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# The determinant representation for quantum correlation functions of the sinh-Gordon model


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**Abstract**

We consider the quantum sinh-Gordon model in this paper. Form factors in this model were calculated in [1]–[3]. We sum up all contributions of form factors and obtain a closed expression for a correlation function. This expression is a determinant of an integral operator. Similar determinant representations were proven to be useful not only in the theory of correlation functions [4]–[12], but also in the matrix models [13]–[16].

# 1 Introduction

The theory of massive, relativistic, integrable models is an important part of modern quantum field theory [17]–[24]. Scattering matrices in these models factorize into a product of two-body S-matrices [17]. Form factors can be calculated on the basis of a bootstrap approach [17]–[24].

The purpose of this paper is to calculate correlation functions. As usual correlation function can be represented as an infinite series of form factors contributions. In this paper we sum up all these contributions and obtain a closed expression for correlation functions of local operators (4.8). The idea of this summation is the following. We introduce an auxiliary Fock space and auxiliary Bose fields (we shall call them dual fields). These fields help us to represent the form factor decomposition of a correlation function in a form similar to "free fermionic" case. This approach was developed in [25], [26] and [6]. Finally a correlation function is represented as a vacuum mean value (in the auxiliary Fock space) of a determinant of an integral operator (5.1). This representation was proven to be useful [6], [8], [11]. It helps for asymptotical analysis of quantum correlation functions. Among other things this approach helped to calculate asymptotic of time and temperature dependent correlation function in Nonlinear Schrödinger equation [9].

In this paper we consider the sinh-Gordon model. It is the model of one (real) relativistic Bose field $\phi$ in $2D$. The action is

$$S = \int_{-\infty}^{\infty} d^2x \left[\frac{1}{2}(\partial_\mu\phi(x))^2 - \frac{m_0^2}{g^2}\cosh g\phi(x)\right]. \tag{1.1}$$

It is the simplest example of the affine Toda field theories [27] with $Z_2$ symmetry $\phi \to -\phi$. The model has only one massive particle. The two-body scattering matrix [18], [28] is given by an expression

$$\begin{aligned} S(\beta, B) &= \frac{\tanh\frac{1}{2}\left(\beta - \frac{i\pi B}{2}\right)}{\tanh\frac{1}{2}\left(\beta + \frac{i\pi B}{2}\right)}, \\ B &= \frac{2g^2}{8\pi + g^2}. \end{aligned} \tag{1.2}$$

We shall consider a real $g$, which corresponds to a positive $B$. Later we shall use a variable

$$x = e^\beta \tag{1.3}$$

instead of rapidity $\beta$ .

We shall use a representation for form factors found in [1]–[3] (another representation can be found in [23])

$$F_n(\beta_1,\ldots,\beta_n)=\langle 0|\mathcal{O}(0)|\beta_1,\ldots,\beta_n\rangle = H_n Q_n(s)\prod_{i>j}^{n}\frac{F_{\min}(\beta_{ij})}{x_i+x_j}. \tag{1.4}$$

Here $\beta_{ij}=\beta_i-\beta_j$. A function $F_{\min}(\beta)$ is holomorphic for real $\beta$

$$F_{\min}(\beta)=\mathcal{N}(B)\Xi(\beta), \tag{1.5}$$

where

$$\Xi(\beta)=\exp\left[8\int_0^\infty\frac{dx}{x}\frac{\sinh\left(\frac{1}{4}xB\right)\sinh\left(\frac{1}{2}x\left(1-\frac{1}{2}B\right)\right)\sinh\frac{1}{2}x}{\sinh^2 x}\sin^2\left(\frac{x\hat{\beta}}{2\pi}\right)\right], \tag{1.6}$$

$$\mathcal{N}(B)=\exp\left[-4\int_0^\infty\frac{dx}{x}\frac{\sinh\left(\frac{1}{4}xB\right)\sinh\left(\frac{1}{2}x\left(1-\frac{1}{2}B\right)\right)\sinh\frac{1}{2}x}{\sinh^2 x}\right], \tag{1.7}$$

and $\hat{\beta}=i\pi-\beta$. The function $F_{\min}(\beta)$ has a simple zero at $\beta=0$ and no poles at the strip $0\le \operatorname{Im}\beta\le\pi$. At $\beta\to\infty$ it goes to one: $F_{\min}(\beta)\to 1$. Functions $Q_n(x_1,\ldots,x_n)$ are symmetric polynomials of variables $x_1,\ldots,x_n$ given by [1]

$$Q_n(s)=\det{}_{n-1}M_{ij}(s),\qquad i,j=1,\ldots,n-1, \tag{1.8}$$

where

$$M_{ij}(s)=\sigma_{2i-j}[i-j+s]. \tag{1.9}$$

Let us explain the notations. Here and later we suppress the dependency of $Q_n(s)$ on variables $x_j$. The index $n-1$ in the expression $\det_{n-1}$ denotes the dimension of the matrix $M_{ij}(s)$. The functions $\sigma_k$ are elementary symmetric polynomials of the $k$-th order of the variables $x_1,\ldots,x_n$:

$$\sigma_k\equiv\sigma_k^{(n)}(x_1,\ldots,x_n)=\sum_{i_1<i_2<\ldots<i_k}^{n}x_{i_1}x_{i_2}\ldots x_{i_k}, \tag{1.10}$$

and $\sigma_k=0$ if $k<0$ or $k>n$. Here we also suppressed the dependency of $\sigma_k$ on $x_j$. The symbol $[m]$ is a "$q$-number" defined by

$$[m]=\frac{\sin\frac{mB}{2}}{\sin\frac{B}{2}}=\frac{q^m-q^{-m}}{q-q^{-1}}, \tag{1.11}$$

where $q = \exp\{iB/2\}$. The number $s$ in (1.8), (1.9) is an arbitrary integer, depending on specific choice of an operator $\mathcal{O}$ in (1.4).

Finally, the constants $H_k$ in (1.4) are normalization constants

$$H_{2n+1} = H_1 \mu^n, \qquad H_{2n} = H_0 \mu^n, \qquad \mu = \frac{4 \sin \frac{\pi B}{2}}{F_{\min}(i\pi)}, \tag{1.12}$$

where $H_0$ and $H_1$ also depend on specific operator $\mathcal{O}$. For instance, the form factor of local field is given by (1.4) with $s = 0$ and

$$\begin{aligned} H_0 &= \langle 0|\phi(0)|0\rangle = 0, \\ H_1 &= \langle 0|\phi(0)|\beta\rangle = \frac{1}{\sqrt{2}}. \end{aligned} \tag{1.13}$$

A correlation function of an operator $\mathcal{O}$ can be presented as an infinite series of form factors contributions

$$\langle 0|\mathcal{O}(0,0)\mathcal{O}(x,t)|0\rangle = \sum_{n=0}^{\infty} \int \frac{d^n\beta}{n!(2\pi)^n} |F_n(\beta_1, \ldots, \beta_n)|^2 \prod_{j=1}^{n} e^{-mr\cosh\beta_j}. \tag{1.14}$$

In the present paper we sum up this series explicitly. Now let us present a plan of the paper. Section 2 is devoted to a transformation of the determinants (1.8), (1.9) to a form, which is convenient for summation. In section 3 we introduce auxiliary quantum operators—dual fields—in order to factorize an expression for a correlation function and to represent it in a form similar to "free fermionic case". In section 4 we sum up the series (1.14) into a Fredholm determinant. In section 5 we use the Fredholm determinant representation for derivation of an asymptotic behavior of correlation functions.

## 2 A transformation of the form factor

A determinant of a linear integral operator $I + V$ can be written as

$$\det(I+V) = \sum_{n=0}^{\infty} \int \frac{dx_1 \cdots dx_n}{n!} \det_n \begin{pmatrix} V(x_1,x_1) & \cdots & V(x_1,x_n) \\ V(x_2,x_1) & \cdots & V(x_2,x_n) \\ \cdot & \cdot & \cdot \\ V(x_n,x_1) & \cdots & V(x_n,x_n) \end{pmatrix}. \tag{2.1}$$

Thus, in order to obtain a determinant representation for correlation functions one need to represent the form factor expansion (1.14) in the form (2.1). Determinants of integral operators, which we consider also can be called Fredholm determinants.

The form factors (1.4) are proportional to polynomials $Q_n(s)$, which in turn are equal to determinants of $(n-1)\times(n-1)$ matrices (1.8)

$$Q_n(s) = \mathrm{det}_{n-1} M_{ij}(s). \tag{2.2}$$

The matrix $M_{ij}(s)$ consists of $(n-1)^2$ different functions, depending on the same set of arguments $x_1, \dots, x_n$:

$$M_{ij}(s) = \sigma_{2i-j}[i-j+s], \qquad i,j = 1,\dots,n-1. \tag{2.3}$$

The main goal of this and next section is to transform the matrix (2.3) to such a form, that entries of a new matrix will be parameterized by a single function, depending on different sets of variables, (like $V(x_i, x_j)$ in (2.1))

$$M_{ij} \to \hat{D}_{ij}, \qquad \hat{D}_{ij} = \hat{D}(x_i, x_j). \tag{2.4}$$

Here $\hat{D}(x,y)$ is a function of two arguments. The element $\hat{D}_{ij}$ depends on $i$ and $j$ only by means of its arguments $x_i$ and $x_j$.

First, it is useful to rewrite the representation (2.2) in terms of a determinant of a matrix $n \times n$ . To do this, notice that $\sigma_{2n-j} = 0$, if $j < n$, so $M_{nj} = \delta_{nj}[s]\prod_{m=1}^{n} x_m$. Thus, we obtain

$$Q_n(s) = [s]^{-1}\prod_{m=1}^{n} x_m^{-1}\mathrm{det}_n M_{ij}(s), \qquad i,j = 1,\dots,n. \tag{2.5}$$

The r.h.s. of (2.5) is well defined for all $s \neq 0$ and $n \neq 0$. These two cases should be considered separately. It is easy to see that for $s = 0$ and $n \neq 0$ one have to understand (2.5) as a limit $s \to 0$, because the determinant is proportional to $[s]$. However, for $n = 0$, the original representation (2.2) is not well defined, while it is natural to define the determinant $\mathrm{det}_0 M_{ij} = 1$ in (2.5). So, we obtain $Q_0(s) = [s]^{-1}$ for $s \neq 0$. On the other hand the case $s = 0$ corresponds to the form factor of local field. In this case we have $H_0 = 0$, and the form factor is equal to zero $H_0 Q_0(0) = F_0 = 0$. Thus, we define $Q_0(s) = [s]^{-1}$ for $s \neq 0$. We do not define $Q_0(0)$, but we simply put $F_0 = 0$ for $s = 0$.

In order to study correlation functions we need to find the square of polynomials $Q_n(s)$.

$$Q_n^2(s) = [s]^{-2} \prod_{m=1}^{n} x_m^{-2} \det_n C_{jk}, \tag{2.6}$$

where

$$C_{jk} = (M^T \cdot M)_{jk} = \sum_{i=1}^{n} [i-j+s][i-k+s]\sigma_{2i-j}\sigma_{2i-k}. \tag{2.7}$$

One can calculate the sum in (2.7) using an integral representation for elementary symmetric polynomials

$$\sigma_k = \frac{1}{2\pi i} \oint \frac{dz}{z^{n-k+1}} \prod_{m=1}^{n} (z + x_m). \tag{2.8}$$

Here the integral is taken in positive direction with respect to an arbitrary circle $|z| = \rho$ around the origin. Choosing the radius of the circle $\rho > 1$ and using (A.5) from Appendix A, we find

$$C_{jk} = \frac{1}{(2\pi i)^2} \oint d^2 z \frac{\prod_{m=1}^{n} (z_1 + x_m)(z_2 + x_m)}{(q - q^{-1})^2} z_1^{n-j+1} z_2^{n-k+1}$$

$$\times \left\{ \frac{q^{2n+2+2s-j-k}}{q^2 z_1^2 z_2^2 - 1} + \frac{q^{-2n-2-2s+j+k}}{q^{-2} z_1^2 z_2^2 - 1} - \frac{q^{k-j}}{z_1^2 z_2^2 - 1} - \frac{q^{j-k}}{z_1^2 z_2^2 - 1} \right\}. \tag{2.9}$$

In order to get a common denominator we make replacements of variables in the braces: $z_1 q^{1/2} = w_1$, $z_2 q^{1/2} = w_2$ in the first term; $z_1 q^{-1/2} = w_1$, $z_2 q^{-1/2} = w_2$ in the second term; $z_1 q^{1/2} = w_1$, $z_2 q^{-1/2} = w_2$ in the third term and $z_1 q^{-1/2} = w_1$, $z_2 q^{1/2} = w_2$ in the fourth term. After simple algebra we arrive at

$$C_{jk} = \frac{1}{(2\pi i)^2} \oint d^2 w \frac{w_1^{n-j+1} w_2^{n-k+1}}{w_1^2 w_2^2 - 1} G^{(j)}(w_1) G^{(k)}(w_2), \tag{2.10}$$

where

$$G^{(\ell)}(w) = \frac{1}{q - q^{-1}} \left( q^{s+\frac{n-\ell}{2}} \prod_{m=1}^{n} (w q^{-1/2} + x_m) - q^{-s-\frac{n-\ell}{2}} \prod_{m=1}^{n} (w q^{1/2} + x_m) \right). \tag{2.11}$$

The matrix $C$ still depends on $n^2$ different functions $C_{jk}$. However, this matrix can be transformed to a more convenient form. Let us introduce matrix $A_{jk}$ (it is studied in Appendix B)

$$A_{jk} = \frac{1}{(n-j)!}\,\frac{d^{n-j}}{dx^{n-j}}\prod_{m\neq k}^{n}(x+x_m)\Big|_{x=0}, \tag{2.12}$$

with a determinant

$$\det A = \prod_{a<b}^{n}(x_a - x_b). \tag{2.13}$$

Instead of matrix $C$ it will be convenient to introduce matrix $D$

$$D = A^T C A. \tag{2.14}$$

Determinants of matrices $C$ and $D$ are related by

$$\det_n C = \prod_{a>b}^{n}(x_a - x_b)^{-2}\det_n D, \tag{2.15}$$

The calculation of the explicit expression for matrix $D$ in (2.14) reduces to the summation of the Taylor series (see (B.8)), so we have

$$D_{jk} = \oint d^2w \frac{w_1 w_2}{(2\pi i)^2(w_1^2 w_2^2 - 1)} Y(w_1, x_j) Y(w_2, x_k), \tag{2.16}$$

where

$$Y(w,x) = \frac{J(w)}{q-q^{-1}}\left(\frac{q^s}{wq^{1/2}+x} - \frac{q^{-s}}{wq^{-1/2}+x}\right), \tag{2.17}$$

and

$$J(w) = \prod_{m=1}^{n}(wq^{1/2}+x_m)(wq^{-1/2}+x_m). \tag{2.18}$$

Taking the integral, for instance, with respect to $w_2$ (recall that $|w_1 w_2| > 1$), we have after the symmetrization of the integrand

$$D_{jk} = \frac{1}{8\pi i}\oint \frac{dw}{w}\Big(Y(w,x_j) + Y(-w,x_j)\Big)\Big(Y(w^{-1},x_k) + Y(-w^{-1},x_k)\Big). \tag{2.19}$$

Thus, we obtain a new representation for the square of the polynomial $Q_n(s)$:

$$Q_n^2(s) = \frac{\det_n D}{[s]^2 \prod\limits_{m=1}^{n} x_m^2 \prod\limits_{a>b}^{n}(x_a - x_b)^2}. \tag{2.20}$$

This brings us closer to (2.1).

# 3 Dual fields

The entries of the matrix $D_{jk}$ are parameterized now by a single function $D$ (2.19). However, an element $D_{jk}$, is still not a function of two arguments only, because of the product $J(w) = \prod_{m=1}^{n}(wq^{1/2} + x_m)(wq^{-1/2} + x_m)$. This product depends on all $x_m$. In order to get rid of these products we introduce auxiliary Fock space and auxiliary quantum operators—dual fields. Dual fields are linear combinations of canonical Bose fields, see page 210 of [6] .

Let us define

$$\begin{aligned} \Phi_1(x) &= q_1(x) + p_2(x), \\ \Phi_2(x) &= q_2(x) + p_1(x), \end{aligned} \tag{3.1}$$

where operators $p_j(x)$ and $q_j(x)$ act in the canonical Bose Fock space in a following way

$$(0|q_j(x) = 0, \qquad p_j(x)|0) = 0. \tag{3.2}$$

Non-zero commutation relations are given by

$$[p_1(x), q_1(y)] = [p_2(x), q_2(y)] = \xi(x, y) = \log\Big((x + yq^{1/2})(x + yq^{-1/2})\Big). \tag{3.3}$$

Due to the symmetry of the function $\xi(x, y) = \xi(y, x)$, all fields $\Phi_j(x)$ commute with each other

$$[\Phi_j(x), \Phi_k(y)] = 0, \qquad j, k = 1, 2. \tag{3.4}$$

However, despite of these simple commutation relations, the vacuum expectation value of the dual fields may be non-trivial, for example:

$$(0|\Phi_1(x)\Phi_2(y)|0) = (0|p_2(x)q_2(y)|0) = \xi(x, y). \tag{3.5}$$

It is easy to show that an exponent of dual field acts like a shift operator. Namely, if $f\left(\Phi_1(y)\right)$ is a function of $\Phi_1(y)$ then

$$\begin{aligned} &(0|\prod_{m=1}^{n} e^{\Phi_2(x_m)} f\left(\Phi_1(y)\right)|0) = (0|\prod_{m=1}^{n} e^{p_1(x_m)} f\left(q_1(y)\right)|0) \\ &= f\left(q_1(y) + \sum_{m=1}^{n} \log \xi(x_m, y)\right)|0) = f(\log J(y)). \end{aligned} \tag{3.6}$$

Using this property of dual fields one can remove the products $J(w)$ from the matrix $D_{jk}$.

Let us define

$$\hat{Y}(w,x) = \frac{e^{\Phi_1(w)}}{q-q^{-1}} \left( \frac{q^s}{wq^{1/2}+x} - \frac{q^{-s}}{wq^{-1/2}+x} \right), \tag{3.7}$$

and

$$\hat{D}_{jk} = \frac{1}{8\pi i} \oint \frac{dw}{w} \Big( \hat{Y}(w,x_j) + \hat{Y}(-w,x_j) \Big) \Big( \hat{Y}(w^{-1},x_k) + \hat{Y}(-w^{-1},x_k) \Big). \tag{3.8}$$

Then, due to (3.6), we have

$$\det{}_n D = (0| \prod_{m=1}^{n} e^{\Phi_2(x_m)} \det{}_n \hat{D} |0), \tag{3.9}$$

or

$$\det{}_n D = (0| \det{}_n \left( \hat{D}(x_j,x_k) e^{\frac{1}{2}\Phi_2(x_j)+\frac{1}{2}\Phi_2(x_k)} \right) |0). \tag{3.10}$$

The entries of the matrix $\hat{D}_{jk}$ depend on $x_j$ and $x_k$ only, and they do not depend on other variables $x_m$. Thus, we have presented the square of the polynomial $Q_n(s)$ in terms of a vacuum expectation value of a determinant of a matrix $n \times n$, similar to one of the terms in the r.h.s of (2.1). Entries of matrix $D$ are parameterized by the single two-variable function $\hat{D}(x,y)$. Let us emphasize again that as an operator in the auxiliary Fock space $\hat{D}(x,y)$ belongs to an Abelian sub algebra.

Besides the polynomial $Q_n(s)$ the form factor (1.4) is proportional to a double product $\prod_{a>b}^{n} F_{\min}\left(\beta_{ab}\right)\left(x_a+x_b\right)^{-1}$. In order to transform (1.14) to (2.1) it is necessary to factorize this product. To do this we introduce another dual field

$$\tilde{\Phi}_0(x) = \tilde{q}_0(x) + \tilde{p}_0(x). \tag{3.11}$$

As usual

$$(0|\tilde{q}_0(x) = 0, \qquad \tilde{p}_0(x)|0) = 0. \tag{3.12}$$

Operators $\tilde{q}_0(x)$ and $\tilde{p}_0(y)$ commute with all $p_j$ and $q_j$ $(j=1,2)$. The only non-zero commutation relation is

$$[\tilde{p}_0(x), \tilde{q}_0(y)] = \eta(x,y), \tag{3.13}$$

where

$$\eta(x,y) = \eta(y,x) = 2\log\left|\frac{F_{\min}\left(\log\frac{x}{y}\right)}{x^2-y^2}\right|. \tag{3.14}$$

Here we have used the fact that $|F_{\min}(z)| = |F_{\min}(-z)|$. It is worth mentioning also that the r.h.s. of (3.14) has no singularity at $x = y$, because $F_{\min}(x)$ has the first order zero at $x = 0$ and $F'_{\min}(0) = \left(i\sin\frac{\pi B}{2}F_{\min}(i\pi)\right)^{-1}$ (see [1]). Hence

$$\eta(x,x) = -2\log\left|2x^2\sin\frac{\pi B}{2}F_{\min}(i\pi)\right|. \tag{3.15}$$

Newly introduced dual fields also commute

$$[\tilde{\Phi}_0(x), \tilde{\Phi}_0(y)] = 0 = [\tilde{\Phi}_0(x), \Phi_j(y)]. \tag{3.16}$$

However, due to the Campbell–Hausdorff formula, we have

$$(0|\prod_{m=1}^{n} e^{\tilde{\Phi}_0(x_m)}|0) = \prod_{a,b=1}^{n} e^{\frac{1}{2}\eta(x_a,x_b)} = \lambda^n \prod_{m=1}^{n} x_m^{-2} \prod_{a>b}^{n}\left|\frac{F_{\min}\left(\log\frac{x_a}{x_b}\right)}{x_a+x_b}\right|^2, \tag{3.17}$$

where

$$\lambda = |2\sin\frac{\pi B}{2}F_{\min}(i\pi)|. \tag{3.18}$$

Combining the last formula and the representations (2.20), (3.10) for $Q_n^2(s)$ we find

$$Q_n^2(s)\prod_{a>b}^{n}\left|\frac{F_{\min}\left(\log\frac{x_a}{x_b}\right)}{x_a+x_b}\right|^2 = \frac{\lambda^n}{[s]^2}(0|\det{}_n\hat{V}(x_j,x_k)|0). \tag{3.19}$$

Here

$$\hat{V}(x_j,x_k) = \hat{D}_{jk}e^{\frac{1}{2}\Phi_0(x_j)+\frac{1}{2}\Phi_0(x_k)}, \tag{3.20}$$

and

$$\Phi_0(x) = \tilde{\Phi}_0(x) + \Phi_2(x). \tag{3.21}$$

So we managed to represent a square of an absolute value of the form factor as a determinant, similar to one of the terms in the r.h.s of (2.1). In the next section we shall sum up all contributions of the form factors and obtain a determinant representation for a correlation function.

# 4 The determinant representation for a correlation function

In the previous sections we have obtained the representation for a square of an absolute value of the form factor

$$F_n(\beta_1, \dots, \beta_n) = \langle 0|\mathcal{O}(0,0)|\beta_1, \dots, \beta_n\rangle, \tag{4.1}$$

in terms of a determinant

$$|F_n(\beta_1, \dots, \beta_n)|^2 = \frac{1}{[s]^2}|H_n|^2\lambda^n(0|\det{}_n\hat{V}(x_j, x_k)|0), \tag{4.2}$$

Here constants $H_n$ are equal to

$$H_{2n+1} = H_1\mu^n, \qquad H_{2n} = H_0\mu^n, \qquad \mu = \frac{4\sin\frac{\pi B}{2}}{F_{\min}(i\pi)}, \tag{4.3}$$

and

$$\frac{1}{[s]}H_0 = F_0 = \langle 0|\mathcal{O}(0,0)|0\rangle, \qquad H_1 = F_1 = \langle 0|\mathcal{O}(0,0)|\beta\rangle. \tag{4.4}$$

We have the representation for a correlation function of operators $\mathcal{O}$ in terms of form factors:

$$\langle 0|\mathcal{O}(0,0)\mathcal{O}(x,t)|0\rangle = \sum_{n=0}^{\infty}\int \frac{d^n\beta}{n!(2\pi)^n}|F_n(\beta_1, \dots, \beta_n)|^2 \prod_{j=1}^{n} e^{-\theta(x_j)}, \tag{4.5}$$

where

$$\theta(x) = \frac{mr}{2}(x + x^{-1}). \tag{4.6}$$

Substituting here (4.2)–(4.3) we arrive at a following representation

$$\langle 0|\mathcal{O}(0,0)\mathcal{O}(x,t)|0\rangle = (0|\frac{1}{[s]^2}\left\{\frac{|H_0|^2+|H_1|^2|\mu|^{-1}}{2}\sum_{n=0}^{\infty}\int_0^{\infty}\frac{d^nx}{n!}\left(\frac{|\lambda\mu|}{2\pi}\right)^n\right.$$

$$\times\mathrm{det}_n\left[\frac{\hat{V}(x_j,x_k)}{\sqrt{x_jx_k}}e^{-\frac{1}{2}(\theta(x_j)+\theta(x_k))}\right]$$

$$+\frac{|H_0|^2-|H_1|^2|\mu|^{-1}}{2}\sum_{n=0}^{\infty}\int_0^{\infty}\frac{d^nx}{n!}\left(-\frac{|\lambda\mu|}{2\pi}\right)^n$$

$$\left.\times\mathrm{det}_n\left[\frac{\hat{V}(x_j,x_k)}{\sqrt{x_jx_k}}e^{-\frac{1}{2}(\theta(x_j)+\theta(x_k))}\right]\right\}|0). \tag{4.7}$$

Both of these series have the form (2.1), so they can be summed up and written as determinants of integral operators (Fredholm determinants)

$$\langle 0|\mathcal{O}(0,0)\mathcal{O}(x,t)|0\rangle = (0|\frac{1}{[s]^2}\left\{\frac{|H_0|^2+|H_1|^2|\mu|^{-1}}{2}\det(I+\gamma\hat{U})\right.$$

$$\left.+\frac{|H_0|^2-|H_1|^2|\mu|^{-1}}{2}\det(I-\gamma\hat{U})\right\}|0), \tag{4.8}$$

where

$$\hat{U}(x,y)=\frac{\hat{V}(x,y)}{\sqrt{xy}}e^{-\frac{1}{2}(\theta(x)+\theta(y))}, \tag{4.9}$$

and

$$\gamma=\frac{4}{\pi}\sin^2\frac{\pi B}{2}. \tag{4.10}$$

The determinant representation (4.8) is the main result of the paper, therefore we summarize here the basic definitions.

The integral operators $I\pm\gamma\hat{U}$ act on a trial function $f(x)$ as

$$[(I\pm\gamma\hat{U})f](x)=f(x)\pm\gamma\int_0^{\infty}\hat{U}(x,y)f(y)dy. \tag{4.11}$$

The kernel $\hat{U}(x,y)$ is equal to

$$\hat{U}(x,y)=\frac{\hat{D}(x,y)}{\sqrt{xy}}e^{-\frac{1}{2}(\theta(x)+\theta(y))}e^{\frac{1}{2}(\Phi_0(x)+\Phi_0(y))}, \tag{4.12}$$

where

$$\hat{D}(x,y)=\frac{1}{8\pi i}\oint\frac{dw}{w}\Big(\hat{Y}(w,x)+\hat{Y}(-w,x)\Big)\Big(\hat{Y}(w^{-1},y)+\hat{Y}(-w^{-1},y)\Big),\quad (4.13)$$

and

$$\hat{Y}(w,x)=\frac{e^{\Phi_1(w)}}{q-q^{-1}}\left(\frac{q^s}{wq^{1/2}+x}-\frac{q^{-s}}{wq^{-1/2}+x}\right).\qquad (4.14)$$

The dual fields $\Phi_0(x)$ and $\Phi_1(x)$ were defined in the section 3 (see (3.1) and (3.11)). The main property of these dual fields is, that they commute with each other, so the Fredholm determinants $\det(I\pm\gamma\hat{U})$ are well defined. Certainly $\det(I\pm\gamma\hat{U})$ are operators in auxiliary Fock space, but they belong to the Abelian sub algebra. On the other hand, the vacuum expectation value of these operators is non-trivial. It follows from commutation relations (3.3), (3.13), that in order to calculate the vacuum expectation value, one should use the following prescription

$$(0|\prod_{a=1}^{M_1}e^{\Phi_0(x_a)}\prod_{b=1}^{M_2}e^{\Phi_1(x_b)}|0)=\prod_{a=1}^{M_1}\prod_{b=1}^{M_1}e^{\frac{1}{2}\eta(x_a,x_b)}\prod_{a=1}^{M_1}\prod_{b=1}^{M_2}e^{\xi(x_a,x_b)}.\qquad (4.15)$$

Here

$$\eta(x,y)=2\log\left|\frac{F_{\min}\left(\log\frac{x}{y}\right)}{x^2-y^2}\right|,\qquad (4.16)$$

and

$$\xi(x,y)=\log\Big((x+yq^{1/2})(x+yq^{-1/2})\Big).\qquad (4.17)$$

Recall also, that the determinant representation (4.8) is valid for an arbitrary $s$. If $s=0$, one should understand the r.h.s. as a limit $s\to 0$, taking into account that $H_0=0$.

Similar Fredholm determinant representations were useful not only in the theory of correlation functions [4]–[12], but also in matrix models [13]–[16]. A work on determinant representation for correlation functions led to the discovery of a determinant formula for a partition function of the six-vertex model with domain wall boundary conditions [29]. In the paper [30] it was shown that this partition function satisfies Hirota equation. In the paper [31] it was shown that the determinant formula for the partition function of the six-vertex model helps to solve a long-standing mathematical problem—to prove the alternating sign matrix conjecture.

## 5 Large $r$-asymptotic

In this section we shall demonstrate, how one can find a long distance asymptotic of a correlation function starting from the Fredholm determinant. We shall reproduce some known results.

The kernel of the integral operator $\hat{U}(x,y)$ can be written in the form

$$\hat{U}(x,y) = \oint dw P_1(w,x)P_2(w,y), \tag{5.1}$$

where projectors $P$ are

$$P_1(w,x) = \frac{1}{8\pi i w\sqrt{x}}\Big(\hat{Y}(w,x) + \hat{Y}(-w,x)\Big)e^{\frac{1}{2}\Phi_0(x)-\frac{1}{2}\theta(x)}, \tag{5.2}$$

$$P_2(w,y) = \frac{1}{\sqrt{y}}\Big(\hat{Y}(w^{-1},y) + \hat{Y}(-w^{-1},y)\Big)e^{\frac{1}{2}\Phi_0(y)-\frac{1}{2}\theta(y)}. \tag{5.3}$$

Let us remind here that $w$ integration goes along a contour around zero in positive direction. A radius of the contour should be greater then 1. The Fredholm determinants of the kernels of type (5.1) can be written as determinants of operators acting in the space of variables "$w$"

$$\det(I \pm \gamma\hat{U}(x,y)) = \det(I \pm \gamma\tilde{U}(w_1,w_2)), \tag{5.4}$$

where

$$\tilde{U}(w_1,w_2) = \int_0^\infty dx P_1(w_1,x)P_2(w_2,x). \tag{5.5}$$

The integral operator $\tilde{U}(w_1,w_2)$ acts on a trial function $f(w)$ as

$$[(I+\tilde{U})f](w_1) = f(w_1) + \oint \tilde{U}(w_1,w_2)f(w_2)\,dw_2. \tag{5.6}$$

Consider the case $r\to\infty$. Then the value of the integral in (5.5) can be estimated by means of a steepest descent method. The saddle point of the function $\theta(x)$ is $x = x_0 = 1$. Examination of commutation relations of dual fields (4.15) shows that dual fields can be considered as analytic functions in the vicinity of real axis. Hence, we can estimate the integral in (5.5) as

$$\tilde{U}(w_1,w_2) = P_1(w_1,1)P_2(w_2,1)\left(\sqrt{\frac{2\pi}{mr}} + O(r^{-3/2})\right). \tag{5.7}$$

Thus, for the large $r$ asymptotics the kernel $\tilde{U}(w_1, w_2)$ becomes a one-dimensional projector, and its Fredholm determinant is equal to

$$\det(I \pm \gamma \tilde{U}) \to 1 \pm \gamma \oint dw \tilde{U}(w, w). \tag{5.8}$$

In order to calculate a vacuum expectation value of $\tilde{U}(w, w)$ one can use prescription (4.15), however it is better to write down the dual field $\Phi_0(x)$ in terms of the original fields $\Phi_0(x) = \tilde{\Phi}_0(x) + \Phi_2(x)$. Then the contribution of the fields $\tilde{\Phi}_0(1)$ gives

$$(0|e^{\tilde{\Phi}_0(1)}|0) = e^{\frac{1}{2}\eta(1,1)} = \lambda^{-1}. \tag{5.9}$$

To find a contribution of the fields $\Phi_1(w)$ and $\Phi_2(x)$ we can use (3.6) and (2.18)

$$(0|\tilde{U}(w, w)|0) = \frac{e^{-mr}}{8\pi i \lambda w} \sqrt{\frac{2\pi}{mr}} \Big( Y_1(w, 1) + Y_1(-w, 1) \Big)$$

$$\times \Big( Y_1(w^{-1}, 1) + Y_1(-w^{-1}, 1) \Big), \tag{5.10}$$

where

$$Y_1(w, 1) = \frac{(wq^{1/2} + 1)(wq^{-1/2} + 1)}{q - q^{-1}} \left( \frac{q^s}{wq^{1/2} + 1} - \frac{q^{-s}}{wq^{-1/2} + 1} \right)$$

$$= [s] + [s - 1/2]w. \tag{5.11}$$

After substituting this into (5.8) it becomes clear that only a pole at $w = 0$ contributes into the integral, so we arrive at

$$(0| \oint dw \tilde{U}(w, w)|0) = [s]^2 \lambda^{-1} \sqrt{\frac{2\pi}{mr}} e^{-mr}, \tag{5.12}$$

and hence

$$(0| \det(I \pm \gamma \tilde{U})|0) \to 1 \pm [s]^2 \frac{\gamma}{\lambda} \sqrt{\frac{2\pi}{mr}} e^{-mr}. \tag{5.13}$$

Finally, substituting this into (4.8), and using explicit expressions for $\lambda$ (3.18), $\mu$ (4.3) and $\gamma$ (4.10) we obtain the correct asymptotical expression

$$\langle 0|\mathcal{O}(0,0)\mathcal{O}(x,t)|0\rangle \to \frac{|H_0|^2}{[s^2]} + |H_1|^2(2\pi mr)^{-1/2}e^{-mr}. \tag{5.14}$$

Recall, that for the correlation function of local fields one should put $H_0 = 0 = H_0/[s]$, therefore we see, that asymptotic formula (5.14) is well defined for arbitrary $s$. If $H_1 = 0$ (for the stress-energy tensor), then (5.14) gives a constant for an asymptotic. However, in this case one had to estimate the kernel $\tilde{U}(w_1, w_2)$ more accurately. Namely, one should take into account explicit expression for corrections of order $r^{-3/2}$ in (5.7). In this case the kernel $\tilde{U}$ turns into two-dimensional projector and it is easy to show, that exponentially decreasing term behaves like $\exp(-2mr)$.

# Summary

We was able to sum up contributions of all the form factors and to obtain the closed expression for correlation functions (4.8). We believe that it will be possible to do not only for Toda models [27] but also for all models of integrable massive relativistic field theory.

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# A Elementary symmetric polynomials

Consider an integral representation (2.8) for the elementary symmetric polynomials

$$\sigma_k^{(n)}(x_1, \ldots, x_n) \equiv \sigma_k = \frac{1}{2\pi i} \oint_{|z|=\rho} \frac{dz}{z^{n-k+1}} \prod_{m=1}^{n} (z + x_m), \tag{A.1}$$

where $\rho$ is arbitrary positive. Notice, that the representation (A.1) holds for arbitrary integer $k$ including $k < 0$ and $k > n$.

Here we are deriving auxiliary formula, which is used in the section 2. Namely, let us consider a sum

$$T_{jk}(\alpha) = \sum_{i=1}^{n} \alpha^{2i} \sigma_{2i-j} \sigma_{2i-k}, \qquad j, k = 1, \ldots, n, \tag{A.2}$$

where $\alpha$ is an arbitrary complex number. One can extend the summation in (A.2) from $-\infty$ to $n$. Then we have

$$T_{jk}(\alpha) = \sum_{i=-\infty}^{n} \alpha^{2i} \sigma_{2i-j} \sigma_{2i-k} = \sum_{l=0}^{\infty} \alpha^{2(n-l)} \sigma_{2n-2l-j} \sigma_{2n-2l-k}. \tag{A.3}$$

Using the integral representation (A.1) we find

$$T_{jk}(\alpha) = \frac{1}{(2\pi i)^2} \sum_{l=0}^{\infty} \oint \frac{d^2 z \alpha^{2n-2l}}{z_1^{2l+j-n+1} z_2^{2l+k-n+1}} \prod_{m=1}^{n} (z_1 + x_m)(z_2 + x_m). \tag{A.4}$$

We can choose the integration contour in such a way that $|\alpha z_1 z_2| > 1$ at the contour. Then one can sum up the series with respect to $l$:

$$T_{jk}(\alpha) = \frac{\alpha^{2n+2}}{(2\pi i)^2} \oint d^2 z \frac{z_1^{n-j+1} z_2^{n-k+1}}{\alpha^2 z_1^2 z_2^2 - 1} \prod_{m=1}^{n} (z_1 + x_m)(z_2 + x_m). \tag{A.5}$$

The integrand contains only two simple poles $\alpha z_1 z_2 = \pm 1$, therefore one can take the integral with respect to $z_1$ or $z_2$ and obtain a single integral expression for $T_{jk}$.

## B Properties of the Vandermonde matrix

Consider a Vandermonde matrix $W_{jk}$:

$$W_{jk} = z_j^{k-1}, \qquad j,k = 1,\dots,n, \tag{B.1}$$

with

$$\det_n(W_{jk}) = \prod_{a>b}^{n} (z_a - z_b). \tag{B.2}$$

The inverse matrix $W^{-1}$ can be written in a form

$$(W^{-1})_{jk} = \frac{1}{(j-1)!} \frac{d^{j-1}}{dx^{j-1}} \prod_{m\neq k}^{n} \left( \frac{x - z_m}{z_k - z_m} \right) \Bigg|_{x=0}. \tag{B.3}$$

Indeed

$$\sum_{l=1}^{n} W_{jl}(W^{-1})_{lk} = \sum_{l=0}^{n-1} \frac{z_j^l}{l!} \frac{d^l}{dx^l} \prod_{m\neq k}^{n} \left( \frac{x - z_m}{z_k - z_m} \right) \Bigg|_{x=0} = \prod_{m\neq k}^{n} \left( \frac{z_j - z_m}{z_k - z_m} \right) = \delta_{jk}. \tag{B.4}$$

Here we have used the fact, that the r.h.s. of (B.4) is a Tailor series for the polynomial of the $(n-1)$ degree $\prod_{m\neq k}^{n}(x - z_m)(z_k - z_m)^{-1}$.

In the section 2 we used the matrix $A_{jk}$:

$$A_{jk} = \frac{1}{(n-j)!} \frac{d^{n-j}}{dx^{n-j}} \prod_{m\neq k}^{n} (x + x_m) \Bigg|_{x=0}. \tag{B.5}$$

The determinant of this matrix is equal to

$$\det_n A = \prod_{a\neq b}^{n} (x_a - x_b) \det \left[ A_{jk} \prod_{m\neq k} (x_m - x_k)^{-1} \right]. \tag{B.6}$$

It is easy to see that the matrix in the r.h.s. of (B.6) coincide with inverse Vandermonde matrix $W^{-1}$ up to replacement $x_m = -z_m$ and a permutation of rows. Thus, we obtain

$$\det_n A = \prod_{a<b}^{n} (x_a - x_b). \tag{B.7}$$

A calculation of products of the matrix $A$ and matrices, containing powers of some complex numbers $w$, is simple. For example, deriving (2.16) we used

$$\sum_{l=1}^{n} \frac{1}{(n-l)!} \frac{d^{n-l}}{dx^{n-l}} \prod_{m \neq j}^{n} (x+x_m) \Big|_{x=0} \cdot w_1^{n-l} G^{(l)}(w_1)$$

$$= \frac{q^s}{q-q^{-1}} \prod_{m=1}^{n} (w_1 q^{-1/2} + x_m) \sum_{l=1}^{n} \frac{(w_1 q^{1/2})^{n-l}}{(n-l)!} \frac{d^{n-l}}{dx^{n-l}} \prod_{m \neq j}^{n} (x+x_m) \Big|_{x=0}$$

$$- \frac{q^{-s}}{q-q^{-1}} \prod_{m=1}^{n} (w_1 q^{1/2} + x_m) \sum_{l=1}^{n} \frac{(w_1 q^{-1/2})^{n-l}}{(n-l)!} \frac{d^{n-l}}{dx^{n-l}} \prod_{m \neq j}^{n} (x+x_m) \Big|_{x=0}$$

$$= \frac{\prod\limits_{m=1}^{n} (w_1 q^{1/2} + x_m)(w_1 q^{-1/2} + x_m)}{q-q^{-1}} \left[ \frac{q^s}{w_1 q^{1/2} + x_j} - \frac{q^{-s}}{w_1 q^{-1/2} + x_j} \right]. \quad \text{(B.8)}$$

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