Fundamental solutions of gauge theories

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Summary. We discuss the physical meaning of a recent differential geometric result, also presented here, stating that gauge fields whose symmetry is essentially different from that of nearby (non gauge-equivalent) fields are critical points for *any* gauge invariant action functional, and are the only fields with this property.

Introduction.

A gauge theory is defined by a gauge-invariant lagrangian L, identified by a density $\mathcal{L}$ to be integrated over a spacetime domain D,

$$L = \int_D \mathcal{L}[A;x]\,\mathrm{d}^n x \; ; \tag{1}$$

this is a functional defined over the space $\mathcal{A}$ of gauge fields $A \equiv A_\mu$ ($\mu = 1,...,n$), depending also on derivatives of the A_μ. The $A_\mu(x)$ are fields taking values in the Lie algebra $\mathcal{G}$ of a Lie group G. We denote by Γ the corresponding gauge group.

In mathematical terms, the gauge fields corresponds to the components of a connection form $\alpha = A_\mu\,\mathrm{d}x^\mu$, and are sections of a bundle associated to a principal G-fiber bundle P over D. The gauge group Γ corresponds to the space of sections of P [1-3].

The space $\mathcal{A}$ is not the space of all possible gauge fields, but should be taken to be that corresponding to fields of *finite energy* (this restriction is analogous e.g. to that to the space of square-integrable functions in quantum mechanics). In mathematical terms, this means selecting a Sobolev space of connections [4,5]. The same applies to Γ, i.e. this as well will be a Sobolev space of gauge transformations (sections).

Thus, the functional L is defined on such a space $\mathcal{A}$. However, it is by now customary to consider not the full space $\mathcal{A}$ (of finite energy gauge fields), but only an open and dense set in it, corresponding to what in mathematical terms are *irreducible connections*; in physical terms, we can think of these as follows. After a gauge fixing, a generic gauge field will not have any residual invariance; however, special fields can still be invariant under a compact group $\Gamma_0 \approx G_0 \subseteq G$: these correspond to reducible connections, while the generic ones correspond to irreducible connections. We will correspondingly speak of reducible and irreducible gauge fields. Notice that these fields should be thought of without any gauge fixing.

As the space $\mathcal{A}^*$ of irreducible gauge fields is open and dense in $\mathcal{A}$, by restricting to it one overlooks a set of measure zero; moreover, the topology of $\mathcal{A}^*$ conveys information on the space $\mathcal{A}_0 = \mathcal{A}\backslash\mathcal{A}^*$ (of reducible gauge fields). The space $\mathcal{A}^*$ also has a number of properties – essentially due to the fact that all gauge fields in it are equivalent under symmetry considerations – that make its study technically easier.

All these reasons account for the usual choice to focus attention on irreducible gauge fields. The purpose of this note is to point out, on the basis of recent relatively abstract and general mathematical results [6], that this is not necessarily the best way of attacking gauge theories and Yang-Mills equations, as relevant Physics lies precisely in the "singular part" $\mathcal{A}_0$ of the space $\mathcal{A}$, and actually in the "most singular" (in a way to be explained below) part of $\mathcal{A}_0$.

The abstract mathematical results mentioned above have actually a physical origin, as they represents a generalization to gauge theories of a classical result by L. Michel [7] on spontaneous symmetry breaking in theories described by a G-invariant potential [G a semisimple compact Lie group, e.g. $SU(n)$] defined on a finite dimensional manifold M [e.g. R^m, or the Lie algebra $\mathcal{G}$ of G itself, $M = su(n)$, with the adjoint action]. This theorem, in turns, was motivated by the study of the $SU(3)$ symmetry in hadronic interaction and provided an explanation in general term for the $SU(3)$ octet [8].

However, these results are so far available only in a mathematical language, as they rely heavily on the differential geometry of fiber bundles and on the (geometric) Michel theory; such an heavily mathematical language is probably an hindrance for the understanding of their physical content and for physical applications.

We want here to give an exposition of such results in physical terms, and to discuss its relevance for Physics.

Reducible and irreducible gauge fields.

A gauge transformation $\gamma \in \Gamma$ can be written in local coordinates as $\gamma = g(x) \in G$. If we apply this on the gauge fields $A(x)$, we obtain new gauge fields $B(x) = (\gamma A)(x)$, given explicitely by

$$B_\mu(x) \;=\; g(x)\, A_\mu(x)\, g^{-1}(x) \;-\; (\partial_\mu g)(x)\, g^{-1}(x) \; . \tag{2}$$

We denote the gauge isotropy subgroup of $A(x)$ as Γ_A; it is known [9] that for any A this is equivalent to a compact subgroup of G, but the equivalence depends on A itself. In slightly more precise terms, Γ_A is given by the γ which are constant under the covariant derivative ∇, where

$$\nabla_\mu \;=\; \partial_\mu \,+\, A_\mu \; , \tag{3}$$

and such that at a reference point $x_0 \in D$,

$$g(x_0) \in G_A := C_G[H_A(x_0)] \; ; \tag{4}$$

here $C_G[H]$ is the centralizer of $H \subseteq G$ in G, and $H_A(x_0)$ is the *holonomy group* under ∇ computed in x_0 [1-3].

As already mentioned, if $G_A = \{e\}$ we say that the gauge field A is irreducible, otherways A will be reducible. Generic fields are irreducible, so that reducible ones are "special" and correspond to a degeneracy.

Critical points.

The physical fields corresponds to critical points for the functional L. We want to argue here that there will be fields which are critical for *any* L with the prescribed gauge invariance under Γ, and that these will actually be the "most degenerate" fields.

We will not attempt to give a proof of the result, referring instead to [6] for a complete, detailed and rigorous proof. A discussion of the significance of this result in geometric terms is given in [10].

First of all, as L is gauge invariant, critical fields will come in gauge orbits; thus we will speak of a critical orbit for L if it is a gauge orbit of critical fields. If a gauge orbit $\omega = \Gamma(A)$ is critical for *any* functional L which is gauge invariant under Γ, we say that it is Γ-*critical.*

Now, to any A we associate an isotropy subgroup Γ_A; fields which have isotropy subgroups conjugated in Γ (i.e. such that $\Gamma_B = \gamma \Gamma_A \gamma^{-1}$ for some $\gamma \in \Gamma$) should be seen as having an equivalent symmetry; thus we denote the conjugacy class of Γ_A in Γ as $[A]$, and call it the *isotropy type* of A.

The set of all the gauge fields having the same isotropy type as A will be denoted by $\Sigma(A)$; notice that obviously any fields B obtained from A via a gauge transformation ($B = \gamma A$) has obviously the same isotropy type as A. Thus we can speak of the isotropy type $[\omega]$ of a gauge orbit ω; obviously, if $A \in \omega$ we have $[A] = [\omega]$.

The set $\Sigma(A)$ – which is a smooth manifold, and actually a fiber bundle [11] – is also called the *stratum* of A.

The topology (essentially, the distance) in $\mathcal{A}$ induces a topology (essentially, a distance) in the $\Sigma(A)$ [6], and we can thus speak of an orbit being *isolated in its stratum.* This means that if we take a sufficiently small neighbourhood $\mathcal{U}$ of the gauge orbit of A in $\mathcal{A}$, all the fields $B \neq A$ in $\mathcal{U}$ have isotropy type different from that of A, $[B] \neq [A]$.

We have then the following result, which generalizes the theorem of L. Michel [7,12]:

Theorem. [6] *A Γ gauge orbit ω is Γ-critical if and only if it is isolated in its stratum.*

Thus the fields which are critical for all gauge invariant functionals are those which are "maximally degenerate": all nearby fields which are not obtained simply by a gauge transformation, have a substantially different (actually, smaller [6]) symmetry.

This is in a way natural: it is clear that the property of being critical for any L – i.e. a solution for any Γ gauge theory – is extremely special, and thus should correspond to very special fields; this theorem tells precisely in what this "special" character lies.

Let us add some comments, also to avoid possible misunderstandings. First of all, when we speak of "any" gauge invariant, it should be understood that the gauge group Γ (and thus G) is fixed, as well as the space domain D (and thus its dimension n). We should also stress that here we have not restricted consideration to Yang-Mills theories, i.e. we have not fixed the form nor the order (order of the maximal derivative of A on which it depends) of the functional L.

It should also be stressed that the theorem only deals with critical points, but is not able to distinguish between stable (minima) and unstable (maxima or saddle points) ones.

This result can be extended to include dependence on parameters to describe successive symmetry breakings and phase transitions [6,10].

It can also be extended to include matter fields ϕ^a (in mathematical terms, these are sections of a vector bundle over the spacetime D); the way to obtain such an extension goes through the fact that for a gauge-matter field configuration (A, ϕ), the gauge isotropy subgroup is just

$$\Gamma_{(A,\phi)} \;=\; \Gamma_A \cap \Gamma_\phi \ , \tag{5}$$

so that we can first use this result on the pure gauge sector, and once a Γ-critical gauge orbit ω is determined, we can consider matter fields with isotropy $\Gamma_\phi \subseteq \Gamma_A$ for $A \in \omega$; see [6,10] for detail.

Physical significance.

After describing this recent abstract result, we should now discuss its meaning and significance for Physics.

First of all, the reader could wonder if there really are gauge fields (orbits) like those considered in the theorem, i.e. which are critical for any gauge invariant L.

As discussed in detail in [6], one can easily find explicit (and physically meaningful) examples of them e.g. by considering $L[A,\phi]$ in the form

$$L \; = \; L_g + L_p + L_{ym} \tag{6}$$

and looking for fields which are critical for each of the three parts separately.

In (5) the gradient, potential, and Yang-Mills part of the lagrangian L are, as usual, given by

$$\begin{aligned} L_g = & \quad \int_D |\nabla_\mu \phi^a(x)|^2 \, \mathrm{d}x^n \\ L_p = & \quad \int_D V[\phi(x)] \, \mathrm{d}x^n \\ L_{ym} = & \quad \int_D F^{\mu\nu} F_{\mu\nu} \, \mathrm{d}x^n \ ; \end{aligned} \tag{7}$$

here $\phi(x) \in C^k$ represents matter fields, $V : C^k \to R$ is a potential, ∇_μ is as in (3), $F_{\mu\nu} = [\nabla_\mu, \nabla_\nu]$, indices are raised and lowered with the $g_{\mu\nu}$ metric on D, and the summation convention is implicit.

For $G = SU(3)$ this class of examples contains, in particular, the case of $A = 0$ and constant matter fields $\phi(x)$ corresponding to the critical directions discussed by Michel and Radicati, i.e. to the particles of the octet. This shows that the theorem is far from being empty, and encompasses physical relevant cases.

Thus we see that there can be fields which are solutions to any gauge theory (with given gauge group); it is natural to see these as *the most fundamental solutions of a gauge theory*, as they are implied by the very symmetry properties of the theory alone, and are there idependently of the details of L, i.e. of the specific model under study.

Now, gauge fields whose gauge orbits is isolated in its stratum cannot belong to the set of irreducible ones (as this set is open and dense). Thus, these fundamental solutions will necessarily correspond to reducible fields.

This does not mean, of course, that the usual approach based on the study of the set $\mathcal{A}^*$ (of irreducible connections) is wrong, nor that it cannot cope with the problems considered here: indeed, as mentioned in the introduction, one can obtain informations on the structure of the set $\mathcal{A}_0$ of reducible fields from the structure of $\mathcal{A}^*$.

On the other side, such an approach is probably not the most convenient one if we are looking for the fundamental solutions identified above, as these are in a sense the most different from the generic ones, i.e. from those in $\mathcal{A}^*$: it seems much more convenient to start from consideration of maximally symmetric nontrivial fields, i.e. fields A such that $\Gamma_A \approx G$, and then – if these do not exist or correspond to unstable solutions – pass to consider A such that $\Gamma_A \approx G_1$ is a maximal isotropy subgroup [12] of G, and so on, reducing at each step the symmetry.

The proposal of such a procedure requires maybe some further explanation. Let us first look for maximally symmetric fields $A(x)$ and gauge orbits ω; suppose we have found say only one maximally symmetric gauge orbit ω, which hence is isolated in its stratum, but that we have determined this is an unstable critical point, and thus not interesting for our purposes. We can then eliminate it from $\mathcal{A}$, and consider $\mathcal{A} \backslash \omega$. As there are no other orbits with $[\omega] = G$, we can then pass to consider gauge fields and gauge orbits with $[\omega] = G_i$, where G_i are the possible maximal isotropy subgroups of G, and repeat the procedure.

It should be stressed that the approach based on highly symmetric *ansatzes* has always been used to determine solutions for gauge theories; however, we pointed out here that these highly symmetric solutions are not only the easiest to determine, but also have a fundamental physical (and mathematical) meaning, as they are determined by the symmetry requirements alone.

Similarly, the fact that sufficiently symmetric (e.g. fully symmetric) points are critical for any invariant potential has been known since the work of Michel [7], but the fact that this applies also to full gauge theories has only been proved in [6]

The same theory developed to obtain the theorem quoted above, also leads to consider the restriction L_0 of L to gauge fields of given symmetry $\Gamma_0 \subseteq \Gamma$; the fact that critical points of L_0 will also be critical for the full L is not so obvious (indeed it could fail for general – but luckily unphysical, at least in this context – cases [13,14]), but can be proved to be true under suitable conditions [13,6]. Thus, in looking for critical gauge fields with given symmetry, we can actually use a simplified analysis, as we can use a functional defined on a much smaller space of possible gauge field configurations; this should be seen as an additional and not irrelevant advantage of the approach proposed here.

We would also like to point out some other features of the result quoted here and of the approach based on it we propose for the analysis of gauge theories. Our theorem, and in general the "Michel-gauge" theory of symmetry breaking, does not postulate the specific form (5) of the gauge invariant functional, nor requires it to be first-order; i.e. is valid beyond the realm of proper Yang-Mills theories. This is a strength of the theory, as it is more general; but it is also a weakness, as it fails to use specific features of the Yang-Mills functional.

Also, in this way we can determine critical points which are not necessarily stable (for physically relevant cases, general theorems [15,16] ensure these will correspond to self-dual or anti-self-dual fields). Again, this is a strength as it can deal with more general critical points, but a weakness as it does not use the special features of stable critical points, which are most relevant physically; notice however that this approach can also be applied directly to self-dual or anti-self-dual Yang-Mills functionals.

However, in this context it should be recalled that saddle points are also relevant physically, e.g. when using a WKB (semiclassical) approximation or thanks to localization-like properties.

Our results, and all the considerations so far, only dealt with classical gauge theories; from a physical point of view, however, we are compelled to think of their quantized version. In the quantum theory, we should perform a functional integration

$$S \ = \ \int_{\mathcal{A}} e^{i\mathcal{L}[A]} \, \mathcal{D}A \tag{8}$$

over all field configurations, and we know that the set $\mathcal{A}_0$ of reducible fields has measure zero; thus, it could naively seem that restricting the functional integral to the set of irreducible fields $\mathcal{A}_* \subset \mathcal{A}$ will not cause essential harm. This is of course not true, because of the nontrivial topology of $\mathcal{A}_*$; taking it into account can be a highly nontrivial task.

Our result shows that there is also another reason for which restriction to $\mathcal{A}^*$ is an unconvenient way of dealing with the functional integration required here: generically (i.e. for generic gauge invariant functionals) the critical points of L will lie precisely in the part A_0 which lies outside A^*; thus, the stationary points for $e^{iL[A]}$ are in $A_0 = \partial\mathcal{A}^*$.

This means that in looking for approximate versions of the functional integral (7), we should rather integrate over a neighbourhood of the set $\mathcal{A}_0$; the simplest domain to be considered for a generic (in the same sense as above) gauge theory would thus be a neighbourhood $\mathcal{U}_0$ of the gauge orbits which are isolated in their stratum (i.e. of the fundamental critical points).

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