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CCR-algebra structure of normal k-mode fluctuations

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Abstract

For a quantum lattice system, the k-mode fluctuation of observables in a quasi-local C^*-algebra $\mathcal{A}$ and a state ω are studied, and shown to be normal under mild cluster conditions.

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1 Introduction

Fourier series transforms of operators are widely used in the physics literature. In [1], we gave a meaning of these Fourier transforms for product states via a non-commutative central limit theorem. The case $k = 0$ was treated before [2, 3, 4] and leads to a mathematical setup of the theory of fluctuation operators. For $k \neq 0$, the k-dependence changes the local character of the fluctuations, such that a new proof of the central limit theorem must be conceived. Here we give the theorem of Fourier transforms for non-product states, which however do satisfy mild cluster conditions. It turns out that if $k \neq 0$, the L^1-cluster condition can be weakened.

The paper is set up as follows. First we prove the central limit theorem for $k \neq 0$, but for L^1-clustering. The proof is performed in such a way that it can be repeated after having relaxed the L^1-cluster condition for the state.

The systems under consideration are quantum lattice systems, i.e. we consider the lattice $\mathbb{Z}^\nu$ and a one-site C^*-algebra $\mathcal{A}_x$ of observables in each lattice site $x \in \mathbb{Z}^\nu$. For each finite $\Lambda \subset \mathbb{Z}^\nu$, the tensor product $\bigotimes_{x\in\Lambda} \mathcal{A}_x = \mathcal{A}_\Lambda$ is the C^*-algebra of the observables measurable in the volume Λ. The local algebra of observables of the system is given by $\bigcup_\Lambda \mathcal{A}_\Lambda = \mathcal{A}$. For more details we refer to [5].

The group $\mathbb{Z}^\nu$ of space translations acts as the group $\{\tau_x | x \in \mathbb{Z}^\nu\}$ of $*$-automorphisms of $\mathcal{A} = \bigcup_\Lambda \mathcal{A}_\Lambda$ by

$$A \in \mathcal{A}_\Lambda \to \tau_x A \in \mathcal{A}_{\Lambda+x} \quad , \quad x \in \mathbb{Z}^\nu.$$

We assume that ω is a state on $\mathcal{A} = \bigcup_\Lambda \mathcal{A}_\Lambda$ which is extremal translation invariant, i.e. for all $x \in \mathbb{Z}^\nu$

$$\omega \circ \tau_x = \omega,$$

and for all $A, B \in \mathcal{A}$,

$$\lim_{|x|\to\infty} \omega(A\tau_x B) = \omega(A)\omega(B).$$

The proof of our central limit theorem is an adaptation of the Bernstein argument used in [3, 4].

2 A BCH formula

The first result imposes no conditions on the state ω. It is a statement about convergence *in norm*. Theorem 2.1 below is a consequence of the *normal scaling* $|\Lambda_n|^{-1/2}$ of the local fluctuations $F_n^k(A)$ (2.1). The BCH-type formula (Baker-Campbell-Hausdorff) will be essential in order to determine the limiting algebraic CCR structure of the fluctuations.

We fix a $k \in [0,\pi]^\nu$ and define for every $A \in \mathcal{A}_0$ the local k-mode fluctuations

$$F_n^k(A) = |\Lambda_n|^{-1/2} \sum_{x\in\Lambda_n} \cos kx \left[\tau_x[A] - \omega(\tau_x[A])\mathbb{1}\right]. \tag{2.1}$$

The ω-dependence of the local fluctuation is implicit. We used the notation

$$\Lambda_n = [-n,n]^\nu \cap \mathbb{Z}^\nu \tag{2.2}$$

and $|\Lambda_n|$ is the number of lattice sites in Λ_n.

Theorem 2.1 *Let $A, B \in \mathcal{A}_{0,sa}$ and $F_n^k(A)$, $F_n^k(B)$ their local fluctuations in the arbitrary state ω, then*

$$\lim_{n\to\infty} \left\| e^{iF_n^k(A)}\, e^{iF_n^k(B)} - e^{iF_n^k(A+B)}\, e^{-\frac{1}{2}[F_n^k(A)\,,\,F_n^k(B)]} \right\| = 0\,. \tag{2.3}$$

proof: Without loss of generality, assume that $\omega(A) = \omega(B) = 0$. To arrive at the general case, replace A by $A - \omega(A)\mathbb{1}$ and analogously for B. For such $A, B \in \mathcal{A}_{0,sa}$,

$$\begin{aligned}
&\left\| e^{iF_n^k(A)}\, e^{iF_n^k(B)} - e^{i(F_n^k(A)+F_n^k(B))}\, e^{-\frac{1}{2}[F_n^k(A)\,,\,F_n^k(B)]} \right\| \\
&\quad = \Big\| \prod_{x\in\Lambda_n} \left\{ e^{i|\Lambda_n|^{-1/2}\tau_x[A]\cos kx}\, e^{i|\Lambda_n|^{-1/2}\tau_x[B]\cos kx} \right\} \\
&- \prod_{x\in\Lambda_n} \left\{ e^{i|\Lambda_n|^{-1/2}(\tau_x[A]\cos kx + \tau_x[B]\cos kx)}\, e^{-\frac{1}{2}|\Lambda_n|^{-1}\tau_x[A,B](\cos kx)^2} \right\} \Big\| \\
&\quad = \Big\| \prod_{x\in\Lambda_n} \left\{ e^{i|\Lambda_n|^{-1/2}\mathbf{a}_x}\, e^{i|\Lambda_n|^{-1/2}\mathbf{b}_x} \right\} - \prod_{x\in\Lambda_n} \left\{ e^{i|\Lambda_n|^{-1/2}(\mathbf{a}_x + \mathbf{b}_x)}\, e^{-\frac{1}{2}|\Lambda_n|^{-1}[\mathbf{a}_x,\mathbf{b}_x]} \right\} \Big\|,
\end{aligned}$$

where we introduced the notation

$$\tau_x[A]\cos kx \equiv \mathbf{a}_x \quad \text{and} \quad \tau_x[B]\cos kx \equiv \mathbf{b}_x\,.$$

Now, number the sites in Λ_n from 1 to $(2n+1)^\nu$ and use the rule

$$\prod_{x=1}^{(2n+1)^\nu} f_x - \prod_{x=1}^{(2n+1)^\nu} g_x = \sum_{l=1}^{(2n+1)^\nu} \left\{ \Big(\prod_{x=1}^{l-1} g_x\Big) \Big(f_l - g_l\Big) \Big(\prod_{m=l+1}^{(2n+1)^\nu} f_m\Big) \right\}$$

to rewrite the norm above as the norm of a telescopic sum. Using the triangle and the product inequalities and the unitarity of the exponentials, we find that

$$\begin{aligned}
&\Big\| \prod_{x\in\Lambda_n} \Big\{ e^{i|\Lambda_n|^{-1/2}\mathbf{a}_x} \, e^{i|\Lambda_n|^{-1/2}\mathbf{b}_x} \Big\} - \prod_{x\in\Lambda_n} \Big\{ e^{i|\Lambda_n|^{-1/2}(\mathbf{a}_x + \mathbf{b}_x)} \, e^{-\frac{1}{2}|\Lambda_n|^{-1}[\mathbf{a}_x, \mathbf{b}_x]} \Big\} \Big\| \\
&\quad \le \sum_{x\in\Lambda_n} \Big\| e^{i|\Lambda_n|^{-1/2}\mathbf{a}_x} \, e^{i|\Lambda_n|^{-1/2}\mathbf{b}_x} - e^{i|\Lambda_n|^{-1/2}(\mathbf{a}_x + \mathbf{b}_x)} \, e^{-\frac{1}{2}|\Lambda_n|^{-1}[\mathbf{a}_x, \mathbf{b}_x]} \Big\| \\
&\quad = \sum_{x\in\Lambda_n} \Big\| \sum_{l=0}^{+\infty} \frac{1}{l!}(i|\Lambda_n|^{-1/2}\mathbf{a}_x)^l \sum_{m=0}^{+\infty} \frac{1}{m!}(i|\Lambda_n|^{-1/2}\mathbf{b}_x)^m \\
&\qquad\qquad - \sum_{p=0}^{+\infty} \frac{1}{p!}(i|\Lambda_n|^{-1/2}(\mathbf{a}_x + \mathbf{b}_x))^p \sum_{q=0}^{+\infty} \frac{1}{q!}(-\frac{1}{2}|\Lambda_n|^{-1}[\mathbf{a}_x, \mathbf{b}_x])^q \Big\| \\
&\quad = \sum_{x\in\Lambda_n} \Big\| \mathbb{1} + i\,|\Lambda_n|^{-1/2}\,[\,\mathbf{a}_x + \mathbf{b}_x\,] \\
&\qquad\qquad - |\Lambda_n|^{-1}[\,\mathbf{a}_x\mathbf{b}_x + \frac{1}{2}\mathbf{a}_x^2 + \frac{1}{2}\mathbf{b}_x^2\,] - i\,|\Lambda_n|^{-3/2}\,[\,\ldots\,] + \ldots \\
&\qquad - \Big\{ \mathbb{1} + i\,|\Lambda_n|^{-1/2}\,[\,\mathbf{a}_x + \mathbf{b}_x\,] \\
&\qquad\qquad - \frac{1}{2}|\Lambda_n|^{-1}\,[\mathbf{a}_x\mathbf{b}_x - \mathbf{b}_x\mathbf{a}_x + (\mathbf{a}_x + \mathbf{b}_x)^2] - i\,|\Lambda_n|^{-3/2}\,[\,\ldots\,] + \ldots \Big\} \Big\| \\
&\quad \le \sum_{x\in\Lambda_n} |\Lambda_n|^{-3/2} \Big\{ \exp\|\mathbf{a}_x\| \, \exp\|\mathbf{b}_x\| + \exp\|\mathbf{a}_x + \mathbf{b}_x\| \, \exp\frac{1}{2}\|\,[\mathbf{a}_x, \mathbf{b}_x]\,\| \Big\} \\
&\quad \le |\Lambda_n|^{-1/2} \, \exp\|A\| \, \exp\|B\| \Big(1 + \exp\frac{1}{2}\|\,[A, B]\,\|\Big) \overset{n\to\infty}{\longrightarrow} 0\,.
\end{aligned}$$

■

3 Normal fluctuations

Definition 3.1 *A translationally invariant state $\omega = \omega \circ \tau_x$ on the algebra $\mathcal{A}$ on $\mathbb{Z}^\nu$ has* normal fluctuations in the mode $k \in [0, \pi]^\nu$ *for the subspace $\mathcal{K}$ of $\mathcal{A}_0$ if*

(i) *for all* $A, B \in \mathcal{K}$

$$\sum_{x \in \mathbb{Z}^\nu} \Big| \omega(A \tau_x[B]) - \omega(A)\omega(B) \Big| < \infty \,, \tag{3.1}$$

(ii) *for all* $A, B \in \mathcal{K}_{sa} \subseteq \mathcal{A}_{0,sa}$ *and* $t \in \mathbb{R}$

$$\lim_{n \to \infty} \omega(e^{itF_n^k(A)}) = e^{-\frac{t^2}{2} s_\omega(A,A)} \tag{3.2}$$

with $s_\omega(A,B) = \lim_n \mathrm{Re}\, \omega\Big(F_n^k(A) F_n^k(B)\Big)$.

Note that condition (i) implies that ω is *extremal* translationally invariant or ergodic. Proposition 3.3 below, proves that s_ω is automatically well-defined because of condition (i).

In section 4, we prove that if the state ω satisfies certain *clustering conditions*, then it has normal fluctuations in all k-modes. Here, we *assume* that ω has normal fluctuations in a fixed mode $k \in [0, \pi]$ and we focus on the *algebraic structure* of the local fluctuations $F_n^k(A)$ in the limit $n \to \infty$. A CCR algebra of macroscopic fluctuations is constructed.

We start by proving a lemma. For ease of notation, introduce

$$\Lambda_n(z) = \Lambda_n \cap (\Lambda_n - z) \tag{3.3}$$

for $z \in \Lambda_{2n}$, and its 1-dimensional analog

$$\lambda_n(z^j) = [\max(-n, -n - z^j), \min(n, n - z^j)] \cap \mathbb{Z} \tag{3.4}$$

for $z^j \in [-2n, 2n] \cap \mathbb{Z}$.

Lemma 3.2 *Let* $n \in \mathbb{N}_0$ *and* $z \in \Lambda_{2n}$. *For all* $q \in \mathbb{R}^\nu$ *and* $k \in [0,\pi]^\nu$,

$$\text{(i)} \quad |\Lambda_n|^{-1} \sum_{x \in \Lambda_n(z)} e^{i\, qx} = \prod_{j=1}^{\nu} \delta_{q^j}(n, z^j) \equiv \delta_q(n, z) \tag{3.5}$$

$$\text{(ii)} \quad |\Lambda_n|^{-1} \sum_{x \in \Lambda_n(z)} \cos kx \, \cos k(x+z) = \frac{1}{2}\, \tilde{\delta}_k(n,z)\,, \tag{3.6}$$

where

$$\delta_{q^j}(n,z^j) = \begin{cases} 1-\frac{|z^j|}{2n+1} & \textit{if } q^j = 0 \pmod{2\pi} \\ \frac{\exp -\frac{i}{2}q^j z^j}{2n+1}\ \frac{\sin[\frac{1}{2}q^j(2n+1-|z^j|)]}{\sin[\frac{1}{2}q^j]} & \textit{if } q^j \neq 0 \pmod{2\pi} \end{cases} \tag{3.7}$$

$$\widetilde{\delta}_k(n,z) = \frac{1}{2}\Big\{e^{ikz}[\delta_{2k}(n,z)+\delta_0(n,z)] \\ +e^{-ikz}[\delta_{-2k}(n,z)+\delta_0(n,z)]\Big\} \tag{3.8}$$

proof: The exponential factorizes (cf. definitions (3.3) and (3.4))

$$|\Lambda_n|^{-1}\sum_{x\in\Lambda_n(z)} e^{i\,qx} = \prod_{j=1}^{\nu}\Big\{(2n+1)^{-1}\sum_{x^j\in\lambda_n(z^j)} e^{i\,q^j x^j}\Big\}$$

and each factor is a simple geometric series

$$\sum_{x^j\in\lambda_n(z^j)} e^{i\,q^j x^j} = \begin{cases} 2n+1-|z^j| & \text{if } q^j = 0 \pmod{2\pi} \\ e^{-\frac{i}{2}q^j z^j}\frac{\sin[\frac{1}{2}q^j(2n+1-|z^j|)]}{\sin[\frac{1}{2}q^j]} & \text{if } q^j \neq 0 \pmod{2\pi} \end{cases}$$

which proves (i).
Statement (ii) follows from

$$\cos kx \quad \cos k(x+z) \\ = \frac{1}{4}\left\{e^{ikz}[e^{i2kx}+1]+e^{-ikz}[e^{-i2kx}+1]\right\}.$$

■

The fact that

$$|\delta_{q^j}(n,z^j)| \leq (2n+1)^{-1}\ |\sin\frac{q^j}{2}|^{-1} \overset{n\to\infty}{\longrightarrow} 0 \tag{3.9}$$

when $q^j \neq 0 \pmod{2\pi}$ and

$$|\delta_{q^j}(n,z^j)| \leq 1 \tag{3.10}$$

otherwise will be important in the proof of the proposition below. Also note that for all fixed $z \in \mathbb{Z}^\nu$, $q \in \mathrm{I\!R}^\nu$, and $k \in [0,\pi]^\nu$

$$\lim_{n\to\infty} \delta_q(n,z) = \delta_{q,0 \ (\mathrm{mod}\ 2\pi)} \,, \tag{3.11}$$
$$\lim_{n\to\infty} \widetilde{\delta}_k(n,z) = \widetilde{\delta}_k \cos kz \,, \tag{3.12}$$

where

$$\widetilde{\delta}_{k,k} = \begin{cases} 1 & \text{if } 0 \neq k^j \neq \pi \text{ for some } j = 1,\ldots,\nu \\ 2 & \text{if } k^j = 0 \text{ or } = \pi \text{ for all } j = 1,\ldots,\nu \,. \end{cases} \tag{3.13}$$

At this point we recollect that the factor $1 - |z^j|/(2n+1)$ in equation (3.7) is the *Fejér convergence factor* and that *absolute* summability of a sequence $\big(f(z)\big)_{z\in\mathbb{Z}^\nu}$ implies *Cesàro* summability (cf. appendix A.2)

$$\sum_{z\in\Lambda_{2n}} f(z) \prod_{j=1}^{\nu} \Big[1 - \frac{|z^j|}{2n+1}\Big] = \prod_{j=1}^{\nu} \Big[\frac{1}{2n+1} \sum_{x^j=0}^{2n} \sum_{z^j=-x^j}^{x^j}\Big] f(z) \overset{n\to\infty}{\longrightarrow} \sum_z f(z) \,.$$

Proposition 3.3 *Assume that the state ω on the quasi-local C^*-algebra $\mathcal{A}$ satisfies condition (i) of definition 3.1 of normal fluctuations in the mode $k \in [0,\pi]^\nu$ for $\mathcal{K} \subseteq \mathcal{A}_0$. Then, for all $A, B \in \mathcal{K}$, the limit*

$$\lim_{n\to\infty} \omega\Big(F_n^k(A)^* F_n^k(B)\Big) = \langle A, B\rangle_\omega^k \,, \tag{3.14}$$

defines a positive sesquilinear form $\langle\cdot,\cdot\rangle_\omega^k : \mathcal{K} \to \mathbb{C}$, which satisfies the Cauchy-Schwarz inequality

$$|\langle A, B\rangle_\omega^k|^2 \leq \langle A, A\rangle_\omega^k \, \langle B, B\rangle_\omega^k \,. \tag{3.15}$$

More explicitly

$$\langle A, B\rangle_\omega^k = \frac{1}{2}\, \widetilde{\delta}_k \sum_z \Big(\omega(A^*\tau_z[B]) - \omega(A^*)\omega(B)\Big) \cos kz \,. \tag{3.16}$$

proof: If the limit (3.14) can be proven, then the positivity and the Cauchy-Schwarz inequality are obvious by positivity of ω and the Cauchy-Schwarz inequality for the state ω. Assume that $A, B \in \mathcal{K}$ with $\omega(A) = \omega(B) = 0$, then by lemma 3.2

$$\begin{aligned}\omega\Big(F_n^k(A)^*F_n^k(B)\Big) &= |\Lambda_n|^{-1}\sum_{x,y\in\Lambda_n}\omega(A^*\tau_{y-x}[B])\cos kx\ \cos ky\\ &= |\Lambda_n|^{-1}\sum_{x\in\Lambda_n}\sum_{z\in\Lambda_n-x}\omega(A^*\tau_z[B])\cos kx\ \cos k(x+z)\\ &= |\Lambda_n|^{-1}\sum_{z\in\Lambda_{2n}}\omega(A^*\tau_z[B])\sum_{x\in\Lambda_n(z)}\cos kx\ \cos k(x+z)\\ &= \sum_{z\in\Lambda_{2n}}\omega(A^*\tau_z[B])\,\frac{1}{2}\,\widetilde{\delta}_k(n,z)\,.\end{aligned}$$

Using the absolute summability of $\Big(\omega(A^*\tau_z[B])\Big)_{z\in\mathbb{Z}^\nu}$ (assumption (3.1)) and hence the Cesàro summability, the proposition is a result of lemma 3.2 together with the remarks following it. Remember that $k\in[0,\pi]^\nu$.

- $\underline{0\neq k^j\neq\pi\ \text{ for some }\ j=1,\ldots,\nu}$: By (3.1) and the estimate (3.9), only the second and fourth term of (3.8) contribute in the limit $n\to\infty$

$$\omega\Big(F_n^k(A)^*F_n^k(B)\Big)\overset{n\to\infty}{\longrightarrow}\sum_z\omega(A^*\tau_z[B])\,\frac{1}{2}\cos kz\,. \tag{3.17}$$

- $\underline{k^j=0\ \text{ or }\ =\pi\ \text{ for all }\ j=1,\ldots,\nu}$: In this case, all four terms of (3.8) contribute in the limit $n\to\infty$

$$\omega\Big(F_n^k(A)^*F_n^k(B)\Big)\overset{n\to\infty}{\longrightarrow}\sum_z\omega(A^*\tau_z[B])\ \cos kz\,. \tag{3.18}$$

Putting the two cases together and remembering the definition (3.13) of $\widetilde{\delta}_k$, the proposition is proven. ■

Definition 3.4 *Separating the real and the imaginary part of the restriction of the sesquilinear form to* $\mathcal{K}_{sa}$

$$\langle A,B\rangle_\omega^k = s_\omega^k(A,B)+\frac{i}{2}\sigma_\omega^k(A,B)\qquad\qquad A,B\in\mathcal{K}_{sa} \tag{3.19}$$

defines a real bilinear positive symmetric form s_ω^k *and a symplectic form* σ_ω^k

$$s_\omega^k(A,B)=\frac{1}{2}\,\widetilde{\delta}_k\sum_z\Big(\mathrm{Re}\,\omega(A\tau_z[B])-\omega(A)\omega(B)\Big)\cos kz \tag{3.20}$$

$$\sigma_\omega^k(A,B)=\widetilde{\delta}_k\sum_z\mathrm{Im}\,\omega(A\tau_z[B])\cos kz=\widetilde{\delta}_k\frac{1}{2i}\omega([A,B]) \tag{3.21}$$

for $k \in [0,\pi]^\nu$ and $A, B \in \mathcal{K}_{sa}$.

Note that the simplification in the expression for σ_ω^k arises from the fact that A and $\tau_z[B]$ commute when $z \neq 0$.

The symplectic form σ_ω^k has a stronger characterization.

Lemma 3.5 *The commutator of the local fluctuations of one-site observables in $\mathcal{A}_{0,sa}$ converges strongly when ω is an extremal translationally invariant state on $\mathcal{A}$. Let $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ denote the GNS representation of $\mathcal{A}$ associated with ω, then for all $A, B \in \mathcal{A}_{0,sa}$ and $k \in [0,\pi]^\nu$,*

$$\text{strong} - \lim_{n\to\infty} \pi_\omega\Big(\big[F_n^k(A), F_n^k(B)\big]\Big) = i\, \sigma_\omega^k(A,B)\, \mathbb{1}_\omega \,, \tag{3.22}$$

where $\mathbb{1}_\omega$ is the unity in $\pi_\omega(\mathcal{A})$.

proof: Take $k \in [0,\pi]^\nu$ and $A, B \in \mathcal{A}_{0,sa}$ with $\omega(A) = \omega(B) = 0$, then the commutator of $F_n^k(A)$ and $F_n^k(B)$ is

$$\begin{aligned}
&\Big[F_n^k(A), F_n^k(B)\Big] = |\Lambda_n|^{-1} \sum_{x\in\Lambda_n}\sum_{y\in\Lambda_n} \cos kx\ \cos ky\ \Big[\tau_x[A], \tau_y[B]\Big] \\
&= |\Lambda_n|^{-1} \sum_{x\in\Lambda_n} (\cos kx)^2 \tau_x[A,B] \\
&= \tfrac{1}{4}\, |\Lambda_n|^{-1} \sum_{x\in\Lambda_n} \Big(e^{i2kx} + e^{-i2kx} + 2\Big)\tau_x[A,B]\,.
\end{aligned}$$

The result follows by the mean ergodic theorem. ■

We are now in a position to formulate and prove the *reconstruction theorem.*

Theorem 3.6 (Reconstruction theorem) *Let the extremal translationally invariant state ω on the algebra $\mathcal{A}$ on $\mathbb{Z}^\nu$ have normal fluctuations in the mode $k \in [0,\pi]^\nu$ for the subspace $\mathcal{K}$ of $\mathcal{A}_0$ (cf. definition 3.1). Then*

$$\lim_{n\to\infty} \omega\left(e^{iF_n^k(A)}\right) = \exp\Big[-\frac{1}{2} s_\omega^k(A,A)\Big] \quad \textit{for all } A \in \mathcal{K}_{sa}. \tag{3.23}$$

The state $\tilde{\omega}_k$ is the quasi-free state on the CCR algebra $\mathcal{W}(\mathcal{K}_{sa}, \sigma_\omega^k)$ which is determined by the bilinear form s_ω^k

$$\tilde{\omega}_k(W_k(A)) = \exp\Big[-\frac{1}{2} s_\omega^k(A,A)\Big] \quad \textit{for all } A \in \mathcal{K}_{sa}\,. \tag{3.24}$$

The $W_k(A), A \in \mathcal{K}_{sa}$, are the Weyl operators; the forms σ_ω^k and s_ω^k are defined in equations (3.20) and (3.21).

More explicitly, the theorem states that for all $A_1, A_2, \ldots, A_l \in \mathcal{K}_{sa}$,

$$\lim_{n\to\infty} \omega\Big(e^{iF_n^k(A_1)}\, e^{iF_n^k(A_2)} \ldots e^{iF_n^k(A_l)}\Big) = \widetilde{\omega}_k\Big(W_k(A_1)\, W_k(A_2) \ldots W_k(A_l)\Big)\,. \tag{3.25}$$

The state $\widetilde{\omega}_k$ is regular and hence for every $A \in \mathcal{K}_{sa}$ there exists a self-adjoint Bosonic field $\Phi_{\widetilde{\omega}_k}(A)$ in the GNS representation $(\mathcal{H}_{\widetilde{\omega}}, \pi_{\widetilde{\omega}}, \Omega_{\widetilde{\omega}})$ such that

$$\pi_{\widetilde{\omega}_k}(W_k(A)) = \exp i\, \Phi_{\widetilde{\omega}_k}(A)\,. \tag{3.26}$$

This implies that in the sense of the central limit (3.25), the local fluctuations converge to the bosonic fields associated with $\Big(\mathcal{W}(\mathcal{K}_{sa}, \sigma_\omega^k), \widetilde{\omega}_k\Big)$

$$\mathrm{CLT} - \lim_{n\to\infty} F_n^k(A) = \Phi_{\widetilde{\omega}_k}(A)\,. \tag{3.27}$$

proof: We prove (3.25) by induction on $l \in \mathbb{N}_0$. The basis of the induction is part (ii) of the assumption that the state ω has normal fluctuations in the mode k (cf. definition 3.1).

To prove the induction step, assume that (3.25) holds for some $l \in \mathbb{N}_0$ and fix $A_1, A_2, \ldots, A_{l+1} \in \mathcal{A}_{0,sa}$.

For convenience, write $e^{iF_n^k(A_1)} \ldots e^{iF_n^k(A_{l-1})} \equiv W_n$. By the BCH formula (2.3),

$$\lim_{n\to\infty} \Big|\omega\Big(W_n\Big[e^{iF_n^k(A_l)}\, e^{iF_n^k(A_{l+1})} - e^{i(F_n^k(A_l)+F_n^k(A_{l+1}))}\, e^{-\frac{1}{2}[F_n^k(A_l),F_n^k(A_{l+1})]}\Big]\Big)\Big| = 0\,.$$

Use the Cauchy-Schwarz inequality to derive that

$$\begin{aligned}
&\Big|\omega\Big(W_n e^{iF_n^k(A_l+A_{l+1})}\, e^{-\frac{1}{2}[F_n^k(A_l)\,,\,F_n^k(A_{l+1})]}\Big) - \omega\Big(W_n e^{iF_n^k(A_l+A_{l+1})}\Big) e^{-\frac{i}{2}\sigma_\omega(A_l,A_{l+1})}\Big|^2 \\
&\quad = \Big|\omega\Big(W_n e^{iF_n^k(A_l+A_{l+1})}\Big[e^{-\frac{1}{2}[F_n^k(A_l)\,,\,F_n^k(A_{l+1})]} - e^{-\frac{i}{2}\sigma_\omega^k(A_l,A_{l+1})}\mathbb{1}\Big]\Big)\Big|^2 \\
&\quad \leq \omega(\mathbb{1})\, \omega\Big(\Big[e^{\frac{1}{2}[F_n^k(A_l),F_n^k(A_{l+1})]} - e^{\frac{i}{2}\sigma_\omega^k(A_l,A_{l+1})}\mathbb{1}\Big]\Big[e^{-\frac{1}{2}[F_n^k(A_l),F_n^k(A_{l+1})]} - e^{-\frac{i}{2}\sigma_\omega^k(A_l,A_{l+1})}\mathbb{1}\Big]\Big) \\
&\quad = 2 - \omega\Big(e^{\frac{1}{2}[F_n^k(A_l),F_n^k(A_{l+1})]}\Big) e^{-\frac{i}{2}\sigma_\omega^k(A_l,A_{l+1})} - \omega\Big(e^{-\frac{1}{2}[F_n^k(A_l)\,,\,F_n^k(A_{l+1})]}\Big) e^{\frac{i}{2}\sigma_\omega^k(A_l,A_{l+1})}\,.
\end{aligned}$$

This expression converges to zero as $n \to \infty$ because of the strong convergence of the commutator. Combining the induction hypothesis and the above result, we find that

$$\begin{aligned}\lim_{n\to\infty}\omega\Big(W_n\, e^{iF_n^k(A_l)}\, e^{iF_n^k(A_{l+1})}\Big) &= \lim_{n\to\infty}\omega\Big(W_n\, e^{iF_n^k(A_l+A_{l+1})}\Big)e^{-\frac{i}{2}\sigma_\omega^k(A_l,A_{l+1})}\\ &= \widetilde{\omega}_k\Big(W(A_1)\dots W(A_l+A_{l+1})\Big)e^{-\frac{i}{2}\sigma_\omega^k(A_l,A_{l+1})}\\ &= \widetilde{\omega}_k\Big(W(A_1)\dots W(A_l)\, W(A_{l+1})\Big)\,.\end{aligned}$$

The last equality results from the CCR algebraic structure of $\mathcal{W}\Big(\mathcal{K}_{sa},\sigma_\omega^k\Big)$.

The only thing left to prove is positivity. Take $A,B\in\mathcal{K}_{sa}$ and use the definitions of σ_ω^k and s_ω^k along with the Cauchy-Schwarz inequality to derive that

$$\frac{1}{4}|\sigma_\omega^k(A,B)|^2 \le |\langle A,B\rangle_\omega^k|^2 \le \langle A,A\rangle_\omega^k\,\langle B,B\rangle_\omega^k = s_\omega^k(A,A)\, s_\omega^k(B,B)\,.$$

■

4 A central limit theorem

In this section, we prove that the k-mode fluctuations in a state ω are Gaussian when ω satisfies some clustering condition. The condition essentially requires that, in a sense to be made precise, the state ω carries only weak spatial correlations between far remote regions. Ultimately, one would want a theory that connects the microscopic interactions in a physical system with the clustering properties of its equilibrium states. Unfortunately, it appears to be very hard to make substantial progress in that direction in a rigorous framework. Near phase transitions, renormalization semi-group techniques have provided some tentative, mostly approximative results.

In this section, we impose exactly the same clustering condition on ω as in [3, 4] and generalize to the case $k\neq 0$ the theorems which were proven there. We use essentially the same technique to prove the more general theorem. We wish to point out that the results can be generalized to the case of strictly local observables which do not necessarily have to be one-site observables.

The idea [6] behind the proof is to divide the hypercube Λ_n with edges of 'length' $2n+1$ into smaller hypercubes Γ_n^γ with edges of length $p(n)$. In the state ω, the spatial correlations between any two of these smaller hypercubes should then become negligible as $n\to\infty$ because the minimal distance $q(n)$ between the hypercubes tends to infinity as $n\to\infty$. The hypercubes Γ_n^γ have to represent the bulk and hence the ratio $q(n)/p(n)$ has to tend to zero fast enough as $n \to \infty$. This Bernstein method is a standard technique to prove central limit theorems. In sections 18.4 and 18.5 in [7], this approach is used to prove some central limit theorems for mixing sequences of *classical* random variables.

As in [3, 4], we introduce a clustering function α_ω for the translationally invariant state ω on the quasi-local algebra $\mathcal{A}$ on the ν-dimensional lattice $\mathbb{Z}^\nu$. This provides a rigorous tool to quantify the spatial correlations between disjoint regions. For $N, d \in \mathbb{R}^+$, define the clustering function

$$\alpha_\omega(N,d) = \sup\Big\{\beta_\omega(\Lambda,\Lambda') \;\Big|\; \max\{|\Lambda|,|\Lambda'|\} \le N \ \text{ and } \ d(\Lambda,\Lambda') \ge d\Big\}, \tag{4.1}$$

where $d(\Lambda,\Lambda')$ is the 'taxi' distance between Λ and Λ'

$$d(\Lambda,\Lambda') = \min\Big\{\sum_{j=1}^{\nu} |x^j - y^j| \;\Big|\; x \in \Lambda \ \text{ and } \ y \in \Lambda'\Big\} \tag{4.2}$$

and the function $\beta_\omega(\Lambda,\Lambda')$ is a measure for the correlation between the finite regions Λ and $\Lambda' \in \mathcal{D}(\mathbb{Z}^\nu)$ with respect to the state ω

$$\beta_\omega(\Lambda,\Lambda') = \sup\Big\{\big|\omega(AB) - \omega(A)\omega(B)\big| \;\Big|\; A \in \mathcal{A}(\Lambda), B \in \mathcal{A}(\Lambda'), \|A\| = \|B\| = 1\Big\}. \tag{4.3}$$

The distance of a site $x \in \mathbb{Z}^\nu$ to the origin is

$$|x| = \sum_{j=1}^{\nu} |x^j|. \tag{4.4}$$

The distance of x to a hypercube Λ_D, with $D \in \mathbb{N}_0$, is

$$|x|_D = d(\{x\},\Lambda_D). \tag{4.5}$$

The clustering function satisfies the following simple properties

$$\alpha_\omega(N,d) \le 2 \ \text{ for all } \ N,d \in \mathbb{R}^+, \tag{4.6}$$
$$\alpha_\omega(N,d) \ge \alpha_\omega(N,d') \quad \text{if } \ d \le d', \tag{4.7}$$
$$\alpha_\omega(N,d) \le \alpha_\omega(N',d) \quad \text{if } \ N \le N'. \tag{4.8}$$

Theorem 4.1 (Central limit theorem) *If the ergodic (i.e. extremal translationally invariant) state ω on the quasi-local C^*-algebra $\mathcal{A}$ on $\mathbb{Z}^\nu$ satisfies the clustering condition expressed by the scaling law*

$$\exists\, \eta > 0 \;\Big|\; \lim_{N\to\infty} N^{1/2}\, \alpha_\omega(N, N^{1/2\nu} N^{-\eta}) = 0\,, \tag{4.9}$$

then ω has normal fluctuations in any mode $k \in [0,\pi]^\nu$ for $\mathcal{A}_0$. That is to say, fix $k \in [0,\pi]^\nu$, then

(i) *for all $A, B \in \mathcal{A}_0$*

$$\sum_{x\in\mathbb{Z}^\nu} \Big|\omega(A\tau_x[B]) - \omega(A)\omega(B)\Big| < \infty\,, \tag{4.10}$$

(ii) *for all $A \in \mathcal{A}_{0,sa}$*

$$\lim_{n\to\infty} \omega(e^{iF_n^k(A)}) = e^{-\frac{1}{2}s_\omega^k(A,A)} \tag{4.11}$$

with $s_\omega^k(A,B) = \lim_n \operatorname{Re}\omega\Big(F_n^k(A)F_n^k(B)\Big)$.

Before turning to the proof of the theorem, let us first remark that the clustering condition (4.9) is equivalent to

$$\exists\, \eta' > 0 \;\Big|\; \lim_{d\to\infty} d^{\nu+\eta'}\, \alpha_\omega(d^{2(\nu+\eta')}, d) = 0\,. \tag{4.12}$$

Either of the conditions could be called 'L^1-clustering', because they imply that the clustering function is summable in the following sense

$$\sum_x \alpha_\omega(|x|^{2(\nu+\eta')}, |x|) < \infty\,. \tag{4.13}$$

proof: The proof of (i) is straightforward. Take $A, B \in \mathcal{A}_0$ and assume $\omega(A) = \omega(B) = 0$. Using the clustering condition (4.12) and the property (4.8), it then follows that

$$\sum_x |\omega(A\tau_x[B])| \leq \|A\|\,\|B\| \sum_x \alpha_\omega(1, |x|)) < \infty\,.$$

This proves statement (i) of the theorem and as a result s_ω^k is well-defined by proposition 3.3.

To prove statement (ii), we use a Bernstein argument. Note that the Bernstein-partitioning of Λ_n uses two functions $p(n)$ and $q(n)$ which are of a different order of magnitude but which both tend to infinity as $n \to \infty$. Their ratio $q(n)/p(n)$, however, tends to zero in the limit.

Fix $\eta > 0$ and $\eta' > 0$ such that the clustering conditions (4.9) and (4.12) are satisfied. Take $A \in \mathcal{A}_{0,sa}$ with $\omega(A) = 0$. Define the functions p, q, r, and s for any $n \in \mathbb{N}_0$

$$r(n) = \mathrm{Int}\Big[(2n+1)^{\frac{1}{2}-\epsilon_1}\Big] = \vartheta(n^{\frac{1}{2}-\epsilon_1}) \tag{4.14}$$

$$s(n) = r(n)^{-1}\,(2n+1) = \vartheta(n^{\frac{1}{2}+\epsilon_1}) \tag{4.15}$$

$$q(n) = (2n+1)^{\frac{1}{2}-\epsilon_2} = \vartheta(n^{\frac{1}{2}-\epsilon_2}) \tag{4.16}$$

$$p(n) = s(n) - q(n) = \vartheta(n^{\frac{1}{2}+\epsilon_1}) \tag{4.17}$$

where $\mathrm{Int}[x]$ denotes the integer part of $x \in \mathbb{R}^+$ and

$$0 < \epsilon_1 < \min\{1/6,\, \eta'/(8\nu),\, \frac{1}{4}\eta'/(\eta'+2\nu)\} \tag{4.18}$$

$$2\epsilon_1 < \epsilon_2 < \min\{\nu\eta,\, \frac{1}{2}\eta'/(\eta'+\nu),\, \frac{1}{2}(1-8\nu\epsilon_1/\eta')\}\,. \tag{4.19}$$

The only thing that really matters in these definitions of r, p and q is their order of magnitude as $n \to \infty$. Take note that there is *no contradiction* in the choice of ϵ_1 and ϵ_2. The requirement that $\epsilon_1 < \frac{1}{4}\eta'/(\eta'+2\nu)$ is exactly what is needed to make inequalities (4.18) and (4.19) consistent.

Define $r(n)^\nu$ disjoint hypercubes Γ_n^γ with edges of 'length' $p(n)$

$$\Gamma_n^\gamma = \mathbb{Z}^\nu \cap \otimes_{i=1}^\nu \left[-n - 1/2 + \gamma^i s(n), -n - 1/2 + \gamma^i s(n) + p(n)\right[\,, \tag{4.20}$$

where $\gamma \in R_n = \{0, 1, \ldots, r(n)-1\}^\nu$. Next, define

$$\Gamma_n = \bigcup_{\gamma \in R_n} \Gamma_n^\gamma \qquad\qquad \Delta_n = \Lambda_n \setminus \Gamma_n\,. \tag{4.21}$$

The 'volume' $|\Delta_n|$ is bounded above by

$$\nu\, q(n)\, r(n) (2n+1)^{\nu-1} = \vartheta(n^{\nu-\epsilon_1-\epsilon_2})\,, \tag{4.22}$$

while $|\Gamma_n| = \vartheta(|\Lambda_n|) = \vartheta(n^\nu)$.

For ease of notation, we drop the superscript k which is fixed anyway and use the notation

$$F^n_\Lambda(A) = |\Lambda_n|^{-1/2}|\Lambda|^{1/2}F^k_\Lambda(A) = |\Lambda_n|^{-1/2}\sum_{x\in\Lambda}\tau_x[A]\cos kx\,. \tag{4.23}$$

It is important to note that, in this notation, the scaling $|\Lambda_n|^{-1/2}$ is determined by the superscript n, while x is summed over the region Λ in the subscript. Remember that $\omega(A) = 0$. Now,

$$F^k_n(A) = F^n_{\Gamma_n}(A) + F^n_{\Delta_n}(A)\,, \tag{4.24}$$

$$F^n_{\Gamma_n}(A) = \sum_{\gamma\in R_n} F^n_{\Gamma^\gamma_n}(A)\,, \tag{4.25}$$

$$[F^n_{\Gamma^\gamma_n}(A), F^n_{\Gamma^{\gamma'}_n}(A)] = 0 \tag{4.26}$$

For ease of notation, we will often drop explicit reference to $A \in \mathcal{A}_{0,sa}$

$$F^k_n(A) \equiv F^k_n\;,\ldots$$

The proof of theorem 4.1 proceeds in three steps

step 1:

$$\lim_{n\to\infty}\Big|\omega(e^{iF^k_n}) - \omega(e^{iF^n_{\Gamma_n}})\Big| = 0\,, \tag{4.27}$$

step 2:

$$\lim_{n\to\infty}\Big|\omega(e^{iF^n_{\Gamma_n}}) - \prod_{\gamma\in R_n}\omega(e^{iF^n_{\Gamma^\gamma_n}})\Big| = 0\,, \tag{4.28}$$

step 3:

$$\lim_{n\to\infty}\Big|\prod_{\gamma\in R_n}\omega(e^{iF^n_{\Gamma^\gamma_n}}) - e^{-\frac{1}{2}s^k_\omega(A,A)}\Big| = 0\,. \tag{4.29}$$

proof of step 1: Using equation (A.1) of the appendix A.1

$$e^{iF^k_n} = e^{iF^n_{\Gamma_n}} + \int_0^1 ds\, e^{isF^n_{\Gamma_n}}\, iF^n_{\Delta_n}\, e^{i(1-s)F^k_n}\,,$$

we get

$$
\begin{aligned}
&|\omega(e^{iF^k_n} - e^{iF^n_{\Gamma_n}})| \\
&\qquad \le \int_0^1 ds\, |\omega(e^{isF^n_{\Gamma_n}}\, F^n_{\Delta_n}\, e^{i(1-s)F^k_n})| \\
&\qquad \le \int_0^1 ds\, \Big\{ |\omega([e^{isF^n_{\Gamma_n}}, F^n_{\Delta_n}]\, e^{i(1-s)F^k_n})| + |\omega(F^n_{\Delta_n}\, e^{isF^n_{\Gamma_n}}\, e^{i(1-s)F^k_n})| \Big\} \\
&\qquad \le \int_0^1 ds\, \Big\{ \Big\| [e^{isF^n_{\Gamma_n}}, F^n_{\Delta_n}] \Big\| \times 1 + \sqrt{\omega([F^n_{\Delta_n}]^2)} \times 1 \Big\} \\
&\qquad \le \frac{1}{2} \Big\| [F^n_{\Gamma_n}, F^n_{\Delta_n}] \Big\| + \sqrt{\omega([F^n_{\Delta_n}]^2)}\,,
\end{aligned} \tag{4.30}
$$

where we used the inequality $|\omega(X)| \le \|X\|$, the product inequality, the Cauchy-Schwarz inequality, and inequality (A.2).

The first term in (4.30) is zero, since $[A, \tau_z[A]] = 0$ when $z \neq 0$.

The second term in (4.30) is easily seen to tend to zero as $n \to \infty$ because of part (i) of the theorem

$$
\begin{aligned}
|\omega([F^n_{\Delta_n}]^2)| &\le |\Lambda_n|^{-1} \sum_{x\in\Delta_n} \sum_{y\in\Delta_n} |\omega(A\tau_{y-x}[A])| \\
&\le |\Lambda_n|^{-1} |\Delta_n| \sum_z |\omega(A\tau_z[A])| = \vartheta(n^{-\epsilon_1-\epsilon_2})\,.
\end{aligned} \tag{4.31}
$$

proof of step 2: Because of the commutation properties (4.26)

$$
\exp iF^n_{\Gamma_n} = \prod_{\gamma\in R_n} \exp iF^n_{\Gamma^\gamma_n}\,.
$$

Order the elements γ of R_n from 0 to $r(n)^\nu - 1 \equiv r^\nu - 1$ and use a 'telescopic sum'-trick

$$
\begin{aligned}
&\Big| \omega\big(\prod_{\gamma\in R_n} e^{iF^n_{\Gamma^\gamma_n}}\big) - \prod_{\gamma\in R_n} \omega\big(e^{iF^n_{\Gamma^\gamma_n}}\big) \Big| \\
&\quad \le \sum_{m=0}^{r^\nu-2} \Big| \omega\big(\prod_{\gamma=0}^{r^\nu-1-m} e^{iF^n_{\Gamma^\gamma_n}}\big) \prod_{\zeta=r^\nu-m}^{r^\nu-1} \omega\big(e^{iF^n_{\Gamma^\zeta_n}}\big) - \omega\big(\prod_{\gamma=0}^{r^\nu-2-m} e^{iF^n_{\Gamma^\gamma_n}}\big) \prod_{\zeta=r^\nu-1-m}^{r^\nu-1} \omega\big(e^{iF^n_{\Gamma^\zeta_n}}\big) \Big| \\
&\quad \le \sum_{m=0}^{r^\nu-2} \Big| \omega\big(\prod_{\gamma=0}^{r^\nu-1-m} e^{iF^n_{\Gamma^\gamma_n}}\big) - \omega\big(\prod_{\gamma=0}^{r^\nu-2-m} e^{iF^n_{\Gamma^\gamma_n}}\big)\, \omega\big(e^{iF^n_{\Gamma^{r^\nu-1-m}_n}}\big) \Big| \times 1 \\
&\quad \le r(n)^\nu\, \alpha_\omega\Big((2n+1)^\nu,\ q(n)\Big)\,.
\end{aligned}
$$

Introduce the notation $(2n+1)^\nu \equiv N$ and make the estimate

$$r(n)^\nu \leq (2n+1)^{\nu(\frac{1}{2}-\epsilon_1)} = N^{\frac{1}{2}-\epsilon_1}\,.$$

Also, since $0 < \epsilon_2 < \eta\nu$ (remember equation (4.19)),

$$q(n) = \ (2n+1)^{(\frac{1}{2}-\epsilon_2)} = \ N^{1/(2\nu)}\, N^{-\epsilon_2/\nu} \geq N^{1/(2\nu)} N^{-\eta}\,.$$

end Using the property (4.7) of the clustering function, it is clear that

$$\Big|\omega\big(\prod_{\gamma\in R_n} e^{iF^n_{\Gamma^\gamma_n}}\big) - \prod_{\gamma\in R_n} \omega\big(e^{iF^n_{\Gamma^\gamma_n}}\big)\Big| \leq N^{\frac{1}{2}-\epsilon_1}\alpha_\omega(N, N^{1/(2\nu)}N^{-\eta})\,.$$

Because of the clustering condition (4.9), this last expression converges to zero as $n\to\infty$. Remember that $\epsilon_1 > 0$ and $N = (2n+1)^\nu$.

proof of step 3: Define

$$h^\gamma_n(t) = \omega\big(e^{itF^n_{\Gamma^\gamma_n}(A)}\big)\,.$$

Because of Taylor's theorem, there is a $c^\gamma_n(t) \in [0,t]$ such that

$$\begin{aligned} h^\gamma_n(t) &= h^\gamma_n(0) + t\frac{dh^\gamma_n}{dt}(0) + \tfrac{1}{2}\, t^2\frac{d^2h^\gamma_n}{dt^2}(0) + \tfrac{1}{3!}\, t^3\frac{d^3h^\gamma_n}{dt^3}(c^\gamma_n(t)) \\ &= 1 + 0 - \tfrac{1}{2}\, t^2\omega\Big([F^n_{\Gamma^\gamma_n}]^2\Big) - \tfrac{i}{3!}\, t^3\omega\Big([F^n_{\Gamma^\gamma_n}]^3 e^{ic^\gamma_n(t)F^n_{\Gamma^\gamma_n}}\Big)\,. \end{aligned}$$

Here, $t = 1$ and we write $c^\gamma_n(1) = c^\gamma_n$. As a result of the above

$$\omega\big(e^{iF^n_{\Gamma^\gamma_n}}\big) = 1 - r(n)^{-\nu}\ r(n)^\nu\Big\{+\tfrac{1}{2}\ \omega\Big([F^n_{\Gamma^\gamma_n}]^2\Big) + \tfrac{i}{3!}\ \omega\Big([F^n_{\Gamma^\gamma_n}]^3 e^{ic^\gamma_n F^n_{\Gamma^\gamma_n}}\Big)\Big\}\,.$$

Next, we want to use lemma A.1 from appendix A.4 with the substitutions

$$\begin{aligned} &x \to \gamma\,, \\ &\Lambda_n \to R_n\,, \\ &|\Lambda_n| \to r(n)^\nu = \# R_n\,. \end{aligned}$$

This will allow us to conclude that

$$\lim_{n\to\infty}\Big|\prod_{\gamma\in R_n}\omega\big(e^{iF^n_{\Gamma^\gamma_n}}\big) - e^{-\frac{1}{2}s^k_\omega(A,A)}\Big| = 0\,, \tag{4.32}$$

and complete the proof of theorem 4.1, if we can prove the following three lemmas.

Lemma 4.2 *There is an $M < \infty$ such that for all $n \in \mathbb{N}_0$ and $\gamma \in R_n$*

$$\left| r(n)^\nu \, \omega\Big([F^n_{\Gamma^\gamma_n}]^2\Big) \right| \leq M \, .$$

Lemma 4.3 *For all $\varepsilon > 0$, there is an $n_0 \in \mathbb{N}_0$ such that for all $n \geq n_0$*

$$\left| r(n)^\nu \, \omega\Big([F^n_{\Gamma^\gamma_n}]^3 e^{ic^\gamma_n F^n_{\Gamma^\gamma_n}}\Big) \right| \leq \varepsilon \qquad\qquad \textit{for all } \gamma \in R_n \, .$$

Lemma 4.4

$$\lim_{n\to\infty} \sum_{\gamma \in R_n} \omega\Big([F^n_{\Gamma^\gamma_n}]^2\Big) = s^k_\omega(A, A) \, .$$

proof of lemma 4.2: Reorganizing the scaling factors, we get

$$\left| r(n)^\nu \, \omega\Big([F^n_{\Gamma^\gamma_n}]^2\Big) \right| = r(n)^\nu \, |\Lambda_n|^{-1} |\Gamma^\gamma_n| \, \omega\Big([F^k_{\Gamma^\gamma_n}(A)]^2\Big) \leq \omega\Big([F^k_{\Gamma^\gamma_n}(A)]^2\Big) \, ,$$

where the last expression is nothing but the second moment of the *normally scaled* local fluctuation $F^k_{\Gamma^\gamma_n}(A)$. This second moment is finite because of statement (i) of the theorem (see equation (4.10))

$$\begin{aligned}
\omega([F_{\Gamma^\gamma_n}]^2) &= |\Gamma^\gamma_n|^{-1} \sum_{x \in \Gamma^\gamma_n} \sum_{y \in \Gamma^\gamma_n} \omega(A\tau_{y-x}[A]) \cos kx \, \cos ky \\
&\leq |\Gamma^\gamma_n|^{-1} \sum_{x \in \Gamma^\gamma_n} \sum_{z} |\omega(A\tau_z[A])| \\
&= \sum_z |\omega(A\tau_z[A])| < \infty \, .
\end{aligned}$$

proof of lemma 4.3: ¿From the Cauchy-Schwarz inequality, it follows that

$$\begin{aligned}
\left| r(n)^\nu \, \omega\Big([F^n_{\Gamma^\gamma_n}]^3 e^{ic^\gamma_n F^n_{\Gamma^\gamma_n}}\Big) \right|^2 &= \left| r(n)^\nu \, \omega\Big([F^n_{\Gamma^\gamma_n}]^2 \, e^{ic^\gamma_n F^n_{\Gamma^\gamma_n}} \, F^n_{\Gamma^\gamma_n}\Big) \right|^2 \\
&\leq \underbrace{r(n)^\nu \, \omega\Big([F^n_{\Gamma^\gamma_n}]^4\Big)} \; \underbrace{r(n)^\nu \, \omega\Big([F^n_{\Gamma^\gamma_n}]^2\Big)} \, .
\end{aligned}$$

Because of lemma 4.2, the second part of the last expression is bounded by $\sum_z |\omega(A\tau_z[A])| < \infty$. We still have to prove that the first part tends to zero when $n \to \infty$. One would expect it to tend to zero as $r(n)^{-\nu}$, because one anticipates that the fourth moment of a *normally scaled* local fluctuation converges. A straightforward estimation yields

$$
\begin{aligned}
& r(n)^\nu \, \omega([F^n_{\Gamma^\gamma_n}(A)]^4) \\
& \quad \leq |\Lambda_n|^{-2} \, r(n)^\nu \sum_{x,y,z,u \in \Gamma^\gamma_n} |\omega(A\tau_{y-x}[A]\tau_{z-x}[A]\tau_{u-x}[A])| \\
& \quad \leq |\Lambda_n|^{-2} \, r(n)^\nu \, |\Gamma^\gamma_n| \sum_{x,y,z \in \Lambda_{p(n)}} |\omega(A\tau_x[A]\tau_y[A]\tau_z[A])| \\
& \quad \leq |\Lambda_n|^{-1} \sum_{x,y,z \in \Lambda_{p(n)}} |\omega(A\tau_x[A]\tau_y[A]\tau_z[A])| \, .
\end{aligned}
$$

We formulate the desired result as a lemma where we make the transformation $p(n) \mapsto n$ and correspondingly $n \mapsto n^{2/(1+2\epsilon_1)}$.

Lemma 4.5 *For all A in $\mathcal{A}_0$ with $\omega(A) = 0$*

$$
\lim_{n\to\infty} n^{-2\nu/(1+2\epsilon_1)} \sum_{x_1,x_2,x_3 \in \Lambda_n} |\omega(A\tau_{x_1}[A]\tau_{x_2}[A]\tau_{x_3}[A])| = 0 \, . \tag{4.33}
$$

proof of lemma 4.5: We first mention that because of the clustering condition (4.12), it holds for all $\varepsilon < \eta'$ that

$$
\lim_{d\to\infty} d^\varepsilon \sum_{|x|\geq d} \alpha_\omega(d^{2(\nu+\eta')}, |x|) = 0 \, . \tag{4.34}
$$

This can be seen by making the following estimates

$$
\begin{aligned}
d^\varepsilon \sum_{|x|\geq d} \alpha_\omega(d^{2(\nu+\eta')}, |x|) \; & \leq \; d^\varepsilon \sum_{j=d}^{\infty} c_\nu j^{\nu-1} \alpha_\omega(d^{2(\nu+\eta')}, j) \\
& \leq d^\varepsilon \sum_{j=d}^{\infty} c_\nu j^{\nu-1} \alpha_\omega(j^{2(\nu+\eta')}, j) \leq \vartheta(d^{\varepsilon-\eta'}) \stackrel{d\to\infty}{\longrightarrow} 0 \, .
\end{aligned}
$$

Now, we turn to the proof of equation (4.33). We, again, use a Bernstein-type argument. Partition the hypercube Λ_n

$$
\begin{aligned}
& \Lambda_{q(n)} \, , \\
& \Gamma^2_n \equiv \Lambda_n \setminus \Lambda_{2q(n)} \, , \\
& \Delta^{1,2}_n \equiv \Lambda_{2q(n)} \setminus \Lambda_{q(n)} \, ,
\end{aligned}
$$

where for any real $t \geq 1$, $\Lambda_t = [-t, t]^\nu \cap \mathbb{Z}^\nu$. Also, introduce

$$\Gamma_n^1 \equiv \Lambda_n \setminus \Lambda_{q(n)}\,.$$

Then

$$\sum_{x_1,\, x_2,\, x_3 \,\in\, \Lambda_n} = \sum_{x_1,\, x_2,\, x_3 \,\in\, \Lambda_q} + \sum_{x_1,\, x_2,\, x_3 \,\in\, \Gamma_n^1} + \frac{1}{2} \sum_{\pi \,\in\, \mathcal{P}_3} \Big\{ \sum_{\substack{x_{\pi(1)}, x_{\pi(2)} \in \Lambda_q \\ x_{\pi(3)} \,\in\, \Gamma_n^1}} + \sum_{\substack{x_{\pi(1)} \in \Lambda_q \\ x_{\pi(2)},\, x_{\pi(3)} \,\in\, \Gamma_n^1}} \Big\}\,,$$

where the elements π of $\mathcal{P}_3$ are the six permutations of $(1, 2, 3)$. We examine the four parts in the summation separately.

$\underline{x_1, x_2, x_3 \in \Lambda_{q(n)}\ :}$

$$\begin{aligned} n^{-2\nu/(1+2\epsilon_1)} & \sum_{x_1,x_2,x_3 \in \Lambda_q} |\omega(A\tau_{x_1}[A]\tau_{x_2}[A]\tau_{x_3}[A])| \\ & \leq \|A\|^4\, n^{-2\nu/(1+2\epsilon_1)}\, (2q(n)+1)^{3\nu} \\ & = \vartheta\Big(n^{-2\nu/(1+2\epsilon_1)}\, q(n)^{3\nu}\Big) \leq \vartheta\Big(n^{-\nu\,(1-6\epsilon_1)/(2+4\epsilon_1)}\Big) \overset{n\to\infty}{\longrightarrow} 0\,. \end{aligned}$$

$\underline{x_1, x_2, x_3 \in \Gamma_n^1\ :}$ Since

$$n \geq \frac{1}{4}(2n+1) = \frac{1}{4} N^{1/\nu}\,,$$

where, as before $N = (2n+1)^\nu$, we can equivalently look at

$$\begin{aligned} N^{-2/(1+2\epsilon_1)} & \sum_{x_1,x_2,x_3 \in \Gamma_n^1} |\omega(A\tau_{x_1}[A]\tau_{x_2}[A]\tau_{x_3}[A])| \\ & \leq \|A\|^4\, N^{-2/(1+2\epsilon_1)} \sum_{x_1,x_2,x_3 \in \Gamma_n^1} \alpha_\omega(|\Lambda_n|, \min\{|x_1|, |x_2|, |x_3|\}) \\ & \leq 3\, \|A\|^4\, N^{-2/(1+2\epsilon_1)} |\Lambda_n|^2 \sum_{x \in \Gamma_n^1} \alpha_\omega(|\Lambda_n|, |x|) \\ & \leq 3\, \|A\|^4\, N^{-2/(1+2\epsilon_1)}\, N^2 \sum_{x \in \Gamma_n^1} \alpha_\omega(|\Lambda_n|, |x|) \\ & \leq 3\, \|A\|^4\, N^{4\epsilon_1/(1+2\epsilon_1)} \sum_{d=q}^{+\infty} \sum_{|y|=d} \alpha_\omega(|\Lambda_n|, |y|) \\ & \leq 3\, \|A\|^4\, q(n)^{8\nu\epsilon_1/[(1-2\epsilon_2)(1+2\epsilon_1)]} \sum_{|y| \geq q} \alpha_\omega(|\Lambda_n|, |y|) \\ & \leq 3\, \|A\|^4\, q(n)^{\varepsilon} \sum_{|y| \geq q} \alpha_\omega(|\Lambda_n|, |y|)\,, \end{aligned}$$

where we choose ε such that $8\nu\epsilon_1/(1-2\epsilon_2) < \varepsilon < \eta'$. This is possible because $\epsilon_1 < \eta'/(8\nu)$ and $\epsilon_2 < \frac{1}{2}(1 - 8\nu\epsilon_1/\eta')$. We want to use equation (4.34) to prove that this expression tends to zero and, hence, we need that

$$|\Lambda_n| \leq (q(n))^{2(\nu+\eta')} .$$

This inequality is satisfied since

$$|\Lambda_n| = (2n+1)^\nu = q(n)^{2\nu/(1-2\epsilon_2)} \text{ , and}$$

$2\nu/(1-2\epsilon_2) < 2(\nu+\eta')$ because $\epsilon_2 < \eta'/[2(\nu+\eta')]$ by the choice (4.19) of ϵ_2.

We consider the remaining two parts for a fixed, but unspecified, permutation π.

$\underline{x_{\pi(1)}, x_{\pi(2)} \in \Lambda_{q(n)} \, , \, x_{\pi(3)} \in \Gamma_n^1 \, :}$

$$\begin{aligned} & n^{-2\nu/(1+2\epsilon_1)} \sum_{\substack{x_{\pi(1)},x_{\pi(2)} \in \Lambda_q \\ x_{\pi(3)} \,\in\, \Gamma_n^1}} |\omega(A\tau_{x_1}[A]\tau_{x_2}[A]\tau_{x_3}[A])| \\ & \qquad \leq \|A\|^4 \, n^{-2\nu/(1+2\epsilon_1)} \, (2q(n)+1)^{2\nu} (2n+1)^\nu \\ & \qquad = \vartheta\Big(n^{-2\nu/(1+2\epsilon_1)} \, q(n)^{2\nu} \, n^\nu\Big) \leq \vartheta\Big(n^{-2\nu(\epsilon_2-2\epsilon_1)/(1+2\epsilon_1)}\Big) \, . \end{aligned}$$

Again, this expression tends to zero by the choice (4.19) of ϵ_2. Alternatively, one could make a sharper estimate by means of the clustering function.

$\underline{x_{\pi(1)} \in \Lambda_{q(n)} \, , \, x_{\pi(2)}, x_{\pi(3)} \in \Gamma_n^1 \, :}$

We make a further decomposition of the relevant summation

$$\sum_{\substack{x_{\pi(1)} \in \Lambda_q \\ x_{\pi(2)},x_{\pi(3)} \,\in\, \Gamma_n^1}} = \sum_{\substack{x_{\pi(1)} \in \Lambda_q \\ x_{\pi(2)},x_{\pi(3)} \,\in\, \Delta_n^{1,2}}} + \sum_{\substack{x_{\pi(1)} \in \Lambda_q \\ x_{\pi(2)} \in \Delta_n^{1,2} \\ x_{\pi(3)} \,\in\, \Gamma_n^2}} + \sum_{\substack{x_{\pi(1)} \in \Lambda_q \\ x_{\pi(2)} \in \Gamma_n^2 \\ x_{\pi(3)} \,\in\, \Delta_n^{1,2}}} + \sum_{\substack{x_{\pi(1)} \in \Lambda_q \\ x_{\pi(2)},x_{\pi(3)} \,\in\, \Gamma_n^2}} .$$

The first term can be bounded by

$$\begin{aligned} & \leq \|A\|^4 \, n^{-2\nu/(1+2\epsilon_1)} \, (2q(n)+1)^\nu \, (4q(n)+1)^{2\nu} \\ & \qquad = \vartheta\Big(n^{-2\nu/(1+2\epsilon_1)} \, q(n)^{3\nu}\Big) \\ & \qquad = \vartheta\Big(n^{-2\nu/(1+2\epsilon_1)} \, n^{3\nu(\frac{1}{2}-\epsilon_2)}\Big) \\ & \qquad \leq \vartheta\Big(n^{-\nu\,(1-6\epsilon_1)/(2+4\epsilon_1)}\Big) \overset{n\to\infty}{\longrightarrow} 0 \, . \end{aligned}$$

The second and third term are bounded by

$$
\begin{aligned}
&\|A\|^4\, n^{-2\nu/(1+2\epsilon_1)}\,(2q(n)+1)^\nu\,(4q(n)+1)^\nu\,(2n+1)^\nu\\
&\quad = \vartheta\Big(n^{-2\nu/(1+2\epsilon_1)}\,q(n)^{2\nu}\,n^\nu\Big)\\
&\quad \le \vartheta\Big(n^{-2\nu\,(\epsilon_2-2\epsilon_1)/(1+2\epsilon_1)}\Big) \overset{n\to\infty}{\longrightarrow} 0\,.
\end{aligned}
$$

Finally, let π_2, π_3 be the numerical order of $\pi(2)$, $\pi(3)$ and write for convenience $\pi(1)=\pi_1$. then the fourth term is bounded by

$$
\begin{aligned}
& n^{-2\nu/(1+2\epsilon_1)} \sum_{\substack{x_{\pi_1}\in\Lambda_q\\ x_{\pi_2},\,x_{\pi_3}\,\in\,\Gamma_n^2}} |\omega(A\tau_{x_1}[A]\tau_{x_2}[A]\tau_{x_3}[A])|\\
&\quad = n^{-2\nu/(1+2\epsilon_1)} \sum_{\substack{x_{\pi_1}\in\Lambda_q\\ x_{\pi_2},\,x_{\pi_3}\,\in\,\Gamma_n^2}} |\omega(A\tau_{x_{\pi_1}}[A]\tau_{x_{\pi_2}}[A]\tau_{x_{\pi_3}}[A])|\\
&\quad \le n^{-2\nu/(1+2\epsilon_1)} \sum_{\substack{x_{\pi_1}\in\Lambda_q\\ x_{\pi_2},\,x_{\pi_3}\,\in\,\Gamma_n^2}} \Big\{\Big|\omega(A\tau_{x_{\pi_1}}[A])\,\omega(\tau_{x_{\pi_2}}[A]\tau_{x_{\pi_3}}[A])\Big|\\
&\qquad + \Big|\omega(A\tau_{x_{\pi_1}}[A]\tau_{x_{\pi_2}}[A]\tau_{x_{\pi_3}}[A]) - \omega(A\tau_{x_{\pi_1}}[A])\,\omega(\tau_{x_{\pi_2}}[A]\tau_{x_{\pi_3}}[A])\Big|\Big\}\\
&\quad \le n^{-2\nu/(1+2\epsilon_1)}\|A\|^4 \Big\{\, |\Lambda_n|\,\Big[\sum_{x\in\Lambda_q}\alpha_\omega\Big(1,|x|\Big)\Big]\,\Big[\sum_{y\in\Lambda_{2n}}\alpha_\omega\Big(1,|y|\Big)\Big]\\
&\qquad + |\Lambda_q| \sum_{x_{\pi_2},x_{\pi_3}\in\Gamma_n^2} \alpha_\omega\Big(2,\min\{|x_{\pi_2}|_q,|x_{\pi_3}|_q\}\Big)\Big\}\\
&\quad \le n^{-2\nu/(1+2\epsilon_1)}\|A\|^4\,|\Lambda_n|\,\Big\{|\Lambda_q|\sum_{x\in\Gamma_n^2}\alpha_\omega\Big(2,|x|_q\Big) + \Big[\sum_{x\in\Lambda_{2n}}\alpha_\omega\Big(1,|x|\Big)\Big]^2\Big\}\\
&\quad \le n^{-2\nu/(1+2\epsilon_1)}\|A\|^4\,|\Lambda_n|\,\Big\{\Big[\sum_{x}\alpha_\omega\Big(1,|x|\Big)\Big]^2 + |\Lambda_q|\sum_{|x|\ge 2q}\alpha_\omega\Big(2,|x|\Big)\Big\}\\
&\quad \le n^{-2\nu/(1+2\epsilon_1)}\|A\|^4\,|\Lambda_n|\,\Big\{\Big[\sum_{x}\alpha_\omega\Big(1,|x|\Big)\Big]^2 + |\Lambda_q|\sum_{x}\alpha_\omega\Big(2,|x|\Big)\Big\}\,.
\end{aligned}
$$

The last expression tends to zero because of the summability (4.13) of the clustering function and because

$$
n^{-2\nu/(1+2\epsilon_1)}\,|\Lambda_n|\,|\Lambda_q| \le \vartheta(n^{-\nu(1-6\epsilon_1)/(2+4\epsilon_1)}) \overset{n\to\infty}{\longrightarrow} 0\,.
$$

This concludes the proof of lemma 4.5.

proof of lemma 4.4: It remains to prove lemma 4.4. We already know (see page 13) that s^k_ω is well-defined because of proposition 3.3 and statement (i) of the present theorem (theorem 4.1)

$$s^k_\omega(A,A) = \lim_{n\to\infty} \omega([F^k_n(A)]^2)\,.$$

Now, using the decomposition (4.24).

$$\begin{aligned}\omega([F^k_n(A)]^2) &= \omega([F^n_{\Gamma_n}(A) + F^n_{\Delta_n}(A)]^2) \\ &= \omega\Big([F^n_{\Gamma_n}]^2 + [F^n_{\Delta_n}]^2 + 2F^n_{\Gamma_n}F^n_{\Delta_n}\Big)\,.\end{aligned}$$

It is easy to see that the last two terms converge to zero as $n\to\infty$, because

$$|\omega\Big(F^n_{\Gamma_n}F^n_{\Delta_n}\Big)|^2 \le \omega\Big([F^n_{\Delta_n}]^2\Big)\,\omega\Big([F^n_{\Gamma_n}]^2\Big)\,,$$

and

$$\begin{aligned}\omega\Big([F^n_{\Gamma_n}]^2\Big) &\le |\Lambda_n|^{-1}\sum_{x\in\Gamma_n}\sum_{y\in\Gamma_n}|\omega(A\tau_{y-x}[A])| \\ &\le |\Lambda_n|^{-1}|\Gamma_n|\sum_z|\omega(A\tau_z[A])| \\ &\le \sum_z|\omega(A\tau_z[A])| < \infty\,.\end{aligned}$$

We already proved (see equation (4.31)) that $\lim_{n\to\infty}\omega\Big([F^n_{\Delta_n}]^2\Big) = 0$ because of the scaling. So, using the decomposition (4.25),

$$\begin{aligned}s^k_\omega(A,A) &= \lim_{n\to\infty}\omega\Big([F^k_n]^2\Big) = \lim_{n\to\infty}\omega\Big([F^n_{\Gamma_n}]^2\Big) \\ &= \lim_{n\to\infty}\Big\{\sum_{\gamma\in R_n}\omega\Big([F^n_{\Gamma^\gamma_n}]^2\Big) + \sum_{\gamma\neq\zeta\in R_n}\omega\Big(F^n_{\Gamma^\gamma_n}F^n_{\Gamma^\zeta_n}\Big)\Big\}\,.\end{aligned}$$

The second term converges to zero because of the property (4.34)

$$\begin{aligned}\Big|\sum_{\gamma\neq\zeta\in R_n}\omega\Big(F^n_{\Gamma^\gamma_n}F^n_{\Gamma^\zeta_n}\Big)\Big| &\le |\Lambda_n|^{-1}\sum_{\gamma\neq\zeta\in R_n}\ \sum_{\substack{x\,\in\,\Gamma^\gamma_n\\ y\,\in\,\Gamma^\zeta_n}}\Big|\omega\Big(A\,\tau_{y-x}[A]\Big)\Big| \\ &\le |\Lambda_n|^{-1}\sum_{\gamma\,\in\,R_n}|\Gamma^\gamma_n|\sum_{\substack{\zeta\,\in\,R_n\\ \zeta\,\neq\,\gamma}}\ \sum_{z\in(\Gamma^\zeta_n-\Gamma^\gamma_n)}\Big|\omega\Big(A\,\tau_z[A]\Big)\Big| \\ &\le |\Lambda_n|^{-1}\sum_{\gamma\,\in\,R_n}|\Gamma^\gamma_n|\sum_{z\in(\Lambda_{2n}\setminus\Lambda_q)}\Big|\omega\Big(A\,\tau_z[A]\Big)\Big| \\ &\le \|A\|^2\sum_{z\in(\Lambda_{2n}\setminus\Lambda_q)}\alpha_\omega(1,|z|) \stackrel{n\to\infty}{\longrightarrow} 0\,.\end{aligned}$$

This concludes the proof of the central limit theorem 4.1. ∎

5 The central limiting behavior for $k \neq 0$

The central limit theorem 4.1 was proven under the L^1-clustering condition (4.9) or (4.12). One would expect, however, that a system in an extremal translationally invariant state exhibits Gaussian k-mode fluctuations for $k \neq 0$ even if the state is not L^1-clustering [8, 9].

This is related to the following empirical finding. Consider a physical system in a critical translationally invariant state. Because of long-range correlations in the local order parameter field, the L^1-clustering condition may not be satisfied at criticality. A singularity in a susceptibility (e.g. magnetic susceptibility) of the system will appear at the wavevector $k = 0$. In other words, the response to infinite wavelength perturbations diverges. At wavevectors $k \neq 0$, i.e. finite wavelengths, the susceptibility displays no singularities, however.

As before, $\mathcal{A}$ is a quasi-local algebra on the square lattice $\mathbb{Z}^\nu$. The state ω is an ergodic (i.e. extremal translationally invariant) state on $\mathcal{A}$ and, hence, clusters in the mean. For simplicity, only *one* mode $k \in]0, \pi]^\nu$ is considered. ¿From now on, this mode $k \neq 0$ is fixed. The aim of this section is to find minimal conditions on the state ω such that for $A \in \mathcal{A}_{0,sa}$ the local fluctuation

$$F_n^k(A) = |\Lambda_n|^{-1/2} \sum_{x \in \Lambda_n} \cos kx \left[\tau_x[A] - \omega(\tau_x[A])\mathbb{1}\right] \tag{5.1}$$

displays Gaussian behavior in the limit $n \to \infty$

$$\lim_{n \to \infty} \omega(e^{iF_n^k(A)}) = e^{-\frac{1}{2} s_\omega^k(A,A)} \tag{5.2}$$

with $s_\omega^k(A, B) = \lim_n \mathrm{Re}\, \omega\Big(F_n^k(A) F_n^k(B)\Big)$ (condition (ii) of definition 3.1). Or less ambitiously formulated, in this section we make plausible that one should find Gaussian fluctuations for $k \neq 0$ even in critical states. The argument is an adaption of the proof of the central limit theorem 4.1. In that proof, second and fourth moments of the fluctuations have to be estimated. The principal idea is to improve on the estimation and as a result allow a relaxation of the imposed clustering conditions. The improvement is obtained by 'partially summing' the k-dependent part of the moments before proceeding with the estimation. The remaining sums are sums of *differences* of correlation functions and should be bounded even in instances where sums of the correlation functions themselves are not bounded.

On the clustering function α_ω, which was defined in equation (4.1), the following very weak condition is imposed.

Assumption 1: The state ω satisfies the clustering condition

$$\forall \ 0 < \eta_1 < \frac{1}{2}\,, \quad \exists \ \eta_2 > 0 \, \Big| \lim_{N\to\infty} N^{\frac{1}{2}-\eta_1} \alpha_\omega(N, N^{\frac{1}{2\nu}} N^{-\eta_2}) = 0\,. \tag{5.3}$$

Because the condition has to be satisfied for η_1 close to $1/2$, the clustering condition is much weaker than the L^1-clustering condition (4.9) or (4.12). Along the way, we will derive which conditions have to be imposed on the correlation functions of A (see assumptions 2 and 3 on pages 30 and 35, respectively).

Fix an arbitrary η_1 close to $1/2$ and then an η_2 such that assumption 1 is fulfilled. Take $A = A^* \in \mathcal{A}_{0,sa}$ with $\omega(A) = 0$. Make a Bernstein partitioning of the hypercube Λ_n much as in the proof of theorem 4.1. Define the functions $p(n)$, $q(n)$, $r(n)$ and $s(n)$ (cf. equations (4.14) to (4.17))

$$r(n) = \mathrm{Int}\Big[(2n+1)^{\frac{1}{2}-\epsilon_1}\Big] = \vartheta(n^{\frac{1}{2}-\epsilon_1}) \tag{5.4}$$

$$s(n) = \mathrm{O}-\mathrm{Int}\Big[r(n)^{-1}\,(2n+1)\Big] = \vartheta(n^{\frac{1}{2}+\epsilon_1}) \tag{5.5}$$

$$q(n) = \mathrm{E}-\mathrm{Int}\Big[(2n+1+2D)^{\frac{1}{2}-\epsilon_2}\Big] = \vartheta(n^{\frac{1}{2}-\epsilon_2}) \tag{5.6}$$

$$p(n) = s(n) - q(n) = \vartheta(n^{\frac{1}{2}+\epsilon_1})\,, \tag{5.7}$$

where $\mathrm{Int}(x)$, $\mathrm{O}-\mathrm{Int}(x)$, and $\mathrm{E}-\mathrm{Int}(x)$ are respectively the largest integer, odd integer, and even integer which are smaller than $x \in \mathbb{R}^+$. Take

$$0 < \epsilon_2 < \min\{\nu\eta_2, \frac{1}{2}\}\,, \tag{5.8}$$

$$0 < \max\{\frac{1}{2} - \epsilon_2, \eta_1\} < \epsilon_1 < \frac{1}{2}\,. \tag{5.9}$$

Define $r(n)^\nu$ disjoint hypercubes Γ_n^γ with edges of 'length' $p(n)$

$$\Gamma_n^\gamma = \mathbb{Z}^\nu \cap \otimes_{i=1}^\nu \left[-n + \gamma^i s(n), -n + \gamma^i s(n) + p(n) - 1\right], \tag{5.10}$$

where $\gamma \in R_n = \{0, 1, \ldots, r(n)-1\}^\nu$. Also, define

$$\Gamma_n = \bigcup_{\gamma\in R_n} \Gamma_n^\gamma \qquad\qquad \Delta_n = \Lambda_n \setminus \Gamma_n\,. \tag{5.11}$$

The following estimates hold

$$
\begin{aligned}
|\Delta_n| &\leq \nu\,(q(n)\,+1)\,(r(n)\,+1)\,(2n+1)^{\nu-1} = \vartheta(n^{\nu-\epsilon_1-\epsilon_2})\,, && (5.12)\\
|\Gamma_n| &= \vartheta(|\Lambda_n|) = \vartheta(n^{\nu})\,. && (5.13)
\end{aligned}
$$

Dropping explicit reference to the fixed $k\ \in]0,\pi]^\nu$ and $A = A^* \in \mathcal{A}(\Lambda_D)$, we introduce the notation

$$
F^n_\Lambda = |\Lambda_n|^{-1/2} \sum_{x\in\Lambda} \tau_x[A] \cos kx \tag{5.14}
$$

resembling equation (4.23). The superscript n indicates the scaling while the summation runs over the region Λ in the subscript. Remember that $\omega(A) = 0$. Using this notation

$$
\begin{aligned}
&F^k_n(A) \equiv F_n = F^n_{\Gamma_n} + F^n_{\Delta_n}\,, && (5.15)\\
&F^n_{\Gamma_n} = \sum_{\gamma\in R_n} F^n_{\Gamma^\gamma_n}\,, && (5.16)\\
&[F^n_{\Gamma^\gamma_n}\,, F^n_{\Gamma^{\gamma'}_n}] = 0 && (5.17)
\end{aligned}
$$

The argument basically proceeds through the same three steps as the proof of theorem 4.1 (see page 15). For notational convenience, we work out the details only for the 1-dimensional case $\nu = 1$. The three steps are

step 1:

$$
\lim_{n\to\infty}\Big|\omega(e^{iF_n}) - \omega(e^{iF^n_{\Gamma_n}})\Big| = 0\,, \tag{5.18}
$$

step 2:

$$
\lim_{n\to\infty}\Big|\omega(e^{iF^n_{\Gamma_n}}) - \prod_{\gamma\in R_n}\omega(e^{iF^n_{\Gamma^\gamma_n}})\Big| = 0\,, \tag{5.19}
$$

step 3:

$$
\lim_{n\to\infty}\Big|\prod_{\gamma\in R_n}\omega(e^{iF^n_{\Gamma^\gamma_n}}) - e^{-\frac{1}{2}s^k_\omega(A,A)}\Big| = 0\,. \tag{5.20}
$$

step 1: In exactly the same way as in the proof of step 1 of theorem 4.1, we arrive at the estimate

$$\left|\omega(e^{iF_n} - e^{iF^n_{\Gamma_n}})\right| \leq \frac{1}{2}\left\|[F^n_{\Gamma_n}, F^n_{\Delta_n}]\right\| + \sqrt{\omega([F^n_{\Delta_n}]^2)}\,, \tag{5.21}$$

and by the same token as there, the first term $\|[F^n_{\Gamma_n}, F^n_{\Delta_n}]\|$ is zero.

In the 1-dimensional case $\nu = 1$, the second term is easily seen to converge to zero by the estimate (5.12), provided that $\epsilon_1 + \epsilon_2 > 1/2$ (see condition (5.9))

$$|\,\omega([F^n_{\Delta_n}]^2)\,| \leq |\Lambda_n|^{-1} \sum_{x,y\in\Delta_n} |\,\omega(A\tau_{y-x}[A])\,| \leq \vartheta(n^{1-2\epsilon_1-2\epsilon_2})\,. \tag{5.22}$$

The general case $\nu \geq 1$ would require a more careful estimation. Introducing the notations

$$\overline{\Delta}_n = \bigcup_{x\in\Delta_n} (\Delta_n - x)\,, \tag{5.23}$$

$$\Delta_n(z) = \Delta_n \cap (\Delta_n - z) \qquad \text{with } z \in \overline{\Delta}_n\,, \tag{5.24}$$

$$C^k_2(z, \Delta_n) = \sum_{x\in\Delta_n(z)} \cos kx \;\; \cos k(z+x)\,, \tag{5.25}$$

the square of the second term of (5.21) can be written as

$$\begin{aligned}
\omega([F^n_{\Delta_n}]^2) &= |\Lambda_n|^{-1} \sum_{x\in\Delta_n} \sum_{y\in\Delta_n} \cos kx \;\cos ky \;\; \omega(A\tau_{y-x}[A]) \\
&= |\Lambda_n|^{-1} \sum_{x\in\Delta_n} \sum_{z\in(\Delta_n - x)} \cos kx \;\cos k(z+x) \;\; \omega(A\tau_z[A]) \\
&= |\Lambda_n|^{-1} \sum_{z\in\overline{\Delta}_n} \omega(A\tau_z[A]) \sum_{x\in\Delta_n(z)} \cos kx \;\cos k(z+x) \\
&= |\Lambda_n|^{-1} \sum_{z\in\overline{\Delta}_n} \omega(A\tau_z[A])\, C^k_2(z, \Delta_n)\,.
\end{aligned} \tag{5.26}$$

The estimation proceeds through analogous methods as the estimation of the second moments in step 3 below.

step 2: Because of the commutation properties (5.17),

$$\exp iF^n_{\Gamma_n} = \prod_{\gamma\in R_n} \exp iF^n_{\Gamma^\gamma_n}\,.$$

Proceed along the lines of the proof of step 2 of theorem 4.1 to find that

$$\Big|\omega\big(\prod_{\gamma\in R_n} e^{iF^n_{\Gamma^\gamma_n}}\big) - \prod_{\gamma\in R_n}\omega\big(e^{iF^n_{\Gamma^\gamma_n}}\big)\Big| \leq N^{\frac{1}{2}-\epsilon_1}\,\alpha_\omega(N, N^{\frac{1}{2\nu}}\,N^{-\frac{\epsilon_2}{\nu}})\,, \tag{5.27}$$

where $N=(2n+1)^\nu$ and α_ω is the clustering function (4.1). Using the weak clustering condition (5.3), expression (5.27) goes to zero because of the choices (5.8) and (5.9) for ϵ_1 and ϵ_2. Hence, step 2 (5.19) holds.

step 3: Taylor's theorem gives

$$\omega\big(e^{iF^n_{\Gamma^\gamma_n}}\big) = 1 - r(n)^{-\nu}\; r(n)^\nu\Big\{\tfrac{1}{2}\,\omega\Big([F^n_{\Gamma^\gamma_n}]^2\Big) + \tfrac{i}{3!}\,\omega\Big([F^n_{\Gamma^\gamma_n}]^3 e^{ic^\gamma_n F^n_{\Gamma^\gamma_n}}\Big)\Big\}\,,$$

with $c^\gamma_n\in[0,1]$. Lemma A.1, cast in the form

$$\lim_{n\to\infty}\prod_{\gamma\in R_n}\Big(1-\frac{a_n(\gamma)}{r(n)^\nu}\Big) = \exp -\langle a\rangle\,, \tag{5.28}$$

with

$$a_n(\gamma) = r(n)^\nu\Big\{\tfrac{1}{2}\,\omega\Big([F^n_{\Gamma^\gamma_n}]^2\Big) + \tfrac{i}{3!}\,\omega\Big([F^n_{\Gamma^\gamma_n}]^3 e^{ic^\gamma_n F^n_{\Gamma^\gamma_n}}\Big)\Big\} \tag{5.29}$$

$$\langle a\rangle = \lim_{n\to\infty} r(n)^{-\nu}\sum_{\gamma\in R_n} a_n(\gamma) \tag{5.30}$$

establishes that in the state ω, the fluctuation $F^k_n(A)$ is Gaussian in the limit $n\to\infty$ (5.2) provided that the following three conditions hold.

condition 1 There is an $M<\infty$ such that for all $n\in\mathbb{N}_0$ and $\gamma\in R_n$

$$\Big|r(n)^\nu\,\omega\Big([F^n_{\Gamma^\gamma_n}]^2\Big)\Big| \leq M\,. \tag{5.31}$$

condition 2

$$\lim_{n\to\infty}\sum_{\gamma\in R_n}\omega\Big([F^n_{\Gamma^\gamma_n}]^2\Big) = s^k_\omega(A,A) \quad \in\mathbb{R}\,. \tag{5.32}$$

condition 3 For any $\varepsilon>0$, there is an $n_0\in\mathbb{N}_0$ such that for all $n\geq n_0$

$$\Big|r(n)^\nu\,\omega\Big([F^n_{\Gamma^\gamma_n}]^3 e^{ic^\gamma_n F^n_{\Gamma^\gamma_n}}\Big)\Big| \leq \varepsilon \qquad\qquad \text{for all } \gamma\in R_n\,. \tag{5.33}$$

analysis of condition 1: Reorganizing the scaling factors, we get

$$\Big|r(n)^\nu\,\omega\Big([F^n_{\Gamma^\gamma_n}]^2\Big)\Big| = r(n)^\nu\,|\Lambda_n|^{-1}|\Gamma^\gamma_n|\,\omega\Big([F^k_{\Gamma^\gamma_n}(A)]^2\Big) \leq \omega\Big([F^k_{\Gamma^\gamma_n}(A)]^2\Big)\,,$$

where the last expression is nothing but the second moment of the *normally scaled* local fluctuation $F^k_{\Gamma^\gamma_n}(A) \equiv F_{\Gamma^\gamma_n}$. Similarly to the calculation (5.26) and with the notations

$$\overline{\Gamma}^\gamma_n = \bigcup_{x\in\Gamma^\gamma_n} (\Gamma^\gamma_n - x) \equiv \overline{\Gamma}_n \qquad \text{independent of } \gamma\,, \tag{5.34}$$

$$\Gamma^\gamma_n(z) = \Gamma^\gamma_n \cap (\Gamma^\gamma_n - z) \qquad \text{with } z \in \overline{\Gamma}_n\,, \tag{5.35}$$

$$C^k_2(z,\Gamma^\gamma_n) = \sum_{x\in\Gamma^\gamma_n(z)} \cos kx \;\; \cos k(z+x)\,, \tag{5.36}$$

the second moment of $F_{\Gamma^\gamma_n}$ can be rewritten as

$$\omega([F_{\Gamma^\gamma_n}]^2) = |\Gamma^\gamma_n|^{-1} \sum_{z\in\overline{\Gamma}_n} \omega(A\tau_z[A])\; C^k_2(z,\Gamma^\gamma_n)\,. \tag{5.37}$$

Now, for the 1-dimensional case $\nu = 1$, we have that $k \in]0,\pi]$ and with $p = p(n)$, $q = q(n)$, $r = r(n)$

$$\Gamma^\gamma_n = -n + \gamma(p+q) + \{0,1,\ldots,p-1\} \tag{5.38}$$

$$\Gamma^\gamma_n(z) = \begin{cases} -n + \gamma(p+q) + \{0,1,\ldots,p-1-z\} & \text{if } \overline{\Gamma}_n \ni z \geq 0 \\ -n + \gamma(p+q) + \{-z,\ldots,p,p-1\} & \text{if } \overline{\Gamma}_n \ni z \leq 0 \end{cases} \tag{5.39}$$

$$\overline{\Gamma}_n = \{-p+1, -p+2\ldots, p-1\} \tag{5.40}$$

where $\gamma \in \{0,1,\ldots,r-1\}$. Hence,

$$\omega([F_{\Gamma^\gamma_n}]^2) = \frac{1}{p} \sum_{z=-p+1}^{p-1} \omega(A\tau_z[A])\; C^k_2(z,\Gamma^\gamma_n)\,. \tag{5.41}$$

It follows from a straightforward calculation that (remember that $k \neq 0$)

$$C^k_2(z,\Gamma^\gamma_n) = \begin{cases} p\,(1-|z|/p)\;\; \cos kz & \text{if } \; k=\pi\,, \\ \\ \frac{1}{2}\Big[p\,(1-|z|/p)\;\; \cos kz \\ \qquad + \sin[k(p-|z|)]\; \frac{\cos[k(p-1-2n+2\gamma(p+q))]}{\sin k}\Big] & \text{if } \; k \neq \pi\,. \end{cases} \tag{5.42}$$

Transforming the Fejér convergence factor $(1-|z|/p)$ into a Cesàro average (cf. appendix A.2) and using Abel's formula for partial summation (cf. appendix

A.3), the expression for the second moment—when $k = \pi$—becomes

$$\begin{aligned}
\omega([F_{\Gamma_n^\gamma}]^2) &= \frac{1}{p}\sum_{x=0}^{p-1}\sum_{z=-x}^{x} \omega(A\tau_z[A]) \, \cos kz \\
&= \frac{1}{p}\sum_{x=0}^{p-1}\Big[\omega(A\tau_x[A]) \, C_1^k(x,\Lambda_x) - \sum_{z=-x}^{x-1} \mathbf{D}\omega(A\tau_z[A]) \, C_1^k(z,\Lambda_x)\Big] .
\end{aligned}$$

In the above expression, $\mathbf{D}$ is the *difference operator*

$$\mathbf{D}\omega(A\tau_z[A]) = \omega(A\tau_{z+1}[A]) - \omega(A\tau_z[A]) \tag{5.43}$$

and

$$C_1^k(z,\Lambda_x) = \sum_{y=-x}^{z} \cos ky = \cos[k(z-x)/2] \, \frac{\sin[k(x+z+1)/2]}{\sin k/2} , \tag{5.44}$$

which is bounded independently of x, p, or z

$$|C_1^k(z,\Lambda_x)| \le \frac{1}{\sin k/2} . \tag{5.45}$$

For the particular case $k = \pi$, the above bound is simply 1, of course. The case $k \neq \pi$ is slighly more involved, but is completely analogous to the case $k = \pi$ and leads to bounds similar to (5.45).

Recapitulating

$$\begin{aligned}
&\Big| r(n)^\nu \, \omega\Big([F_{\Gamma_n^\gamma}^n]^2\Big)\Big| \le \Big|\omega\Big([F_{\Gamma_n^\gamma}^k(A)]^2\Big)\Big| \\
&\quad \le \frac{1}{p}\sum_{x=0}^{p-1}\Bigg[\Big|\omega(A\tau_x[A]) \, C_1^k(x,\Lambda_x)\Big| + \sum_{z=-x}^{x-1}\Big|\mathbf{D}\omega(A\tau_z[A]) \, C_1^k(z,\Lambda_x)\Big|\Bigg] .
\end{aligned} \tag{5.46}$$

Assumption 2: As a result of the above, we see that condition 1 (5.31) is satisfied if we *assume* that

$$\sum_x \Big| \mathbf{D}\omega(A\tau_x[A]) \Big| < \infty . \tag{5.47}$$

Note that it is the *difference* of the two-point correlation function which has to be summable. The sum of the two-point correlation function itself may diverge, which would indicate critical behavior. It is important to stress that

l^1-summability of the two-point cluster function (condition (ii) of definition (3.1) of normal fluctuations)

$$\sum_x \left| \omega(A\tau_x[A]) - [\omega(A)]^2 \right| < \infty \tag{5.48}$$

implies that assumption 2 (5.47) is fulfilled. We re-entered the expectation value $\omega(A)$ which was chosen to be zero before.

analysis of condition 2: The analysis of condition 1 showed that the second moment of the local fluctuation $F_n^k(A)$ converges if assumption (5.47) is fulfilled. More in particular, this second moment was rewritten as a series which was assumed to be absolutely convergent. Hence, the symmetric bilinear form s_ω^k is well-defined on A

$$s_\omega^k(A, A) = \lim_{n\to\infty} \omega([F_n^k(A)]^2) \,. \tag{5.49}$$

What remains to be shown is that

$$\lim_{n\to\infty} \sum_{\gamma\in R_n} \omega\Big([F^n_{\Gamma_n^\gamma}]^2\Big) \stackrel{?}{=} \lim_{n\to\infty} \omega([F_n^k(A)]^2) \,. \tag{5.50}$$

This is trivial, however, because the '$\cos kz$'-part of $C_2^k(z, \Gamma_n^\gamma)$ (see equation (5.42)) which contributes to the limit of the second moment (5.41) is independent of γ. The '$\sin[k(p-|z|)]$'-part of $C_2^k(z, \Gamma_n^\gamma)$ does not contribute in the limit because ω clusters in the mean. Hence,

$$\begin{aligned} \lim_{n\to\infty} \sum_{\gamma\in R_n} \omega\Big([F^n_{\Gamma_n^\gamma}]^2\Big) &= \lim_{n\to\infty} \frac{r(n)\, p(n)}{2n+1}\ \omega\Big([F_{\Gamma_n^\gamma}]^2\Big) \\ &= 1 \times s_\omega^k(A, A) \end{aligned} \tag{5.51}$$

since $r(n)\, p(n) \sim 2n+1$ in the limit $n \to \infty$.

analysis of condition 3: ¿From the Cauchy-Schwarz inequality, it follows that

$$\begin{aligned} \Big| r(n)^\nu\, \omega\Big([F^n_{\Gamma_n^\gamma}]^3 e^{ic_n^\gamma F^n_{\Gamma_n^\gamma}}\Big)\Big|^2 &= \Big| r(n)^\nu\, \omega\Big([F^n_{\Gamma_n^\gamma}]^2\, e^{ic_n^\gamma F^n_{\Gamma_n^\gamma}}\ F^n_{\Gamma_n^\gamma}\Big)\Big|^2 \\ &\leq \underbrace{r(n)^\nu\, \omega\Big([F^n_{\Gamma_n^\gamma}]^4\Big)}\ \underbrace{r(n)^\nu\, \omega\Big([F^n_{\Gamma_n^\gamma}]^2\Big)}\,. \end{aligned} \tag{5.52}$$

Because of condition 1, the second part of the last expression is bounded. It remains to check whether the first part with the fourth moment tends to zero when $n \to \infty$. One would expect it to tend to zero like $r(n)^{-\nu}$, because one anticipates that the fourth moment of the *normally scaled* local fluctuation for $k \neq 0$ converges.

The computations and estimations for the fourth moment with the *cosines* are rather cumbersome. The law of conservation of misery ensures that the number of different terms increases from 1 to $2^4 = 16$ when the cosines are written as *complex exponentials*. However, the 16 resulting terms can be treated on the same footing.

Because the state ω is translationally invariant and because $p(n) \in 2\mathbb{N}+1$, the summations over Γ_n^γ can be rewritten as summations over

$$\Lambda_{\frac{p(n)-1}{2}} = \mathbb{Z}^\nu \cap [\tfrac{-p(n)+1}{2}, \tfrac{p(n)-1}{2}]^\nu . \tag{5.53}$$

This creates a γ-dependent phase factor $f_{\gamma,n}(\kappa)$. The notation κ has four components $\kappa = (\kappa_x, \kappa_y, \kappa_z, \kappa_u)$ each of which can take on the values ± 1, resulting in 16 possible combinations for κ.

If we introduce the notation

$$G_n^k(\pm, A) \equiv G_n(\pm) = |\Lambda_n|^{-1/2} \sum_{x \in \Lambda_n} e^{\pm ikx} \Big[\tau_x[A] - \underbrace{\omega(A)\mathbb{1}}_{=0}\Big] , \tag{5.54}$$

then

$$\begin{aligned}
r(n)^\nu\, \omega\Big([F_{\Gamma_n^\gamma}^n]^4\Big) \\
&= \frac{1}{16} \frac{r(n)^\nu}{|\Lambda_n|^2} \sum_\kappa \sum_{\substack{x,y,z,u \\ \in \Gamma_n^\gamma}} e^{ik(\kappa_x x + \kappa_y y + \kappa_z z + \kappa_u u)} \omega\big(\tau_x[A]\tau_y[A]\tau_z[A]\tau_u[A]\big) \\
&= \frac{1}{16} \frac{r(n)^\nu}{|\Lambda_n|^2} \sum_\kappa f_{\gamma,n}(\kappa) \sum_{\substack{x',y',z',u' \\ \in \Lambda_{(p(n)-1)/2}}} e^{ik(\kappa_x x' + \kappa_y y' + \kappa_z z' + \kappa_u u')} \omega\big(\tau_{x'}[A]\tau_{y'}[A]\tau_{z'}[A]\tau_{u'}[A]\big) \\
&= \frac{r(n)^\nu p(n)^{2\nu}}{16\,(2n+1)^{2\nu}} \sum_\kappa f_{\gamma,n}(\kappa)\ \omega\Big(\prod_\zeta\ G^k_{(p(n)-1)/2}(\kappa_\zeta, A)\Big) \\
&\leq \frac{1}{16\, r(n)^\nu} \sum_\kappa f_{\gamma,n}(\kappa)\ \omega\Big(\prod_\zeta\ G^k_{(p(n)-1)/2}(\kappa_\zeta, A)\Big) .
\end{aligned} \tag{5.55}$$

The index ζ in the fourth moment of the 'G-type' fluctuations indicates x, y, z, or u in that order. The γ-dependent phase factor is a result of the transformations

$$x^j = x'^j - n + \gamma^j s(n) + (p(n)-1)/2 \qquad \text{with } j = 1, \ldots, \nu, \tag{5.56}$$

and similarly for y, z, and u. Explicitly,

$$f_{\gamma,n}(\kappa) = \exp\Big[i \sum_{\zeta} \kappa_\zeta \sum_{j=1}^{\nu} k^j \{-n + \gamma^j s(n) + (p(n)-1)/2\}\Big] . \tag{5.57}$$

We continue the computation for a fixed κ and focus on the fourth moment of the 'G-type' fluctuations. For convenience $(p(n)-1)/2$ is rescaled to n, so we have to check whether (cf. the reformulations of condition 3 in (5.52) and (5.55))

$$n^{-\nu\frac{1-2\epsilon_1}{1+2\epsilon_1}}\ \omega\Big(G_n(\kappa_x)\, G_n(\kappa_y)\, G_n(\kappa_z)\, G_n(\kappa_u)\Big) \stackrel{n\to\infty}{\longrightarrow} 0\,, \tag{5.58}$$

where $1 - 2\epsilon_1 > 0$ because of (5.9). With the 4-point correlation function

$$f(x,y,z,u) = \omega(\tau_x[A]\tau_y[A]\tau_z[A]\tau_u[A]) = f(0, y-x, z-x, u-x)\,, \tag{5.59}$$

the fourth moment becomes

$$\omega\Big(\prod_{\zeta} G_n(\kappa_\zeta)\Big) = \frac{1}{|\Lambda_n|^2} \sum_{\substack{x,y,z,u\\ \in\, \Lambda_n}} e^{ik(\kappa_x x + \kappa_y y + \kappa_z z + \kappa_u u)}\, f(x,y,z,u)\,. \tag{5.60}$$

This expression will be partially summed. Note that $[A, \tau_x[A]] = 0$ because $A \in \mathcal{A}_0$, and hence

$$f(x,y,z,u) = f(\pi(x), \pi(y), \pi(z), \pi(u)) \tag{5.61}$$

where $\pi \in \mathcal{P}_4$ is any permutation of (x,y,z,u). Again, we restrict ourselves to the 1-dimensional case $\nu = 1$. For the partial summation, we need the sum $E_n^{\pm}(x)$ of the exponential $e^{\pm ikx}$ with $k \in\,]0,\pi]$ (cf. appendix A.3). For $x \in \Lambda_n$,

$$E_n^{\pm}(x) = \sum_{x'=-n}^{x} e^{\pm ikx'} = e^{\pm ik(x-n)/2}\, \frac{\sin k(x+n+1)/2}{\sin k/2} \tag{5.62}$$

$$= \frac{\pm 1}{2i \sin k/2} (e^{\pm ik(x+\frac{1}{2})} - e^{\mp ik(n+\frac{1}{2})})\,, \tag{5.63}$$

and $E_n^{\pm}(x) = 0$ when $x \notin \Lambda_n$. We have that

$$|E_n^{\pm}(x)| \leq \frac{1}{\sin k/2} \quad \text{independent of } \pm,\ x, \text{ or } n. \tag{5.64}$$

By four times applying Abel's partial summation to expression (5.60) (cf. appendix equation (A.25)), sixteen ($= 2^4$) terms emerge. Because of the symmetry (5.61) these reduce to five distinct terms

$$(2n+1)^2\,\omega\Big(G_n(\kappa_x)\,G_n(\kappa_y)\,G_n(\kappa_z)\,G_n(\kappa_u)\Big) = \tag{5.65}$$

$$\begin{aligned}
&E_n^{\kappa_x}(n)\,E_n^{\kappa_y}(n)\,E_n^{\kappa_z}(n)\,E_n^{\kappa_u}(n)\,f(n,n,n,n)\\
&-4\sum_{x=-n}^{n-1} E_n^{\kappa_x}(x)\,E_n^{\kappa_y}(n)\,E_n^{\kappa_z}(n)\,E_n^{\kappa_u}(n)\,\mathbf{D}_1 f(x,n,n,n)\\
&+6\sum_{x,y=-n}^{n-1} E_n^{\kappa_x}(x)\,E_n^{\kappa_y}(y)\,E_n^{\kappa_z}(n)\,E_n^{\kappa_u}(n)\,\mathbf{D}_1\mathbf{D}_2 f(x,y,n,n)\\
&-4\sum_{x,y,z=-n}^{n-1} E_n^{\kappa_x}(x)\,E_n^{\kappa_y}(y)\,E_n^{\kappa_z}(z)\,E_n^{\kappa_u}(n)\,\mathbf{D}_1\mathbf{D}_2\mathbf{D}_3 f(x,y,z,n)\\
&+\sum_{x,y,z,u=-n}^{n-1} E_n^{\kappa_x}(x)\,E_n^{\kappa_y}(y)\,E_n^{\kappa_z}(z)\,E_n^{\kappa_u}(u)\,\mathbf{D}_1\mathbf{D}_2\mathbf{D}_3\mathbf{D}_4 f(x,y,z,u)\,,
\end{aligned}$$

where $\mathbf{D}_j$, is the difference operator with respect to the j-th argument (cf. appendix equation (A.26)).

We still have to check the condition (5.58). In light of the scaling factor in that condition and because of the bound (5.64), the first 3 terms of (5.65) do not create any problems with respect to condition 3 (equation (5.33)).

The last term of (5.65) can be simplified further by using the translational invariance (5.59) of the state ω

$$\begin{aligned}
&\sum_{x=-n}^{n-1}\ \sum_{\substack{y,z,u=\\-n-x}}^{n-1-x} E_n^{\kappa_x}(x)\,E_n^{\kappa_y}(y+x)\,E_n^{\kappa_z}(z+x)\,E_n^{\kappa_u}(u+x)\,\mathbf{D}_1\mathbf{D}_2\mathbf{D}_3\mathbf{D}_4 f(0,y,z,u)\\
&\quad=\sum_{\substack{y,z,u=\\-2n+1}}^{2n-1} \mathbf{D}_1\mathbf{D}_2\mathbf{D}_3\mathbf{D}_4 f(0,y,z,u) \qquad\qquad (5.66)\\
&\qquad\qquad\times\sum_{x=-n-m}^{n-1-M} E_n^{\kappa_x}(x)\,E_n^{\kappa_y}(y+x)\,E_n^{\kappa_z}(z+x)\,E_n^{\kappa_u}(u+x)\ ,
\end{aligned}$$

where the shorthand

$$M = \max\{0,y,z,u\} \qquad\qquad m = \min\{0,y,z,u\} \tag{5.67}$$

was used. ¿From equation (5.63), we can write out the last factor of (5.66) as sixteen separate terms. It is clear that there will always be terms proportional to $2n - M + m$ as a result of the summation $\sum_{x=-n-m}^{n-1-M}$. These terms 'use up' one scaling factor $2n+1$ of equation (5.65). Looking back at equations (5.58)) and (5.65) and noticing that

$$-1 - \frac{1-2\epsilon_1}{1+2\epsilon_1} = -\frac{2}{1+2\epsilon_1}\,, \tag{5.68}$$

we see that the last term of (5.65) is under control if we require that

$$\lim_{n\to\infty} n^{-\frac{2}{1+2\epsilon_1}} \sum_{y,z,u=-n}^{n} \Big|\mathbf{D}_1\mathbf{D}_2\mathbf{D}_3\mathbf{D}_4 f(0,y,z,u)\Big| = 0\,. \tag{5.69}$$

In equation (5.65), there remains one more term to be estimated—the one with third order differences. Using the translational invariance (5.59) of the state ω, this term is under control as long as the following limit holds

$$\lim_{n\to\infty} n^{-2-\frac{1-2\epsilon_1}{1+2\epsilon_1}} \sum_{x,y,z=-2n}^{0} \Big|\mathbf{D}_1\mathbf{D}_2\mathbf{D}_3 f(x,y,z,0)\Big| = 0\,. \tag{5.70}$$

Comparing conditions (5.69) and (5.70) with lemma 4.5 on page 19, it is clear that the partial summation has indeed relaxed the imposed conditions. Lemma 4.5, which was proved under the assumption of L^1-clustering (4.9), *implies* conditions (5.69) and (5.70). Because of the difference operators $\mathbf{D}_j$, the latter conditions should be satisfied also in situations where the state ω is not L^1-clustering.

Assumption 3: The behavior of the 4-point correlation function

$$f(x,y,z,u) = \omega(\tau_x[A]\tau_y[A]\tau_z[A]\tau_u[A]) \tag{5.71}$$

is such that conditions (5.69) and (5.70) are fulfilled.

Finally, we wish to remark that the approach with the 'G-type fluctuations' can also be applied to the estimation of second moments.

A Some useful relations

A.1 Some estimates for elements of a C^*-algebra

Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{A}_{sa}$ its self-adjoint part. For all $x, y \in \mathcal{A}_{sa}$ and $z \in \mathcal{A}$

$$(1)\quad e^{it(x+y)} = e^{itx} + \int_0^t ds\, e^{isx}\, iy\, e^{i(t-s)(x+y)}\,, \tag{A.1}$$

$$(2)\quad \|[e^{ix}, z]\| \leq \|[x, z]\|\,, \tag{A.2}$$

$$(3)\quad \|e^{i(x+y)} - e^{ix}\| \leq \|y\|\,, \tag{A.3}$$

$$(4)\quad \|[e^{ix}, e^{iy}]\| \leq \|[x, y]\|\,, \tag{A.4}$$

$$(5)\quad \|e^{i(x+y)} - e^{ix}e^{iy}\| \leq \frac{1}{2}\|[x, y]\|\,. \tag{A.5}$$

A.2 Fejér convergence factors and Cesàro averages

The Fejér convergence factor is sometimes used to define Fourier transforms on a set of functions which are not necessarily l^1-summable or L^1-integrable [7].

Λ_n is the centered hypercube in the square lattice $\mathbb{Z}^\nu$ with edges of length $2n+1$

$$\Lambda_n = \mathbb{Z}^\nu \cap [-n, n]^\nu\,. \tag{A.6}$$

For any function f on $\mathbb{Z}^\nu$ (or sequence f when $\nu = 1$)

$$f : \mathbb{Z}^\nu \to \mathbb{C} : z \mapsto f(z)\,, \tag{A.7}$$

the following equality holds

$$\sum_{z \in \Lambda_{n-1}} f(z) \prod_{j=1}^{\nu} \Big[1 - \frac{|z^j|}{n}\Big] = \prod_{j=1}^{\nu} \Big[\,\frac{1}{n} \sum_{x^j=0}^{n-1}\ \sum_{z^j=-x^j}^{x^j}\Big] f(z)\,. \tag{A.8}$$

The left-hand side of this expression is a sum with a Fejér convergence factor

$$\prod_{j=1}^{\nu} \Big[1 - \frac{|z^j|}{n}\Big]\,, \tag{A.9}$$

while the right-hand side is a Cesàro average. Since the argument does not intertwine dimensions, we only write out the proof of equality (A.8) for one dimension

$$\begin{aligned}\frac{1}{n}\sum_{x^j=0}^{n-1}\sum_{z^j=-x^j}^{x^j} f(z^j) &= \frac{1}{n}\sum_{z^j=0}^{n-1}\sum_{x^j=z^j}^{n-1} f(z^j) + \frac{1}{n}\sum_{z^j=-n+1}^{-1}\sum_{x^j=-z^j}^{n-1} f(z^j) \\ &= \frac{1}{n}\sum_{z^j=0}^{n-1}[n-z^j]\, f(z^j) + \frac{1}{n}\sum_{z^j=-n+1}^{-1}[n+z^j]\, f(z^j) \\ &= \sum_{z^j=-n+1}^{n-1} f(z^j)\,[1-\frac{|z^j|}{n}]\,. \end{aligned} \tag{A.10}$$

Next, we recollect that summability of a sequence $\big(f(z)\big)_{z\in\mathbb{Z}^\nu}$ implies *Cesàro* summability. Moreover, the limits are the same

$$\lim_{n\to\infty}\sum_{z=-n}^{n} f(z) = a\in\mathbb{C} \Longrightarrow \lim_{n\to\infty}\frac{1}{n}\sum_{x=0}^{n-1}\sum_{z=-x}^{x} f(z) = a\,. \tag{A.11}$$

Note that the sequence $(f(z))_z$ does not have to be *absolutely* summable (i.e. l^1-summable). For the sake of completeness, the proof of statement (A.11) follows. First, write

$$\Big|a-\frac{1}{n}\sum_{x=0}^{n-1}\sum_{z=-x}^{x} f(z)\Big| = \Big|\frac{1}{n}\sum_{x=0}^{n-1} g(x)\Big|\,, \tag{A.12}$$

with

$$g(x) = a - \sum_{z=-x}^{x} f(z) \overset{x\to\infty}{\longrightarrow} 0\,. \tag{A.13}$$

With $0<\lambda<1$,

$$\Big|\,\frac{1}{n}\sum_{x=0}^{n-1} g(x)\,\Big| \le \Big|\,\frac{1}{n^{1-\lambda}}\frac{1}{n^\lambda}\sum_{x=0}^{[(n-1)^\lambda]} g(x)\,\Big| + \Big|\,\frac{1}{n}\sum_{x=[(n-1)^\lambda]+1}^{n-1} g(x)\,\Big|\,. \tag{A.14}$$

The last expression tends to zero as $n\to\infty$ because $g(x)\overset{x\to\infty}{\longrightarrow} 0$ (A.13) and $\sup_x|g(x)|<\infty$. In the above expression, $[x]$ denotes the integer part of $x\in\mathbb{R}$.

The generalization to higher dimensions is straightforward. For $\xi \in \mathbb{N}^\nu$, denote

$$\Lambda_\xi = \mathbb{Z}^\nu \cap \otimes_{j=1}^{\nu} [-\xi^j, \xi^j] \tag{A.15}$$

then

$$\lim_{\xi} \sum_{z \in \Lambda_\xi} f(z) = a \in \mathbb{C} \Longrightarrow \lim_{n \to \infty} \sum_{z \in \Lambda_{n-1}} f(z) \prod_{j=1}^{\nu} \Big[1 - \frac{|z^j|}{n}\Big] = a\,. \tag{A.16}$$

The convergence in the premise is taken to mean that for all $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $\xi \in \mathbb{N}^\nu$ with $\xi^j > N$, $j = 1, \ldots, \nu$,

$$\Big|a - \sum_{z \in \Lambda_\xi} f(z)\Big| < \varepsilon\,. \tag{A.17}$$

A.3 Abel's partial summation

Abel's partial summation is nothing but a discrete version of partial integration of ordinary calculus.

Let $\Lambda \in \mathcal{D}(\mathbb{Z}^\nu)$ be a *finite* subset of the ν-dimensional square lattice $\mathbb{Z}^\nu$. Consider two functions f and g on the lattice

$$f\,,\, g : \mathbb{Z}^\nu \to \mathbb{C} : z \mapsto f(z)\,,\, g(z) \tag{A.18}$$

and define

$$G_\Lambda(x) = \sum_{z \in \Lambda[x]} g(z) \tag{A.19}$$

$$\Lambda[x] = \{z \in \Lambda \,\big|\, z^j \leq x^j \text{ for all } j = 1, \ldots, \nu\}\,. \tag{A.20}$$

For the 1-dimensional case $\nu = 1$,

$$g(x) = G_\Lambda(x) - G_\Lambda(x-1) \quad \text{if } x \in \Lambda\,. \tag{A.21}$$

While for the 2-dimensional case $\nu = 2$, when $x \in \Lambda$

$$g(x) = G_\Lambda(x) - G_\Lambda(x - e_1) - G_\Lambda(x - e_2) + G_\Lambda(x - e_1 - e_2)\,, \tag{A.22}$$

with $e_1 = (1,0)$ and $e_2 = (0,1)$. The higher dimensional cases produce similar expressions which can be interpreted as discrete versions of partial derivatives. For notational convenience, we will proceed with the 1-dimensional case $\nu = 1$.

When $\nu = 1$, by equation (A.21)

$$\begin{aligned}
\sum_{z\in\Lambda} f(z)\ g(z) &= \sum_{z\in\Lambda} f(z)\,[G_\Lambda(z) - G_\Lambda(z-1)] \\
&= \sum_{z\in\Lambda} f(z)\ G_\Lambda(z) - \sum_{z\in(\Lambda-1)} f(z+1)\ G_\Lambda(z) \\
&= -\sum_{z\in\Lambda\cap(\Lambda-1)} [f(z+1) - f(z)]\ G_\Lambda(z) \\
&\quad + \sum_{z\in\Lambda\setminus(\Lambda-1)} f(z)\ G_\Lambda(z) - \sum_{z\in(\Lambda-1)\setminus\Lambda} f(z+1)\ G_\Lambda(z)\,. \qquad \text{(A.23)}
\end{aligned}$$

When the set Λ is an ‘interval’

$$\Lambda = \mathbb{Z} \cap [a,b] \quad \text{with} \quad a,b \in \mathbb{Z}, \tag{A.24}$$

then

$$\begin{aligned}
\sum_{z=a}^{b} f(z)\ g(z) &= f(b)\ G_\Lambda(b) - \sum_{z=a}^{b-1} [f(z+1) - f(z)]\ G_\Lambda(z) \\
&\equiv f(b)\ G_\Lambda(b) - \sum_{z=a}^{b-1} \mathbf{D}f(z)\ G_\Lambda(z)\,. \qquad \text{(A.25)}
\end{aligned}$$

In this expression, $\mathbf{D}$ is a *difference operator*

$$\mathbf{D}f(z) = f(z+1) - f(z)\,, \tag{A.26}$$

and

$$G_\Lambda(z) = \sum_{y=a}^{z} g(y)\,. \tag{A.27}$$

A.4 A useful lemma

On the complex plane, define the logarithm function with a cut along the negative real axis $]-\infty, 0]$. Then, for $z_i \in \mathbb{C}$ with $|z_i| < 1$,

$$\ln(1 - z_i) = -\sum_{l=1}^{+\infty} \frac{1}{l}\ z_i^l \tag{A.28}$$

$$\ln \prod_{i=1}^{n} (1 - z_i) = \sum_{i=1}^{n} \ln (1 - z_i) \pmod{2\pi i} \,. \tag{A.29}$$

Lemma A.1
Consider a sequence of functions $\big(a_n(x)\big)_n$ *such that* $a_n : x \in \Lambda_n \mapsto a_n(x) \in \mathbb{C}$, $|a_n(x)| \leq M < \infty$ *and*

$$\lim_{n\to\infty} |\Lambda_n|^{-1} \sum_{x\in\Lambda_n} a_n(x) = \langle a \rangle \in \mathbb{C}, \tag{A.30}$$

then

$$\lim_{n\to\infty} \prod_{x\in\Lambda_n} \Big(1 - \frac{a_n(x)}{|\Lambda_n|}\Big) = \exp - \langle a \rangle \,. \tag{A.31}$$

proof: Take n large enough such that $M < |\Lambda_n|$. The properties (A.28) and (A.29) of the logarithm function give the equalities *modulo* $2\pi i$

$$\begin{aligned}
\ln \prod_{x\in\Lambda_n} \Big(1 - \frac{a_n(x)}{|\Lambda_n|}\Big) &= \sum_{x\in\Lambda_n} \ln \Big(1 - \frac{a_n(x)}{|\Lambda_n|}\Big) \\
&= \sum_{x\in\Lambda_n} - \sum_{l=1}^{+\infty} \frac{1}{l} \Big(\frac{a_n(x)}{|\Lambda_n|}\Big)^l \\
&= -|\Lambda_n|^{-1} \sum_{x\in\Lambda_n} a_n(x) \; - \sum_{x\in\Lambda_n} \sum_{l=2}^{+\infty} \frac{1}{l} \Big(\frac{a_n(x)}{|\Lambda_n|}\Big)^l .
\end{aligned}$$

The last term goes to zero as $n \to \infty$

$$\begin{aligned}
\Big|\sum_{x\in\Lambda_n} \sum_{l=2}^{+\infty} \frac{1}{l} \Big(\frac{a_n(x)}{|\Lambda_n|}\Big)^l\Big| &\leq |\Lambda_n| \sum_{l=2}^{+\infty} \frac{1}{l} \Big(\frac{M}{|\Lambda_n|}\Big)^l \\
&\leq M \sum_{l=1}^{+\infty} \Big(\frac{M}{|\Lambda_n|}\Big)^l \\
&= M \lim_{m\to\infty} \frac{M/|\Lambda_n| - (M/|\Lambda_n|)^m}{1 - M/|\Lambda_n|} \\
&= M^2 \, |\Lambda_n|^{-1} \, (1 - M/|\Lambda_n|)^{-1} \;\overset{n\to\infty}{\longrightarrow}\; 0 \,.
\end{aligned}$$

Using this estimate and the continuity of the exponential function, the existence of the limit (A.31) follows

$$\lim_{n\to\infty} \prod_{x\in\Lambda_n} \Big(1 - \frac{a_n(x)}{|\Lambda_n|}\Big) = \lim_{n\to\infty} \exp\Big[-|\Lambda_n|^{-1} \sum_{x\in\Lambda_n} a_n(x) - \sum_{x\in\Lambda_n} \sum_{l=2}^{+\infty} \frac{1}{l} \Big(\frac{M}{|\Lambda_n|}\Big)^l\Big]$$

$$
\begin{aligned}
&= \exp \lim_{n\to\infty} \Big[-|\Lambda_n|^{-1} \sum_{x\in\Lambda_n} a_n(x) - \sum_{x\in\Lambda_n} \sum_{l=2}^{+\infty} \frac{1}{l} \Big(\frac{M}{|\Lambda_n|}\Big)^l \Big] \\
&= \exp - \langle a \rangle \, .
\end{aligned}
$$

■

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