The Renormalization of Self Intersection Local Times
I: The Chaos Expansion

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1 Introduction

The intersections of Brownian motion paths have been investigated since the Forties [20]. One can consider intersections of sample paths with themselves or e.g. with other, independent Brownian motions [36], one can study simple [5] or n-fold intersections [6] and one can ask all of these questions for linear, planar, spatial or - in general - d-dimensional Brownian motion: evidently self-intersections become increasingly scarce as the dimension d increases.

Intersection local times of Brownian motion were studied by many authors, see e.g. [1], [3]-[11], [15], [18]-[38]. A more systematic review than we can give here of the subject will be found e.g. in the recent reference [15].

An informal but rather suggestive definition of self intersection local time of Brownian motion B is in terms of an integral over Dirac's - or Donsker's - δ-function

$$L \equiv \int d^2t\, \delta(B(t_2) - B(t_1)),$$

intended to sum up the contributions from each pair of "times" t_1, t_2 for which the Brownian motion B is at the same point.

In Edwards' modeling of polymer molecules by Brownian motion paths, L is used to model the "excluded volume" effect: different parts of the molecule should not be located at the same point in space. As another application, Symanzik introduced L as a tool for constructive quantum field theory in [31].

A rigorous definition, such as e.g. through a sequence of Gaussians approximating the δ-function or in terms of generalized Brownian functionals [10] [30] [33], will lead to increasingly singular objects and will necessitate various "renormalizations" as the dimension d increases. For $d > 1$ the expectation will diverge in the limit and must be subtracted [18] [32], clearly L will then no more be positive. For $d > 3, 5, 7, \ldots$ further subtractions have been proposed [33] that will make L into a well defined generalized function of Brownian motion.

For $d = 3$ another renormalization has been constructed by Westwater to make the Gibbs factor $e^{-g \cdot L}$ of the polymer model well-defined [35], for yet another, recent approach see [1].

Yor, in [38], first suppresses the short time accumulation of self-intersections by the regularization

$$\delta(\overrightarrow{B}(t_2) - \overrightarrow{B}(t_1)) \to \delta(\overrightarrow{B}(t_2) - \overrightarrow{B}(t_1) + \overrightarrow{\varepsilon})$$

and shows, again for $d = 3$, that a multiplicative renormalization

$$r(\varepsilon)\,(L_\varepsilon - E(L_\varepsilon))$$

gives rise to another, independent Brownian motion as the weak limit of regularized and subtracted approximations to L.

In the present note we study similar limits, for arbitrary $d \geq 3$, using a Gaussian regularization of the δ-function for which the chaos expansion of the corresponding regularized L_ε is available [10]. For a suitably subtracted and renormalized local time, each term in this expansion converges in law to a Brownian motion.

We prepare and state these results in the following section 2, in section 3 we give their proofs. In a companion paper [3] we extend these results to the corresponding series.

2 Definitions and Main Results

2.1 White Noise Analysis and Local Times

We reproduce here some White Noise Analysis concepts as introduced in [10], referring to [14] for a systematic presentation. Brownian motions $B_i, i = 1, ..., d$, have version in terms of white noise ω_i via

$$B_i(t) =< \omega_i, 1_{[0,t]} >= \int_0^t \omega_i(s)ds.$$

Hence we consider independent d-tuples of Gaussian white noise $\omega = (\omega_1, ..., \omega_d)$ and correspondingly, d-tuples of test functions $\mathbf{f} = (f_1, ..., f_d) \in S(R, R^d)$, and introduce the following notation:

$$\vec{n}= (n_1, \ldots, n_d), \quad n = \sum_1^d n_i, \quad \vec{n}! = \prod_1^d n_i!$$

$$< \mathbf{f}, \mathbf{f} > = \sum_{i=1}^d \int dt \ f_i^2(t)$$

$$< F_{\vec{n}}, \mathbf{f}^{\otimes \vec{n}} >= \int d^n t \ F_{\vec{n}}(t_1, , ..., t_n) \ \bigotimes_{i=1}^d f_i^{\otimes n_i}(t_1, , ..., t_n)$$

and similarly for $<: \omega^{\otimes \vec{n}} :, F_{\vec{n}} >$ where for d-tuples of white noise the Wick product $: \ldots :$ [14] generalizes to

$$: \omega^{\otimes \vec{n}} :=\bigotimes_{i=1}^d : \omega_i^{\otimes n_i} : .$$

The vector valued white noise ω has the characteristic function

$$C(\mathbf{f}) = E(e^{i<\omega,\mathbf{f}>}) = \int_{S^*(R,R^d)} d\mu\,[\omega]\, e^{i<\omega,\mathbf{f}>} = e^{-\frac{1}{2}<\mathbf{f},\mathbf{f}>}, \tag{1}$$

where $< \omega, \mathbf{f} > = \sum_{i=1}^d < \omega_i, f_i>$ and $f_i \in S(R, R)$.

The Hilbert space

$$(L^2) \equiv L^2(d\mu).$$

is canonically isomorphic to the $d-$fold tensor product of Fock spaces of symmetric square integrable functions:

$$(L^2) \simeq (\bigoplus_{k=0}^{\infty} SyL^2(R^k, k!d^k t))^{\otimes d} \equiv \mathfrak{F}, \tag{2}$$

for the general element of (L^2) this implies the chaos expansion

$$\varphi(\omega) = \sum_{\vec{n}=0}^{\infty} <: \omega^{\otimes \vec{n}} :, F_{\vec{n}} > \tag{3}$$

with kernel functions F in $\mathfrak{F}$.

It is desirable to introduce regularizations for the intersection local time, with a view towards the construction of well-defined, "renormalized" intersection local times in higher dimensions where the latter do not exist without subtractions. A computationally simple regularization is, for $\varepsilon > 0$,

$$L_\varepsilon \equiv \int_0^t dt_2 \int_0^{t_2} dt_1 \, \delta_\varepsilon \left(\mathbf{B}\left(t_2\right) - \mathbf{B}\left(t_1\right) \right),$$

with

$$\delta_\varepsilon \left(\mathbf{B}\left(t_2\right) - \mathbf{B}\left(t_1\right) \right) \equiv \left(2\pi\varepsilon\right)^{-d/2} e^{-\frac{\left(\mathbf{B}\left(t_2\right) - \mathbf{B}\left(t_1\right)\right)^2}{2\varepsilon}}, \tag{4}$$

It has the following chaos expansion, which we quote here only for $d \geq 3$:

Theorem 2.1 *[10] For any $\varepsilon > 0$, $L_\varepsilon - E\left(L_\varepsilon\right)$ has kernel functions $F \in \mathfrak{F}$ given by*

$$F_{\varepsilon,\vec{n}}(s_1, \ldots, s_n) = (-1)^{\frac{n}{2}} \left(\kappa(\kappa+1)(2\pi)^{d/2} \, 2^{\frac{n}{2}} \, \frac{\vec{n}}{2}! \right)^{-1} \cdot$$

$$\Theta(u)\Theta(t-v) \cdot \left((v-u+\varepsilon)^{-\kappa} + (t+\varepsilon)^{-\kappa} - (v+\varepsilon)^{-\kappa} - (t-u+\varepsilon)^{-\kappa} \right) \tag{5}$$

if all n_i are even, and zero otherwise, with $v(s_1, ..., s_n) \equiv max(s_1, ..., s_n)$, $u(s_1, ..., s_n) \equiv min(s_1, ..., s_n)$, and $\kappa \equiv (n+d)/2 - 2$. Θ is the Heaviside function.

Each kernel function is thus the sum of four terms. The first one is

Definition 2.2

$$\begin{aligned}
M_t(d,\overrightarrow{n},\varepsilon) &\equiv \int_{[0,t]^n} d^n s (v-u+\varepsilon)^{-\varkappa} : \omega^{\otimes \overrightarrow{n}}(s) : && (6)\\
&= \int_{[0,t]^n} d^n s (v-u+\varepsilon)^{-\varkappa} \bigotimes_{i=1}^{d} : \omega_i(s_1^i)...\omega_i(s_{n_i}^i) : \\
&= \sum_{i=1}^{d}\sum_{m=1}^{n_i} \int_0^t ds_m^i \int_0^{s_m^i} d^{n-1}s(v-u+\varepsilon)^{-\varkappa} \bigotimes_{i=1}^{d} : \omega_i(s_1^i)...\omega_i(s_{n_i}^i) : \\
&= \sum_{i=1}^{d} n_i \int_0^t dB_i(\tau) \int_0^{\tau} d^{n-1}s(\tau-u+\varepsilon)^{-\varkappa} : \omega^{\otimes \overrightarrow{n-\delta_i}}(s) : \\
&= \sum_{k=1}^{d} \int_0^t dB_k(v) m_k(v) && (7)\\
&\equiv \sum_{k=1}^{d} M_{k,t} && (8)
\end{aligned}$$

The others give

$$N_t(d,\overrightarrow{n},\varepsilon) \equiv \int_{[0,t]^n} d^n s \left((t+\varepsilon)^{-\kappa} - (v+\varepsilon)^{-\kappa} - (t-u+\varepsilon)^{-\kappa}\right) : \omega^{\otimes \overrightarrow{n}}(s) :$$

All of the above processes are continuous.

Definition 2.3 *We denote the $\overrightarrow{n}^{\,th}$ order contribution to the regularized local time L_ε by*

$$K_t(d,\overrightarrow{n},\varepsilon) \equiv (-1)^{\frac{n}{2}} \left(\kappa(\kappa+1)(2\pi)^{d/2}\, 2^{\frac{n}{2}}\ \frac{\overrightarrow{n}}{2}!\right)^{-1} (M_t(d,\overrightarrow{n},\varepsilon) + N_t(d,\overrightarrow{n},\varepsilon))$$

> **Remark 2.4** *Our key observation is that, as ε goes to zero, M is more singular than N, and that it is a Brownian martingale.*

2.2 The main theorems

Theorem 2.5 *: For $d \geq 3$, the renormalized M converge in distribution to independent Brownian motions β_i:*

$$(rM_i, ; i = 1, ..., d) \underset{\varepsilon \to +0}{\overset{\mathfrak{L}}{\to}} \left(\frac{n_i}{n} k_n \beta_i ; i = 1, ..., d \right)$$

with

$$k_n^2 = \overrightarrow{n}! n(n-1) \begin{cases} 1 & \mathrm{i}f d = 3 \\ \frac{1}{(n-1)n...(n+d-5)} & \mathrm{i}f d > 3 \end{cases} \tag{9}$$

if

$$r(\varepsilon) = \begin{cases} |ln\varepsilon|^{-1/2} \\ \varepsilon^{(d-3)/2} \end{cases}$$

Theorem 2.6 *:For $d \geq 3$, the renormalized $\overrightarrow{n}$ th order contributions to the regularized local time L_ε converge in distribution to Brownian motions β*

$$rK(d, \overrightarrow{n}, \varepsilon;) \underset{\varepsilon \to +0}{\overset{\mathfrak{L}}{\to}} c_{n,d} \beta$$

with

$$c^2_{\overrightarrow{n},d} = k_n^2 \left(\kappa(\kappa+1)(2\pi)^{d/2} \, 2^{\frac{n}{2}} \, \frac{\overrightarrow{n}}{2}! \right)^{-2}$$

3 Proofs

Proposition 3.1 *$M_{k,t}$ are orthogonal Brownian martingales*

Proof: Orthogonality is obvious. For the martingale property see [13] and [2], it is a consequence of the fact that the kernel functions of M_t in (6) do not depend on t (except through the limit of integrations).■

Their limiting behavior, as $\varepsilon \to +0$, is studied in the following lemma (from now on we shall consider only the situations which require renormalisation, i.e., $d \geq 3$.)

Lemma 3.2 *As $\varepsilon \to +0$,*

$$\|M_{k,t}\|_2^2 = \frac{n_k}{n} k_n^2 \left(t + o(1) \right) \begin{cases} |ln\varepsilon| \, \mathrm{f}ord = 3 \\ \varepsilon^{-(d-3)} \mathrm{f}ord > 3 \end{cases}$$

Proof:

$$\|M_{k,t}\|_2^2 = \frac{n_k}{n} \overrightarrow{n}! \left\|(v-u+\varepsilon)^{-\varkappa}\right\|^2_{L^2([0,t]^n)}$$

For $d > 3$

$$\begin{aligned}
\left\|(v-u+\varepsilon)^{-\varkappa}\right\|^2_{L^2([0,t]^n)} &= \int_0^t d^n s (v-u+\varepsilon)^{-2\varkappa} \\
&= n(n-1)\int_0^t dv \int_0^v du \frac{(v-u)^{n-2}}{(v-u+\varepsilon)^{2\varkappa}} \\
&= n(n-1)\varepsilon^{3-d}\int_0^t dv \int_0^{v/\varepsilon} dx \frac{x^{n-2}}{(x+1)^{n+d-4}} \\
&= \varepsilon^{3-d} t \left(\frac{n(n-1)}{(n-1)n...(n+d-5)} + o(1) \right)
\end{aligned}$$

while for $d = 3$

$$\left\|(v-u+\varepsilon)^{-\varkappa}\right\|^2_{L^2([0,t]^n)} = n(n-1)t\left|\ln\varepsilon\right|(1+o(1))$$

■

To show convergence of these martingales by Theorem VIII.2.17 of [16] we must show convergence of characteristics. Since the processes M are continuous this reduces to showing convergence in probability of $\langle rM_i, rM_k\rangle_t$ as $\varepsilon \to +0$. This will be taken care of by proposition 3.7 which we prepare now:

$$\langle rM_i, rM_k\rangle_t = r^2\delta_{ik}\int_0^t dv\,(m_i(v))^2$$

and we need to estimate

$$\int_0^t dv\,(m_i(v))^2 \equiv \sum_{\overrightarrow{k}} \langle :\omega^{\otimes \overrightarrow{k}}:, G^{(i)}_{\overrightarrow{k}}\rangle$$

Let us note that

$$E\left(\langle rM_i, rM_k\rangle_t\right) = E\left(r^2 M_i M_k\right) = \delta_{ik} r^2 \frac{n_k}{n}\overrightarrow{n}! \left\|(v-u+\varepsilon)^{-\varkappa}\right\|^2_{L^2([0,t]^n)}$$

$$E\left(\langle rM_i, rM_k\rangle_t\right) = \frac{n_k}{n}k_n^2\left(t+o(1)\right)\begin{cases} |ln\varepsilon|\, for d=3 \\ \varepsilon^{-(d-3)} for d>3\end{cases}$$

Next we intend to show that the rhs gives indeed that the rest of $\langle rM_i, rM_k\rangle_t$ goes to zero. The kernel of the highest order is

$$G^{(i)}_{2\overrightarrow{n}-2\overrightarrow{\delta}_i}(s_1,s_1';,\dots;s_{n-1},s_{n-1}') = \int_0^t d\tau\Theta\left(\tau - v\vee v'\right)(\tau - u + \varepsilon)^{-\varkappa}(\tau - u' + \varepsilon)^{-\varkappa} \tag{10}$$

$v^{(\prime)}$ and $u^{(\prime)}$ are the largest and the smallest of the $s_i^{(\prime)}$, and $\left(\overrightarrow{\delta}_i\right)_k \equiv \delta_{ik}$.

(For $n = 2$: $u^{(\prime)} = v^{(\prime)} = s^{(\prime)}$).

Remark 3.3 *The integral (10) can be calculated in closed form (using e.g. [12] nos. 2.15 and 2.263.4), but one gets a useful approximation by introducing the following auxiliary functions:*

$$\begin{aligned}
& H_{2n-2}(s_1,...,s_{n-1};s_1',...,s_{n-1}';\varepsilon;d) \\
\equiv\ & (v\vee v' - u\vee u' + \varepsilon)^{-(n+d)/2+3}(v\vee v' - u\wedge u' + \varepsilon)^{-(n+d)/2+2}
\end{aligned}$$

where $v \equiv max(s_i), u \equiv min(s_i)$ and $v' \equiv max(s_i'), u' \equiv min(s_i')$. These functions majorize the kernel functions

Lemma 3.4 *For $n > 2$, and $\kappa > 1$*

$$\begin{aligned}
0 &\le G^{(i)}_{2\overrightarrow{n}-2\overrightarrow{\delta}_i}(s_1,s_1';,\dots;s_{n-1},s_{n-1}') \\
&\le \frac{1}{\kappa-1}H_{2n-2}(s_1,...,s_{n-1};s_1',...,s_{n-1}';\varepsilon;d)
\end{aligned} \tag{11}$$

Proof:

$$\begin{aligned}
0 &\le \int_0^t d\tau\Theta\left(\tau - v\vee v'\right)(\tau - u + \varepsilon)^{-\varkappa}(\tau - u' + \varepsilon)^{-\varkappa} \\
&\le \int_{v\vee v'}^t d\tau(\tau - u\vee u' + \varepsilon)^{-\varkappa}(v\vee v' - u\wedge u' + \varepsilon)^{-\varkappa} \\
&\le \frac{1}{\kappa-1}(v\vee v' - u\vee u' + \varepsilon)^{-\varkappa+1}(v\vee v' - u\wedge u' + \varepsilon)^{-\varkappa}
\end{aligned}$$

This estimate suffices to show

Lemma 3.5 *For $2\overrightarrow{n} - \overrightarrow{e}_i - \overrightarrow{e}_k \neq 0$*

$$r^2 G^{(i)}_{2\overrightarrow{n}-2\overrightarrow{\delta}_i} \rightarrow 0 in L^2(R^{2n-2})$$

as ε goes to $+0$.

Proof: Consider first $n \geq 3$. By the above estimate it is sufficient to show that

$$\lim_{\varepsilon \to +0} r^4 \, \|H_{2n-2}\|^2_{L^2([0,t]^{2n-2})} = 0.$$

$$\begin{aligned}
\|H_{2n-2}\|^2_{L^2(R^{2n-2})} &= \int d^{n-1}s \int d^{n-1}s' (v \vee v' - u \vee u' + \varepsilon)^{2-2\kappa} (v \vee v' - u \wedge u' + \varepsilon)^{-2\kappa} \\
&= c_n \int_0^t dv \int_0^v dv' \int_0^v du \int_0^{v'} du' \\
&\quad \frac{(v-u)^{n-3}(v'-u')^{n-3}}{(v \vee v' - u \vee u' + \varepsilon)^{n+d-6}(v \vee v' - u \wedge u' + \varepsilon)^{n+d-4}} \\
&\leq c_n \varepsilon^{8-2d} \int_0^{t/\varepsilon} dy \int_0^y dy' \int_0^y dx \int_0^{y'} dx' \\
&\quad \frac{1}{(y - x \vee x' + 1)^{d-3}(y - x \wedge x' + 1)^{d-1}}
\end{aligned}$$

and then decompose the x-integration into the following two domains

$$x' < x and x < x'$$

In the first case

$$\begin{aligned}
I &= \varepsilon^{8-2d} \int_0^{t/\varepsilon} dy \int_0^y dy' \int_0^{y'} dx' \int_{x'}^y dx \frac{1}{(y-x+1)^{d-3}(y-x'+1)^{d-1}} \\
&= \varepsilon^{8-2d} \int_0^{t/\varepsilon} dy \int_0^y dy' \int_0^{y'} dx' \frac{1}{(y-x'+1)^{d-1}} \int_{x'}^y dx \frac{1}{(y-x+1)^{d-3}}
\end{aligned}$$

We first estimate the last integral

$$\int_{x'}^y \frac{1}{(y-x+1)^{d-3}} dx \leq \begin{cases} y - x' \mathrm{if}\, d = 3 \\ \ln{(y - x' + 1)} \,\mathrm{if}\, d = 4 \\ \frac{1}{d-4} \mathrm{if}\, d > 4 \end{cases}$$

and one finds

$$I = \begin{cases} O\,(1) \,\mathrm{if}\, d = 3 \\ O(\varepsilon^{7-2d}) \mathrm{if}\, d > 3 \end{cases}$$

Hence, in both cases, $r^4\ I$ vanishes as $\varepsilon \to 0$.
The second case is

$$I = \varepsilon^{8-2d} \int_0^{t/\varepsilon} dy \int_0^y dx \frac{1}{(y-x+1)^{d-1}} \int_x^y dy' \int_x^{y'} dx' \frac{1}{(y-x'+1)^{d-3}} \quad (12)$$

Estimating the last two integrals we find

$$\int_x^y dy' \int_x^{y'} dx' \frac{1}{(y-x'+1)^{d-3}} \leq \begin{cases} \frac{(y-x)^2}{2} for d = 3 \\ const. \, (y-x)^{5-d} for d > 3 \end{cases}$$

Substitution of these estimates into the integrals over x and y gives for $d > 3$

$$I \leq const. \varepsilon^{8-2d} \int_0^{t/\varepsilon} dy \int_0^y dx \frac{(y-x)^{n+2-d}}{(y-x+1)^{n+d-4}}$$

where the integral over x is bounded, so that again $I = O(\varepsilon^{7-2d})$. For $d = 3$ one finds similarly $I = O(\varepsilon^0)$ so in both cases, $r^4\ I$ vanishes as $\varepsilon \to 0$. For $n = 2$ it is sufficient to use the estimate

$$\begin{aligned} 0 \ &\leq \ G_2 = \int_0^t d\tau \Theta\left(\tau - u \vee u'\right) (\tau - u + \varepsilon)^{1-d/2} (\tau - u' + \varepsilon)^{1-d/2} \\ &\leq \ \int_{u \vee u'}^t d\tau (\tau - u \vee u' + \varepsilon)^{1-d/2} (u \vee u' - u \wedge u' + \varepsilon)^{1-d/2} = H_2(u, u') \end{aligned}$$

and to verify that

$$\lim_{\varepsilon \to +0} r^4 \left\| H_2 \right\|^2_{L^2([0,t]^2)} = 0.$$

■

With this lemma we have established that the highest order term of the (renormalized) quadratic variation goes to zero in quadratic mean for any t¿0. The kernels G_k of the other terms are obtained by integrating over pairs of s, s' such as e.g. in

$$Sym \int_0^t ds G^{(i)}_{2(\overrightarrow{n} - \overrightarrow{e}_i)}(s, s; s_2, s'_2; , \ldots ; s_{n-1}, s'_{n-1}),$$

these new functions are in fact also bounded by an expression like (10), and hence by (11), so that for all $\varepsilon > 0$,

$$\|G_k\| \leq const.\, \|H_k\|$$

With this goal we show

Lemma 3.6 *Let $n \geq 2$ and $m > 1$, and let $F_{n,m}$ be a function symmetric in the variables s and in the variables s', with*

$$\begin{aligned} 0 \;&\leq\; F_{n,m}(s_1, \ldots, s_{n-1}, s_n, s'_1, \ldots, s'_{n-1}, s'_n) \qquad (13) \\ &\leq\; c \int_0^t d\tau \Theta\left(\tau - \underset{i\leq n}{max}\,(s, s')\right)(\tau - \underset{i\leq n}{min}\,(s_i) + \varepsilon)^{-m}(\tau - \underset{i\leq n}{min}\,(s'_i) + \varepsilon)^{-m} \end{aligned}$$

Then $\exists c_m < \infty$ such that

$$\begin{aligned} 0 \;&\leq\; \int_0^t ds F_{n,m}(s_1, \ldots, s, s'_1, \ldots, s) \\ &\leq\; c_m \int_0^t d\tau \Theta\left(\tau - \underset{i<n}{max}\,(s, s')\right)(\tau - \underset{i<n}{min}\,(s_i) + \varepsilon)^{-m+1/2}(\tau - \underset{i<n}{min}\,(s'_i) + \varepsilon)^{-m+1/2} \end{aligned}$$

Proof: Under the assumption of the lemma

$$\int_0^t ds F_{n,m}\left(s_1, ..., s_{n-1}, s, s'_1, ..., s'_{n-1}, s\right)$$

$$\leq \int_0^t ds \left(c\int_0^t d\tau \Theta\left(\tau - \max_{0\leq i\leq n-1}(s_i, s'_i, s)\right)\left(\tau - \min_{0\leq i\leq n-1}(s_i, s) + \varepsilon\right)^{-m}\left(\tau - \min_{0\leq i\leq n-1}(s'_i, s) + \varepsilon\right)^{-m}\right).$$

Using

$$\begin{aligned} u &= \min_{0\leq i\leq n-1} s_i \quad , \quad u' = \min_{0\leq i\leq n-1} s'_i \\ v &= \max_{0\leq i\leq n-1} s_i \quad , \quad v' = \max_{0\leq i\leq n-1} s'_i \end{aligned}$$

(For $n = 2$: $u^{(\prime)} = v^{(\prime)} = s_1^{(\prime)}$).

Assuming without loss of generality that $u < u'$ we can decompose the s-integration of our estimate as follows

$$c\int_0^t ds\int_0^t d\tau\Theta\left(\tau-\max\left(v,v',s\right)\right)\left(\tau-\min\left(u,s\right)+\varepsilon\right)^{-m}\left(\tau-\min\left(u',s\right)+\varepsilon\right)^{-m}$$

$$\begin{aligned}
&= c\int_0^t d\tau\left(\int_0^u+\int_u^{u'}+\int_{u'}^{v\vee v'}+\int_{v\vee v'}^{t}\right)ds \\
&\quad\cdot\Theta\left(\tau-\max\left(v,v',s\right)\right)\left(\tau-\min\left(u,s\right)+\varepsilon\right)^{-m}\left(\tau-\min\left(u',s\right)+\varepsilon\right)^{-m} \\
&= c\int_{v\vee v'}^{t} d\tau\int_0^u ds\left(\tau-s+\varepsilon\right)^{-2m}+\int_u^{u'} ds\left(\tau-u+\varepsilon\right)^{-m}\left(\tau-s+\varepsilon\right)^{-m} \\
&\quad+\int_{u'}^{v\vee v'} ds\left(\tau-u+\varepsilon\right)^{-m}\left(\tau-u'+\varepsilon\right)^{-m}+\int_{v\vee v'}^{\tau} ds\left(\tau-u+\varepsilon\right)^{-m}\left(\tau-u'+\varepsilon\right)^{-m}.
\end{aligned}$$

We now show that each of the four terms obeys the postulated estimate (14)

$$\begin{aligned}
\int_{v\vee v'}^{t} d\tau\int_0^u ds\left(\tau-s+\varepsilon\right)^{-2m} &\leq \frac{1}{2m-1}\int_{v\vee v'}^{t} d\tau\left(\tau-u+\varepsilon\right)^{-2m+1} \\
&\leq \frac{1}{2m-1}\int_0^t d\tau\Theta\left(\tau-v\vee v'\right)\left(\tau-u+\varepsilon\right)^{-m+\frac{1}{2}}\left(\tau-u'+\varepsilon\right)^{-m+\frac{1}{2}}
\end{aligned}$$

since $u<u'$ and $m>1/2$. The second term

$$\begin{aligned}
&\int_{v\vee v'}^{t} d\tau\int_u^{u'} ds\left(\tau-u+\varepsilon\right)^{-m}\left(\tau-s+\varepsilon\right)^{-m} \\
\leq\ & \frac{1}{m-1}\int_{v\vee v'}^{t} d\tau\left(\tau-u+\varepsilon\right)^{-m}\left(\tau-u'+\varepsilon\right)^{-m+1} \\
\leq\ & \frac{1}{m-1}\int_0^t d\tau\Theta\left(\tau-v\vee v'\right)\left(\tau-u+\varepsilon\right)^{-m+\frac{1}{2}}\left(\tau-u'+\varepsilon\right)^{-m+\frac{1}{2}}.
\end{aligned}$$

using again $u < u'$. The third term

$$\begin{aligned}
& \int\limits_{v\vee v'}^{t} d\tau \int\limits_{u'}^{v\vee v'} ds\, (\tau - u + \varepsilon)^{-m} (\tau - u' + \varepsilon)^{-m} \\
= & \int\limits_{v\vee v'}^{t} d\tau\, (\tau - u + \varepsilon)^{-m} (\tau - u' + \varepsilon)^{-m} (v \vee v' - u') \\
\leq & \int\limits_{v\vee v'}^{t} d\tau\, (\tau - u + \varepsilon)^{-m} (\tau - u' + \varepsilon)^{-m} (\tau - u') \\
\leq & \int\limits_{0}^{t} d\tau \Theta (\tau - v \vee v') (\tau - u + \varepsilon)^{-m+\frac{1}{2}} (\tau - u' + \varepsilon)^{-m+\frac{1}{2}}.
\end{aligned}$$

Finally

$$\begin{aligned}
& \int\limits_{v\vee v'}^{t} d\tau \int\limits_{v\vee v'}^{\tau} ds\, (\tau - u + \varepsilon)^{-m} (\tau - u' + \varepsilon)^{-m} \\
= & \int\limits_{v\vee v'}^{t} d\tau\, (\tau - u + \varepsilon)^{-m} (\tau - u' + \varepsilon)^{-m} (\tau - v \vee v') \\
\leq & \int\limits_{v\vee v'}^{t} d\tau\, (\tau - u + \varepsilon)^{-m} (\tau - u' + \varepsilon)^{-m} (\tau - u') \\
\leq & \int\limits_{0}^{t} d\tau \Theta (\tau - v \vee v') (\tau - u + \varepsilon)^{-m+\frac{1}{2}} (\tau - u' + \varepsilon)^{-m+\frac{1}{2}}
\end{aligned}$$

■

Combining this Lemma with the previous one we conclude that for all kernel functions G_k with $k \geq 2$ arguments

$$\lim_{\varepsilon \to +0} r^4 \|SymG_k\|^2 \leq \lim_{\varepsilon \to +0} r^4 \|G_k\|^2 \leq \lim_{\varepsilon \to +0} r^4 \|H_k\|^2 = 0$$

i.e. we have shown

Proposition 3.7

$$ms-\lim_{\varepsilon\to+0}\langle rM_i, rM_k\rangle_t=\frac{n_k}{n}k_n^2 t$$

Proof of Theorem 2.5: The above limit is clearly (up to constants) the quadratic variation of a Brownian motion. Theorem 2.5 is then a consequence of Theorem VIII.2.17 in [16], which in the present case of continuous martingales requires convergence in probability of quadratic variations for a dense set of t.

To control the remaining terms $N_t(d,\vec{n},\varepsilon;)$ in the chaos expansion we observe that

$$\begin{aligned}\|N_t(d,\vec{n},\varepsilon)\|^2_{(L^2)} &= \vec{n}!\left\|(t+\varepsilon)^{-\kappa}-(v+\varepsilon)^{-\kappa}-(t-u+\varepsilon)^{-\kappa}\right\|^2_{L^2([0,t]^n)}\\ &\leq \vec{n}!\left(\left\|(t+\varepsilon)^{-\kappa}\right\|^2_{L^2}+\left\|(v+\varepsilon)^{-\kappa}\right\|^2_{L^2}+\left\|(t-u+\varepsilon)^{-\kappa}\right\|^2_{L^2}\right)\end{aligned}$$

The first of these three norms is equal to $t^n(t+\varepsilon)^{-2\kappa}$, i.e. $O(1)$. The second one is

$$\begin{aligned}\int_{[0,t]^n} d^n s(v+\varepsilon)^{-2\kappa} &= n\int_0^t dv\frac{v^{n-1}}{(v+\varepsilon)^{n+d-4}}=n\varepsilon^{4-d}\int_0^{t/\varepsilon}dx\frac{x^{n-1}}{(x+1)^{n+d-4}}\\ &= \begin{cases} O(1) \text{f}ord=3\\ O(\ln\varepsilon) \text{f}ord=4\\ O(\varepsilon^{4-d}) \text{f}ord>4\end{cases}\end{aligned}$$

which are suppressed by the renormalization

$$r^2(\varepsilon)=\begin{cases}|ln\varepsilon|^{-1}\text{f}ord=3\\ \varepsilon^{d-3}\text{f}ord>3\end{cases}.$$

A similar estimate holds for the third term of N, so that we have shown

Lemma 3.8

$$ms-\lim_{\varepsilon\to+0}r(\varepsilon)N_t(d,\vec{n},\varepsilon)=0.$$

In fact the convergence is uniform in any finite t-interval. Next we show

Lemma 3.9 *The processes* $\{r(\varepsilon)N.(d,\overrightarrow{n},\varepsilon;):\varepsilon>0\}$, $\{r(\varepsilon)M.(d,\overrightarrow{n},\varepsilon;):\varepsilon>0\}$ *and their linear combinations are tight.*

Proof: A criterion for tightness of M (following [17], p.64) is

$$\sup_{\varepsilon>0} E\left|rM_t - rM_s\right|^{\alpha} \leq C_T\left(t-s\right)^{1+\beta}, \tag{14}$$

$\forall T>0$ and $0\leq s<t\leq T$ and for some positive constants α and β and C_T.

As a first step we show

$$\sup_{\varepsilon>0} E\left|rM_t - rM_s\right|^{2} \leq C_T\left(t-s\right) \tag{15}$$

by direct calculation:

$$\begin{aligned} E\left|M_t - M_s\right|^2 &= n\int_s^t dv \int_0^v d^{n-1}s\left(v-u\right)^{-2\kappa} \\ &\leq (t-s)n\int_0^t d^{n-1}s\left(v-u+\varepsilon\right)^{-2\kappa} \end{aligned}$$

This integral may be estimated as follows

$$\begin{aligned} \int_0^t d^{n-1}s\left(v-u+\varepsilon\right)^{-2\kappa} &= c\int_0^t du\frac{(v-u)^{n-2}}{\left(v-u+\varepsilon\right)^{n+d-4}} \\ &\leq c\varepsilon^{3-d}\int_0^{T/\varepsilon} dx\frac{1}{\left(x+1\right)^{d-2}} \\ &= \begin{cases} O(1)\cdot\ln T ford=3 \\ O(\varepsilon^{3-d}) for d>3 \end{cases} \end{aligned}$$

Renormalization of this estimate by the factor r^2 gives the desired estimate (15), i.e.

$$\left\|rM_t - rM_s\right\|_2^2 \leq c_T\left(t-s\right) \tag{16}$$

By the hypercontractivity of the Ornstein-Uhlenbeck semigroup (see e.g.[14], p.235), one has for nth order white noise monomials $\varphi\in(L^2)$, and any $\alpha>2$

$$\left\|\varphi\right\|_{\alpha}^{\alpha} \leq c_{n,\alpha}\left\|\varphi\right\|_2^{\alpha}$$

For $\varphi = rM_t - rM_s$,and using the above estimate for the 2-norm, we get

$$E\left|rM_t - rM_s\right|^{\alpha} \leq C_T\left(t-s\right)^{\alpha/2}$$

as required to ensure tightness. The estimates for rN etc. are of the same kind.■

Proof of Theorem 2.6: We need to consider

$$rK = rM + rN$$

knowing that, as $\varepsilon \to +0$, the rK are tight by Lemma 3.9, the rM converge in law, and the rN_t go to zero in mean square. The latter two facts are sufficient, via the Cramér-Wold device (see e.g. [17] p.61), for finite dimensional convergence of K; tightness then implies convergence in law.

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