Quantum Invariants*

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Abstract

Consider the partition function $\mathfrak{Z}^Q(a,g) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{Tr}_{\mathcal{H}}(\gamma U(g) a e^{-Q^2+itda}) e^{-t^2} dt$. In this paper we give an elementary proof that this is an invariant. This is what we mean: assume that Q is a self-adjoint operator acting on a Hilbert space $\mathcal{H}$, and that the operator Q is odd with respect to a $\mathbb{Z}_2$-grading γ of $\mathcal{H}$. Assume that a is an operator that is even with respect to γ and whose square equals I. Suppose further that e^{-Q^2} has a finite trace, and that $U(g)$ is a unitary group representation that commutes with γ, with Q, and with a. Define the differential $da = [Q, a]$. Then $\mathfrak{Z}^Q(a,g)$ is an invariant in the following sense: if the operator $Q(\lambda)$ depends differentiably on a parameter λ, and if $d_\lambda a$ satisfies a suitable bound, (we specify these regularity conditions completely in §XI) then $\mathfrak{Z}^{Q(\lambda)}(a,g)$ is independent of λ. Once we have set up the proper framework, a short calculation in §IX shows that $\partial \mathfrak{Z}^{Q(\lambda)}/\partial\lambda = 0$. These considerations apply to non-commutative geometry, to super-symmetric quantum theory, to string theory, and to generalizations of these theories to underlying quantum spaces.

I Introduction

In an earlier paper [QHA] we have studied a class of geometric invariants that arise within the framework of differential geometry and its non-commutative generalization [C1, C2, JLO]. By pairing a cocycle τ with an operator-valued, even, square-root of unity a, we obtained a specific formula for an invariant $\mathfrak{Z}^Q(a,g) = \left\langle \tau^{\mathrm{JLO}}, a \right\rangle$. In case that τ is the JLO-cochain [JLO], this invariant has the numerical value $\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{Tr}_{\mathcal{H}}(\gamma U(g) a e^{-Q^2+itda}) e^{-t^2} dt$, see [QHA]. Here Q is a self adjoint operator and Hilbert space $\mathcal{H}$, and $da = [Q, a]$. We assume that the differential Q is odd with respect to the $\mathbb{Z}_2$-grading γ, while a is even. We also assume that g is an element of a group G of symmetries of Q and of a. The invariant $\mathfrak{Z}$ is not necessarily integer, but it is an integer-valued when g equals the identity element.

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In this note we present an alternative point of view to [QHA]. Rather than relating $\mathfrak{Z}$ to a general theory of invariants, to entire cyclic cohomology, and to K-theory, we start from the formula for $\mathfrak{Z}^{Q(\lambda)}(a,g)$ above and ask the basic question: can one see directly, when $Q = Q(\lambda)$ depends on a parameter λ, that $\mathfrak{Z}^{Q(\lambda)}(a,g)$ is actually independent of λ?

We answer this question affirmatively, by studying a new auxilliary Hilbert space $\hat{\mathcal{H}}$ containing $\mathcal{H}$, and defining a new representation for $\mathfrak{Z}$ as an expectation $\mathfrak{J}(\lambda,a,g)$ on $\hat{\mathcal{H}}$. We replace the operator $Q(\lambda)$ on $\mathcal{H}$ with the operator $q(\lambda,a) = Q(\lambda) + \eta a$ on $\hat{\mathcal{H}}$. We call q the *extended supercharge.* Here $\hat{\mathcal{H}}$ differs from $\mathcal{H}$ by also containing the additional independent fermionic coordinate η chosen so that $\eta^2 = I$ and $\eta Q(\lambda) + Q(\lambda)\eta = 0$. We may interpret ηa as a connection associated with the translation in the auxiliary direction t, paired with η. In §VIII we define an expectation on $\hat{\mathcal{H}}$, namely

$$\mathfrak{J}(\lambda,a,g) = \langle\langle Ja\rangle\rangle \ , \tag{I.1}$$

where the notation is explained in (VIII.1–2). We also show that $\mathfrak{J}(\lambda,a,g)$ and $\mathfrak{Z}^{Q(\lambda)}(a,g)$ agree.

Once we have defined the proper framework, we show with a short calculation in §IX that $\mathfrak{Z}$ is constant. We present the algebraic aspects of the proof (because these are new) without any analytic details. Of course the analytic details are absolutely crucial. Taking $\lambda \in [0,1]$, we could set $Q(\lambda) = \lambda Q + (1-\lambda)\tilde{Q}$ and interpolate between any Q and any other $\tilde{Q}$, as long as they both commute with the symmetry group $U(g)$. Thus were algrbraic arguments presented here valid without any further assumptions, we would be in the unfortunate position of showing that all invariants for a given a agree!

Though the analytic part of the argument is crucial, it also remains identical to our paper [QHA]. In that other work, we formulate precisely two regularity conditions: first the regularity of $Q(\lambda)$ with respect to λ, and secondly the regularity of a with respect to $Q(\lambda)$. We call the latter the fractional differentiability properties of a. The conditions we give are useful because they are easy to verify in a large set of examples.

Under these regularity conditions, $\mathfrak{Z}^{Q(\lambda)}(a,g)$ is once-differentiable in λ. Furthermore, the resulting λ-derivative of $\mathfrak{Z}$ equals the expression that we would obtain by interchanging the order of differentiating and the order of taking traces or integrals in the definition of $\mathfrak{Z}$. The analysis to establish these facts is lengthy, but can be taken over directly from [QHA]. For the convenience of the reader, in §XI, we summarize the analytic hypotheses used in argument.

In §X we consider a different but related case with two differentials Q_1 and Q_2, but where only Q_1 is invariant under the symmetry group G. Instead we assume that $Q_1^2 + Q_2^2$ is also invariant, and that $Q_1^2 - Q_2^2$ commutes with all relevant operators. We show in this case that an expectation (X.4) has a reprsentation similar to (I.1) and also is an invariant with respect to λ.

II The Supercharge

Our basic framework involves an odd, self adjoint operator Q on a $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H}$. This means we have a self-adjoint operator γ on $\mathcal{H}$ for which $\gamma^2 = I$. Thus $\mathcal{H}$ splits into the direct sum $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ of eigenspaces of γ. The statement that Q is odd means $Q\gamma + \gamma Q = 0$. In terms of the direct sum decomposition,

$$Q = \begin{pmatrix} 0 & Q_+^* \\ Q_+ & 0 \end{pmatrix} \text{ and } \gamma = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} . \tag{II.1}$$

The opeartor Q is the supercharge[1] and its square

$$H = Q^2 = \begin{pmatrix} Q_+^* Q_+ & 0 \\ 0 & Q_+ Q_+^* \end{pmatrix} \tag{II.2}$$

will be referred to as the Hamiltonian. We let $x^\gamma = \gamma x \gamma$ denote the action of γ on operators. We say that the operator x is even (bosonic) if $x = x^\gamma$ and odd (fermionic) if $x^\gamma = -x$. We define the graded differential

$$dx = Qx - x^\gamma Q . \tag{II.3}$$

We suppose that there is a compact Lie group G with a continuous unitary representation $U(g)$ on $\mathcal{H}$ such that

$$U(g)\gamma = \gamma U(g) , \qquad \text{and} \qquad U(g)Q = QU(g) . \tag{II.4}$$

Denote the action of $U(g)$ on the operator x by

$$x \to x^g = U(g) x U(g)^{-1} \tag{II.5}$$

III The Observables

We also consider an algebra of bounded operators $\mathfrak{A}$ on $\mathcal{H}$ with the properties that each $a \in \mathfrak{A}$ is even and invariant. In other words, each $a \in \mathfrak{A}$ commutes with γ and with $U(g)$ for all $g \in G$. We also consider $\mathrm{Mat}_n(\mathfrak{A})$, the set of $n \times n$ matrices with matrix elements (or entries) in $\mathfrak{A}$. If $a, b \in \mathrm{Mat}_n(\mathfrak{A})$ are matrices with entries $a_{ij}, b_{ij} \in \mathfrak{A}$, we use the shorthand ab to denote the matrix with entries $\sum_{k=1}^n a_{ik} b_{kj} \in \mathfrak{A}$.

The differential of elements of $\mathfrak{A}$ is

$$da = Qa - aQ = [Q, a] \tag{III.1}$$

is always defined as a quadratic form on $\mathcal{H}$. We make precise the nature of the differentiabiliy of $\mathfrak{A}$, in §XI. The operators $\mathfrak{A}$ and the derivative d are the fundamental building blocks of non-commutative geometry; here $\mathfrak{A}$ generalizes the notion of functions on a manifold M, and Q generalizes a Dirac operator on a bundle over M.

[1] We are not concerned with the basic structure of $\mathcal{H}$ or Q, aside from the possibility to perform the construction in §V.

IV The Invariant $\mathfrak{Z}^{Q(\lambda)}(a,g)$

In [QHA] we gave a simple formula for an invariant. Let $Q(\lambda)$ depend on a real parameter λ. We denote the graded commutator (II.3) of $Q(\lambda)$ with x by

$$d_\lambda x = Q(\lambda)x - x^\gamma Q(\lambda)\ . \tag{IV.1}$$

For $a \in \mathfrak{A}$ define

$$\mathfrak{Z}^{Q(\lambda)}(a,g) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{Tr}_{\mathcal{H}} \left(\gamma U(g) a e^{-Q(\lambda)^2 + itd_\lambda a} \right) e^{-t^2} dt\ . \tag{IV.2}$$

More generally, we let $a \in \mathrm{Mat}_n(\mathfrak{A})$. In this case

$$\mathfrak{Z}^{Q(\lambda)}(a,g) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{Tr}_{\mathcal{H}\otimes\mathbb{C}^{n^2}} \left(\gamma U(g) a e^{-Q(\lambda)^2 + itd_\lambda a} \right) e^{-t^2} dt\ , \tag{IV.3}$$

where we extend $\gamma, U(g), Q(\lambda)^2$ to diagonal $n \times n$ matrices.

Theorem I. *For $a \in \mathfrak{A}$, assume $a^2 = I$. Furthermore assume that $Q = Q(\lambda)$ and $d_\lambda a = [Q(\lambda), a]$ satisfy the regularity hypotheses given in §XI. Then $\mathfrak{Z}^{Q(\lambda)}(a,g)$ is independent of λ.*

The main point of this paper is to present a new, elementary proof of Theorem I.

V The Extended Supercharge q

In order to exhibit our proof, we introduce a new Hilbert space $\hat{\mathcal{H}}$ on which the opeartors $Q, \gamma, \mathfrak{A}$, and $U(g)$ also act. In addition, on $\hat{\mathcal{H}}$ there are two additional self adjoint operators η and J, both of which have square one,

$$\eta^2 = J^2 = I\ , \tag{V.1}$$

and such that

$$[\eta, x] = [J, x] = 0 \ \text{ for } \ x = \gamma, a \in \mathfrak{A}, \ \text{ or } \ U(g)\ , \tag{V.2}$$

and also

$$\eta J + J\eta = \eta Q + Q\eta = [J, Q] = 0\ . \tag{V.3}$$

Let $\Gamma = \gamma J$ denote a $\mathbb{Z}_2$-grading on $\hat{\mathcal{H}}$, and for x acting on $\hat{\mathcal{H}}$ let

$$x^\Gamma = \Gamma x \Gamma\ . \tag{V.4}$$

The operator η is our auxillary fermionic coordinate, and $J = (-I)^{N_\eta}$ is the corresponding $\mathbb{Z}_2$ grading. [2]

[2] Suppose that $\mathcal{H} = \mathcal{H}_b \otimes \mathcal{H}_f$ is a tensor product of bosonic and fermionic Fock spaces, that Q is linear in fermionic creation or annihilation operators, and that $\gamma = (-I)^{N_f}$. This would be standard in the physics of supersymmetry. Suppose in addition that $\eta = b + b^*$ denotes one fermionic degree of freedom independent of those in $\mathcal{H}_f$ and acting on the two-dimensional space $\mathcal{H}_\eta$. Then take $\hat{\mathcal{H}} = \mathcal{H}_b \otimes (\mathcal{H}_f \wedge \mathcal{H}_\eta)$ and $J = (-I)^{b^* b}$, with Q, γ, a, and $U(g)$ acting on $\hat{\mathcal{H}}$ in the natural way. This gives a realization of (V.1–3) on $\hat{\mathcal{H}}$.

Given $a \in \mathfrak{A}$, define the extended supercharge $q = q(\lambda, a)$ by

$$q = q(\lambda, a) = Q(\lambda) + \eta a \ , \tag{V.5}$$

and also let

$$h = h(\lambda, a) = q(\lambda, a)^2 = Q(\lambda)^2 + a^2 - \eta d_\lambda a \ . \tag{V.6}$$

Note that

$$q^\Gamma = -q \ , \ \text{ and } \ h^\Gamma = h \ . \tag{V.7}$$

We use the notation d_q to denote the Γ-graded commutator on $\hat{\mathcal{H}}$,

$$d_q x = qx - x^\Gamma q \ . \tag{V.8}$$

If we need to emphasize the dependence of q on λ or a, then we write $d_{q(\lambda,a)}x$. We continue to reserve d or d_λ to denote the γ-graded commutator (IV.1).

VI Heat Kernel Regularization on $\hat{\mathcal{H}}$

Let us introduce the heat kernel regularizations $\hat{X}_n$ of X_n on $\hat{\mathcal{H}}$. Let $X_n = \{x_0, \ldots, x_n\}$ denote an ordered set of $(n+1)$ linear operators x_j acting on $\hat{\mathcal{H}}$. We call the x_j *vertices* and X_n a set of vertices. Choose $a \in \mathfrak{A}$ and let $q(\lambda, a) = Q(\lambda) + \eta a$, and $h = h(\lambda, a) = q(\lambda, a)^2$. Define the heat kernel regularization $\hat{X}_n(\lambda, a)$ of $X_n = \{x_0, \ldots, x_n\}^\wedge(\lambda, a)$ by

$$\hat{X}_n(\lambda, a) = \int_{s_j > 0} x_0 e^{-s_0 h} x_1 e^{-s_1 h} \cdots x_n e^{-s_n h} \delta(1 - s_0 - \cdots - s_n) ds_0 \cdots ds_n \ . \tag{VI.1}$$

Note that if T is any operator on $\hat{\mathcal{H}}$ that commutes with $h = q^2$, then

$$\{x_0, \ldots, x_j T, x_{j+1}, \ldots, x_n\}^\wedge(\lambda, a) = \{x_0, \ldots, x_j, T x_{j+1}, \ldots, x_n\}^\wedge(\lambda, a) \ . \tag{VI.2}$$

Furthermore $T = J\eta$ anticommutes with $q(\lambda, a)$ and commutes with $h(\lambda, a)$ for all a.

Proposition II. (Vertex Insertion) *Let $X_n = \{x_0, \ldots, x_n\}$ denote a set of vertices possibly depending on λ. Then with the notation $\dot{Q} = \partial Q(\lambda)/\partial\lambda$, we have*

$$\begin{aligned} \frac{\partial}{\partial\lambda}\{x_0, \ldots, x_n\}^\wedge(\lambda, a) &= -\sum_{j=0}^{n}\{x_0, \ldots, x_j, d_q\dot{Q}, x_{j+1}, \ldots, x_n\}^\wedge(\lambda, a) \\ &\quad + \sum_{j=0}^{n}\{x_0, \ldots, \frac{\partial x_j}{\partial\lambda}, \ldots, x_n\}^\wedge(\lambda, a) \ . \end{aligned} \tag{VI.3}$$

Here

$$d_q\dot{Q} = d_{q(\lambda,a)}\dot{Q} = d_\lambda\dot{Q} + \eta[a, \dot{Q}] \ . \tag{VI.4}$$

Proof. By differentiating $\hat{X}_n$ defined in (VI.1), we obtain two types of terms. Differentiating the x_j's gives the second sum in (VI.3). (This sum is absent if the x_j's are λ-independent.) The other terms arise from differentiating the heat kernels. We use the identity

$$\frac{\partial}{\partial\lambda}e^{-sh} = -\int_0^s e^{-uh}\frac{\partial h}{\partial\lambda}e^{-(s-u)h}du\ . \tag{VI.5}$$

Note that

$$\frac{\partial h}{\partial\lambda} = \frac{\partial}{\partial\lambda}(q^2) = q\frac{\partial q}{\partial\lambda} + \frac{\partial q}{\partial\lambda}q = d_q\left(\frac{\partial q}{\partial\lambda}\right) = d_q\dot{Q}\ .$$

Explicitly

$$d_q\dot{Q} = (Q+\eta a)\dot{Q} + \dot{Q}(Q+\eta a) = d_\lambda\dot{Q} + \eta[a\dot{Q}]\ .$$

Inserted back into the definition of $\hat{X}_n$, we observe that the differentiation of the heat kernel between vertex j and vertex $j+1$ produces one new $-d_q\dot{Q}$ vertex at position $j+1$. This completes the proof of (VI.3).

Define the action of the grading Γ on sets of vertices X_n by

$$X_n \to X_n^\Gamma = \{x_0^\Gamma, x_1^\Gamma, \ldots, x_n^\Gamma\}\ . \tag{VI.6}$$

Since $q^2 = (q^\Gamma)^2$, the regularization $X_n \to \hat{X}_n$ commutes with the action of Γ, namely

$$\left(\hat{X}_n(\lambda,a)\right)^\Gamma = \left(X_n^\Gamma\right)^\wedge(\lambda,a)\ . \tag{VI.7}$$

It is also convenient to write explicitly differential of $\hat{X}_n$,

$$d_q\hat{X}_n(\lambda,a) = q\hat{X}_n(\lambda,a) - \hat{X}_n(\lambda,a)^\Gamma q = \{qx_0, x_1, \ldots, x_n\}^\wedge(\lambda,a) - \{x_0^\Gamma, \ldots, x_n^\Gamma q\}^\wedge(\lambda,a)\ . \tag{VI.8}$$

Note that

$$d_q\hat{X}_n(\lambda,a) = \sum_{j=0}^n \{x_0^\Gamma, x_1^\Gamma, \ldots, x_{j-1}^\Gamma, d_qx_j, \ldots, x_n\}^\wedge(\lambda,a)\ . \tag{VI.9}$$

One other identity we mention is

Proposition III. (Combination Identity) *The heat kernel regularizations satisfy*

$$\{x_0, x_1, \ldots, x_n\}^\wedge(\lambda,a) = \sum_{j=0}^n \{x_0, x_1, \ldots, x_j, I, x_{j+1}, \ldots, x_n\}^\wedge(\lambda,a)\ . \tag{VI.10}$$

Proof. The jth term on the right side of (VI.9) is

$$\begin{aligned}&\{x_0,\ldots,x_j,I,x_{j+1},\ldots,x_n\}^\wedge(\lambda,a)\\&\qquad=\int_{s_j>0} x_0e^{-s_0h}\cdots x_je^{-(s_j+s_{j+1})h}\cdots x_ne^{-s_{n+1}h}\delta(1-s_0-\cdots-s_{n+1})ds_0\cdots ds_{n+1}\,. \end{aligned}\qquad\text{(VI.11)}$$

Change the s-integration variables to $s'_0=s_0, s'_1=s_1,\ldots,s'_j=s_j+s_{j+1}$, $s'_{j+1}=s_{j+2},\ldots,s'_n=s_{n+1}$, and $s'_{n+1}=s_j$. This change has Jacobian 1, and the resulting integrand has the form of the integrand for $\{x_0,\ldots,x_n\}^\wedge$ with variables $s'_0,\ldots,s'_n$, namely

$$\int_{s'_0,s'_1,\ldots,s'_n>0} ds'_0\cdots ds'_n\left(\int ds'_{n+1}x_0e^{-s'_0h}\cdots x_ne^{-s'_nh}\delta(1-s'_0-\cdots-s'_n)\right)\,, \qquad\text{(VI.12)}$$

with the integrand depending on the variable s'_{n+1} only through the restriction of the range of the s'_{n+1} integral. The original domain of integration restricts s'_{n+1} to the range $0\le s'_{n+1}\le s'_j$, so the dependence of the integrand on s'_{n+1} is the characteristic function of the interval $[0,s'_j]$. Thus performing the s'_{n+1} integration produces a factor s'_j in the $s'_0,\ldots,s'_n$-integrand. Add the similar results for $0\le j\le n$ to give the factor $s'_0+s'_1+\cdots s'_n$. But the delta function in (VI.12) restricts this sum to be 1, so the integral of the sum is exactly $\{x_0,\ldots,x_n\}^\wedge(\lambda,a)$.

VII Expectations on $\hat{\mathcal{H}}$

Let $a\in\mathfrak{A}$ satisfy $a^2=I$, and let $\hat{X}_n=\hat{X}_n(\lambda,a)$ denote the heat kernel regularization of X_n. We define the expectation

$$\left\langle\left\langle\hat{X}_n\right\rangle\right\rangle_{\lambda,a,g}=\frac{1}{\sqrt{4\pi}}\int_{-\infty}^{\infty}\mathrm{Tr}_{\hat{\mathcal{H}}}\left(\Gamma U(g)\hat{X}_n(\lambda,ta)\right)dt\,. \qquad\text{(VII.1)}$$

Here we choose $a^2=I$ to ensure that the t^2 term in h provides a gaussian convergence factor to the t-integral. This integral represents averaging over a's whose squares are multiples of the identity.

These expectations can be considered as $(n+1)$-multilinear expectations on sets X_n of vertices. We sometimes suppress the λ- or a- or g-dependence of the expectations, or the n-dependence of sets of vertices. Furthermore, where confusion does not occur we omit the $^\wedge$ that we use to distinguish a set of vertices X from the heat kernel regularization of the set. Thus at various times we denotes $\left\langle\left\langle\hat{X}_n\right\rangle\right\rangle_{a,g}$ by $\langle\langle X\rangle\rangle$, or when we wish to clarify the dependence on n,a, or g with some subset of these indices, or even as one of the following:

$$\left\langle\left\langle\hat{X}_n\right\rangle\right\rangle_{\lambda,a,g}=\langle\langle X\rangle\rangle=\langle\langle X\rangle\rangle_n=\langle\langle X\rangle\rangle_{n,a}=\langle\langle X\rangle\rangle_{n,a,g}\,, \qquad\text{(VII.2)}$$

etc.

Proposition IV. *With the above notation, we have the identities*

(Γ-invariance) $$\langle\langle X\rangle\rangle_n=\left\langle\left\langle X^\Gamma\right\rangle\right\rangle_n\,, \qquad\text{(VII.3)}$$

(differential) $$\langle\langle d_q X\rangle\rangle_n = \sum_{j=0}^{n} \left\langle\left\langle\left\{x_0^\Gamma, x_1^\Gamma, \ldots, x_{j-1}^\Gamma, d_q x_j, \ldots, x_n\right\}\right\rangle\right\rangle \ , \qquad \text{(VII.4)}$$

(cyclic symmetry) $$\langle\langle\{x_0, x_1, \ldots, x_n\}\rangle\rangle = \left\langle\left\langle\left\{x_n^{g^{-1}\Gamma}, x_1, x_2, \ldots, x_{n-1}\right\}\right\rangle\right\rangle \ , \qquad \text{(VII.5)}$$

and

(combination identity)
$$\langle\langle\{x_0, x_1, \ldots, x_n\}\rangle\rangle_n = \sum_{j=0}^{n} \langle\langle\{x_0, x_1, \ldots, x_j, I, x_{j+1}, \ldots, x_n\}\rangle\rangle_{n+1} \ . \qquad \text{(VII.6)}$$

Also, in case $Q = Q^g$ and $a = a^g$, then $q = q^g$ and we have

(ininfitesimal invariance) $$\left\langle\left\langle d_{q(\lambda,ta)} X\right\rangle\right\rangle = 0 \ . \qquad \text{(VII.7)}$$

Proof. The symmetry (VII.3) is a consequence of the fact that $\Gamma^2 = I$, and Γ commutes with $U(g)$ and with q^2. The expectation of (VI.9) completes the proof of (VII.4). The proof of (VII.5) involves cyclicity of the trace. The identity (VII.6) is the expectation of (VI.10). To establish (VII.7), note that every $\hat{X}_n$ can be decomposed uniquely as $\hat{X}_n = \hat{X}_n^+ + \hat{X}_n^-$, where $\left(\hat{X}_n^\pm\right)^\Gamma = \pm\hat{X}_n^\pm$. The symmetry (VII.3) ensures that $\left\langle\left\langle d_{q(\lambda,ta)} X_n^+\right\rangle\right\rangle = 0$. On the other hand, $q^\Gamma = -q$, together with cyclicity of the trace and $q^g = q$ ensures that

$$\begin{aligned}
\left\langle\left\langle d_{q(\lambda,ta)} X_n^-\right\rangle\right\rangle &= \left\langle\left\langle q(\lambda, ta) X_n^-\right\rangle\right\rangle + \left\langle\left\langle X_n^- q(\lambda, ta)\right\rangle\right\rangle \\
&= \left\langle\left\langle q(\lambda, ta) X_n^-\right\rangle\right\rangle + \left\langle\left\langle q(\lambda, ta)^{g^{-1}\Gamma} X_n^-\right\rangle\right\rangle = 0 \ .
\end{aligned}$$

Except in (VII.7), we have implicitly assumed that the vertices x_j in X_n are t-independent. In case that X_n has one factor linear in t, the heat kernel regularizations of the following agree,

$$\{tx_0, x_1, \ldots, x_n\}^\wedge(\lambda, ta) = \{x_0, x_1, \ldots, tx_j, \ldots, x_n\}^\wedge(\lambda, ta) \ , \qquad \text{(VII.8)}$$

for any $j = 0, 1, \ldots, n$. We then obtain an interesting relation for expectations,

Proposition V. (Integration by parts) *Let $a^2 = I$. Then*

$$\langle\langle\{tx_0, x_1, \ldots, x_n\}\rangle\rangle_n = \sum_{j=0}^{n} \frac{1}{2} \langle\langle\{x_0, \ldots, x_j, \eta d_\lambda a, x_{j+1}, \ldots, x_n\}\rangle\rangle_{n+1} \ . \qquad \text{(VII.9)}$$

Proof. In order to establish (VII.9), we collect together the terms $\exp(-s_j t^2)$ that occur in $\{x_0, \ldots, x_n\}^\wedge(ta)$. Since the integrand for the heat kernel regularization has a δ-function restricting the variables s_j to satisfy $s_0 + \cdots + s_n = 1$, we obtain the factor $\exp(-t^2)$. Write

$$te^{-t^2} = -\frac{1}{2}\frac{d}{dt}\left(e^{-t^2}\right)$$

and integrate by parts in t. The resulting derivative involves the t-derivative of each heat kernel $\exp -(s_j q(\lambda, ta)^2)$ with the quadratic term in t removed from q^2. Note that

$$\begin{aligned} e^{-st^2}\frac{d}{dt}e^{-s(q^2-t^2)} &= -\int_0^s e^{-uq^2}\left(\frac{d}{dt}(q^2-t^2)\right)e^{-(s-u)q^2}du \\ &= \int_0^s e^{-uq^2}\eta d_\lambda a e^{-(s-u)q^2}du\ . \end{aligned}$$

Here we use (V.6) with ta replacing a and with $a^2 = I$ in order to evaluate the t derivative of $q^2 - t^2$. Thus each derivative introduces a new vertex equal to $\frac{1}{2}\eta d_\lambda a$, and the proof of (VII.9) is complete.

VIII The Functional $\mathfrak{J}(\lambda, a, g)$

Let us consider a single vertex and $X_0 = x_0 = Ja$, where $a \in \mathfrak{A}$, and its expectation

$$\mathfrak{J}(\lambda, a, g) = \langle\langle Ja \rangle\rangle\ . \tag{VIII.1}$$

Explicitly

$$\mathfrak{J}(\lambda, a, g) = \frac{1}{\sqrt{4\pi}}\int_{-\infty}^{\infty} \mathrm{Tr}_{\hat{\mathcal{H}}}\left(\gamma U(g) a e^{-q(\lambda,ta)^2}\right) dt\ . \tag{VIII.2}$$

This functional allows us to recover the functional $\mathfrak{Z}$.

Theorem VI. *Let a satisfy $a^2 = I$. Then*

$$\mathfrak{J}(\lambda, a, g) = \mathfrak{Z}^{Q(\lambda)}(a, g)\ . \tag{VIII.3}$$

Proof. Let $h = h_0 - t\eta da$, where $h_0 = Q(\lambda)^2 + t^2$. The Hille-Phillips perturbation theory for semi-groups can be written

$$\begin{aligned} e^{-q(\lambda,ta)^2} &= e^{-h_0 + t\eta da} \\ &= e^{-h_0} + \sum_{n=1}^{\infty} t^n \int_{s_j>0} e^{-s_0 h_0}\eta d_\lambda a e^{-s_1 h_0}\eta d_\lambda a \cdots \eta d_\lambda a e^{-s_n h_0}\delta(1 - s_0 - s_1 - \cdots - s_n) ds_0 ds_1 \cdots ds_n\ . \end{aligned} \tag{VIII.4}$$

In the nth term we collect all factors of η on the left. Note that η commutes with a and h_0, and it anticommutes with $d_\lambda a$. Therefore the result of collecting the factors of η on the left is

$\eta^n(-1)^{n(n-1)/2}$. If n is odd, then $\eta^n = \eta$ and $\mathrm{Tr}_{\mathcal{H}_\eta}(\eta) = 0$. Thus only even n terms contribute to (VIII.2). For even n, $\eta^n(-1)^{n(n-1)/2} = (-1)^{n/2}I$ and $\mathrm{Tr}_{\mathcal{H}_\eta}(I) = 2$. Thus (VIII.2) becomes

$$\mathfrak{J}(\lambda, a, g) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dt \sum_{n=0}^{\infty} (-t^2)^n \left\langle \{a, d_\lambda a, \ldots, d_\lambda a\} \right\rangle_{2n} e^{-t^2} , \tag{VIII.5}$$

where we use expectations $\langle\ \rangle_n$ on $\mathcal{H}$ similar to $\langle\langle\ \rangle\rangle_n$ on $\hat{\mathcal{H}}$ (but without the t-integration) and defined by

$$\langle\{x_0, \ldots, x_n\}\rangle_n = \int_{s_j>0} \mathrm{Tr}_{\mathcal{H}} \left(\gamma U(g) x_0 e^{-s_0 Q(\lambda)^2} \cdots x_n e^{-s_n Q(\lambda)^2}\right) \delta(1 - s_0 - s_1 - \cdots - s_n) ds_0 ds_1 \cdots ds_n . \tag{VIII.6}$$

But using the Hille-Phillips formula once again, (VIII.6) is just

$$\frac{1}{\sqrt{\pi}} \int_{\infty}^{\infty} dt \mathrm{Tr}_{\mathcal{H}} \left(\gamma U(g) a e^{-Q(\lambda)^2 + it d_\lambda a}\right) e^{-t^2} = \mathfrak{Z}^{Q(\lambda)}(a, g) . \tag{VIII.7}$$

(Here we use the symmetry of (VIII.7) under γ to justify vanishing of terms involving odd powers of $d_\lambda a$.) Thus we can prove that $\mathfrak{Z}^{Q(\lambda)}(a, g)$ is independent of λ by showing that $\mathfrak{J}(\lambda, a, g)$ is constant in λ.

IX $\mathfrak{J}(\lambda, a, g)$ Does Not Depend on λ

We now prove Theorem I. Calculate $\partial\mathfrak{J}/\partial\lambda$ using (VI.3), in the simple case of one vertex independent of λ. Thus

$$\frac{\partial}{\partial\lambda} \mathfrak{J}(\lambda, a, g) = \frac{\partial}{\partial\lambda} \langle\langle Ja \rangle\rangle = -\left\langle\left\langle\left\{ Ja, d_{q(\lambda,ta)} \dot{Q} \right\}\right\rangle\right\rangle . \tag{IX.1}$$

Using the identity (VII.7) in the form

$$0 = \left\langle\left\langle d_{q(\lambda,ta)} \left\{ Ja, \dot{Q} \right\}\right\rangle\right\rangle = \left\langle\left\langle\left\{ d_{q(\lambda,ta)}(Ja), \dot{Q} \right\}\right\rangle\right\rangle + \left\langle\left\langle\left\{ Ja, d_{q(\lambda,ta)} \dot{Q} \right\}\right\rangle\right\rangle , \tag{IX.2}$$

we have

$$\frac{\partial}{\partial\lambda} \mathfrak{J}(\lambda, a, g) = \left\langle\left\langle\left\{ d_{q(\lambda,ta)}(Ja), \dot{Q} \right\}\right\rangle\right\rangle . \tag{IX.3}$$

It is at this point that we have used $q^g = q$, namely the invariance of both Q and a under $U(g)$. To evaluate (IX.3), note that

$$d_{q(\lambda,ta)}(Ja) = [q(\lambda, ta), Ja] = J d_\lambda a - 2tJ\eta . \tag{IX.4}$$

Here we use the assumption $a^2 = I$. From Proposition V we therefore infer

$$\begin{aligned} \frac{\partial}{\partial\lambda} \mathfrak{J}(\lambda, a, g) &= \left\langle\left\langle\left\{ J d_\lambda a, \dot{Q} \right\}\right\rangle\right\rangle - 2\left\langle\left\langle\left\{ tJ\eta, \dot{Q} \right\}\right\rangle\right\rangle \\ &= \left\langle\left\langle\left\{ J d_\lambda a, \dot{Q} \right\}\right\rangle\right\rangle - \left\langle\left\langle\left\{ J\eta, \eta d_\lambda a, \dot{Q} \right\}\right\rangle\right\rangle - \left\langle\left\langle\left\{ J\eta, \dot{Q}, \eta d_\lambda a \right\}\right\rangle\right\rangle . \end{aligned} \tag{IX.5}$$

Since $J\eta$ commutes with $h = q^2$, and since $J\eta\dot{Q} = -\dot{Q}J\eta$, use (VI.2) to establish

$$\begin{aligned}\left\langle\left\langle\left\{J\eta, \eta d_\lambda a, \dot{Q}\right\}\right\rangle\right\rangle + \left\langle\left\langle\left\{J\eta, \dot{Q}, \eta d_\lambda a\right\}\right\rangle\right\rangle &= \left\langle\left\langle\left\{I, J d_\lambda a, \dot{Q}\right\}\right\rangle\right\rangle - \left\langle\left\langle\left\{I, \dot{Q}, J d_\lambda a\right\}\right\rangle\right\rangle \\ &= \left\langle\left\langle\left\{J d_\lambda a, \dot{Q}, I\right\}\right\rangle\right\rangle + \left\langle\left\langle\left\{J d_\lambda a, I, \dot{Q}\right\}\right\rangle\right\rangle \ . \end{aligned} \quad \text{(IX.6)}$$

In the last step we also use $\dot{Q}^\Gamma = -\dot{Q}$ and the cyclic symmetry (VII.5). Hence we can simplify (IX.6) to $\left\langle\left\langle\left\{J d_\lambda a, \dot{Q}\right\}\right\rangle\right\rangle$, by applying the combination identity (VII.6). Substituting this back into (IX.5), we end up with

$$\frac{\partial}{\partial\lambda}\mathfrak{J}(\lambda, a, g) = \left\langle\left\langle\left\{J d_\lambda a, \dot{Q}\right\}\right\rangle\right\rangle - \left\langle\left\langle\left\{J d_\lambda a, \dot{Q}\right\}\right\rangle\right\rangle = 0 \ . \quad \text{(IX.7)}$$

Thus $\mathfrak{J}(\lambda, a, g)$ is invariant under change of λ, and the demonstration is complete.

X Independent Supercharges $Q_j(\lambda)$

Let us generalize our consideration to the case that there are two self-adjoint operators $Q_1 = Q_1(\lambda)$ and $Q_2 = Q_2(\lambda)$ on $\mathcal{H}$ such that

$$Q_1\gamma + \gamma Q_1 = Q_2\gamma + \gamma Q_2 = Q_1 Q_2 + Q_2 Q_1 = 0 \ . \quad \text{(X.1)}$$

Thus we have two derivatives $d_j a = [Q_j, a]$. We assume that the energy operator on $\mathcal{H}$ is defined by

$$H = H(\lambda) = \frac{1}{2}\left(Q_1 + Q_2\right)^2 = \frac{1}{2}\left(Q_1^2 + Q_2^2\right) \quad \text{(X.2)}$$

and that the operator

$$P = \frac{1}{2}\left(Q_1^2 - Q_2^2\right) \quad \text{(X.3)}$$

has the properties:

i) P does not depend on λ.

ii) P commutes with Q_1, Q_2 and with each $a \in \mathfrak{A}$.

iii) $U(g)$ commutes with Q_1 and with $H(\lambda)$.

Assumption (i) corresponds to a common situation where P can be interpreted as a "momentum" operator. Then the energy, but not the momentum is assumed to depend on λ. Assumption (ii) says that Q_1, Q_2 are translation invariant, and that $\mathfrak{A}$ is a "zero-momentum" or translation-invariant subalgebra. According to assumption (iii), $U(g)$ commutes with Q_2^2, but $U(g)$ may not commute with Q_2. Under these hypotheses, and with appropriate regularity assumptions, we showed in [QHA] that for $a = a^g$ and $a^2 = I$,

$$\mathfrak{Z}^{Q_j(\lambda)}(a, g) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \mathrm{Tr}\left(\gamma U(g) a e^{-H + it d_1 a - t^2}\right) dt \quad \text{(X.4)}$$

is independent of λ. Here we show how the framework above can also be used to show that (X.4) is constant.

We introduce on $\hat{\mathcal{H}}$ two extended supercharges $q_1 = q_1(\lambda, a) = Q_1 + \eta a$ and $q_2 = Q_2$. Thus with η as before, $\eta Q_1 + Q_1 \eta = \eta Q_2 + Q_2 \eta = 0$. Define

$$h = h(\lambda, ta) = H(\lambda) + t^2 a^2 - t\eta d_1 a \, . \tag{X.5}$$

Note that

$$h = q_1(\lambda, ta)^2 - P = Q_1(\lambda)^2 + t^2 a^2 - t\eta d_1 a - P \, , \tag{X.6}$$

so we can eliminate $Q_2(\lambda)$ from h by introducing the operator P, that commutes with a, γ, J, $U(g)$, η, and $Q_j(\lambda)$.

Thus P commutes with all operators that we consider on $\hat{\mathcal{H}}$, so we repeat the constructions of §V–IX. However, we replace $q(\lambda, ta)^2$ in the previous construction with $h(\lambda, ta)$ defined by (X.5). Also we replace $d_q x$ with $d_{q_j} x = q_j x - x^\Gamma q_j$. We use the heat kernel $\exp(-sh)$ to define the heat kernel regularization. Then define the expectation $\langle\langle \cdot \rangle\rangle$ by the formula (VII.1) with this new $h(\lambda, ta)$. As $q_1 = q_1^g$, therefore we have

$$\left\langle \left\langle d_{q_1(\lambda, ta)} X \right\rangle \right\rangle = 0 \, . \tag{X.7}$$

However it may not be true that $q_2 = q_2^g$, so it may not be true that $\left\langle \left\langle d_{q_2(\lambda, ta)} X \right\rangle \right\rangle$ vanishes. As before, with $a^2 = I$, define

$$\mathfrak{J}(\lambda, a, g) = \langle\langle Ja \rangle\rangle \, . \tag{X.8}$$

In this case, we establish as in the proof of Theorem VI that

$$\mathfrak{J}(\lambda, a, g) = \mathfrak{Z}^{Q_j(\lambda)}(a, g) \, . \tag{X.9}$$

Thus the proof of Theorem I shows:

Theorem VII. *Let $a \in \mathfrak{A}$, assume $a^2 = I$, and also assume the regularity hypotheses on $Q_j(\lambda)$ and $d_1 a = [Q_1(\lambda), a]$, stated in §XI. Then the expectation $\mathfrak{Z}^{Q_j(\lambda)}(a, g)$ is independent of λ.*

XI Regularity Hypotheses

As explained in the introduction, our results depend crucially on some regularity hypotheses. In order for $\mathfrak{Z}$ to exist, we assume $e^{-H(\lambda)} = e^{-Q(\lambda)^2}$ exists and is trace class on $\mathcal{H}$. We give sufficient conditions to ensure this, as well as to ensure the validity of the results claimed in §I — IX. The content of §X require only minor modification of these hypotheses. We have explored the consequences of these hypotheses in [QHA].

1. The operator Q is self-adjoint operator on $\mathcal{H}$, odd with respect to γ, and $e^{-\beta Q^2}$ is trace class for all $\beta > 0$.

2. For $\lambda \in \Lambda$, where Λ is a open interval on the real line, the operator $Q(\lambda)$ can be expressed as a perturbation of Q in the form
$$Q(\lambda) = Q + W(\lambda) \ . \tag{XI.1}$$
Each $W(\lambda)$ is a symmetric operator on the domain $\mathcal{D} = C^\infty(Q)$.

3. Let λ lie in any compact subinterval $\Lambda' \subset \Lambda$. The inequality
$$W(\lambda)^2 \leq aQ^2 + bI \ , \tag{XI.2}$$
holds as an inequality for forms on $\mathcal{D} \times \mathcal{D}$. The constants $a < 1$ and $b < \infty$ are independent of λ in the compact set $\Lambda' \subset \Lambda$.

4. Let $R = (Q^2 + I)^{-1/2}$. The operator $Z(\lambda) = RW(\lambda)R$ is bounded uniformly for $\lambda \in \Lambda'$, and the difference quotient
$$\frac{Z(\lambda) - Z(\lambda')}{\lambda - \lambda'} \tag{XI.3}$$
converges in norm to a limit as $\lambda' \to \lambda \in \Lambda' \subset \Lambda$.

5. The bilinear form $d_\lambda a$ satisfies the bound
$$\|R^\alpha d_\lambda a R^\beta\| < M \ , \tag{XI.4}$$
with a constant M independent of λ for $\lambda \in \Lambda'$. Here α, β are non-negative constants and $\alpha + \beta < 1$.

In certain examples we are interested in the behavior of $\mathfrak{J}(\lambda, a, g)$ as λ tends to the boundary of Λ. In this case, we may establish the constancy of $\mathfrak{J}$ with estimates that are weaker than (1–5) at the endpoint of Λ, by directly proving the existence and continuity of $\mathfrak{J}$ at the endpoint. We study one such an example in [J], though other types of endpoint singularities are also of interest (often involving a $\lambda \to \infty$ limit).

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