$1_{k<n}$

BASES IN (UMD)-SPACES: APPLICATIONS TO THE NON-COMMUTATIVE SYMMETRIC SPACES

0. Introduction.

The present paper deals with the problems of existence of the Schauder bases and unconditional finite dimensional decompositions (UFDD) in some non-commutative symmetric spaces. In order to solve these problems we study the arbitrary (UMD)-spaces with strongly continuous representations of compact abelian groups. It turns out that under the suitable conditions the eigenvectors of such representations form the bases in (UMD)-spaces and eigenspaces form the unconditional finite dimensional decompositions in ones. It may be emphasized that the trigonometrical system and Walsh system in functional spaces $L^p(0,1)$, $1<p<\infty$ as well as the matrix identities in von Neumann-Schatten p-classes C^p, $1<p<\infty$ are such bases.

Usually the Hilbert transform (actually the Riesz projection) properties are used for demonstration that trigonometrical system is the basis in suitable functional spaces. S. Kwapien and A. Pelczynski (see [KP]) used the triangle projection for investigation of bases and UFDD in operator ideals. We need of a projection which can be considered as generalization of all these projections. For this purpose we define the "projection along a set" in section 1.

There are two somewhat different approaches to the problem of construction of bases in (UMD)-space, say X, in this work. The first one uses the property of boundness of vector-valued Hilbert transform in $L^2([0,1],X)$, requires the representation of connected group on space X and allows to construct the "generalized trigonometrical bases" in X. The second one uses the unconditionality property for martingale differences of $L^2([0,1],X)$, requires the representation of some group like dyadic group and allows to construct the "generalized Walsh bases" in X. In spite of the differences these approaches both are based on the application of transference method of [BGM1].

Because of length of this paper we wish to provide a section-by-section survey of the results. In the next section we establish the notations and briefly describe necessary results on non-commutative symmetric spaces, (UMD)-spaces, compact abelian groups and their representations and eigenspaces, transference method and basic definitions of the theory of bases. In this section we also define the projections along a set and applicate the transference method to these projections (Theorem 1.2).

In section 2 we prove a main result (Theorem 2.1) about unconditional finite dimensional decomposition in (UMD)-spaces.

In section 3 we study bases in (UMD)-spaces. There are two main results in this section. The first one is Theorem 3.1 about a "generalized trigonometrical basis" and the second one is Theorem 3.2 about a "generalized Walsh basis" in (UMD)-spaces.

Section 4 is devoted to examples. We would like to apply our results to the theory of bases in the non-commutative symmetric spaces. In section 4.0 we consider the same preliminaries for examples. In section 4.1 we show that a number of known bases and UFDD in $L^p([0,1])$, $1<p<\infty$ can be generated using Theorem 2.1 and 3.1. We study the generalized Walsh system and prove the analog of Paley theorem for this system (Theorem 4.2) as the application of Theorem 2.1. In section 4.2 we briefly show how a classical theorems about bases and UFDD in operator ideals follow from the results of sections 2–3. In section 4.3 we study the non-commutative L^p-spaces, $1<p<\infty$ associated with the von Neumann algebras of type I concentrating mainly on the most interesting example of such algebra, namely $L^\infty([0.1])\overline{\otimes}B(H)$ and construct examples of basis and (UFDD) (Theorem 4.3) in these spaces. Section 4.4 contains, perhaps, our most interesting contributions to the theory of bases in the non-commutative L^p-spaces, $1<p<\infty$. Namely we exhibit some "non-commutative Walsh basis" and UFDD in L^p-spaces, $1<p<\infty$ associated with the hyperfinite II_1 factor R (Theorem 4.4). There is an extension of these results to arbitrary symmetric spaces $E(R)$ associated with R and rearrangement-invariant function space E whose Boyd indexes are non-trivial in Theorem 4.4.

Finally remark that methods employed here suggest other important applications such as construction of Schauder bases in $L_p(M)$, $1<p<$

∞ where M is an injective factor of type II_∞ or type III_λ, $0<\lambda<1$, investigations of RUC-bases in non-commutative L_p-spaces and others. Several of these will be treated in future works.

1. Preliminaries.

Notations. As usual, $\mathbb{Z}, \mathbb{R}, \mathbb{C}$ and $\mathbb{Z}_n$ where n is an arbitrary natural number denote locally compact abelian groups of integers, real numbers, complex numbers and the additive finite group $\mathbb{Z}/n\mathbb{Z}$ respectively, $\mathbb{N}$ is the positive integers and $\mathbb{T}=\mathbb{R}/\mathbb{Z}$ is compact abelian connected circle group. For arbitrary compact abelian group G the dual group is denoted by $\widehat{G}$.

Let X_α, $\alpha\in I$ are subspaces of some Banach space X. The closed linear span of $\{X_\alpha\}_{\alpha\in A}$ in X we denote by $[X_\alpha]_{\alpha\in A}$.

The vector-valued L^p-space, $1\le p\le\infty$ constructed by the probability space (A,Σ_A,μ_A) and Banach space X is the Banach space

$$L^p(A,X)=\left\{f: AX \mid \int_A \|f(t)\|_X^p d\mu_A<\infty\right\} \text{ with norm } \|f\|_p=\left(\int_A \|f(t)\|_X^p d\mu_Z\right)^{1/p}.$$

Let E be a symmetric sequence space. C_E will denote the space of compact operators A on a separable HIlbert space H whose sequence of s-numbers $(s_n(A))_{n=1}^\infty$ belongs to E with norm $\|A\|_{C_E}=\|(s_n(A))_{n=1}^\infty\|_E$. In case $E=1_p$ this space is called “the von Neumann-Schatten p-class” and written as C_p. It is convenient to work with the matrix representation of the elements of C_E with respect to a fixed orthonormal basis $(f_i)_{i=1}^\infty$ of H. The matrix $(a_{ij})_{i,j=1}^\infty$ representing an element $A\in C_E$ is defined by $a_{ij}=(Af_j,f_i)$, $1\le i,j\le\infty$. The notion diag $(b_i)_{i=1}^\infty$ means the element of C_E whose matrix $(a_{ij})_{i,j=1}^\infty$ is such that $a_{ii}=b_i$ for every integer i and $a_{ij}=0$ for every unequal integers i and j. For every integers i,j define the matrix identity $E_{ij}\in C_E$ as operator whose matrix has only one non-zero entry, namely, this is 1 in the (i,j)th place. See [LTI] and [GK] for additional information on symmetric sequence space and ideals of compact operators respectively.

The algebra of all bounded linear operators on the Hilbert space H with canonical trace Tr is denoted by $B(H)$.

Non-commutative symmetric spaces. We denote by M an arbitrary semi-finite von Neumann algebra in the Hilbert space H with given normal faithful semi-finite trace τ. Let $\tau(1)=a$, $0<a\le\infty$, where 1 is the unit of M and let $E(0,a)$ be a rearrangment-invariant function space over $[0,a)$ (cf. [LTII] for a definition and basic properties). The set of all τ-measurable operators (see [FK]) will be denoted

by $K(M,\tau)$. For $x \in K(M,\tau)$ the generalized singular value function $\tau(x)$ of x is defined by

$$\tau_t(x) = \inf\{s \geq 0 \mid \tau(\{|x| > s\}) \leq t\}, \quad 0 \leq t < a,$$

where $|x| > s$ is a spectral projection for $|x| = (x^*x)^{1/2}$ corresponding to set (s,∞). We define the non-commutative symmetric space $E(M,\tau)$ of measurable operators associated with $E(0,a)$ and (M,τ) by setting

$$E(M,\tau) = \{x \in K(M,\tau) \mid \tau_t(x) \in E(0,a)\}$$

and

$$\|x\|_{E(M,\tau)} = \|\tau_t(x)\|_{E(0,a)}, \ x \in E(M,\tau).$$

Details can be found in [DDP1], [CS].

It should be noticed, that

$$L^p(M,\tau) = \left\{A \in M + L^1(M,\tau) \mid \tau(|A|^p) < \infty\right\} \text{ with norm } \|A\|_{L^p(M)} = \tau\left(|A|^p\right)^{1/p}.$$

Besides, if $M = L^\infty(0,a)$ and $\tau(f) = \int_0^a f(t)dt$ then $E(M,\tau) = E(0,a)$ and if $M = B(H)$ and τ is usual standard trace then obviously $a = \infty$ and $E(M,\tau) = C_{1_E}$, where $1_E = \left\{(x_n) \mid \sum_{n=1}^{\infty} x_n \chi_{[n-1,n)} \in E(0,\infty)\right\}$ with the norm $\|(x_n)\|_{1_E} = \left\|\sum_{n=1}^{\infty} x_n \chi_{[n-1,n)}\right\|_{E(0,\infty)}$.

UMD spaces. The Banach space X has the unconditional property for martingale differences (written as "X is (UMD)-space" or "$X \in$ (UMD)") if for $1 < p < \infty$ there are constants $C(p,X)$ such that for every martingale differences sequence $\{d_i\}_{i=1}^\infty$ in $L^p([0,1],X)$ and every sequence $\{\epsilon_i\}_{i=1}^\infty, \epsilon_i = \pm 1$

$$\left\|\sum_{i=1}^{\infty} \epsilon_i d_i\right\|_p \leq C(p,X)\left\|\sum_{i=1}^{\infty} d_i\right\|_p. \tag{1.1}$$

Obviously, we can replace signs $\{\epsilon_i\}_{i=1}^\infty$ in (1.1) on any complex numbers with module 1.

It is easy to see, that if (A, Σ_A, μ_A) is a probability space isomorphic to measure space $[0,1]$ with Lebesgue measure, and $X \in$ (UMD), so that (1.1) is true, then (1.1) also appeared to be true for every martingale differences sequence $\{d_i\}_{i=1}^\infty$ in $L^p(A,X)$.

We refer to the review [Bukh], papers [B1] and [Bur] for more details on (UMD)-spaces. The class (UMD) contains a number of classical Banach spaces, in particular, the arbitrary non-commutative L^p-spaces, $1 < p < \infty$ (see [BGM2]), and, therefore, the L^p-spaces of an arbitrary measure and von Neumann-Schatten classes C^p for $1 < p < \infty$. If

$X \in$ (UMD), then for all $1 < p < \infty$ the space $L^p([0,1], X)$ is also (UMD)-space (see [Bukh]).

Compact abelian groups. We will use compact abelian groups only. It is known that arbitrary compact abelian group G has the finite translation-invariant regular normed borelian measure (the Haar measure), which we denote by dt. The category of all compact abelian groups with the Haar measures and continuous homomorphisms we denote by $\mathbf{CAG}$. By $G \in \mathbf{CAG}$ we mean $G \in \mathrm{Obj}\ \mathbf{CAG}$. It is clear (see [Bu1]) that dual category $\mathbf{CAG}^\circ$ is the category of discrete abelian groups with the Haar measures and $G \in \mathbf{CAG}$ iff $\widehat{G} \in \mathbf{CAG}^\circ$. In the sequel we suppose $\widehat{G}$ to be countable.

Let $\{G_k, \pi_{kl}\}_{k,l=1}^{\infty}$ be the projective family in $\mathbf{CAG}$ and $\{\widehat{G}_k, \widehat{\pi}_{kl}\}_{k,l=1}^{\infty}$ is associate injective family in $\mathbf{CAG}^\circ$. We remind that the projective limit $\underset{\longrightarrow}{\lim} G_k$ belongs to $\mathbf{CAG}$ as well as the injective limit $\underset{\longleftarrow}{\lim} \widehat{G}_k$ belongs to $\mathbf{CAG}^\circ$ and, moreover, if $G = \underset{\longrightarrow}{\lim} G_k$ then $\widehat{G}_k \underset{\longleftarrow}{\lim} \widehat{G}_k$.

The following lemma is most likely to be well-known (see for example [PP], ch. 6).

Lemma 1.1. *If $G \in \mathbf{CAG}$ and $\widehat{G}$ is countable, then G has countable basis of topology. If G is infinite, then G with Haar measure dt is isomorphic (as measure space) to space $[0,1]$ with the Lebesgue measure (and therefore dt is separable atomless measure).*

Spectral projections and eigenspaces. Let $G \in \mathbf{CAG}$ and $\{R_t\}_{t\in G}$ be a strongly continuous representation of G on a Banach space X, where R_t are isomorphisms of X satisfy $\sup\limits_{t\in G} \|R_t\|_{X\to X} = c < \infty$. The spectral projection associated with $\tau \in \widehat{G}$ is the operator $E_\gamma : X \to X$ defined by

$$E_\gamma x = \int_G \gamma(t) R_{-t} x\, dt, \tag{1.2}$$

where integral exists as the Bochner integral by the Haar measure dt. It is easily seen, that E_γ are bounded projections, $\|E_\gamma\|_{X\to X} \le c$, and corresponding eigenspaces $X_\gamma = E_\gamma X$ are

$$X_\gamma = \{x \in X \mid R_t x = \gamma(t) x\}. \tag{1.3}$$

The following properties of spectral projections and eigenspaces are marked in [BGM1]:

(1.4)

$$\text{if } \gamma \neq \nu \text{ then } E_\gamma E_\nu = 0; \qquad [X_\gamma]_{\gamma\in\widehat{G}} = X; \qquad \bigcap_{\gamma\in\widehat{G}} \mathrm{Ker}\, E_\gamma = \{0\}.$$

In this paper, we denote by U such subset of $\widehat{G}$, that $\gamma \in U$ if and only if $X_\gamma \neq \{0\}$. Before proceeding it is convenient to introduce the following

Definition. Let $A \subset \widehat{G}$ be some subset of $\widehat{G}$. $T_A : X \to X$ is the projection along A (constructed by some strongly continuous representation $\{R_t\}_{t\in G}$ of some $G \in$ **CAG** if T_A is a bounded operator such that $T_A\mid_{X_\gamma}=$ Id for all $\tau \in A$ and $T_A\mid_{x_\gamma}= 0$ for all $\tau \notin A$.

We remark that if T_A exists then it uniquely determinates and, moreover, for any $A_1, A_2 \subset \widehat{G}$

$$T_{A_1\cap A_2} = T_{A_1}T_{A_2}, \qquad T_{A_1\cup A_2} = T_{A_1} + T_{A_2} - T_{A_1}T_{A_2}. \tag{1.5}$$

On the other hand even if $X = L^p(0,1), p \neq 2$, and G, $\{R_t\}_{t\in G}$ are "nice" there are many $A \subset \widehat{G}$ such that T_A does not exist. Although, in [BGM1] the following theorem was proved:

Theorem 1.1. *Let G be connected and $\widehat{G}$ be ordered groups, $A = \{\gamma \in \widehat{G} \mid \gamma > 0\}$. Then for arbitrary $X \in$ (UMD) and strongly continuous representation $\{R_t\}_{t\in G}$ exists the projection along A.*

If $X = C_p$, $1 < p < +\infty$, $R_t x = U_t^* x U_t$, where $U_t =$ diag $(e^{int})_{n=1}^{\infty}$ for any $t \in \mathbb{T}$, $x \in X$ and $\mathbb{Z} = \widehat{\mathbb{T}}$ have the natural order then projection along $A = \{\gamma \in \mathbb{Z} \mid \gamma > 0\}$ (of course, it is the set of natural numbers) is the classical triangle projection (see [KP], [Ar]). By this the projection along the positive characters would be called the generalized triangle projection.

Fejer kernels.

Lemma 1.2. *For all $G \in$ **CAG** there exist function $\{k_n\}_{n=1}^{\infty} \subset L^1(G)$ such that for all $n \in \mathbb{N} : \|k_n\|_1 \leq 2$, for all $\gamma \in \widehat{G} : \hat{k}_n(\gamma) = (k_n * \gamma)(0)$ tends to 1 if $n \to \infty$ and $\hat{k}_n(\gamma) \neq 0$ for only finitely many $\gamma \in \widehat{G}$.*

Proof. Let $\{D_n\}_{n=1}^{\infty}$, $D_{n+1} \subset D_n$ be the basis of neighborhoods of zero which exists by Lemma 1.1. Since $dt(D_n) > 0$ we can define $f_n = \chi_{(-D_n)}/dt(D_n) \in L_1(G)$. It is clear, that $\|f_n\|_1 = 1, n = 1, 2, \ldots$. Also, if $\varphi(t)$ is a continuous function in $L^1(G)$, then we have $(f_n * \varphi)(0) = \int_G f_n(t)\varphi(-t)dt = \int_{-D_n} \varphi(-t)dt/dt(D_n) = \int_{D_n} \phi(t)dt/dt(D_n) \to \varphi(0)$.

Now choose $k_n \in L_1(G)$ such that $\|f_n - k_n\|_1 < 1/n$ and $k_n = \sum\limits_{\gamma\in G} \hat{k}_n(\gamma)\gamma(t), \hat{k}_n(\gamma) \neq 0$ for only finitely many $\gamma \in \widehat{G}$ (recall that such functions are dense in $L^1(G)$, see, for example, [Bu2]). Then we have $\|k_n\|_1 \leq \|f_n\|_1 + \|f_n - k_n\|_1 \leq 2$ and $\hat{k}_n(\gamma) = (k_n * \gamma)(0) = (f_n *$

$\gamma)(0) + ((k_n - f_n) * \gamma)(0)$. Since $\left|\int_G (k_n - f_n)(-t)\gamma(t)dt\right| \leq \int_G \left|(k_n - f_n)(-t)\gamma(t)\right|dt = \int_G \left|(k_n - f_n)(-t)\right|dt = \int_G \left|(k_n - f_n)(t)\right|dt = \|k_n - f_n\|_1 < 1/n$, we obtain $\hat{k}_n(\gamma) \to 1$.

Functions $\{k_n\}_{n=1}^{\infty}$ are called "the Fejer kernels" and $k_n -$ "the n-th Fejer kernel" respectively. □

Transference method. The authors of [BGM1] have created the useful variant of the transference method. We could applicate this method to the projections along A. Namely, let $G \in \mathbf{CAG}$, X be an arbitrary Banach space and let $\{R_t\}_{t\in G}$ be a strongly continuous representation G on X.

Fix some $p \in [1, +\infty)$ and consider the Banach space $L^p(G, X)$ with canonical representation G on this space as forward translations group. Let A be an arbitrary subset of $\widehat{G}$. There exists the following connection between projection along A in X and projection along A in $L^p(G, X)$:

Theorem 1.2. *Assume that $G \in \mathbf{CAG}$ and fix some $A \subset \widehat{G}$. If projection along A constructed by representation G on $L^p(G, X)$ as forward translation group, say P_A exists then projection along A constructed by arbitrary representation $\{R_t\}_{t\in G}$ in X, say T_A exists and moreover*

$$\|T_A\|_{x\to x} \leq 2c^2 \|P_A\|_{L^p(G,X)\to L^p(G,X)}, \tag{1.6}$$

where $c = \sup\limits_{t\in G} \|R_t\|_{x\to x}$.

Proof. It is mild adaptation of [BGM1] results.

First, remark that for every $x \in X$ one has $P_A(\gamma(t)x) = \gamma(t)x$ if $\gamma \in A$ and $P_A(\gamma(t)x) = 0$ if $\gamma \notin A$.

Let k_n be the n-th Fejer kernel and $h_n = \sum\limits_{\gamma\in A} \hat{k}_n(\gamma)\gamma(t) \in L^p(G)$ (remember that $\hat{k}_n(\gamma) \neq 0$ for only finitely many $\gamma \in \widehat{G}$ so that previous sum is finite). Define transferred operator $T_n : X \to X$ by

$$T_n x = \int_G h_n(t) R_{-t} x dt.$$

Fix some $x = \sum\limits_{\gamma} x_\gamma$, $x_\gamma \in X_\gamma$, $x_\gamma \neq 0$ for only finitely many $\gamma \in \widehat{G}$. Notice that in this case $T_n x = \sum\limits_{\gamma\in A} \hat{K}_n(\gamma) x_\gamma$ and $T_n R_s = R_s T_n$ for all $s \in G$. Then $\|T_n X\|_x^p \leq c^p \|R_s T_n x\|_x^p \leq c^p \|T_n R_s x\|_x^p$ and averaged this inequality we obtain

$$\|T_n x\|_x^p \le c^p \|T_n R_s x\|_x^p ds \le c^p \int_G \left\| \int_G h_n(t) R_{s-t} x dt \right\|_x^p ds = c^p \|h_n(t)^* R_t x\|_p^p$$

$$= c^p \left\| \left(\sum_{\gamma \in A} \hat{k}_n(\gamma)\gamma(t) \right) * \left(\sum_{\gamma} x_\gamma \gamma(t) \right) \right\|_p^p = c^p \left\| \sum_{\gamma \in A} \hat{k}_n(\gamma)(\gamma * x_\gamma, \gamma') \right\|_p^p =$$

$$= c^p \left\| \sum_{\gamma \in A} \hat{k}_n(\gamma) x_\gamma \gamma(t) \right\|_p^p = c^p \left\| \sum_{\gamma \in A} \hat{k}_n(\gamma)(\gamma * x_\gamma, \gamma') \right\|_p^p =$$

$$= c^p \left\| \left(\sum_{\gamma} \hat{k}_n(\gamma)\tau(t) \right) * \left(\sum_{\gamma \in A} x_\gamma \gamma(t) \right) \right\|_p^p = c^p \left\| k_n(t) * \left(\sum_{\gamma \in A} x_\gamma \gamma(t) \right) \right\|_p^p =$$

$$c^p \|k_n(t) * P_A R_t x\|_p^p \le c^p \|k_n(t)\|_1^p \, \|P_A R_t x\|_p^p \le c^p \|k_n(t)\|_1^p \cdot$$

$$\cdot \|P_A\|_{L^p(G,X)\to L^p(G,X)}^p \|R_t\|_{x\to x}^p \|x\|_x^p \le (2c^2)^p \|P_A\| A_{L^p(G,X)\to L^p(G,X)}^p \|x\|_x^p.$$

and we can conclude that $\|T_n x\|_x \le 2c^2 \|P_A\|_{L^p(G,X)\to L^p(G,X)} \|x\|_x$ for all $n \in \mathbb{N}$ and $x \in X$ (recall that by (1.4) elements $\sum_\gamma x_\gamma$ are dense in x).

For such x as above we have $T_n \left(\sum_\gamma \right) = \sum_{\gamma \in A} \hat{k}_n(\gamma) x_\gamma \to \sum_{\gamma \in A} x_\gamma$, hence there exists the strong limit of sequence $\{T_n\}_{n=1}^\infty$ which is T_A by definition. Therefore, T_A exists and by Banach-Steinhaus theorem

$$\|T_A\|_{x\to x} \le \sup_n \|T_n\|_{x\to x} \le 2c^2 \|P_A\|_{L^p(G,X)\to L^p(G,X)}.$$

□

Bases and (UFDD). The next notions are well-known ones. Let X be an arbitrary Banach space.

A sequence $(x_n)_{n=1}^\infty \subset X$ is a (Schauder) basis in X if for any $x \in X$ there exists unique sequence $(\alpha_n)_{n=1}^\infty \subset \mathbb{C}$ such that

$$x = \sum_{n=1}^\infty \alpha_n x_n.$$

A finite-dimensional subspaces $\{X_n\}_{n=1}^\infty$ of X are unconditional finite dimensional decompositions (UFDD) of X if for any $x \in X$ exists unique sequence $(x_n)_{n=1}^\infty$ where $x_n \in X_n$ for every integer $\mathbb{N}$ such that $x = \sum_{n=1}^\infty x_n$, and there is constant $C(X)$ satisfies

$$\left\| \sum_{n=1}^\infty \epsilon_n x_n \right\|_x \le C(X) \left\| \sum_{n=1}^\infty x_n \right\|_x$$

for any choice of signs $\{\epsilon_n\}_{n=1}^{\infty}$. There are many equivalent formulations of (1.7), see [LTI]. For example, (1.7) is true iff there is constant $K(X)$ such that for any $A \subset \mathbb{N}$, Card $A < \infty$:

$$\Big\| \sum_{n \in A} x_n \Big\|_x \leq C(X) \Big\| \sum_{n=1}^{\infty} x_n \Big\|_x .$$

A basis $\{x_n\}_{n=1}^{\infty}$ of X is an unconditional basis of X if one-dimensional spaces $\{[x_n]\}_{n=1}^{\infty}$ forms (UFDD) of X.

2. Unconditional Finite Dimensional Decompositions in (UMD)-spaces.

Here and henceforth let $(G_k, \pi_{Lkl})_{l=1}^{\infty}$ be a projective family in **CAG**, $G = \lim\limits_{\longrightarrow} G \in \mathbf{CAG}$ is a projective limit of this family and $\pi_k : G \to G_k$ are canonical projections. Respectively, let $\left(\widehat{G}_k, \hat{\pi}_{kl}\right)_{k,l=1}^{\infty}$ be the associate injective family in $\mathbf{CAG}^\circ$, $\widehat{G} = \lim\limits_{\longleftarrow} \widehat{G}_k \in \mathbf{CAG}^\circ$ is the injective limit of this family and $\hat{\pi}_k : \widehat{G}_k \to \widehat{G}$ are canonical morphisms. Let X be a (UMD)-space, $\{R_t\}_{t \in G}$ be a strongly continuous representation of G on X. Let as above $U = \left\{ \gamma \in \widehat{G} \mid X_\gamma \neq \{0\} \right\} \subset \widehat{G}$ and for every $n \in \mathbb{N}$, $n > 1$ let $U_n = U \cap (\mathrm{Im}\ \hat{\pi}_n / \mathrm{Im}\ \hat{\pi}_{n-1})$, and $U_1 = U \cap (\mathrm{Im}\ \hat{\pi}_1)$. It is clear that for all $k, l \in \mathbb{N} : U_k \cap U_1 = \emptyset$ and $U = U \bigcup\limits_{k=1}^{\infty} U_k$. Let Σ be the σ-algebra of all measurable subsets of G. If B is a σ-subalgebra of Σ, then $E^B : L^p(G, X) \to L^p(G, X)$ means the conditional expectation on B.

Lemma 2.1. *Let* $B_n = \{B \in \Sigma \mid B + \mathrm{Ker}\ \pi_n = B\}$, *where* $n \in \mathbb{N}$. *Then*

(i) $\{B_n\}_{n=1}^{\infty}$ is increasing family of σ-algebras in Σ,

(ii) for all integers $k \leq n$ if $\gamma \in U_k$ then $\left(E^{B_n}\right)(\tau x) = \gamma x$, for every $x \in X$.

(iii) for all integers $k > n$ if $\gamma \in U_k$ then $(E^{B_n})(\gamma x) = 0$, for every $x \in X$.

(iv) E^{B_n} is projection along $\bigcup\limits_{k=1}^{n} U_k$ in $L^p(G, X)$ with canonical representation of G as forward translations groups.

(here γx means the functions from $L^p(G, X)$ which maps an arbitrary $t \in G$ to the point $\gamma(t) x \in X$)

Proof.

(i) It is clear that $\{B_n\}_{n=1}^{\infty}$ are σ-algebras. Since $\text{Ker } \pi_n = \{x \in G \mid \pi_n x = 0\} = \{x \in G \mid \pi_{n+1,n}\pi_{n+1}x = 0\} \supset \{x \in G \mid \pi_{n+1}x = 0\} = \text{Ker } \pi_{n+1}$ we obtain for all $B \in \Sigma$ and natural n if $B + \text{ Ker } \pi_n = B$ then obviously $B + \text{ Ker } \pi_{n+1} = B$ therefore $B_n \subset B_{n+1}$.

(ii) If $\gamma \in U_k$, $k \leq n$, then since $\gamma \in \text{ Im } \hat{\pi}_n$ there exists such $\gamma_0 \in \widehat{G}_n$, that $\gamma(t) = \gamma_0(\pi_n(t))$. Assume that $s \in \text{ Ker } \pi_n$. Then for every $x \in G$ we have $\gamma(t+s) = \gamma_0(\pi_n(t+s)) = \gamma_0(\pi_n(t)) = \gamma(t)$. Let x be an arbitrary element of X, let $\theta(t) = \gamma(t)x \in L^p(G, X)$. Then for every borelian $A \subset X$ and $s \in \text{ Ker } \pi_n$ if $t \in \theta^{-1}(A)$ then $\theta(t) \in A$ i.e. $(\gamma(t)x) \in A$ that follows $(\gamma(t+s)x) \in A$ and $(t+s) \in \theta^{-1}(A)$. Therefore $\theta^{-1}(A) + \text{ Ker } \pi_n = \theta^{-1}(A)$ i.e. θ is B_n-measurable and $(E^{B_n})(\gamma x) = \gamma x$.

(iii) Let B be a B_n-measurable subset of G. Then for every $s \in \text{ Ker } \pi_n$ we have $B + s = B$. Next, if $\gamma \in U_{n+1}$ then there is $s \in \text{ Ker } \pi_n$ such that $\gamma(s) \neq 1$. Fix such s, we obtain

$$\int_B \gamma(t)xdt = \int_{B+s} \gamma(t-s)xdt = \gamma(-s)\int_{B+s} \gamma(t)xdt = \gamma(-s)\int_B \gamma(t)xdt$$

and hence $\int_B \gamma(t)xdt = 0$ for every $B \in B_n$. Therefore, $(E^{B_n})(\gamma x) = 0$.

(iv) For every $\gamma \in \widehat{G}$ we have

$$(L^p(G,X))_\gamma = \{f \in L^p(G,X) \mid f(x+t) = \gamma(t)f(x) \text{ for all } t, x \in G\} = \{f \in L^p(G,X) \mid f(t) = \gamma(t)f(0) \text{ for all } t \in G\}.$$

Thus, if $\gamma \in \bigcup_{k=1}^{n} U_k$ and $f \in (L^p(G,X))_\gamma$ then by (ii)

$$(E^{B_n})(f(t)) = (E^{B_n})(\gamma(t)f(0)) = \gamma(t)f(0) = f(t),$$

and if $\gamma \notin \bigcup_{k=1}^{n} U_k$ and $f \in (L^p(G,X))_\gamma$ then by (iii)

$$(E^{B_n})(f(t)) = (E^{B_n})(\gamma(t)f(0)) = 0$$

and proof is completed. $\square$

The following theorem describes the unconditional finite dimensional decompositions in (UMD)-spaces with strongly continuous representation of compact abelian group.

Theorem 2.1. *Let X be an (UMD)-space, let $(G_k, \pi_{kl})_{k,l=1}^{\infty}$ be projective family in* **CAG**, *$G = \lim_{longrightarrow} G_k$, $\{R_t\}_{t\in G}$ is a strongly continuous representation of G on X, $\sup_{t\in G}\|R_t\|_{x\to x} = c < \infty$ and all spaces*

X_γ are finite dimensional. Let $U = \left\{\gamma \in \widehat{G} \mid X_\gamma \neq \{0\}\right\} \subset \widehat{G}$ and for every $n \in \mathbb{N}$, $n > 1$ let $U_n = U \cap (Im\ \hat{\pi}_n /\ Im\ \hat{\pi}_{n-1})$, $U_1 = U \cap (Im\ \hat{\pi}_1)$. If each U_n is finite set then subspaces $X_n = [X_\gamma]_{\gamma \in U_n}$ form the (UFDD) of X.

Proof. We preserve the notations of Lemma 2.1. Fix some $A \subset \mathbb{N}$, Card $A < \infty$ and consider an operator $P = \sum\limits_{k \in A}(E^{B_k} - E^{B_{k-1}})$ in $L^p(G, X)$. By Lemma 2.1 and (1.5), P is projection along $U_A = \underset{k\in A}{\cup} U_k$. Since X is (UMD)-space and by Lemma 1.1 G is isomorphic as measurable space to space $[0,1]$ with Lebesgue measure, we have that (1.1) is true for every sequence of martingale differences in $L_p(G, X)$. This fact implies that

$$\|P\|_{L^p(G,X) \to L^p(G,X)} \leq C(p, X),$$

where $C(p, X)$ is the constant from (1.1) which not dependent on A. By Theorem (1.2), for any $A \subset \mathbb{N}$, Card $A < \infty$ there exist projection T_{U_A} along U_A and

$$\|T_{U_A}\|_{x \to x} \leq K = 2c^2 C(p, X),$$

where K is constant which not dependent on A.

Now consider finite dimensional subspaces $X_n = [X_\gamma]_{\gamma \in U_n}$ in X. It is clear that $[X_n]_{n=1}^\infty = [[X_\gamma]_{\gamma \in U_n}]_{n=1}^\infty = [X_\gamma]_{\gamma \in U} = [X_\gamma]_{\gamma \in G} = X$. Consider a finite sum $\sum\limits_n x_n \in X$ where $x_n \in X_n$ for each $n \in \mathbb{N}$. There exist $x_n\gamma \in X_\gamma$, $\gamma \in U_n$ such that $x_n = \sum\limits_{\gamma \in U_n} x_n\gamma$. Then

$$\Big\|\sum_{n\in A} x_n\Big\|_x = \Big\|\sum_{n\in A}\sum_{\gamma\in U_n} x_n\gamma\Big\|_x = \Big\|\Big(T_{U_A}\sum_{n}\sum_{\gamma\in U_n} x_n\gamma\Big)\Big\|_x \leq K\Big\|\sum_{n} x_n\Big\|_x.$$

Therefore, subspaces $\{X_n\}_{n=1}^\infty$ form (UFDD) of X. □

Corollary 2.1. *Let $G = \underset{\longrightarrow}{\lim} G_k$ where each G_k is finite. If G acts strongly continuously on (UMD)-space X such that all eigenspaces are finite dimensional then X have some (UFDD).*

Proof. It is sufficient to remark that if $G = \lim\limits_{longrightarrow} G_k$ where each G_k is finite then Im $\hat{\pi}_n$ are finite for all $n \in \mathbb{N}$ and U_n are obviously finite. Using Theorem 2.1 one completes the proof.

The following corollary is much more useful for examples construction (see section 4). □

Corollary 2.2. *Let X be an (UMD)-space, $G_k \in$* **CAG** *for all $k \in \mathbb{N}$, $G = G_k$ $\{R_t\}_{t\in G}$ is strongly continuous representation of G on X, $\sup\limits_{t\in G}\|R_t\| < \infty$ and for all $\gamma = (\gamma_i)_{i=1}^{\infty} \in \widehat{G}$, $\gamma_i \in \widehat{G}_i$, $i \in \mathbb{N}$ the space X_γ is a finite dimensional. Let $U = \{\gamma \in \widehat{G} \mid X_\gamma \neq \{0\}\} \subset \widehat{G}$ and for every $n \in \mathbb{N}$ let $U_n = \{\gamma \in U \mid \gamma_n \neq 0, \gamma_k = 0, k > n\}$, $U_0 = U \cap \{0\}$. If each U_n is finite set then subspaces $X_n = [X_\gamma]_{\gamma\in U_n}$, $n \in \{0\} \cup \mathbb{N}$ form the (UFDD) in X.*

Proof. Remember that G represents as projective limit of family $H_n = G_k$, $n \in \mathbb{N}$, $H_0 = \{0\}$ with natural system of projections such that $\operatorname{Im} \hat{\pi}_n = \{\gamma \in \widehat{G} \mid \gamma_k = 0, k > n\}$, $n \in \mathbb{N}$, $\operatorname{Im} \hat{\pi}_0 = \{0\}$. Now the using of Theorem 2.1 for projective family $\{H_n\}_{n=0}^{\infty}$ completes the proof. □

3. Bases in (UMD)-spaces

Let X be an (UMD)-space, let $\{R_t\}_{t\in T^n}$ be a strongly continuous representation of group $\mathbb{T}^n$, $n \in \mathbb{N}$, $\sup\limits_t \|R_t\|_{x\to x} = c < \infty$ and suppose that the dimensions of eigenspaces are less or equal one and $\geq$ is some total ordering on $\widehat{\mathbb{T}^n} = \mathbb{Z}^n$ with respect to the additive group structure. By Theorem 1.1, in (UMD)-space $L^p(\mathbb{T}^n, X) 1 < p < \infty$ with canonical representations $\mathbb{T}^n$ as forward translation group the generalized triangle projection P exists.

Put

$$A_\gamma : L^p(\mathbb{T}^n, X) \to L^p(\mathbb{T}^n, X), (A_\gamma f)(t) = \gamma(t) f(t).$$

Notice that A_γ is an isometry of $L^p(\mathbb{T}^n, X)$ for all $\gamma \in \mathbb{Z}^n$.

Now we need the next definitions. Let $\gamma_1, \gamma_2 \in \mathbb{Z}^n$. Denote:

$$(\gamma_1, \gamma_2) \stackrel{\text{def}}{=} \{\gamma \in \mathbb{Z}^n \mid \gamma_1 \leq \gamma \leq \gamma_2\},$$
$$(\gamma_1, \infty) \stackrel{\text{def}}{=} \{\gamma \in \mathbb{Z}^n \mid \gamma_1 \leq \gamma\}, (\infty, \gamma_2) \stackrel{\text{def}}{=} \{\gamma \in \mathbb{Z}^n \mid \gamma \leq \gamma_2\}$$

("$\leq$" is the order in $\mathbb{Z}^n$). These sets are called "intervals (finite or infinite) in $\mathbb{Z}^n$".

Lemma 3.1. *For any order on $\mathbb{Z}^n$ and any interval A (finite or infinite) in $\mathbb{Z}^n$ on any (UMD)-space X there exists the projection along A. Furthermore, exists a constant $C(X)$ dependent on X but not on A and order on G such that $\|T_A\|_{x\to x} \leq C(X)$.*

Proof. It is sufficient to consider the infinite intervals (it follows from equality $(\gamma_1, \gamma_2) = (\gamma_1, \infty) \cap (\infty, \gamma_2)$ and (1.5)). Since interval (∞, γ_2) becomes to (γ_2, ∞) for reverse order in $\mathbb{Z}^n$, we can suppose that $A = (\gamma_0, \infty)$.

Now consider operator $A_{\gamma_0} P A_{-\gamma_0} : L^p(\mathbb{T}^n, X) \to L^p(\mathbb{T}^n, X)$. If $f_\gamma \in (L^p(\mathbb{T}^n, X))_\gamma$ then for all $t \in \mathbb{T}^n$ we have

$$(A_{-\gamma_0} f_\gamma)(x+t) = (-\gamma_0)(x+t) f_\gamma(x+t) = (-\gamma)(x)(-\gamma_0)(t)\gamma(t) f_\gamma(x) = \\ (\gamma - \gamma_0)(t)((-\gamma_0)(x) f_\gamma(x)) = (\gamma - \gamma_0)(t)(A_{-\gamma_0} f_\gamma)(x),$$

i.e., $(A_{-\gamma_0} f_\gamma) \in (L^p(\mathbb{T}^n, X))_{(\gamma-\gamma_0)}$ (see section 1), and therefore $(A_{\gamma_0} P A_{-\gamma_0}) f_\gamma = A_{\gamma_0} P(A_{-\gamma_0} f_\gamma) = \begin{cases} f_\gamma, & \gamma \geq \gamma_0 \\ 0, & \text{otherwise} \end{cases}$. Moreover,

$$\|A_{\gamma_0} P A_{-\gamma_0}\|_{L^p(T^n,x) \to L^p(T^n,x)} = \|P\|_{L^p(T^n,x) \to L^p(T^n,x)}$$

and with respect to the definition we can conclude that $A_{\gamma_0} P A_{-\gamma_0}$ is the projection along A. By Theorem 1.2, there exists $T_A : X \to X$ – projection along A constructed by representation $\{R_t\}_{t \in T^N}$ on X and

$$\|T_A\|_{x \to x} \leq 2c^2 \|P\|_{L^p(T^n,x) \to L^p(T^n,x)}.$$

Since $L^p(\mathbb{T}^n, X) \in$ (UMD) , norm of generalized triangle projection P is not dependent on the order in the $\mathbb{Z}^n$ and the proof is completed. $\square$

In fact, we generalized Theorem 1.1 (from $A = (0, \infty)$ to arbitrary interval). Nevertheless, for progress of theory of bases on (UMD)-spaces, a more generalization is needed.

Let $(f_i)_{i=1}^n, f_i(x) = a_{i0} + \sum_{k=1}^n a_{ik} x_k, x \in \mathbb{Z}^n, a_{ik} \in Z, i, k = 1, 2 \ldots, n$ be a system of forms on $\mathbb{Z}^n$ such that $f_i(x) = 0$ for all $i = 1, 2, \ldots, n$ implies $x = 0$.

Definition. For every form $f : \mathbb{Z}^n \to \mathbb{Z}$ let semi-space H_f, closed semi-space $\overline{H}_f$ and border ∂H_f mean the sets $\{\gamma \in \mathbb{Z}^n \mid f(\gamma) > 0\}$, $\{\gamma \in \mathbb{Z}^n \mid f(\gamma) \geq 0\}$ and $\{\gamma \in \mathbb{Z}^n \mid f(\gamma) = 0\}$ respectively.

Definition. The order constructed by system $(f_i)_{i=1}^n$ is such order "$\leq$" on $\mathbb{Z}^n$ that $\{\gamma_1 \leq \gamma_2$ iff $f_i(\gamma_1) = f_i(\gamma_2)$ for every $i \leq n$ or there exists $k_0 \leq n$ such that $f_i(\gamma_1) = f_i(\gamma_2)$ for all $i \leq k_0 - 1$ and $f_{k_0}(\gamma_1) < f_{k_0}(\gamma_2)$.

Standard procedure shows that $\mathbb{Z}^n$ with this order becomes total ordered group.

Lemma 3.2. *Let $A = H_f$ or $A = \overline{H}_f$ or $A = \partial H_f$. Then for any order, any form $f : \mathbb{Z}^n \to \mathbb{Z}$ and for any $X \in$ (UMD) there exists the projection along A. Furthermore, exists a constant $C_1(X)$ dependent on X but not on f and order on G such that $\|T_A\|_{x \to x} \leq C_1(X)$.*

Proof. We may and do assume that $f_1 \neq 0$. Then there exists system $(f_i)_{i=1}^n$ as in Definition above with $f_1 = f$. Let γ_0 be an element of $\overline{H}_f$ such that $f(\gamma_0) = \inf\limits_{\gamma \in \overline{H}_f} f(\gamma)$. Let us consider two orders on $\mathbb{Z}^n :\leq_1$, defined by system $f_1, f_2, \ldots, f_n$ and $\leq_2$, by system $f_1, -f_2, \ldots, -f_n$. In this case, it is easily to see that $\overline{H}_f = \{\gamma \mid \gamma_0 \leq_1 \gamma\} \cup \{\gamma \mid \gamma_0 \leq_2 \gamma\}$. Now using notations of Lemma 3.1 we have by (1.5) and Lemma 3.1 that $\|T_{\overline{H}_f}\| \leq C_1(X)$ where $C_1(X) = C(X)^2 + 2C(X)$.

Similar estimations for H_f and ∂H_f follows from equalities $\partial H_f = \overline{H}_f \cap \overline{H}_{-f}, H_f = \overline{H}_f / \partial H_f$ and (1.5). □

Definition. $A \subset \mathbb{Z}^n$ is said to be set of type $(k,l), k, l \in \mathbb{N}$ if there exist a semi-spaces $A_{ij}, i = 1, 2, \ldots k, j = 1, 2 \ldots l$ such that $A = \cup_{i=1}^k \cap_{j=1}^l A_{ij}$.

Next lemma follows directly from previous definition, Lemma 3.2 and (1.5).

Lemma 3.3. *Let $A \subset \mathbb{Z}^n$ be some set of type (k,l). Then for any order and (UMD)-space X there exists the projection along A. Furthermore, there is a constant $C_2(k,l,X)$ dependent on X, k and l but not on order on G such that $\|T_A\|_{x \to x} \leq C_2(k,l,X)$.*

Choose for every $\gamma \in U$ element $0 \neq x_\gamma \in X_\gamma$. Let θ be an arbitrary bijection from $\mathbb{Z}^n$ onto $\mathbb{N}$ (numeration). The numeration induced by θ on U is the unique bijection ϑ from U onto $\mathbb{N}$ such that for all $x, y \in U : \theta(x) \leq \theta(y)$ iff $\vartheta(x) \leq \vartheta(y)$.

We arrive at the first principal result of this section.

Theorem 3.1. *Let $X \in$ (UMD), let $\{R_t\}_{t \in T^n}$ be a strongly continuous representation of group $\mathbb{T}^n$ on X, $n \in \mathbb{N}$, $\sup\limits_t \|R_t\|_{x \to x} < \infty$, such that the dimension of eigenspaces is less or equal one. For all $\gamma \in U = \{\gamma \mid X_\gamma \neq \{0\}\}$ we choose an $0 \neq x_\gamma \in X_\gamma$. Fix some $k, l \in \mathbb{N}$ and let θ be such bijection from $\mathbb{Z}^n$ onto $\mathbb{N}$ that $\theta^{-1}([1,n])$ are sets of type (k,l) for all $n \in \mathbb{N}$ and ϑ is induced numeration on U. Then $\{y_m = x_{\vartheta^{-1}(m)}\}_{m=1}^\infty$ is basis in X.*

Proof. Since $[y_m]_{m=1}^\infty = [x_\gamma]_{\gamma \in U} = X$, it is sufficient to prove the existence of constant C such that for any $m \leq M : \left\|\sum\limits_{k=1}^m y_k\right\|_x \leq C \left\|\sum\limits_{k=1}^M y_k\right\|_x$. It is easily seen that for any $m \leq M T_{\vartheta^{-1}([1,m])} \left(\sum\limits_{k=1}^M y_k\right) = \sum\limits_{k=1}^m y_k$. But simple calculation shows that $\vartheta^{-1}([1,m]) = \theta^{-1}([1, \theta(\vartheta^{-1}(m))]) \cap U$ i.e. $T_{\vartheta^{-1}([1,m])} = T_{\theta^{-1}([1,\theta(\vartheta^{-1}(m))])}$. Since $\theta^{-1}([1,n])$ is a set of type (k,l) for all $n \in \mathbb{N}$ we have that $\theta^{-1}([1, \theta(\vartheta^{-1}(m))])$ is also the set of type (k, l) and by Lemma 3.3 there is constant $C(k, 1, X)$ such that

$$\|T_{\theta^{-1}([1,\theta(\vartheta^{-1}(m))])}\|_{x \to x} = \|T_{\vartheta^{-1}([1,m])}\|_{x \to x} \leq C(k, l, X).$$

Therefore for any $m \leq M$ we obtain $\Big\|\sum\limits_{k=1}^{m} y_k\Big\|_x \leq C(k, l, X)\Big\|\sum\limits_{k=1}^{M} y_k\Big\|_x$. □

Remark 3.1. *In section 4 we use this result to exhibit examples of bases in non-commutative symmetric spaces which are similar to well-known rectangular basis in C_p (see [KP]).*

Remark 3.2. *An attentive reader could notice, that we obtain an upper estimation of basis constant through norm of some generalized triangle projection (and norm of representation $\{R_t\}_{t \in T^n}$ which often is one). Possibly this result may be used for an estimation of basis constants.*

There was the "generalized trigonometrical system" described in Theorem 3.1. Now consider the "generalized Walsh system" constructed by representations of "dyadic group" $G = \mathbb{Z}_2$ in (UMD)-spaces.

Let $G = \mathbb{Z}_2$, $\widehat{G} = \mathbb{Z}_2$, $n \in \mathbb{N}$. Put

$$U_n = \{\gamma = (\gamma_k)_{k=1}^{\infty} \in \widehat{G} \mid \gamma_n = 1, \gamma_k = 0 \text{ for every } k > n\}, U_0 = \{0\}.$$

We define numeration $\theta : \widehat{G} \to \mathbb{N} \cup \{0\}$ by setting $\theta(\gamma) = \sum\limits_{k=1}^{\infty} \gamma_k 2^{k-1}$.

For each $\gamma \in \widehat{G}$ we define two subsets of $\widehat{G}$:

$$V_\gamma = \{\gamma' \in \widehat{G} \mid \theta(\gamma') \leq \theta(\gamma)\} \text{ and } W_\gamma = \cup_{\gamma_k=1} U_k \cup U_0 \tag{3.1}$$

The following simple lemma takes place:

Lemma 3.4. *For every $\gamma \in \widehat{G} : V_\gamma + \gamma = W_\gamma, W_\gamma + \gamma = V_\gamma$.*

Proof. Let's prove the first equality. It is clear that if $\gamma = 0$ then assertion of Lemma is true. For $\gamma \neq 0$ we apply induction by n so that $\gamma \in U_n$. If $n = 1$ then $\gamma = (1, 0, 0 \dots)$ and $V_\gamma = W_\gamma = \{0, \gamma\}$ i.e. our equality is true. We denote by $\gamma_{[n]}$ the character $(0, \dots, 0, 1, 0, \dots)$.

For every $\gamma \in U_n$ we have

$$V_\gamma + \gamma = \left(\overset{n-1}{\underset{k=1}{\cup}} U_k \cup \{\gamma' \in U_n \mid \theta(\gamma') \leq \theta(\gamma)\} \right) + \gamma =$$

$$\left(\overset{n-1}{\underset{k=1}{\cup}} U_k + \gamma \right) \cup (\{\gamma' \in U_n \mid \theta(\gamma') \leq \theta(\gamma)\} + \gamma) =$$

$$U_n \cup \left((\{\gamma' \in U_n \mid \theta(\gamma') \leq \theta(\gamma)\} + \gamma_{[n]}) + (\gamma + \gamma_{[n]}) \right) =$$

$$U_n \cup \left(\{\gamma' \mid \theta(\gamma') \leq \theta(\gamma + \gamma_{[n]})\} + (\gamma + \gamma_{[n]}) \right).$$

Since $(\gamma + \gamma_{[n]}) \in U_k$ for some $k < n$, by induction we obtain

$$\{\gamma' \mid \theta(\gamma') \leq \theta(\gamma + \gamma_{[n]})\} + (\gamma + \gamma_{[n]}) = W_{\gamma+\gamma_{[n]}} = \bigcup_{\substack{\gamma_k=1 \\ k<n}} U_k \cup U_0$$

and therefore $V_\gamma + \gamma = U_n \cup \bigcup\limits_{\substack{\gamma_k=1 \\ k<n}} U_k \cup U_0 = W_\gamma$.

Finally, $W_\gamma + \gamma = (V_\gamma + \gamma) + \gamma = V_\gamma + (\gamma + \gamma) = V_\gamma$ and proof is completed. □

Theorem 3.2. *Let $X \in$ (UMD), $\{R_t\}_{t \in T^n}$ be a strongly continuous representation of group $G = \mathbb{Z}_2$ on X, $n \in \mathbb{N}$, $\sup\limits_t \|R_t\|_{x \to x} < \infty$, such that the dimension of eigenspaces is less or equal one. For all $\gamma \in U = \{\gamma \mid X_\gamma \neq \{0\}\}$ choose an $0 \neq x_\gamma \in X_\gamma$. Let $\theta : \widehat{G} \to \mathbb{N}$, $\theta(\gamma) = \sum\limits_{k=1}^{\infty} \gamma_k 2^{k-1}$-numeration on $\widehat{G}$ and ϑ is induced numeration on U. Then $\{y_m = x_{\vartheta^{-1}(m)}\}_{m=1}^{\infty}$ is basis in X.*

Proof. The same arguments as in the proof of Theorem 3.1 show that it is sufficient to prove that there exists constant $C(X)$ such that for any $n \in \mathbb{N}$ the norm of projections along $\theta^{-1}([1,n])$ are not greater than $C(X)$. In order to establish this for $\gamma = \theta^{-1}(n) \in \widehat{G}$ we consider an operator $A_\gamma P_{W_\gamma} A_\gamma$, where $P_{W_\gamma} : L^p(G,X) \to L^p(G,X)$ is the projection along W_γ defined in (3.2) and

$$A_\gamma : L^p(G,X) \to L^p(G,X), (A_\gamma f)(t) = \gamma(t) f(t)$$

are isometries in $L^p(G,X)$. Remember that for any $\gamma \in \widehat{G}$ projection P_{W_γ} exists by Corollary 2.2 and, moreover, there exists constant $K(X)$ such that for any $\gamma \in \widehat{G} \|P_{W_\gamma}\|_{L^p(G,X)} \leq K(X)$. It is clear that $\|A_\gamma P_{W_\gamma} A_\gamma\|_{L^p(G,X) \to L^p(G,X)} \leq K(X)$. For each $f_{\gamma'} \in (L^p(G,X))_{\gamma'}$ we have $(A_\gamma f_{\gamma'}) \in (L^p(G,X))_{(\gamma'+\gamma)}$ (see (3.1)). So, by Lemma 3.4 we obtain

$$(A_\gamma P_{W_\gamma} A_\gamma) f_{\gamma'} = A_\gamma P_{W_\gamma}(A_\gamma f_{\gamma'}) = \begin{cases} f_{\gamma'} & \text{if } \gamma' \in V_\gamma \\ 0, & \text{otherwise} \end{cases}.$$

Therefore $A_\gamma P_{W_\gamma} A_\gamma$ is the projection along $V_\gamma = \theta^{-1}([1,n])$ in $L^p(G,X)$. By Theorem 1.2, for every $n \in \mathbb{N}$ there exists $T_{\theta^{-1}([1,n])}$-projection along $\theta^{-1}([1,n])$ in X

$$\|T_{\theta^{-1}([1,n])}\|_{x\to x} \le C(X) = 2c^2 K(X).$$

□

Finally let's consider the "algebraic version" of Theorem 3.1.

Theorem 3.3. *Let $1 < p < \infty$, let M be a semifinite von Neumann algebra with faithful normal semifinite trace τ and $L^p(M,\tau)$ be a non-commutative L^p-space. Let $n \in \mathbb{N}$ and $\{\alpha_t\}_{t\in T^n}$ be a flow on M (i.e., σ-weakly continuous representation $\mathbb{T}^n$ on M which preserves τ) such that the dimension of eigenspaces*

$$(M \cap L^1(M,\tau))_\gamma = \{x \in M \mid \alpha_t x = \gamma(t)x\}$$

is less or equal one. For all $\gamma \in U = \{\gamma \mid (M \cap L^1(M,\tau))_\gamma \ne \{0\}\}$ choose an $0 \ne x_\gamma \in (M \cap L^1(M,\tau))_\gamma$. Fix any $k, l \in \mathbb{N}$ and let θ be a biection from $\mathbb{Z}^n$ to $\mathbb{N}$ such that $\theta^{-1}([1,m])$ for all $m \in \mathbb{N}$ are sets of type $(k,1)$ and ϑ be induced numeration on U. Then $\{y_m = x_{\vartheta^{-1}(m)}\}_{m=1}^\infty$ is basis in $L^p(M,\tau)$.

Proof. By lemma 2.2 from [BGM2] for any flow on M there exists strongly continuous representation $\{R_t\}_{t\in T^n}$ on $L^p(M,\tau)$ such that $R_t \mid_{MnL^1(M,\tau)} = \alpha_t$. Since $M \cap L^p(M,\tau)$ is dense in $L^p(M,\tau)$ and E_γ are continuous operators on $L^p(M,\tau)$, we have

$$(L^p(M,\tau))_\gamma = (\overline{M \cap L^p(M,\tau)})_\gamma = \overline{(M \cap L^p(M,\tau))_\gamma}$$

i.e. the dimension of $(L^p(M,\tau))_\gamma$ is less or equal one. Since space $L^p(M,\tau) \in$ (UMD) for $1 < p < \infty$, we can use Theorem 2.1 for $X = L^p(M,\tau)$ with representation $\{R_t\}_{t\in T^n}$ in order to complete this proof. □

4. Applications to the non-commutative symmetric space

In this section we survey a number of concrete applications of theorems which are proved in previous sections. Throughout this section we denote by M a semifinite von Neumann algebra with faithful, normal, semifinite trace τ, p is arbitrary real number belonging to interval $(1,\infty)$. Because the non-commutative symmetric spaces $E(M)$ are principal objects in this section the examples fall naturally into disjunct classes dependent on M.

4.0 Preliminaries for examples.

For Theorem 3.1 using we must present a numeration θ_n of elements of $\mathbb{Z}^n$ such that for every $n \in \mathbb{N}$ the set $\theta_n^{-1}([1,n])$ is set of type (k_n, l_n) for some fixed $k_n, l_n \in N$. In cases $n = 1,2$ these numerations are well-known. For $n = 1$ it is numeration in sequence $0, -1, 1, -2, 2, \ldots$. For $n = 2$ it is just "numeration in rectangular order" (see [KP], [E1]). Now for arbitrary natural n we define the "numeration by n-cubes" by inductively.

First, define

$$K_m = \{x = (x_k) \in \mathbb{Z}^n \mid -m \leq x_k \leq m \text{ for any } k \leq n\} - n - \text{cube of size } m$$

and

$$\Gamma_m^{2l-1} = \{x = (x_k) \in K_m \mid x_l = -m, |x_k| < m \text{ for } k < l\},$$
$$\Gamma_m^{2l-1} = \{x = (x_k) \in K_m \mid x_l = m, |x_k| < m \text{ for } k < l\}, l \in [1,n] - \text{faces of } K_m.$$

There are canonical projections $\pi_m^k : \Gamma_m^k \to \mathbb{Z}^{n-1}, k \in [1, 2n]$, defined by $\pi_m^k((x_k)) = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)$. We shall define inductively an order "$<_n$" on $\mathbb{Z}^n$ sequences θ_n, k_n, l_n. Order "$<_1$" is the order of $\mathbb{Z}$ such that $x <_1 y$ iff $|x| < |y|$ or $y = -x, y > 0$. The numeration θ_1 is defined by $\theta_1(x) = \text{ Card } \{y \in \mathbb{Z}^n \mid y <_1 x\}$. It is easy to see that we can choose $k_1 = 1, l_1 = 2$. If "$<_{n-1}$', $\theta_{n-1}, k_{n-1}, l_{n-1}$ have been already defined, we say that $x <_n y, x, y \in \mathbb{Z}^n$ if and only if one of the following conditions is satisfied:

1) there is $m \in \mathbb{N}$ such that $x \in K_m$ and $y \notin K_m$,
2) there are $m, k, l \in \mathbb{N}$ such that $x \in \Gamma_m^k$ and $y \in \Gamma_m^1$,
3) there are $m, k \in \mathbb{N}$ such that $x, y \in \Gamma_m^k$ and $\pi_m^k(x) <_{n-1} \pi_m^k(y)$.

Next , put $\theta_n(x) = \text{ Card } \{y \in \mathbb{Z}^n \mid y <_n x\}$. It is straightforward that θ_n is numeration of $\mathbb{Z}^n$ and next recursive equation takes place $\theta_n^{-1}([1,N]) = K_{m-1} \cup \bigcup_{j=1}^{k-1} \Gamma_m^j \cup (\pi_m^k)^{-1}(\theta_{n-1}^{-1}([1, \theta_{n-1}(\pi_m^k(\theta_n^{-1}(N)))]))$, where $k, m \in \mathbb{N}$ such that $\theta_n^{-1}(N) \in \Gamma_m^k$. Notice that K_{m-1} is the set of type $(1, 2n)$, every $\Gamma_m^j, j = \overline{1, 2n}$ has type $(3, 2n-2)$ and by induction assumption there exists k_{n-1}, l_{n-1} not dependent on N such that $(\pi_m^k)^{-1}(\theta_{n-1}^{-1}([1, \theta_{n-1}(\pi_m^k(\theta_n^{-1}(N)))]))$ is set of type (k_{n-1}, l_{n-1}). Thus $\theta_n^{-1}([1,N])$ is the set of type (k_n, l_n) where k_n, l_n depends on n but not on N. This completes the inductive construction of numeration θ_n (further, numeration by n-cubes).

Since every θ_n satisfies to hypotesis of Theorem 3.1 we have

Theorem 4.1. *Let $X \in$ (UMD), $\{R_t\}_{t\in T^n}$ be a strongly continuous representation of group $\mathbb{T}^n$ on X, $n \in N \sup\limits_t \|R_t\|_{x\to x} < \infty$, such that the dimension of eigenspaces is less or equal one. For all $\gamma \in U = \{\gamma | X_\gamma \neq \{0\}\}$ choose an $0 \neq x_\gamma \in X_\gamma$. Then system $\{x_\gamma\}$ taking in numeration induced on U by numeration by n-cubes is basis in X.*

In examples below we need of strongly continuity of the representation $\{R_t\}_{t\in G}$. The checking of this property is straightforward using the following simple lemma and often omitted.

Lemma 4.1. *Let $\{R_t\}_{t\in G}$ be the representation $G \in$ **CAG** on Banach space $X, \sup\limits_t \|R_t\|_{x\to x} < \infty$, and exists such X_0 which is dense in X and $\{R_t\}_{t\in G}$ is strongly continuous representation of G on X_0. Then $\{R_t\}_{t\in G}$ is strongly continuous representation of G on X.*

4.1 The case $M = L^\infty([0,1])$

Let $M = L^\infty(\mathbb{T})$ with faithful normal semifinite trace $\tau(f) = \displaystyle\int_T f(t)dt, f \in M$. Remember that $L^p(M,\tau) = L^p(\mathbb{T})$.

Trigonometrical system in $X = L^p(\mathbb{T})$. Let group $\mathbb{T}$ acts on $L^p(\mathbb{T})$ as $\{R_t\}_{t\in G}$, where $(R_t f)(x) = f(x+t)$. For arbitrary $\gamma \in \widehat{\mathbb{T}} = \mathbb{Z}$ we have: $X_\gamma = \{f \in X \mid f(x+t) = \gamma(t)f(x)$ for all $x,t \in \mathbb{T}\} = \{f \in X \mid f(t) = f(0)\gamma(t)$ for all $t\} = \{Ce^{i\gamma t} \mid C \in \mathbb{C}\} = [e^{i\gamma t}]$. Therefore, all X_γ are one-dimensional spaces. By Theorem 4.1 this system forms basis in $X = L^p(\mathbb{T})$ in numeration by 1-cubes (i.e., in numeration $1, e^{-it}, e^{it}, e^{-2it}, e^{2it}, \ldots$). Of course, it is well-known result (see, for example, [LTII]).

Generalized Walsh system in $X = L^p([0,1])$. Let $G = \mathbb{Z}_{p_k}$, where $\{p_k\}_{k=1}^\infty$ is prime numbers, $t = (t_k)_{k=1}^\infty \in G, t_k \in \mathbb{Z}_{p_k}, m_k = p_1 p_2 \ldots p_k, k = 1, 2, \ldots$. For every $x \in [0,1]$ there exists such $x_k \in Z_{p_k}$ that $x = \displaystyle\sum_{k=1}^\infty x_k/m_k$. If we will "take twice" p_k-rational numbers from $[0,1]$, the map $x \to (x_k)_{k=1}$ becomes a bijection. Define the representation $\{r_t\}_{t\in G}$ of group G on $[0,1]$ by rule $r_t\left(\displaystyle\sum_k x_k/m_k\right) =$

$\sum_n (x_k \oplus_k t_k)/m_k$, where $x_k, t_k \in Z_{p_k}$ and $\oplus_k$ is the operation in Z_{p_k}. It is known that $\{r_t\}_{t\in G}$ preserves the Lebesgue measure on $[0,1]$ (see [Mo]). The representation of G on X is defined by rule $(R_t f)(x) = f(r_t(x))$, for all $f \in X, x \in [0,1]$. It is clear, that $\{R_t\}_{t\in G}$ are isometries in X. Now calculate X_γ : $X_\gamma = \{f \in X \mid f(r_t(x)) = \gamma(t)f(x)$ for all $x \in [0,1], t \in G\} = \{f \in X \mid f(r_t(O)) = \gamma(t)f(0)$ for all $t \in G\} = \{f \in X \mid f\left(\sum_k t_k/m_k\right) = f(0)\gamma(t)$ for all $t \in G\} = [f_\gamma]$, where f_γ is such a function that $f_\gamma\left(\sum_k t_k/m_k\right) = \gamma(t)$. Therefore, $\{X_\gamma\}_{\gamma\in\widehat{G}}$ are one-dimensional spaces.

Notice that if $\gamma = \gamma_1 + \gamma_2$ then, by definition $f_\gamma = f_{\gamma_1} f_{\gamma_2}$, i.e. the system $\{f_\gamma\}_{\gamma\in\widehat{G}}$ is multiplicative system. Let us put $f_{[n]} = f_\gamma$, where $\gamma = (0,0,\ldots,0,n,0,0,\ldots)$. Since for this γ by definition $\gamma(t) = \exp(2\pi \mathrm{i} t_n/p_n)$ we can write $f_{[n]}\left(\sum_k t_k/m_k\right) = \exp(2\pi \mathrm{i} t_n/p_n)$. For arbitrary $\gamma = (\gamma_k)_{k=1}^\infty$ we obtain:

$$f_\gamma = f_{[k]}^{\gamma_k} = \exp(2\pi i_{\gamma_k} t_k/p_k),$$

(where always only finitely many multipliers are not equal to 1).

System $\{f_\gamma\}_{\gamma\in\widehat{G}}$ is known as "generalized Walsh system" which studied by many authors (see [Mo], [Wa1], [Wa2]). In case $p_k = 2$ for all $k \in \mathbb{N}$ this system appears to be the classical Walsh system (it is easy to verify that $f_{[n]}$ is the n-th Rademacher function). Using Corollary 2.2 in this situation we obtain, in fact, the "generalized Paley theorem".

Theorem 4.2. *Fix some sequence of prime numbers $\{p_k\}_{k=1}^\infty$ and consider the generalized Walsh system $\{f_\gamma\}_{\gamma\in\widehat{G}}$ constructed by this sequence in $L^p([0,1]), 1 < p < \infty$. Let $U_0 = \{0\}, U_n = \{\gamma = (\gamma_k)_{k=1}^\infty \mid \gamma_n \neq 0, \gamma_k = 0, k > n\}$. Then subspace $X_n = [f_\gamma]_{\gamma\in U_n}$ forms (UFDD) of $L^p([0,1])$. In other words, for every $a_{(\gamma_1,\ldots,\gamma_n,0,0\ldots)} \in \mathbb{C}, n \in \mathbb{N}$ put*

$$\delta_n = \sum_{\gamma_n=1}^{p_0-1}\sum_{\gamma_1=0}^{p_1-1}\cdots\sum_{\gamma_{n-1}=0}^{p_{n-1}-1} a_{(\gamma_1,\ldots,\gamma_n,0,0\ldots)} f_{(\gamma_1,\ldots,\gamma_n,0,0\ldots)},$$

$\delta_0 = 1$. Then there exists such constant $C(p)$, that for every signs $\{\epsilon_k\}_{k=1}^\infty$ the following inequality holds:

$$\left\|\sum_{n=0}^\infty \epsilon_n\delta_n\right\|_p \le C(p)\left\|\sum_{n=0}^\infty \delta_n\right\|_p.$$

Remark 4.1. *Notice that if all $p_k = 2$ then it is just classical Paley theorem for Walsh system.*

Remark 4.2. *Author [Wa2] also obtained the generalized Paley theorem in slightly different form. In case $p_k = 0(1)$ our result is equivalent to Theorem 4.2; but otherwise Theorem 4.2 is also valid unlike the [Wa2] formulation of generalized Paley theorem.*

4.2. Case $M = B(H)$

Let $M = B(H)$, where H is the separable Hilbert space and $\tau = \text{ Tr}$ is canonical trace on $B(H)$. It is known that in this case $L^p(M,\tau) = C^p$.

(UFDD) in $X = C^p$. Let $G = \mathbb{T}, t = (t_k)_{k=1}^{\infty} \in G, U_t = \text{ diag }(\exp(\text{it}_{2n}))_{n=1}^{\infty}, V_t = mboxdiag(\exp(\text{it}_{2n-1}))_{n=1}^{\infty}$ is unitary operators in $B(H)$. Define $\{R_t\}_{t\in G}$ by rule $R_t x = U_t x V_t$, for every $x \in X$. It is clear that R_t are isometries of X. We draw now our attention to the calculation of eigenspaces. Let

$$X_n = \left\{ \sum_{k,l} a_{kl}E_{kl} \mid a_{kl} = 0 \text{ if } k > n \text{ or } l > n \right\}, X_0 = \bigcup_{n=1}^{\infty} X_n.$$

It is clear that X_0 is dense in X. Next, for every integer n $R_t X_n \subset X_n$ follows for every $\gamma \in \widehat{G} E_\gamma X_n \subset X_n$ and therefore $E_\gamma X_0 \subset X_0$. It means by (1.3) that for every $\gamma \in \widehat{G}$

$$X_\gamma = E_\gamma \overline{X}_0 = \overline{E_\gamma X_0} = \overline{\{y \in X_0 \mid R_t y = \gamma(t)y \text{ for every } t \in G\}}.$$

Let y be an element X_0 such that $R_t \left(\sum_{k,l} y_{kl}E_{kl} \right) = \exp\left(\sum_j i_{\gamma j} t_j \right) \sum_{k,l} y_{kl}E_{kl}$

for all $t \in G$. Then $\sum_{k,l} \exp(it_{2k}) y_{kl} \exp(it_{2l-1}) E_{kl} = \exp\left(\sum_j i_{\gamma j} t_j \right) \sum_{k,l} y_{kl}E_{kl}$

for all $t \in G$, i.e. $\exp(\text{it}_{2k}) y_{kl} \exp(\text{it}_{2l-1}) = \exp\left(\sum_j i_{\gamma j} t_j \right) y_{kl}$ for all $k, l \in \mathbb{N}, t \in G$. If such k_o, l_0 exist that $y_{k_0 l_0} \neq 0$, then $\gamma_{2k_0} = \gamma_{2l_0-1} = 1$ and $\gamma_j = 0$ otherwise and hence $y_{kl} = 0$ for $k \neq k_0, l \neq l_o$. Therefore, we obtain that

$$X_\gamma = \begin{cases} [E_{k_0 l_0}], \gamma_{2k_0} = \gamma_{2l_0-1} = 1, \gamma_j = 0 & \text{for } j \neq 2k_0, j \neq 2l_0 + 1 \\ \{0\} & \text{otherwise.} \end{cases}$$

The straightforward calculations show that

$$U_n = \{\gamma \mid \gamma_n = \gamma_m = 1, m \neq n \pmod 2, m < n, \gamma_k = 0 \text{ for } k \neq n, m\}$$

and by Corollary 3.2 subspaces $X_{2n} = [E_{nk}]_{k\leq n}, X_{2n-1} = [E_{kn}]_{k<n}$ form (UFDD) in C^p. Notice that (UFDD) $Y_n = X_n \cup X_{2n-1}$ is well-known and useful one (see [KP] and [Ar] for additional information).

Basis of matrix identities in $X = C^p$. Similarly to the previous example it could be noticed that if $G = \mathbb{T}^2, t = (t_1, t_2) \in G, \gamma = (\gamma_1, \gamma_2) \in \widehat{G}$ and $R_t x = U_t x V_t$, for every $x \in X, U_t = \text{diag }(\exp(\text{i}t_1 n))_{n=1}^\infty, V_t = \text{diag }(\exp(\text{i}t_2 n))_{n=1}^\infty$, then

$$X_\gamma = \begin{cases} [E_{\gamma_1\gamma_2}], \tau_1 = \tau_2 \in \mathbb{N} \\ \{0\} \text{ otherwise} \end{cases}$$

and therefore by Theorem 4.1 the matrix identities in 2-cubes numeration form basis of C^p. It is easy to see that 2-cubes numeration in this case is standard "square numeration" and this basis is classical basis in C^p (see, for example, [KP], [Ar]).

4.3. Case $M = L^\infty(\mathbb{T})\overline{\otimes}B(H)$

Let M be the tensor product $L^\infty(\mathbb{T})\overline{\otimes}B(H)$ with trace $\tau\otimes$ Tr, where τ and Tr are the canonical traces on the $L^\infty(\mathbb{T})$ and $B(H)$, respectively. The following lemma says that non-commutative L^p-space $L^p(M, \tau\otimes \text{Tr})$ can be considered as a vector-valued L^p-space.

Lemma 4.2. *The space $L^p(\mathbb{T}, C^p)$ is isometric to space $L^p(M, \tau\otimes$ Tr$)$ for all $p \in [1, +\infty)$.*

Proof. We claim that linear operator which transforms function $f(t)A \in L^p(\mathbb{T}, C^p)$ (where $f(t) \in L^p(\mathbb{T}), A \in C^p$) to the tensor $f\otimes A \in L^p(M, \tau\otimes \text{Tr})$ is extended to isometry between $L^p(\mathbb{T}, C^p)$ and $L^p(M, \tau \otimes \text{Tr})$.

First, assume that $p = 2s, s \in \mathbb{N}$. For every $f_k(t) \in L^p(\mathbb{T}), A_k \in C^p, k = \overline{1,n}$ we have

$$\left\| \sum_{k=1}^{n} f_k(t) A_k \right\|^p_{L^p(T,C^p)} = \int_T \left\| \sum_{k=1}^{n} f_k(t) A_k \right\|^p_{C^p} dt = \int_T \text{ Tr } \left(\left| \sum_{k=1}^{n} f_k(t) A_k \right|^p \right) dt =$$

$$\int_T \text{ Tr } \left(\sum_{k_i,l_i=1}^{n} \bar{f}_{k_1}(t) A^*_{k_1} f_{l_1}(t) A_{l_1} \dots \bar{f}_{k_s}(t) A^*_{k_s} f_{l_s}(t) A_{l_s} \right) dt =$$

$$\int_T \sum_{k_i,l_i=1}^{n} \bar{f}_{k_1}(t) f_{l_1}(t) \dots \bar{f}_{k_s}(t) f_{l_s}(t) \cdot \text{ Tr } (A^*_{k_1} A_{l_1} \dots A^*_{k_s} A_{l_s}) dt =$$

$$\sum_{k_i,l_i=1}^{n} \int_T \bar{f}_{k_1}(t) f_{l_1}(t) \dots \bar{f}_{k_s}(t) f_{l_s}(t) dt \cdot \text{ Tr } (A^*_{k_1} A_{l_1} \dots A^*_{k_s} A_{l_s}) =$$

$$\sum_{k_i,l_i=1}^{n} \tau \left(\bar{f}_{k_1}(t) f_{l_1}(t) \dots \bar{f}_{k_s}(t) f_{l_s}(t) \right) \cdot \text{ Tr } (A^*_{k_1} A_{l_1} \dots A^*_{k_s} A_{l_s}) =$$

$$\sum_{k_i,l_i=1}^{n} \tau \otimes \text{ Tr } \left(\bar{f}_{k_1}(t) f_{l_1}(t) \dots \bar{f}_{k_s}(t) f_{l_s}(t) \overline{\otimes} (A^*_{k_1} A_{l_1} \dots A^*_{k_s} A_{l_s} \right) =$$

$$\tau \otimes \text{ Tr } \left(\sum_{k_i,l_i=1}^{n} \bar{f}_{k_1}(t) f_{l_1}(t) \dots \bar{f}_{k_s}(t) f_{l_s}(t) \overline{\otimes} A^*_{k_1} A_{l_1} \dots A^*_{k_s} A_{l_s} \right) =$$

$$\tau \otimes \text{ Tr } \left(\left| \sum_{k=1}^{n} f_k \overline{\otimes} A_k \right|^p \right) = \left\| \sum_{k=1}^{n} f_k \overline{\otimes} A_k \right\|^p_{L^p(M, \tau \otimes \text{ Tr })}.$$

Since finite sums $\sum_{k=1}^{n} f_k(t) A_k, \sum_{k=1}^{n} f_k \overline{\otimes} A_k, f_k(t) \in L^p(\mathbb{T}), A_k \in C^p$ are dense in $L^p(\mathbb{T}, C^p)$ and $L^p(M, \tau \otimes \text{ Tr })$ respectively we can conclude that there is isometry (namely, the linear continuous extension of map $L^p(\mathbb{T}, C^p) \ni \sum_{k=1}^{n} f_k(t) A_k \longrightarrow \sum_{k=1}^{n} f_k \overline{\otimes} A_k \in L^p(M, \tau \otimes \text{ Tr }))$ between these spaces. For the case $p = 1$ the assertion of lemma is also true (see [Ta]). Using of these facts together with the complex interpolation method allow to complete the proof. □

Basis in $\boldsymbol{X = L^p(M, \tau \otimes}$ Tr $\boldsymbol{)}$. Consider X as vector-valued space. Let $G = \mathbb{T}^3, t = (t_1, t_2, t_3) \in G, \gamma = (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{Z}^3 = \widehat{G}$. For every $f \in X, t \in G$ define $(R_t f)(x) = U_t f(x + t_1) V_t$, where $U_t = \text{diag}(\exp(\text{i} t_2 n))_{n=1}^{\infty}, V_t \text{diag}(\exp(\text{i} t_3 n))_{n=1}^{\infty}$-unitary operators in C^p. It is clear that $\{R_t\}_{t \in G}$ is isometries. Then, $X_\gamma = \{f \in X \mid U_t f(x +$

$t_1)V_t = \gamma(t)f(x)$ for all $x \in \mathbb{T}, t \in G\} = \{f \in X \mid U_t f(t_1)V_t = \gamma(t)f(0)$ for all $t \in G\} = \{f \in X \mid U_t f(t_1)V_t = e^{i\gamma_1 t_1 e^{i(\gamma_2 t_2+\gamma_3 t_3)}} f(0)$ for all $t \in G\}$.

Fix $t_1 = 0$ we obtain that $U_t f(0)V_t = e^{i(\gamma_2 t_2+\gamma_3 t_3)} f(0)$ it follows (similarly to calculations in section 4.2) $f(0) = CE_{\gamma_2\gamma_3}, C \in \mathbb{C}$ and $X_\gamma = [e^i\gamma_1 t E_{\gamma_2\gamma_3}]$. By Theorem 4.1, system $\{e^{i\gamma_1 t_1} E_{\gamma_2\gamma_3}\}_{\gamma\in\widehat{G}}$ in numeration by 3-cubes is basis in $L^p(\mathbb{T}, C^p)$, or, in other words, system $\{e^{i\gamma_1 t_1} \overline{\otimes} E_{\gamma_2\gamma_3}\}_{\gamma\in\widehat{G}}$ in such numeration is basis in $L^p(M, \tau \otimes \text{ Tr })$. Similar result about system $\{e^{i\gamma_1 t_1} E_{\gamma_2\gamma_3}\}_{\gamma\in\widehat{G}}$ (however, in different numeration) in the space $L^p(\mathbb{T}, C^p)$ was proved earlier in [E1].

(UFDD) in $X = L^p(M, \tau \otimes \text{ Tr})$. Again we use that $X = L^p(\mathbb{T}, C^p)$. Let $G = \mathbb{T}$. Consider finite sums $\sum\limits_{n,k,l} a_{nkl}\chi_n(x)E_{kl}$, ($\{\chi_n\}$ is the Haar system in $L^p(\mathbb{T})$ (see [LTI])), which are dense in X. Define for every $t - (t_k)_{k=1}^\infty \in G, t_k \in \mathbb{T}$:

$$R_t\left(\sum_{n,k,l} a_{nkl}\chi_n(x)E_{kl}\right) = \sum_{n,k,l} a_{nkl}\chi_n(x)E_{kl} \cdot \exp(i(t_{3n} + t_{3k+1} + t_{3l+2})).$$

Let $U_t = \text{diag}(\exp(it_{3n+1}))_{n=1}^\infty, V_t \text{diag}(\exp(it_{3n+2}))_{n=1}^\infty$, then we have

$$\begin{aligned}\left\|R_t\left(\sum_{n,k,l} a_{nkl}\chi_n(x)E_{kl}\right)\right\|_p &= \left\|U_t\left(\sum_{n,k,l} a_{nkl}\chi_n(x)E_{kl}\exp(it_{3n})\right)V_t\right\|_p \leq \\ &\leq \left\|\sum_{n,k,l} a_{nkl}\chi_n(x)E_{kl}\exp(it_{3n})\right\|_p.\end{aligned}$$

For every integer N denote by B_N such a minimal σ-algebra, that all functions $\chi_0, \chi_1, \ldots, \chi_N$ are B_N-measurable. Since the system $\left(\sum\limits_{n=0}^N \chi_n(x)\sum\limits_{k,l} a_{nkl}E_{kl}, B_N\right)$ is martingal in $L^p(\mathbb{T}, C^p)$ with respect to $\{B_N\}_{N=1}^\infty$ we have by (1.1):

$$R_t\left\|\left(\sum_{n,k,l} a_{nkl}\chi_n(x)E_{kl}\right)\right\|_p \leq K\left\|\left(\sum_{n,k,l} a_{nkl}\chi_n(x)E_{kl}\right)\right\|_p.$$

Hence $\{R_t\}_{t\in G}$ can be extended to linear bounded strongly continuous representation of G on $X, \sup\limits_t\|R_t\|_{x\to x} \leq K$.

Directly from definition of $\{R_t\}_{t\in G}$ we have

$$X_\gamma = \begin{cases} [\chi_{n_0}(t)E_{k_0 l_0}], \gamma_{3n_0} = \gamma_{3k_0+1} = \gamma_{3l_0+2} = 1, \gamma_j = 0 \text{ for } j \neq 3n_0 + 3l_0 + 1, 3k_0 + 2 \\ \{0\} \text{ otherwise} \end{cases}.$$

and $U_n = \{\gamma \mid \gamma_n = \gamma_k = \gamma_l = 1, k < n, l < n, n \not\equiv k \not\equiv l \pmod 3), \gamma_j = 0$ for $j \not\equiv n, k, l\}$. Next theorem is direct consequence of these facts and Corollary 2.2.

Theorem 4.3. *Let $X = L^p(L^\infty \overline{\otimes} B(H), \tau \otimes \ Tr)$. Then subspaces of X $X_{3n} = [\chi_n \overline{\otimes} E_{kl}]_{k<n,l<n}, X_{3n+1} = [\chi_k \overline{\otimes} E_{nl}]_{k \le n,l<n}, X_{3n+2} = [\chi_k \overline{\otimes} E_{ln}]_{k \le n,l \le n}$ form (UFDD) in X.*

4. L^p-spaces associated with II_1 type hyperfinite factor.

Instead of considering the von Neumann algebras of type I we now deal with factors of type II_1. The simplest and well-known one is the hyperfinite Murray-von Neumann factor R (see [Sa], [Ta]). Remember that R could be considered as infinite tensor product of 2×2 matrix algebras $M_k = B(H_2), \dim H_2 = 2$ with traces $\varphi_k(x) = 1/2$ Tr (x) so that $R = \bigotimes\limits_{k=1}^{\infty} M_k$ have the faithful normal finite trace $\varphi = \bigotimes\limits_{k=1}^{\infty} \varphi_k$. It is easy to see (with using condition expectations techniques) that set X_0 of finite sums of finite tensors (i.e., tensors $x_1 \otimes \cdots \otimes x_n \otimes 1 \otimes 1 \ldots$, where 1 is the unit of $B(H_2), x_k \in M_k, k, n \in \mathbb{N}$) are dense in $L_p(R, \varphi), 1 \le p < \infty$.

Basis and (UFDD) for $X = L_p(R, \varphi)$. First, let $t = (t_1, t_2) \in \mathbb{Z}_2 \otimes \mathbb{Z}_2$, let $\{r_t\}$ be the representation of group $\mathbb{Z}_2 \otimes \mathbb{Z}_2$, on M_2 defined by

$$r_t x = U(t_1)V(t_2)xV(t_2)U(t_1) \text{ for every } x \in M_2,$$

where $U(0) = V(0) = 1, U(1) = U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, V(1) = V = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
Now consider such representation of group $G = \mathbb{Z}_2$ on X_0:

$$R_t x = \left(\bigotimes_{k=1}^{\infty} r_{(t_{2k-1}, t_{2k})} \right) x \text{ for every } x \in X_0, t = ((t_{2k-1}, t_{2k}))_{k=1}^{\infty}$$

where, in fact, only finitely many multipliers are not equal to Id. It is clear that $\{R_t\}_{t \in G}$ are isometries of X_0. Moreover, $\{R_t\}_{t \in G}$ is the strongly continuous representation. Indeed, if $G \ni t^{(n)} - 0$ then for every $k_0 \in \mathbb{N}$ there exist $N \in \mathbb{N}$ such that $t_k^{(n)} = 0$ for every $k \le k_0, n > N$. Therefore, for every $x \in X_0$ there exist $N \in \mathbb{N}$ such that $R_t^{(n)} x = x$ for all $> N$ i.e. $\lim\limits_{n \to \infty} R_{t^{(n)}} x = x$. Because of density X_0 in X and by Lemma 4.1 $\{R_t\}_{t \in G}$ extends to the strongly continuous representation of isometries on whole X.

Let $\gamma = (\gamma_k)_{k=1}^{\infty} \in \widehat{G}$, let x be a finite tensor. It is clear that there exists an integer number m such that $\gamma_l = 0$ for all $1 > 2m$ and $x_k = 1$

for all $k > m$. We have

$$E_\gamma x = \int_G \gamma(t) R_{-t} x dt =$$

$$\frac{1}{2^{2m}} \cdot \sum_{\substack{t_l \in \mathbb{Z}_2 \\ 1 \leq 2m}} \left((\bigotimes_{k=m}^{m} r_{(t_{2k-1}, t_{2k})} x_k)(\bigotimes_{k>m} 1) \right) \gamma_1(t_1) \dots \gamma_{2m}(t_{2m}) =$$

$$\frac{1}{2^{2m}} \cdot \sum_{\substack{t_l \in \mathbb{Z}_2 \\ 1 \leq 2m}} \left(\bigotimes_{k=m}^{m} (r_{(t_{2k-1}, t_{2k})} x_k \gamma_{2k-1}(t_{2k-1}) \gamma_{2k}(t_{2k})) \otimes (\bigotimes_{k>m} 1) \right) =$$

$$\frac{1}{2^{2m}} \cdot \sum_{\substack{t_l \in \mathbb{Z}_2 \\ 1 \leq 2m}} \left(\bigotimes_{k=m}^{m} (r_{(t_{2k-1}, t_{2k})} x_k \gamma_{2k-1}(t_{2k-1}) \gamma_{2k}(t_{2k})) \right) \otimes (\bigotimes_{k>m} 1) =$$

$$\sum_{\substack{t_l \in \mathbb{Z}_2 \\ 1 \leq 2m}} \left(\bigotimes_{k=m}^{m} \left(\frac{1}{4} r_{(t_{2k-1}, t_{2k})} x_k \gamma_{2k-1}(t_{2k-1}) \gamma_{2k}(t_{2k}) \right) \right) \otimes (\bigotimes_{k>m} 1) =$$

$$\bigotimes_{k=m}^{m} \sum_{t_{2k-1}, t_{2k} \in \mathbb{Z}_2} \left(\frac{1}{4} r_{(t_{2k-1}, t_{2k})} x_k \gamma_{2k-1}(t_{2k-1}) \gamma_{2k}(t_{2k}) \right) \otimes (\bigotimes_{k>m} 1) =$$

$$\bigotimes_{k=m}^{m} e_{(\gamma_{2k-1}, \gamma_{2k})}(x_k) \otimes (\bigotimes_{k>m} 1),$$

where $e_{(\gamma_{2k-1}, \gamma_{2k})} : M_2 \to M_2$ is the projection onto eigenspace of representation $\{r_t\}$ associated with character $(\gamma_{2k-1}, \gamma_{2k}) \in \mathbb{Z}_2 \otimes \mathbb{Z}_2$. Now our objective is to calculate all these eigenspaces, namely, $e_{(0,0)} M_2$, $e_{(1,0)} M_2$, $e_{(0,1)} M_2$, and $e_{(1,1)} M_2$. It is convenient to use (1.3) in this calculation. For example, $\gamma = (1, 1) \in \widehat{\mathbb{Z}}_2 \otimes \widehat{\mathbb{Z}}_2$ acts on $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ as follows: $\gamma((0, 0)) = \gamma((1, 1)) = 1, \gamma((1, 0)) = \gamma((0, 1)) = -1$, and by (1.3) we have

$$\begin{aligned} e_{(1,1)} M_2 &= \{x \in M_2 \mid r_{(0,0)} x = r_{(1,1)} x = x, r_{(0,1)} x = r_{(1,0)} x = -x\} = \\ &= \{x \in M_2 \mid UVxVU = x, UxU = VxV = -x\} = \\ &= \{x = (x_{ij})_{i,j=1}^2 \in M_2 \mid x_{11} = -x_{11}, x_{22} = -x_{22}, x_{12} = -x_{21}\} = [e_{1,1}], \end{aligned}$$

where $e_{1,1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Analogous calculations show that for each $(\gamma_1, \gamma_2) \in \mathbb{Z}_2 \otimes \mathbb{Z}_2$ we have $e_{(\gamma_1, \gamma_2)} M_2 = [e_{\gamma_1, \gamma_2}]$, where

$e_{0,0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $e_{1,0} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $e_{0,1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $e_{1,1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

So, using (4.3) and taking into account that $e_{\gamma_{2k-1}\gamma_{2k}} = 1$ for every $k > m$, we have

$$E_\gamma x = \lambda \bigotimes_{k=1}^{\infty} e_{\gamma_{2k-1}\gamma_{2k}} \otimes (\bigotimes_{k>m} 1) = \lambda \bigotimes_{k=m}^{m} e_{\gamma_{2k-1}\gamma_{2k}}$$

for some $\lambda \in \mathbb{C}$.

Therefore $E_\gamma x \in [\overset{\infty}{\underset{k=1}{\otimes}} e_{\gamma_{2k-1}\gamma_{2k}}]$ and $\overline{E_\gamma X_0 = [\overset{\infty}{\underset{k=1}{\otimes}} e_{\gamma_{2k-1}\gamma_{2k}}]}$. Thus, $X_\gamma = E_\gamma X = E_\gamma \overline{X}_0 = \overline{E_\gamma X_0} = [\overset{\infty}{\underset{k=1}{\otimes}} e_{\gamma_{2k-1}\gamma_{2k}}] = [\overset{\infty}{\underset{k=1}{\otimes}} e_{\gamma_{2k-1}\gamma_{2k}}]$ is one-dimension space for any $\gamma \in \widehat{G}$. Let us calculate the finite dimensional spaces $X_m = [X_\gamma]_{\gamma \in U_m}, m \in \{0\} \cup \mathbb{N}$. Clearly $X_0 = \mathbb{C} \cdot 1$, where 1 is unit of R. Next, if m is odd number, say $2n-1$, we have

$$X_{2n-1} = [X_\gamma]_{\gamma \in U_{2n-1}} = [X_\gamma \mid (\tau_1, \dots, \gamma_{2n-2}, 1, 0, 0, \dots)] =$$
$$= [e_{(\gamma_1,\gamma_2)} \otimes \dots \otimes e_{(\gamma_{2n-3},\gamma_{2n-2})} \otimes e_{(1,0)} \otimes 1 \otimes 1 \dots] =$$
$$= \{x_1 \otimes \dots \otimes x_{n-1} \otimes \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \otimes 1 \otimes 1 \dots \mid x_k \in M_2, k < n, \lambda \in C\},$$

and if m is even number, say $2n$, then

$$X_{2n} = [X_\gamma]_{\gamma \in U_{2n}} = [X_\gamma \mid (\gamma_1, \dots, \gamma_{2n-1}, 1, 0, 0, \dots)] =$$
$$= [e_{(\gamma_1,\gamma_2)} \otimes \dots \otimes e_{(\gamma_{2n-3},\gamma_{2n-2})} \otimes e_{(\gamma_{2n-1},1)} \otimes 1 \otimes 1 \dots] =$$
$$= \{x_1 \otimes \dots \otimes x_{n-1} \otimes \begin{pmatrix} 0 & \lambda \\ \mu & 0 \end{pmatrix} \otimes 1 \otimes 1 \dots \mid x_k \in M_2, k < n, \lambda, \mu \in C\}.$$

The next theorem is the obvious application of Corollary 2.2 and Theorem 3.2 to (UMD)-space $L^p(R, \varphi), 1 < p < \infty$.

Theorem 4.4. *Let R be hyperfinite Murray-vonNeumann factor representing as infinite tensor product of I_2-type factors with finite trace φ such that $\varphi(1) = 1$. Let $1 < p < \infty$ and $L^p(R, \varphi)$ be the non-commutative L^p-space associated with (R, φ). Then*

a) *The subspaces of $L^p(R, \varphi) X_0 = \mathbb{C} \cdot 1 X_{2n-1} = \{x_1 \otimes \dots \otimes x_{n-1} \otimes \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \otimes 1 \otimes 1 \dots \mid x_k \in M_2, k < n, \lambda \in C\}$ and (4.4) $X_{2n} = \{x_1 \otimes \dots \otimes x_{n-1} \otimes \begin{pmatrix} 0 & \lambda \\ \mu & 0 \end{pmatrix} \otimes 1 \otimes 1 \dots \mid x_k \in M_2, k < n, \lambda, \mu \in C\}$ where $n \in \mathbb{N}$ form the (UFDD) of $L^p(R, \varphi)$;*

b) *$L^p(R, \varphi)$ have the Schauder basis $x_n = \overset{\infty}{\underset{k=1}{\otimes}} e_{\gamma_{2k-1},\gamma_{2k}}$, where $n = \sum_{k=1}^{\infty} \gamma_k 2^{k-1}, \gamma_k \in \{0, 1\}, k \in \mathbb{N}$ and*

$$e_{0,0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, e_{1,0} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e_{0,1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } e_{1,1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Remark 4.3. *It is well-known that if (A_1, A_2) is a compatible couple of the Banach spaces and A is an interpolation space between A_1 and A_2*

such that $A_1 \cap A_2$ is dense in A then every system $(x_n)_{n=1}^{\infty} \subset A_1 \cap A_2$ (respectively, $(X_n)_{n=1}^{\infty} \subset A_1 \cap A_2$) which forms basis (respectively (UFDD)) simultaneously in A_1 and A_2 is also basis (respectively (UFDD)) in A. Hence (with non-commutative interpolation using, see [DDP2]) one can generalizes Theorems 4.1–4.4 on the non-commutative spaces associated with functional space $E(0, \infty)$ whose Boyd indexes p_E, q_E are non-trivial (i.e., $1 < p_E < q_e < \infty$). For example, the generalization of Theorem 4.4 is as follows.

Theorem 4.5. *Let $E(0, \infty)$ be a functional space whose the Boyd indexes p_E, q_E are non-trivial, let $E(R, \varphi)$ be the non-commutative symmetric space associated with $E(0, \infty)$ and (R, φ). Then*

a) *The subspaces (4.4) form the (UFDD) of $E(R, \varphi)$;*
b) *$E(R, \varphi)$ has the Schauder basis (4.5).*

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