambda^{(j)} \in M_\xi(\Lambda)$$

Then the equation (4.30) implies that the limit,

$$\mu_{\infty,\xi}(\sigma_\Lambda) = \lim_{\beta\to\infty} \mu_{\beta,\xi}(\sigma_\Lambda)$$

exists and is concentrated on the ground state configurations. In addition, by (4.30)

$$\mu_{\infty,\xi}(\sigma_\Lambda) = Z^{-1} \exp H_1(\sigma_\Lambda) \tag{4.33}$$

In the case when $\Lambda = \{x\}$, (4.31) reduces to

$$H_0(\sigma_\Lambda) = \left(\xi_x + \sum_{<x,y>} g_{yx}(\infty,\xi)\right) \sigma_x$$

hence if $x \in \Lambda_\pm(\xi)$, i.e.

$$\pm\left(\xi_x + \sum_{<x,y>} g_{yx}(\infty,\xi)\right) > 0,$$

then

$$\mu_{\infty,\xi}(\sigma_x = \pm 1) = 1, \qquad x \in \Lambda_\pm(\xi).$$

Let now $\Lambda_k(\xi)$ be a dipole configuration and let

$$B_k(\xi) = \{x \in \Lambda_k^c(\xi) : d(x, \Lambda_k(\xi)) = 1\},$$

be the boundary of $\Lambda_k(\xi)$. Then $B_k(\xi) \subset \Lambda_+ \cup \Lambda_-$. Therefore, for all ground states $\sigma_\Lambda^{(j)}$ on the set $\Lambda = \Lambda_k(\xi) \cup B_k(\xi)$ the values $\sigma_x = \pm 1$, $x \in B_k(\xi)$. By (4.33)

$$\mu_{\infty,\xi}(\sigma_\Lambda) = Z^{-1} \exp\left\{ \sum_{\substack{<x,y> \\ x\in B_k(\xi), y\in\Lambda^c}} c_{xy}\sigma_x \right\}$$

This expression does not depend on $\{\sigma_x, x \in \Lambda_k(\xi)\}$, hence $\mu_{\infty,\xi}$ is a uniform measure, i.e.,

$$\mu_{\infty,\xi}(\sigma_\Lambda) = \frac{1}{|M_k(\xi)|},$$

where $|M_k(\xi)|$ is the number of ground states configurations of the Hamiltonian

$$H_{\Lambda_k(\xi)}(\sigma_\Lambda) = \sum_{\substack{<x,y> \\ x,y\in\Lambda_k(\xi)}} \sigma_x\sigma_y + \sum_{x\in\Lambda_k(\xi)} \xi_x\sigma_x + \sum_{\substack{x\in\Lambda_k(\xi) \\ y\in B_k(\xi)}} \sigma_x\bar{\sigma}_y$$

where $\bar{\sigma}_y = \pm 1$ for $y \in \Lambda_\pm$, respectively. Let us show that the set $M_k(\xi)$ of ground states configurations coincides with the set of dipole configurations.

Since the Gibbs measure $\mu_{\beta,\xi}$ has the Markov property, the measure $\mu_{\infty,\xi}$ has it as well. This implies that

$$\mu_{\infty,\xi}(\sigma_{\Lambda_k(\xi)}) = \mu_{\infty,\xi}(\sigma_{y_0}) \prod_{<x,y>\in L_k} \mu_{\infty,\xi}(\sigma_x \mid \sigma_y) \tag{4.34}$$

where L_k is the set of directed edges, which starts at some point $y_0 \in \Lambda_k(\xi)$ and which has the property that for every $x \in \Lambda_k(\xi)$, there exists a unique path by L_k from y_0 to x. Notice that

$$\mu_{\infty,\xi}(\sigma_x \mid \sigma_y) = \frac{\mu_{\infty,\xi}(\sigma_x, \sigma_y)}{\mu_{\infty,\xi}(\sigma_y)} \tag{4.35}$$

hence we deduce from (4.34) that $\mu_{\infty,\xi}(\sigma_{\Lambda_k(\xi)}) \neq 0$ for all dipole configurations and only for dipole configurations. Hence the set $M_k(\xi)$ of ground state configurations coincides with the set of dipole ground state configurations and Theorem 4.1 is proven.

5 Residual entropy

In this section we will assume that $k = 2$ and $2 \leq \alpha \leq 3$. By Theorem 2.4 (a) this ensures that the Gibbs state $\mu_{\beta,\xi}$ is unique for all $\beta < \infty$ and almost all ξ. We will derive some general formula for the entropy S_β of $\mu_{\beta,\xi}$. As it is well–known, on the Bethe lattice the entropy depends on the boundary conditions. and S_β is the entropy on the "interior" spins (see, e.g., a discussion in [Ba]). We shall show that the entropy density S_β for "interior" spins (see, e.g., [Ba] for discussion) is a "self-averaging" quantity, i.e., it is independent of ξ for *almost all* ξ. Then we shall calculate the residual entropy $S_\infty = \lim_{\beta\to\infty} S_\beta$ and show that $S_\infty > 0$.

Consider the stochastic recursive equation

$$g_x = f_\beta\Big(\xi_x + \sum_{y\in S(x)} g_y\Big) \tag{5.1}$$

It is understood as follows. Let $\{g_y, y \in S(x)\}$ be independent random variables with some distribution $\nu(dg)$, the same for all g_y's. Then we denote by $Q_\beta(\nu)(dg)$ the distribution of $f_\beta\Big(\xi_x + \sum_{y\in S(x)} g_y\Big)$, i.e., $Q_\beta(\nu)(dg)$ is the distribution of g_x in (5.1). A measure $\nu(dg)$ is called *invariant* with respect to Q_β if

$$\nu = Q_\beta(\nu)$$

Let $\mu^\pm_{\beta,\xi}$ be the Gibbs states with $(\pm)$–boundary conditions, respectively, and let $g^\pm(\beta,\xi) = \{g^\pm_x(\beta,\xi), x \in V\}$ be corresponding effective fields. Let $\nu^\pm_\beta(dg)$ be a probability distribution of $g^\pm_x(\beta,\xi)$. Then $\nu^\pm_\beta(dg)$ is independent of x.

¿From the definition of $\mu^\pm_{\beta,\xi}$ it follows that

$$\nu^\pm_\beta = \lim_{n\to\infty} Q^n_\beta(\nu^\pm) \tag{5.2}$$

where $\nu^\pm = \delta(g \pm 1)dg$. This implies that

$$\nu^\pm_\beta = Q_\beta(\nu^\pm_\beta) \tag{5.3}$$

i.e., $\nu^\pm_\beta$ are invariant measures. In addition,

$$\nu^-_\beta = S\nu^+, \qquad S : g \to -g \tag{5.4}$$

In the case when $\mu^+_{\beta,\xi} = \mu^-_{\beta,\xi}$ for almost all ξ, $\nu^+_\beta = \nu^-_\beta$ and it is symmetric.

Proposition 5.1 *Assume that for a given $\beta < \infty$, $\mu^+_{\beta,\xi} = \mu^-_{\beta,\xi}$ for almost all ξ. Let $\nu = \nu^+ = \nu^-$. Then for all probability measures $\nu_0(dg)$ on $\mathbb{R}$,*

$$\lim_{n\to\infty} Q^n_b(\nu_0) = \nu \tag{5.5}$$

Proof. Consider a half–tree V_0 with a root at x. ¿From F.K.G.,

$$\langle\sigma_x\rangle_{\mu^-_n} \le \langle\sigma_x\rangle_{\mu_n(\bar\sigma)} \le \langle\sigma_x\rangle_{\mu^+_n}$$

where $\mu_n^{\pm}$, $\mu_n(\bar\sigma)$ are finite Gibbs distributions on V^0 with boundary conditions $\pm$ and $\bar\sigma$, respectively. Since

$$\langle\sigma_x\rangle_{\mu_n(\bar\sigma)} = \tanh\Big(\beta(\xi_x + h_x)\Big), \tag{5.6}$$

this implies that

$$h_{xn}^- \le h_{xn}(\bar\sigma) \le h_{xn}^+$$

Since $g_x = f_\beta(\xi_x + h_x)$ this, in turn, implies that

$$g_{xn}^- \le g_{xn}(\bar\sigma) \le g_{xn}^+ \tag{5.7}$$

Consider random boundary conditions $\bar\sigma = \{\bar\sigma_x, x \in W_{n+1}\}$ where $\bar\sigma_x$ are independent random variables with the distribution ν_0. Then averaging with respect to $\bar\sigma$ we obtain that

$$g_{xn}^- \le g_{xn}(\nu_0) \le g_{xn}^+ \tag{5.8}$$

These inequalities hold for all ξ. Since $\mu_{\beta,\xi}^+ = \mu_{\beta,\xi}^-$ *a.e.* ξ, then taking in (4.8) $n \to \infty$, we obtain that

$$g_x^- = g_x(\nu_0) = g_x^+ \qquad a.e.\ \xi,$$

and hence the distribution of $g_x(\nu_0) = \lim_{n\to\infty} g_{xn}(\nu_0)$ coincides with ν. Since the distribution of $g_{xn}(\nu_0)$ is nothing else than $Q^n(\nu_0)$, we obtain that $\nu = \lim_{n\to\infty} Q^n(\nu_0)$. Proposition 5.1 is proved. *QED*

Now we turn to calculation of the entropy S_β. Consider the partition function of $\mu_{\beta,\xi}(\sigma_n)$,

$$Z_n^{-1}(\beta,\xi) = \sum_\sigma \exp\Big(\beta \sum_{<x,y>\in L_n} \sigma_x\sigma_y + \beta\sum_{x\in V_n}\xi_x\sigma_x + \beta\sum_{x\in W_n} h_x\sigma_x\Big) \tag{5.9}$$

The free energy (density) is defined as

$$F(\beta) = \lim_{n\to\infty} -\frac{1}{\beta 2^{n+1}} \log Z_n(\beta,\xi) \tag{5.10}$$

Theorem 5.1 *Assume that for a given $\beta < \infty$, $\mu_{\beta,\xi}^+ = \mu_{\beta,\xi}^-$ for almost all ξ. Then the free energy exists for almost all ξ, it is independent of ξ, and it is given by the formula*

$$F(\beta) = -\sum_{\xi=\pm\alpha}\int d_\beta(\xi + \sum_{y\in S(x)} g_y) \prod_{y\in S(x)} \nu_\beta(dg_y), \tag{5.11}$$

where $\nu_\beta(dg)$ is the invariant measure of the stochastic equation (5.1).

Proof. ¿From the formula (1.7), we obtain the recursive equation

$$Z_n(\beta,\xi) = \exp\left(\beta \sum_{x\in W_n} d_\beta(\xi_x + h_x)\right) Z_{n-1}(\beta,\xi) \tag{5.12}$$

This gives that

$$F(\beta,\xi) = \lim_{n\to\infty} -\frac{1}{2^{n+1}} \sum_{k=0}^{n} \sum_{x\in W_{n-k}} d_\beta(\xi_x + h_x)$$

Observe that $|d_\beta(\xi_x + h_x)| \le C_\beta$ for all ξ_x, h_x, hence

$$\frac{1}{2^{n+1}} \sum_{k=\ell+1}^{n} \sum_{x\in W_{n-k}} d_\beta(\xi_x + h_x) \le \frac{C_\beta}{2^{n+1}} \left(\sum_{k=\ell+1}^{n} 2^{n-k}\right) \le C_\beta \cdot 2^{-\ell}$$

Therefore,

$$F(\beta,\xi) = \lim_{\ell\to\infty} \lim_{n\to\infty} -\frac{1}{2^{n+1}} \sum_{k=0}^{\ell} \sum_{x\in W_{n-k}} d_\beta(\xi_x + h_x)$$

Since the random variables $\xi_x + h_x$, $x \in W_n$, are independent, and the distribution of $\xi_x + h_x$ is the same for all $x \in V$, we obtain, by the law of large numbers, that for a fixed k, for almost all ξ,

$$\lim_{n\to\infty} -\frac{1}{2^{n-k}} \sum_{x\in W_{n-k}} d_\beta(\xi_x + h_x) = C = -\sum_{\xi=\pm\alpha} \int d_\beta(\xi + h)\nu(dh) \tag{5.13}$$

where $\nu(dh)$ is the distribution of h. This implies that for almost all ξ,

$$F(\beta,\xi) = \lim_{\ell\to\infty} \sum_{k=0}^{\ell} \frac{1}{2^{k+1}} C = C$$

Since $h_x = \sum_{y\in S(x)} g_y$, the formula (5.11) follows from (5.13). Theorem 5.1 is proved. *QED*

The entropy S_β can be obtained from (5.11) as

$$S_\beta = -\frac{dF}{dT}(\beta), \qquad \beta = T^{-1}$$

The residual entropy at $T=0$ is then

$$S_\infty = -\lim_{\beta\to\infty} \frac{F(\beta)-F(\infty)}{(1/\beta)} \tag{5.14}$$

where $F(\infty) = \lim_{\beta\to\infty} F(\beta)$. By Lemma 4.1, $\lim_{\beta\to\infty} g_x(\beta,\xi) = g_x(\infty,\xi)$ takes values in $\{\pm1, \pm\varepsilon\}$. Thus the distribution $\nu_\infty = \lim_{\beta\to\infty} \nu_\beta(dg)$ has the form

$$\nu_\infty = [p\delta(g+1) + q\delta(g+\varepsilon) + q\delta(g-\varepsilon) + p\delta(g-1)]\, dg, \qquad \varepsilon = \alpha - 2 \tag{5.15}$$

The weights p, q satisfy $p+q = \frac{1}{2}$ and they are determined from the fixed point stochastic equation

$$\nu = Q_\infty(\nu) \tag{5.16}$$

Assume that $0 < \varepsilon < 1$. Then (5.16) reduces to the equations

$$q = \frac{1}{2}\cdot p^2, \quad p+q = \frac{1}{2}$$

This gives

$$p = \sqrt{2}-1, \qquad q = \frac{3}{2} - \sqrt{2} \tag{5.17}$$

To derive the residual entropy S_∞ from (5.14) we need the linear term in the asymptotic of $\nu_\beta(dg)$ as $\frac{1}{\beta}\to 0$. By Lemma 4.1, $\forall N > 1$,

$$g_x(\beta,\xi) = g(\infty, (\xi) + c(\xi)\cdot\frac{1}{\beta} + O(\beta^{-N})$$

where $g_\infty(\xi)$ takes values in the set $\{\pm1, \pm\varepsilon\}$ and $c(\xi)$ takes values in the set

$$M = \left\{ \frac{\log\frac{m}{n}}{2}, m, n = 1, 2, \ldots \right\}.$$

Let $x_0 = 0, x_1, x_2, \ldots$ be an enumeration of the points in M. Then we obtain that at $\beta\to\infty$, $\nu_\beta(dg)$ is approximated by the distribution

$$\nu_\beta^{\text{asymp}}(dg) = \left[\sum_{a=\pm1,\pm\varepsilon}\sum_{j=p}^{\infty} p\cdot\delta\left(g - a - \frac{x_j}{\beta}\right)\right] dg, \qquad x_j \in M \tag{5.18}$$

in the sense that for any smooth test function $\varphi(g)$,

$$\int \varphi(g)\left[\nu_\beta(dg) - \nu_\beta^{\text{asymp}}(dg)\right] = O(\beta^{-N}) \tag{5.19}$$

The weights $p_{a,j}$ are found from the fixed point equation $\nu = Q_\beta(\nu)$. They satisfy the equations

$$\sum_{j=0}^{\infty} p_{1,j} = p + O(\beta^{-N}), \quad \sum_{j=0}^{\infty} p_{\varepsilon,j} = q + O(\beta^{-N}), \tag{5.20}$$

The formula (5.11) for the free energy can be written as

$$F(\beta) = -\int d_\beta(s) W_\beta(ds) \tag{5.21}$$

where $W_\beta(ds)$ is the probability distribution of

$$\sigma_x = \xi_x + \sum_{y \in S(x)} g_y \tag{5.22}$$

¿From (5.18), (5.19) we obtain that $W_\beta(ds)$ is approximated by the distribution

$$W_\beta^{\text{asymp}}(ds) = \left[\sum_{a\in A}\sum_{j=0}^{\infty} w_{a,j}\delta\left(s - a - \frac{x_j}{\beta}\right)\right] dg, \quad x_j \in M, \tag{5.23}$$

where

$$A = \{\xi + g_y + g_z \mid \xi = \pm(2+\varepsilon) : g_y, g_z = \pm 1, \pm\varepsilon\}$$

so that

$$\int \varphi(s)\left[W_\beta(ds) - W_\beta^{\text{asymp}}(ds)\right] = O(\beta^{-N})$$

The function

$$d_\beta(s) = \frac{1}{2\beta}\log[\cosh(2\beta) + \cosh(2\beta s)]$$

has the following asymptotic as $\beta \to \infty$:

$$d_\beta(s) = \begin{cases} 1 + O(\beta^{-N}) & \text{if} \quad 0 < s < 1 \\ s + O(\beta^{-N}) & \text{if} \quad s > 1 \end{cases} \tag{5.24}$$

and

$$d_\beta\left(1 + \frac{s}{\beta}\right) = 1 + \frac{1}{2\beta}\log(1 + e^{2\beta}) + O(\beta^{-N}) \tag{5.25}$$

Combining these asymptotic formulas with (5.23), we derive from (5.21) that

$$\begin{aligned} F(\beta) &= -2\sum_{0<a<1}\sum_{j=0}^{\infty} w_{a,j} - 2\sum_{a>1}\sum_{j=0}^{\infty}\left(a+\frac{x_j}{\beta}\right) w_{a,j} \\ &\quad -2\sum_{j=0}^{\infty}\left[1+\frac{1}{2\beta}\log(1+e^{2x_j})\right] w_{1,j} + O(\beta^{-N}) \end{aligned} \tag{5.26}$$

hence

$$F(\beta) = F(\infty) - \frac{1}{\beta}\left[2\sum_{a>1}\sum_{j=0}^{\infty} x_j w_{a,j} + \sum_{j=0}^{\infty}\log(1+e^{2x_j})w_{1,j}\right] + O(\beta^{-N}) \tag{5.27}$$

¿From (5.14) we obtain the residual entropy at $T=0$ as

$$S_\infty = 2\sum_{a>1}\sum_{j=0}^{\infty} x_j w_{a,j} + \sum_{j=0}^{\infty}\log(1+e^{2x_j})w_{1,j} \tag{5.28}$$

This is an exact formula. Observe that S_∞ does not depend on $\alpha = 2+\varepsilon$, $0<\varepsilon<1$. We don't have an analytic expression for $w_{a,j}$ but some estimates and numerics shows that for every $a \in A$, the weight $w_{a,0}$ corresponding to $x_0 = 0$ is considerably larger than the other weights, corresponding to $x_j \neq 0$ This can be explained as follows. The weights $p_{a,j}$ in (5.18) can be found from the iterations

$$\nu_n = Q_\beta^n \nu_0$$

where we take $\nu_0 = \frac{1}{2}[\delta(g-1)+\delta(g+1)]dg$. Initially, at $n=0$, we have weights only at $g=\pm 1$. Under iterations new weights appear. If a weight appear at the n-th iteration, we call it a weight of n-th level. Some simple estimates show that the total weight of the weight of n-th level decreases exponentially in n, as λ^n, and the number λ is rather small. For instance, one can show that

$$\frac{\sum_{j\geq 1} p_{1,j}}{\sum_{j\geq 0} p_{1,j}} = q$$

where $q = 3/2-\sqrt{2} \approx 0.085$. This shows that in the series of the weights $p_{i,j}$, the weight $p_{0.1}$ dominates. Similar domination persists for the weights $w_{a,j}$ in (5.23).

Taking only $x_j = 0$ in (5.28) leads to an approximation for the residual entropy,

$$S_\infty^{(0)} = w_{1,0}\log 2$$

In addition, since $w_{1,0}$ is essentially larger than $\sum_{j=1}^{\infty} w_{1,j}$, we can replace $w_{1,0}$ in the last formula by $w_1 = \sum_{j=0}^{\infty} w_{1,j}$. This gives an approximation

$$S_{\infty}^{(0)} = w_1 \log 2 \tag{5.29}$$

¿From (5.23) we find $w_1 = pq$, hence

$$S_{\infty}^{(0)} = pq \log 2 = \frac{5\sqrt{2} - 7}{2} \log 2 \approx 0.035 \log 2, \qquad 2 < \alpha < 3 \tag{5.30}$$

There is no small parameter in (5.28) and the closeness of $S_{\infty}^{(0)}$ in (5.28) is understood in the sense that numerically $|S_{\infty} - S_{\infty}^{(0)}|$ is noticeably smaller than $S_{\infty}^{(0)}$.

The formula (5.29) was given by Bruinsma [Br]. The formulae (5.28), (5.29), can be extended to $\alpha = 2$ and $\alpha = 3$. In the approximation (5.29) it gives (see [Br]):

$$S_{\infty}^{(0)} = \begin{cases} \frac{1}{8} \log 2 & \text{if} \quad \alpha = 3 \\ \frac{1}{16} \log 2 & \text{if} \quad \alpha = 2 \end{cases}$$

The values $S_{\infty}^{(0)}$ at the spikes $\alpha = 2, 3$ are higher than the values (5.30) of $S_{\infty}^{(0)}$ on the plateau $2 < \alpha < 3$.

We don't know the uniqueness of the Gibbs state $\mu_{\infty,\xi}$ when $1 < \alpha < 2$. Our numerical simulations strongly suggest the uniqueness, because they show the symmetry of the invariant probability distribution $\nu(dg)$. This indicates on the possible applicability of the formulae (5.28), (5.29) in the interval $1 < \alpha < 2$ as well. An analysis of the residual entropy in the interval $1 < \alpha < 2$, based on the approximate formula (5.29), is done by Bruinsma [Br].

6 Proof of Theorems 2.5 and 2.6

Proof of the Theorem 2.5 *from the Theorem* 2.6. Note that by the symmetry

$$\mathbb{E}_{\xi} \int \sigma_x \, \mu_{\beta,\xi}^{-}(\sigma) = - \mathbb{E}_{\xi} \int \sigma_x \, \mu_{\beta,\xi}^{+}(\sigma) = -M(\beta, \alpha)$$

hence by Theorem 2.6

$$\mathbb{E}_{\xi} \left(\int \sigma_x \, \mu_{\beta,\xi}^{+}(\sigma) - \int \sigma_x \, \mu_{\beta,\xi}^{-}(\sigma) \right) = 2M(\beta, \alpha) > 0$$

In addition by F.K.G.,

$$\int \sigma_x \, \mu^+_{\beta,\xi}(\sigma) - \int \sigma_x \, \mu^-_{\beta,\xi}(\sigma) \geq 0$$

We would like to prove that for almost all ξ we actually have a strict inequality, at least for one x. Define the random variable

$$F(\xi) = \sum_{x \in V} a(x) \left(\int \sigma_x \, \mu^+_{\beta,\xi}(\sigma) - \int \sigma_x \, \mu^-_{\beta,\xi}(\sigma) \right)$$

where $a(x) > 0$ are arbitrary numbers such that

$$\sum_{x \in V} a(x) < \infty$$

Then $F(\xi) > 0$ and $\mathbb{E}_\xi \, F(\xi) > 0$. Define

$$A = \{\xi \; : \; F(\xi) > 0\}$$

Then $\Pr A > 0$, because otherwise $\mathbb{E}_\xi \, F(\xi) = 0$. Let $T : \; V \to V$ be a shift of the Bethe lattice. Notice that

$$\int \sigma_{Tx} \, \mu^+_{\beta,T\xi}(\sigma) = \int \sigma_x \, \mu^+_{\beta,\xi}(\sigma)$$

Hence, if $F(\xi) > 0$, then $F(T\xi) > 0$. Therefore, $A = TA$. Since $\{\xi_x\}$ are independent , the shift T is ergodic and consequently $\Pr A = 0$ or 1. Since $\Pr A > 0$, actually, $\Pr A = 1$. Hence, for almost all ξ

$$\int \sigma_x \, \mu^+_{\beta,\xi}(\sigma) - \int \sigma_x \, \mu^-_{\beta,\xi}(\sigma) > 0$$

for at least one x. This proves that $\mu^+_{\beta,\xi} \neq \mu^-_{\beta,\xi}$ for almost all ξ and ends the proof of Theorem 2.5.

Proof of Theorem 2.6. Let g_x satisfy the basic equation

$$g_x = f_\beta(\xi_x + g_y + g_z) \tag{6.1}$$

Assuming that g_y and g_z are independent and have the same distribution $\rho(dg)$, the equation (6.1) determines a distribution of g_x, which we denote by $Q\rho(dg)$. We are interested in the behavior of $\rho_n = Q^n \rho_0$ for $n \to \infty$, assuming that

$$\rho_0(dg) = \delta(g-1)\, dg \tag{6.2}$$

which corresponds to (+)–boundary conditions, see (1.2) and (3.3).

The key point is some inductive assumptions on ρ_n which holds for ρ_0 and which are reproducible when we pass from ρ_n to ρ_{n+1}. To formulate these inductive assumptions we need some definitions. Let $\alpha = H^{\mathrm{F}}(T)+\varepsilon$. We will assume that $\varepsilon > 0$ is sufficiently small, so that it satisfies some conditions formulated below. Let

$$f_{\pm}(g) = f_{\beta}(\pm\alpha + 2g)\,, \tag{6.3}$$

and let $a > 0$ be the point where

$$f'_{-}(a) = 1; \qquad f'_{-}(g) < 1 \qquad \forall g > a \tag{6.4}$$

(see Fig. 6.1).

Near a, there is a narrow corridor of the width of the order of ε, between the diagonal $y = g$ and the graph of equation $y = f_{-}(g)$. This implies that we have a long intermittent trajectory $\{g_n = f_{-}(g_{n-1})\}$ near a.

Consider some points b, c, d such that $a < b < c < d$ and such that when $\varepsilon \to 0$,

$$\varepsilon \ll c - b \ll b - a = \frac{d-c}{2} \ll 1 \tag{6.5}$$

Define a sequence $b_0, b_1, \ldots, b_N$ by the recursive equation

$$b_n = f_{-}(b_{n-1}), \qquad N \geq n \geq 1, \qquad b_0 = b \tag{6.6}$$

We assume that

$$a = b_N \tag{6.7}$$

(see Fig. 6.2). Define then $b_{N+1}, b_{N+2}, \ldots$ by the equation

$$b_n = b_{n-1} + \varepsilon, \qquad n \geq N + 1 \tag{6.8}$$

Let

$$\begin{aligned} p^{+}(k) &= \int_{d}^{\infty} \rho_k(dg) \\ p_0(k) &= \int_{b}^{c} \rho_k(dg) \\ p_n(k) &= \int_{b_n}^{b_{n-1}} \rho_k(dg), \qquad n \geq 1 \end{aligned} \tag{6.9}$$

To avoid technical difficulties we will assume that $\rho_k(\{b_n\}) = \rho_k(\{b_n\}) = \rho_k(\{b_n\}) = 0$ for all k.

Inductive assumption I_k

(i) $p^+(k) \geq 0.499$;

(ii) $p_0(k) \leq 0.13$;

(iii) $p_n(k) \leq \frac{1}{8} \cdot 2^{-n}, \qquad n = 1, 2, \ldots, N.$

Main Lemma *There exists* $\varepsilon(T) > 0$, *which depends continuously on* $0 < T < T_c$, *such that for* $0 < \varepsilon < \varepsilon(T)$, $I_k \Rightarrow I_{k+1}$, $k = 0, 1, 2, \ldots$.

Proof. Assume that (i)–(iii) hold for ρ_k and prove that (i)–(iii) hold for ρ_{k+1}

Proof of (i). Observe that by (6.5), $0 < d - a \ll 1$. This implies that

$$f_-(g) > d \qquad \text{if} \qquad g > a.$$

(see Fig. 6.1). Therefore

$$\begin{aligned} \Pr\{g_x \geq \alpha\} &\geq \Pr\{\xi_x = -\alpha\} \cdot \Pr\{g_y \geq a\} \cdot \Pr\{g_z \geq a\} \\ &\geq \frac{1}{2} \cdot \left(1 - 2^{-N}\right)^2 \geq 0.499 \end{aligned}$$

if N is sufficiently large. This proves (i).

Proof of (ii). To evaluate $p_0(k+1)$ consider how can g_x in (6.1) get into $[b, c]$. One possibility is that $\xi_x = \alpha$ and

$$\frac{g_y + g_z}{2} \in f_+^{-1}([b, c])$$

¿From Fig. 6.1 it is clear that $f_+^{-1}(g) < a$ if $d > g > a$, hence either $g_y < a$ or $g_z < a$. The probability of this possibility is evaluated by

$$\delta = 2 \Pr\{\xi_x = \alpha\} \cdot \Pr\{g_y < \alpha\} \leq 2^{-N}$$

Since δ can be made as small as we want by taking N sufficiently large, we will call such probability negligibly small.

Another possibility for $g_x \in [b, c]$ is that $\xi_x = -\alpha$ and

$$\frac{g_y + g_z}{2} \in f_+^{-1}([b, c]) = [b_{-1}, c_{-1}] \tag{6.10}$$

where $b_{-1} = f_+^{-1}(b)$, $c_{-1} = f_+^{-1}(c)$. ¿From Fig. 6.2 it is clear that $|b - b_{-1}|$, $|c - c_{-1}| \leq C\varepsilon$, hence

$$|c_{-1} - b_{-1}| \ll d - c\,.$$

Consider two cases for g_y, g_z.

Case (a) $g_y, g_z < d$. By (i),

$$\Pr\{g_y < a\} \leq 0.501$$

hence the probability of this case is estimated by

$$\delta_1 = \Pr\{\xi_x = -\alpha\} \cdot \Pr\{g_y < d\} \cdot \Pr\{g_z < d\} \leq \frac{1}{2} \cdot 0.501^2 < 0.126 \tag{6.11}$$

Case (b) either $g_y \geq d$ or $g_z \geq d$. Let, say $g_y \geq d$. Then by (6.5) and (6.10),

$$g_z < a\,,$$

hence probability of this case is negligibly small.

Thus, δ_1 in (6.11) gives the main contribution to $p_0(k+1)$, hence $p_0(k+1) < 0.13$. This proves (ii).

Let us prove (iii) for $p_n(k+1)$. First we consider $n = 1, 2$, then $2 < n \leq N$, and then $n > N$.

Proof of (iii) for $p_1(k+1)$. If $g_x \in [b_1, b_0]$ and $\xi_x = \alpha$, then

$$\frac{g_y + g_z}{2} \in f_+^{-1}([b, b_0]) < a$$

hence the probability of this case is negligibly small. Assume that $\xi_x = -\alpha$. Then

$$\frac{g_y + g_z}{2} \in f_-^{-1}([b, b_0]) = [b_0, b_{-1}]$$

Consider two cases

Case (a), $g_y, g_z \geq b_0 = b$. Then $g_y, g_z < c$, because otherwise $\frac{g_y+g_z}{2} > b_{-1}$. Hence the probability of this case is estimated by

$$\delta_1 = \Pr\{\xi_x = -a\} \left(\Pr\{g_y \in [b,c]\}\right)^2 \leq \frac{1}{2} \cdot 0.13^2$$

Case (b), either $g_y < b_0$ or $g_z < b_0$. Let, say, $g_y < b_0$. Consider two subcases,

Subcase (b_1), $b_0 < g_z < c$. The probability of this subcase is estimated by

$$\delta_2 = 2\Pr\{\xi_x = -a\} \cdot \Pr\{g_y < b_0\} \cdot \Pr\{g_z \in [b,c]\}$$

where factor 2 comes from the possibility to exchange g_y and g_z. Since

$$\Pr\{g_y < b_0\} = \sum_{n=1}^{\infty} \Pr\{g_y \in (b_n, b_{n-1}]\} \leq \frac{1}{8} + 2^{-N}$$

we obtain that

$$\delta_2 \leq \frac{1}{8} \cdot 0.13 + 2^{-N}$$

Subcase (b_2), $g_z > c$. Then

$$g_y = 2 \cdot \frac{g_y + g_z}{2} - g_z < b_{-1} - c$$

Observe that

$$\Pr\{g_y < b_{-1} - c\} \leq 2^{-N_0}$$

where $N_0 = \dfrac{c-b}{b_1 - b_0} \geq C\dfrac{c-b}{\varepsilon}$, hence probability of this subcase is negligibly small (c.f. (6.11)). Thus,

$$p_1(k+1) < \frac{1}{2} \cdot 0.13^2 + \frac{1}{8} \cdot 0.13 \leq 0.026$$

plus negligibly small terms. This proves that $p_1(k+1) < \frac{1}{16}$, hence (iii) for $n = 1$.

Proof of $p_2(k+1) < \frac{1}{32}$. As before, the case $\xi_x = \alpha$ has a negligibly small probability. Let $\xi_x = -\alpha$. Then

$$\frac{g_y + g_z}{2} \in f_-^{-1}([b_2, b_1]) = [b_1, b_0]$$

Consider two cases.

Case (a), $g_y, g_z \geq b_1$. Observe that either $g_y \leq b_0$ or $g_z \leq b_0$, hence the probability of this case is estimated by

$$\begin{aligned}\delta_1 &= 2\Pr\{\xi_x = -a\} \cdot \Pr\{g_y \in [b_1, b_0]\} \cdot \Pr\{g_z \in [b_1, c]\} \\ &\leq \frac{1}{16}\left(\frac{1}{16} + 0.13\right) \leq \frac{1}{32} \cdot 0.4\end{aligned}$$

Case (b), either $g_y < b_1$ or $g_z < b_1$. Let, say, $g_y < b_1$. Consider two subcases,

Subcase (b$_1$), $g_z < c$. The probability of this subcase is estimated by

$$\delta_2 = 2\Pr\{\xi_x = -a\} \cdot \Pr\{g_y < b_1\} \cdot \Pr\{g_z \in [b_1, c]\}$$

Since

$$\Pr\{g_y < b_1\} \leq \frac{1}{16} + 2^{-N}$$

we get that

$$\delta_2 \leq \frac{1}{16}\left(\frac{1}{16} + 0.13\right) + 2^{-N} < \frac{1}{32} \cdot 0.4$$

Subcase (b$_2$), $g_z \geq c$. Then $g_y \leq b_{N_0}$ and probability of this subcase is negligibly small. Thus, $p_2(k+1) < \frac{1}{32} \cdot 0.8$. This proves (iii) for $n = 2$.

Estimate of $p_n(k+1)$ for $3 \leq n \leq N$. As before, the case $\xi_x = \alpha$ has negligibly small probability. Let $\xi_x = -\alpha$. Then

$$\frac{g_y + g_z}{2} \in f_-^{-1}([b_n, b_{n-1}]) = [b_{n-1}, b_{n-2}]$$

Consider two cases.

Case (a), $g_y, g_z \geq b_{n-1}$. Observe that either $g_y \leq b_{n-2}$ or $g_z \leq b_{n-2}$. Let, say $g_y \leq b_{n-2}$. Then $g_z \in [b_{n-1}, b_{n-3}]$, because

$$b_{n-2} - b_{n-3} > b_{n-1} - b_{n-2}$$

due to the expansion of f_-^{-1} for $g > a$. Hence the probability of this case is estimated by

$$\begin{aligned}\delta_1 &= 2\Pr\{\xi_x = -a\} \cdot \Pr\{g_y \in [b_{n-1}, b_{n-2}]\} \cdot \Pr\{g_z \in [b_{n-1}, b_{n-3}]\} \\ &\leq \frac{1}{8} \cdot 2^{-(n-1)} \cdot \frac{1}{16}\left(2^{-(n-1} + 2^{-(n-2)}\right) < \frac{1}{4} \cdot 2^{-2n} \leq \frac{1}{4} \cdot 2^{-n} \cdot \frac{1}{4}\end{aligned}$$

Case (b), either $g_y < b_{n-1}$ or $g_z < b_{n-1}$. The probability of this subcase is estimated by

$$\delta_2 = \Pr\{g_y < b_{n-1}\} \cdot \Pr\{g_z \in [b_{n-2}, c]\} \leq \frac{1}{8} \cdot 2^{-n}(0.25 + 0.51) < \frac{1}{8} \cdot 2^{-n}$$

This finishes the proof of (iii) for $n \leq N$.

Proof of (iii) for $n > N$. The case $\xi_x = -a$ gives a negligibly small probability. Let Let $\xi_x = -\alpha$. Then

$$\frac{g_y + g_z}{2} \in f_-^{-1}([b_n, b_{n-1}])$$

Since $f'(g) < 2\tanh\beta < 2$ and $f(a) = a + O(\varepsilon)$, we obtain that

$$f_-^{-1}(b_{n-1}) > b_m, \qquad m = \frac{n}{2\tanh\beta} - 2$$

This implies that

$$f_-^{-1}([b_n, b_{n-1}]) \subset [b_j, b_{j-\ell}]$$

where

$$j \geq \frac{n}{2\tanh\beta} - 2$$

and ℓ is a number that does not depend on ε. Hence

$$\frac{g_y + g_z}{2} \in ([b_j, b_{j-\ell}]$$

By (iii), the probability distribution of g_y, g_z on $[b_i, b_{i-1}]$ is bounded by the exponential function $\frac{1}{8} \cdot 2^{-i}$. This implies that the distribution of $\frac{g_y+g_z}{2}$ on $[b_i, b_{i-1}]$ is bounded by a convolution of the exponential function $\frac{1}{8} \cdot 2^{-i}$ with itself, with subsequent scaling $g \to 2g$. This gives the estimate

$$\begin{aligned} \Pr\left\{\frac{g_y + g_z}{2} \in [b_j, b_{j-\ell}]\right\} &\leq C \cdot j \cdot 2^{-2j} \\ &\leq C_0 \cdot n \cdot 2^{-2n/(2\tanh\beta)} = C_0 \cdot n \cdot 2^{-n/\tanh\beta} \end{aligned}$$

Since $\tanh\beta < 1$, this implies that

$$\Pr\left\{\frac{g_y + g_z}{2} \in [b_j, b_{j-\ell}]\right\} \leq \frac{1}{10} \cdot 2^{-n}$$

and $p_n(k+1) \le \frac{1}{8} \cdot 2^{-n}$. This finishes the proof of the inductive assumption I_{k+1}. Main Lemma is proven. *QED*

Completion of the proof of Theorem 2.6. We have that $h_x = \sum_{y \in S(x)} g_y$, hence by (1.3),

$$\mu_n(\sigma_n \mid h) = Z_n^{-1} \exp\Big\{\beta \sum_{<x,y> \in L_n} \sigma_x \sigma_y + \beta \sum_{x \in V_n} \xi_x \sigma_x + \beta \sum_{\substack{x \in W_n \\ y \in S(x)}} \sigma_x g_y\Big\} \quad (6.12)$$

For the + boundary conditions, $\bar{\sigma}_y = 1$, the equation (1.2) reads

$$\mu_n(\sigma_n \mid \{+1\}) = Z_n^{-1} \exp\Big\{\beta \sum_{<x,y> \in L_n} \sigma_x \sigma_y + \beta \sum_{x \in V_n} \xi_x \sigma_x + \beta \sum_{\substack{x \in W_n \\ y \in S(x)}} \sigma_x\Big\}$$

Comparing these two equations we see that the + boundary conditions correspond to $g_y = 1$, $y \in W_{n+1}$. This means that the probability distribution of g_y is $\rho_0(dg) = \delta(g-1)dg$ when $y \in W_{n+1}$. The probability distribution $\rho_0(dg)$ satisfies the inductive assumption I_0. Hence by Main Lemma, $\rho_k(dg)$ satisfies I_k. By I_k,

$$\Pr\{g_x \ge a = b_N\} \ge 1 - \frac{1}{8} \sum_{j=N+1}^{\infty} 2^{-j} = 1 - \frac{2^{-N}}{8} \quad (6.13)$$

where $N = N(\varepsilon) \to \infty$ as $\varepsilon \to \infty$.

Let $x_0 \in V$. By (6.12)

$$\mu(\sigma_{x_0} \mid \{+\}) = Z^{-1} \exp\Big\{\beta(\xi_{x_0} + \sum_{<x_0,y>} g_y)\sigma_{x_0}\Big\} \quad (6.14)$$

Assume that $g_y \ge a$, then

$$\xi_{x_0} + \sum_{<x_0,y>} g_y \ge \xi_{x_0} + 3a \ge -\alpha + 3a$$

Since $f_\beta(-\alpha + 2a) = a + O(\varepsilon)$ and $f'_\beta(g) < \tanh\beta$, we obtain that

$$-\alpha + 2a = f_\beta^{-1}(a + O(\varepsilon)) > a$$

hence $\xi_{x_0} + \sum_{<x_0,y>} g_y > 2a$ and from (6.14) one gets :

$$\langle \sigma_{x_0} \rangle = \tanh\Big(\beta(\xi_{x_0} + \sum_{<x_0,y>} g_y)\Big) > \tanh 2\beta a$$

On the other hand by (6.13) we have :

$$\Pr\{g_y \geq a, d(y, x_0 = 1\} = \left(1 - \frac{2^{-N}}{8}\right)^3$$

This probability insures that

$$\mathbb{E}_\xi \langle \sigma_{x_0} \rangle > 0$$

This finishes the proof of the Theorem 2.6.

Acknowledgements

The first version of the manuscript reached its final form during our visit to *KU–Leuven* funded by the *Instituut voor Theoretische Fysica*, which we would like to thank for warm hospitality and finacial support. P.M. Bleher acknowledges the *Centre de Physique Théorique de Marseille* and *Université de Provence* for kind hospitality and financial support during his stay in Marselle-Luminy.

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Figure Captions

- Figure 1.1: Phase diagram of ferromagnetic RFIM on Bethe lattice for $k = 2$. In domain I one has two extreme Gibbs states corresponding to $(\pm)$ boundary conditions for *all* ξ, while in domain II they are different for *almost all* configurations of external field ξ. The boundary of domain I is the line of Giffiths singularities corresponding to non–analytic C^{∞} transition. Domain $\{\alpha > 3 \text{ or } T > T_c\}$ corresponds to uniqueness of the Gibbs state for *all* ξ. For $\{2 \le \alpha \le 3 \text{ and } T < T_c\}$ the Gibbs state is unique for *almost all* ξ and residual entropy $S_{\rm res}$ is a positive constant for $2 < \alpha < 3$ with spikes at $\alpha = 2$ and $\alpha = 3$. We guess that in fact the Gibbs state is unique for *all* ξ in domain $\{\alpha > H^{AF}(T), 0 < T < T_c\}$ and unique for *almost all* ξ in domain IV, where $H^{AF}(T)$ is again a line of Giffiths singularities. We have no guess for the rest of the domain III. Numerical simulations strongly suggest uniqueness for $\{1 < \alpha < 2, T < T_c\}$. Approximating formula for residual entropy (see Sect. 5) gives $S_{\rm res} > 0$ for $\{1 < \alpha < 2, T = 0\}$.

- Figure 1.2: Asymmetric distribution of effective field in domain I : $\beta = 1.0$, $\alpha = 0.4$.

- Figure 1.3: Asymmetric distribution of effective field in domain II : $\beta = 1.0$, $\alpha = 0.8$.

- Figure 1.4: Symmetric distribution of effective field in domain III. $\beta = 1.0$, $\alpha = 1.0$.

- Figure 4.1: Ground state of dipole $\{x, t\}$ configuration with $x, t \in V_0$. Signes $(+, -)$ correspond to distribution of the external field (charges). "Charge" $\xi_x = 2 + \varepsilon$ polarizes the neighbours to take opposite signs. The same for the charge $\xi_t = -(2 + \varepsilon)$. Dashed line corresponds to the "current" $0 < g_{xt} = -g_{tx} = \varepsilon < 1$, while solid lines to "currents" $= \pm 1$. Positive currents are directed from positive to negative charges.

- Figure 4.2: Configuration of "dipole polymer" corresponding to a ground–state configuration of charges $\{\xi_x = \pm(2 - \varepsilon)\}$ and "currents" : $\{g_{xy} = \pm\varepsilon\}$ (dashed lines) and $\{g_{uv} = \pm 1\}$ (solid lines).

- Figure 4.3: Illustration of the property (4): $\xi_x + g_{xy} + g_{zx} \ge 2$.

- Figure 6.1: Narrow corridor of the order ε in the vicinity of $g = a$ is reponsible for appearence of a long intermittent trajectory near a and finally for asymmetric distribution of effective field, cf. Fig. 1.3. The same phenomenon one has in the vicinity of $g = -a$.

- Figure 6.2: Illustration of the sequence b_n, $n = 0, \ldots, N$ which we choose for the proof of Theorem 2.6: $\varepsilon \ll c - b \ll b - a = \frac{d-c}{2} \ll 1$