Dyson's Model of Interacting Brownian Motions at Arbitrary Coupling Strength

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Abstract. For Dyson's model of Brownian motions we prove that the fluctuations are of order one and, in a scaling limit, are governed by an infinite dimensional Ornstein-Uhlenbeck process. This extends a previous result valid only at the free Fermion point $\beta = 2$. Dyson's model can also be interpreted as a random surface. Our result implies that the surface statistics is governed by a massless Gaussian field in the scaling limit.

1 Introduction

We consider N "particles" with positions $x_1(t), \ldots, x_N(t)$ at time t on the circle $[0, \ell]$. These particles diffuse and are governed by the stochastic differential equations

$$dx_j(t) = a_j(x(t))dt + \sqrt{2}db_j(t), \quad j = 1, \ldots, N, \tag{1.1}$$

with $\{b_j(t), j = 1, \ldots, N\}$ independent standard Brownian motions and with the drift

$$a_j(x) = \beta \sum_{i=1, i \neq j}^{N} \frac{\pi}{\ell} \cot(\pi(x_j - x_i)/\ell), \quad \beta > 0. \tag{1.2}$$

The diffusion process $x(t) = (x_1(t), \ldots, x_N(t))$ is assumed to be stationary. Then $x(t)$ is a reversible process with stationary measure

$$|\psi(x)|^2 = Z^{-1} \prod_{i<j=1}^{N} |\sin(\pi(x_i - x_j)/\ell)|^\beta. \tag{1.3}$$

The model (1.1) was introduced by Dyson in 1962 [1] in the context of random matrices. He considered $N \times N$ real orthogonal and unitary random matrices. Their eigenvalues $\exp[i2\pi x_j/\ell], j = 1, \ldots, N$, have distribution $|\psi(x)|^2$ for $\beta = 1$, resp. for $\beta = 2$. The eigenvalues are very stiffly arranged and have small fluctuations. The distance between neighboring eigenvalues has a distribution which vanishes at the origin. In constructing the reversible diffusion process (1.1) one obtains a dynamical interpretation of this level repulsion.

The case $\beta = 2$ has been studied before [2, 3]. We generalize here to arbitrary $\beta > 0$, which requires a novel technique, since the free Fermion method of $\beta = 2$ is no longer applicable.

The generator for (1.1) reads, with $\partial_j = \partial/\partial x_j$,

$$L_N = \sum_{j=1}^{N} a_j(x)\partial_j + \sum_{j=1}^{N} \partial_j^2. \tag{1.4}$$

Since L_N is reversible, we can associate to it the self-adjoint hamiltonian H_N by

$$-\psi L_N \psi^{-1} = H_N - E_N, \tag{1.5}$$

which turns out to be the famous Calogero-Sutherland hamiltonian [4, 5]

$$H_N = -\sum_{j=1}^{N} \partial_j^2 + \frac{\pi^2\beta(\beta-2)}{2\ell^2} \sum_{i<j=1}^{N} |\sin(\pi(x_i - x_j)/\ell)|^{-2} \tag{1.6}$$

and its ground state energy

$$E_N = \beta^2\pi^2 N(N^2-1)/12\ell^2. \tag{1.7}$$

For $\beta = 2$ the potential vanishes and one has the model of free Fermions, for which an analytic machinery is available. For $\beta = 1, 4$ exact solutions become already painful, even more so for general β, cf. Remark (iii) after Theorem 1. In this paper we focus on the low energy behavior and the statistics of small excitations. Such a problem can be tackled by general methods. The proof will exploit the analogy to the hydrodynamic fluctuation theory of Brownian particles interacting through short range forces [6].

The proper definition of (1.1) and (1.6) requires some thought, as can be seen from the coupling constant in (1.6) which for β and $2-\beta$ formally defines the same hamiltonian. Let us first discuss the hamiltonian H_2 for two particles. Subtracting the center of mass yields the operator $-2\partial_x^2 + (\pi^2\beta(\beta-2)/2\ell^2)|\sin(\pi x/\ell)|^{-2}$ which in the short distance approximation $-2\partial_x^2 + (\beta(\beta-2)/2)x^{-2}$, $x \geq 0$, is well studied [7, 8]. The minimal domain of H_2 consists of smooth functions vanishing at $\{x_1 = x_2\}$. For $\beta \geq 3$, H_2 has a unique self-adjoint extension. For $1 < \beta < 3$, H_2 has a one-parameter family of self-adjoint extensions, but only for two of them $\exp[-tH_2]$ has a positive kernel. These are labeled by β and $2-\beta$ with ground state $|\sin(\pi x/\ell)|^{\beta/2}$, respectively $|\sin(\pi x/\ell)|^{(2-\beta)/2}$. For $\beta = 1$ there is only one self-adjoint extension with positive transition kernel. For $\beta \leq -1$ the formal ground state $|\sin(\pi x/\ell)|^{\beta/2}$ is not square integrable. Thus the maximal range of admissible β is $\beta > -1$. For $N > 2$ to have a square integrable ground state requires $\beta > -2/N$. In the interval $1 < \beta < 2 + (2/N)$ H_N has two self-adjoint extensions with a positive transition kernel. These are labeled by β and $2-\beta$ with corresponding ground state as defined in (1.3).

For our enterprise the stochastic differential equations (1.1) will be more natural. We require $\beta > 0$. Then $|\psi(x)|^2$ is normalizable for all N. Let $\Omega = \{x, x_i = x_j, i \neq j, i, j = 1, \ldots, N\}$. We start the process at $x \in [0,\ell]^N \backslash \Omega$, since Ω has $|\psi|^2$- measure zero. To reach a point of Ω where three or more particles coincide has probability zero. The probability for two particles to coincide depends on β. Close to Ω the drift a_j diverges as $\beta(x_j - x_i)^{-1}$. Thus the relative distance between two particles behaves as the Bessel process of order $(\beta - 1)/2$ for small separation. We conclude that if $\beta \geq 1$, then with probability one Ω is never reached. Thus for all t we have $x_i(t) \neq x_j(t)$, $i \neq j$. There is no boundary condition at Ω. If $0 < \beta < 1$, then $x(t)$ will hit Ω with a non-zero probability and $x_i(t) = x_j(t)$ at some time t for some pair (i, j). Since the solution to (1.1) is well-defined even for $x(0) \in \Omega$, we can continue either keeping the order (reflecting boundary conditions) or interchanging the order (transmitting boundary conditions) of the pair of particles. Later on we will consider only functions symmetric in the particle label. Thus either convention is fine. However, as to be explained in the following paragraph, for the surface picture reflecting boundary conditions are more natural. Thus at time $t = 0$ we label the particles such that $0 \leq x_1(0) < \ldots < x_N(0) < \ell$. Then almost surely (1.1) has a unique solution with continuous sample paths and at any time t we have

$$x_1(t) \leq x_2(t) \leq \ldots \leq x_N(t) \mod \ell. \tag{1.8}$$

We refer to [9] for a more detailed discussion.

The hamiltonian (1.6) suggests a physical application of interest [10, 11]. We regard $t \mapsto x_j(t)$ as a step of unit height in a vicinal surface. The height function, h, of the surface takes then integer values. We define h first for $t = 0$ by

$$h(x,0) = \begin{cases} 0 & \text{for } \ 0 \leq x < x_1(0), \\ j & \text{for } \ x_j(0) \leq x < x_{j+1}(0)\ , j = 1, \ldots, N-1, \\ N & \text{for } \ x_N(0) \leq x < \ell \end{cases} \tag{1.9}$$

and extend h to all $x \in \mathbb{R}$ by $h(x + n\ell, 0) = h(x, 0) + nN$ with integer n. Let $J([t_1, t_2])$ be the current through the origin integrated over the time span $[t_1, t_2]$, i.e. $J([t_1, t_2])$ is the number of particles crossing forwards minus those crossing backwards the origin during $[t_1, t_2]$. Then for all $x, t \in \mathbb{R}^2$

$$h(x,t) = \begin{cases} h(x,0) - J([0,t]) & \text{for } \ t > 0, \\ h(x,0) + J([t,0]) & \text{for } \ t < 0. \end{cases} \tag{1.10}$$

In the transverse (x) direction the surface has period ℓ and a miscut angle determined through the step density $\rho = N/\ell$.

For the physical interpretation we associate to H_N the formal Boltzmann weight

$$\exp\quad\Big[-\sum_{j=1}^N\int\dot{x}_j(t)^2dt-\tfrac{\pi^2\beta(\beta-2)}{2\ell^2}\sum_{i<j=1}^N\int|\sin(\pi(x_i(t)-x_j(t))/\ell)|^{-2}dt\Big]$$
$$\prod_j\prod_t dx_j(t) \tag{1.11}$$

for the ensemble of step configurations. We only allow those steps which satisfy the non-crossing constraint (1.8) with strict inequalities. The first term in (1.11) is the energy of a single step with line tension normalized to one. The second term is a step-step interaction which decays as $1/x^2$. Physically such a potential comes from elastic forces mediated through the bulk of the crystal [12] and from electrostatic interactions [13]. The constraint (1.8) with strict inequalities causes entropic repulsion which effectively corresponds to a potential also decreasing as $1/x^2$. In this interpretation, for $\beta=2$ there is purely entropic repulsion. For $1\le\beta<2$, the potential is attractive but still outweighed by the entropic repulsion. For $\beta<1$ the strict non-crossing condition can no longer be maintained and one continues to the branch where neighboring steps may touch each other.

For surfaces one natural problem is to identify its statistics on large scales. Since the x-gradient of the surface is the density of the Brownian particles, equivalently one wants to understand the density fluctuations when averaged over suitable space-time regions. In the x-direction we sum over a number of steps which is of the order N. For surfaces one expects, up to some anisotropy, both coordinate axes to live on the same footing. Thus in the t-direction we should sum also over an interval of order N. We will prove that on such a large scale the density fluctuations of the particle system (equivalently the surface fluctuations) are Gaussian and we determine explicitely the covariance.

2 Main Result

At this point it is convenient to rewrite (1.1) on the macroscopic scale. Without loss of generality we set $\ell=2\pi N$ and $\rho=1/2\pi$. We rescale through the substitution $x\rightsquigarrow Nx$, $t\rightsquigarrow Nt$. Then $x_j(t)\in[0,2\pi]$ with periodic boundary conditions. $x(t)$ is governed by

$$dx_j(t)=a_j^N(x(t))dt+\sqrt{2/N}db_j(t),\quad j=1,\dots,N, \tag{2.1}$$

with drift

$$a_j^N(x) = \beta \frac{1}{2N} \sum_{i=1, i \neq j}^{N} \cot((x_j - x_i)/2) \tag{2.2}$$

and stationary distribution

$$|\psi_N(x)|^2 = Z^{-1} \prod_{i<j=1}^{N} |\sin((x_i - x_j)/2)|^\beta \tag{2.3}$$

on $[0, 2\pi]^N$ relative to the Lebesgue measure. The path measure of this stationary process is denoted by P^N, expectation by $\mathbb{E}_N$. By a law of large numbers [14, 15] we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} f(x_j(t)) = \frac{1}{2\pi} \int_0^{2\pi} f(x)dx. \tag{2.4}$$

Our interest are the fluctuations around this deterministic density.

As usual, let $\hat{f}_k = (1/2\pi) \int_0^{2\pi} e^{-ikx} f(x)dx$ be the Fourier coefficients of f, $k \in \mathbb{Z}$. For $\alpha \in \mathbb{R}$ we introduce the real Hilbert spaces $\mathcal{H}_\alpha = \{f | \hat{f}_0 = 0, \hat{f}_k^* = \hat{f}_{-k}, \sum_{k\neq 0} |k|^\alpha |\hat{f}_k|^2 < \infty\}$. Clearly $\mathcal{H}_\alpha^* = \mathcal{H}_{-\alpha}$. For $f \in \mathcal{H}_{3+\kappa}$ with some $\kappa > 0$ we define the fluctuation field

$$\xi^N(f, t) = \sum_{j=1}^{N} f(x_j(t)). \tag{2.5}$$

By definition $\mathbb{E}_N(\xi^N(f,t)) = N\hat{f}_0 = 0$. Note that the prefactor in (2.5) is one. This reflects that fluctuations are severely suppressed because of the strong repulsive forces. By linear duality we regard $t \mapsto \xi^N(\cdot, t)$ as Hilbert space valued stochastic process. As to be shown, $\xi^N(\cdot, t)$ almost surely takes values in $\mathcal{H}_{-3-\kappa}$ and is weakly continuous in t. Let Q^N be the path measure on $C(\mathbb{R}, \mathcal{H}_{-3-\kappa})$ induced by P^N.

In the limit $N \to \infty$, Q^N is well approximated by an infinite-dimensional Ornstein-Uhlenbeck process $\xi(f, t)$, which is governed by the stochastic partial differential equation

$$d\xi(f,t) = \xi(-(\beta/2)\sqrt{-(\partial/\partial x)^2} f, t)dt + dW(f', t) \tag{2.6}$$

with $f \in \mathcal{H}_{3+\kappa}$ and $dW(f,t)$ white noise normalized as $\mathbb{E}(dW(f,t)dW(g,s)) = 2\delta(t-s)dtds(1/2\pi)\int_0^{2\pi} f(x)g(x)dx$. The stationary measure for $\xi(f,t)$ is Gaussian with covariance

$$\mathbb{E}(\xi(f,t)\xi(g,t)) = \frac{2}{\beta}\sum_k |k|\hat{f}_k^*\hat{g}_k. \tag{2.7}$$

We denote by Q the path measure on $C(\mathbb{R}, \mathcal{H}_{-3-\kappa})$ of the stationary Ornstein-Uhlenbeck process (2.6).

As our main result we state

Theorem 1. *Let Q^N be the induced path measure for (2.5) and let Q be the path measure of the stationary Ornstein-Uhlenbeck process (2.6) on $C(\mathbb{R}, \mathcal{H}_{-3-\kappa})$. Then in the sense of weak convergence of measures*

$$\lim_{N\to\infty} Q^N = Q. \tag{2.8}$$

Remarks. (i) In the surface picture the steps are the x-derivative of the surface height $h(x,t)$, cf. Eq. (1.8). If we scale the surface according to (2.1), then the step height is $1/N$ and the average surface $< h^N(x,t) >= x/2\pi$. Theorem 1 (together with a bound on the large values of ξ^N) implies that

$$\lim_{N\to\infty} \mathbb{E}_N(\xi^N(f,t)\xi^N(g,0)) = \frac{2}{\beta}\sum_k \hat{f}_k^* \tag{2.9}$$

Therefore the surface fluctuations, $h^N(x,t) - x/2\pi$, are of order $1/N$ and in the limit $N \to \infty$ are Gaussian with covariance $8/(\beta^2k^2 + 4\omega^2)$ in Fourier space representation with respect to x, t. This means that the surface has a large scale statistics governed by a massless free field, which is the generally expected behavior if the surface tension depends smoothly on the slope. In our case the surface tension is the ground state energy $e(\rho)$ per unit volume of the Calogero-Sutherland model. By (1.7) $e(\rho) = \beta^2\pi^2\rho^3/12$ which satisfies the hypothesis on smooth dependence on the slope. The massles Gaussian limit has been established only for a few surface models, amongst them the Ginzburg-Landau $\nabla\phi$ surface model [16, 17].

(ii) One would like to take first the limit $\ell, N \to \infty$, $\rho = N/\ell$ fixed, in (1.1).

The force between the Brownian particles is then repulsive and given by $\beta/(x_j - x_i)$. Except for $\beta = 2$, one does not even know the existence of the infinite volume stationary measure, which is a prerequisite for constructing the process. Also the Johansson result used below is not available in this setting.

(iii) Over the years there has been a considerable effort to compute ground state correlations in the Calogero-Sutherland model. In our context the results on the density-density correlations are of interest. On the microscopic scale (1.1) let

$$\rho(x,t) = \sum_{i=1}^{N} \delta(x - x_j(t)) \tag{2.10}$$

and let

$$< \rho(x,t)\rho(0,0) >_{N,\ell} - < \rho(0,0) >_{N,\ell}^2 = S_{N,\ell}(x,t), \tag{2.11}$$

average in the stationary process. For rational $\beta = p/q$ the infinite volume limit

$$S_\rho(x,t) = \lim_{N,\ell\to\infty, N/\ell=\rho} S_{N,\ell}(x,t) \tag{2.12}$$

is computed explicitely in [18] and expressed as a $p+q$-fold Dotsenko-Fateev integral. The asymptotics of these integrals are studied in [19] with the result

$$S_\rho(x,t) = \frac{1}{2\beta\pi^2}((x - i\pi\rho\beta|t|)^{-1} + (x + i\pi\rho\beta|t|)^{-1}) \tag{2.13}$$

for large x, t. Defining the structure function

$$\hat{S}_\rho(k,t) = \int e^{ikx} S_\rho(x,t)dx, \tag{2.14}$$

we obtain

$$\hat{S}_\rho(k,t) = \frac{1}{\pi\beta}|k|e^{-\pi\rho\beta|k||t|}. \tag{2.15}$$

As expected this result agrees with (2.8), except for the prefactor $1/2\pi$ which results from the conventions in defining Fourier transforms. To us it seems prohibitively complicated to establish a Gaussian limit with similar kind of techniques.

(vi) Physically it would be of great interest to prove Theorem 1 for a diffusion process where in the Hamiltonian H_N the long range potential $1/x^2$ is replaced by a short range potential and where Dirichlet boundary conditions are imposed on the crossing set Ω.

(v) If the drift is given by

$$a_j(x) = -\sum_{i=1, i\neq j}^{N} \partial_j U(x_j - x_i) \tag{2.16}$$

with a sufficiently weak finite range potential U, then a result analogous to Theorem 1 has been proved in [6]. If we reintroduce the density in (2.6), then in both cases the noise strength is $\sqrt{\rho}$. However the drift term is now local and given by $D\partial^2/\partial x^2$ with $D = \rho/\chi$ and χ the static compressibility.

3 Convergence to an Ornstein-Uhlenbeck Process

We follow the standard route of hydrodynamic fluctuation theory [20]. We have

$$\xi^N(f,t) = \xi^N(f,0) + \int_0^t \gamma^N(f)(x(s))ds + M^N(f,t). \tag{3.1}$$

$M^N(f,t)$ is a martingale and

$$M^N(f,t)^2 - \int_0^t \gamma_1^N(f)(x(s))ds = M_1^N(f,t), \tag{3.2}$$

where $M_1^N(f,t)$ is again a martingale. Here $\gamma^N(f)$ and $\gamma_1^N(f)$ are given by

$$\begin{aligned} \gamma^N(f)(x) &= L_N\xi^N(f)(x) = \beta\frac{1}{2N}\sum_{i=1, i\neq j}^{N} \cot((x_j - x_i)/2)f'(x_j) + \frac{1}{N}\sum_{j=1}^{N} f''(x_j), \\ \gamma_1^N(f)(x) &= (L_N\xi^N(f)^2 - 2\xi^N(f)L_N\xi^N(f))(x) = \frac{2}{N}\sum_{j=1}^{N} f'(x_j)^2. \end{aligned} \tag{3.3}$$

According to Theorem 1 we have to establish that in the limit $N \to \infty$ the fluctuation field satisfies

$$\xi(f,t) = \xi(f,0) + \int_0^t \xi(-(\beta/2)\sqrt{-(\partial^2/\partial x^2)}f, s)ds + W(f',t). \tag{3.4}$$

To our own surprise, the proof requires only static estimates which we state first. Let $\{x_j, j = 1, \ldots, N\}$ be distributed according to (2.3). We have

Lemma 1. *Let $f : [0, 2\pi] \to \mathbb{R}$ be continuous. Then in distribution*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} f(x_j) = \hat{f}_0. \tag{3.5}$$

Proof: [14, 15]. □

Lemma 2. *Let $f \in \mathcal{H}_1$. Then*

$$\lim_{N\to\infty} \mathbb{E}_N(\xi^N(f)^2) = \frac{2}{\beta} \sum_k |k||\hat{f}_k|^2. \tag{3.6}$$

Proof: In [21, 22] Johansson proves

$$\lim_{N\to\infty} \mathbb{E}_N(\exp[\xi^N(f)]) = \exp[\frac{1}{\beta} \sum_k |k||\hat{f}_k|^2], \tag{3.7}$$

which means that the static fluctuations of $\xi^N(f)$ are Gaussian in the limit. In [21] only the case $\beta = 2$ is stated. The proof extends to all β as explained in [22], where the more difficult case of a non-uniform background density is handled. □

With such an input we have

Lemma 3. *For $f \in \mathcal{H}_2$*

$$\lim_{N\to\infty} \gamma_1^N(f) = 2 \sum_k k^2 |\hat{f}_k|^2 \tag{3.8}$$

in distribution.

Lemma 4. *Let $f \in \mathcal{H}_4$ and let A be the integral operator defined by the Hilbert transform*

$$Af(x) = \frac{\beta}{4\pi} \int_0^{2\pi} \cot((y - x)/2) f'(y) dy \tag{3.9}$$

in the sense of the Cauchy principal part. Then

$$\lim_{N\to\infty} \mathbb{E}_N((\gamma^N(f) - \xi^N(Af))^2) = 0. \tag{3.10}$$

Proof: We work out the square and use that

$$\begin{aligned}
\widehat{Af}_k &= \frac{\beta}{2}(2\pi)^{-2}\int_0^{2\pi}\int_0^{2\pi} e^{-ikx}\cot((y-x)/2)f'(y)dxdy \\
&= -(\beta/2)|k|\hat{f}_k. \qquad (3.11)
\end{aligned}$$

(i) First term. By Lemma 2 and (3.11)

$$\lim_{N\to\infty}\mathbb{E}_N(\xi^N(Af)^2) = \frac{2}{\beta}\sum_k |k||\widehat{Af}_k|^2 = \frac{\beta}{2}\sum_k |k|^3|\hat{f}_k|^2. \qquad (3.12)$$

(ii) Second term. We use that $\partial_j|\psi_N(x)|^2 = Na_j^N(x)|\psi_N(x)|^2$ and partially integrate. Then

$$\begin{aligned}
&- \quad 2\mathbb{E}_N(\gamma^N(f)\xi^N(Af)) \\
&= \quad -2\sum_{i,j=1}^N \left\{\mathbb{E}_N[a_j^N(x)f'(x_j)Af(x_i)] + \frac{1}{N}\mathbb{E}_N[f''(x_j)Af(x_i)]\right\} \\
&= \quad \frac{2}{N}\sum_{i,j=1}^N \{\mathbb{E}_N[\partial_j(f'(x_j)Af(x_i)] - \mathbb{E}_N[f''(x_j)Af(x_i)]\} \\
&= \quad \frac{2}{N}\mathbb{E}_N[\sum_{j=1}^N f'(x_j)Af'(x_j)], \qquad (3.13)
\end{aligned}$$

which by Lemma 1 converges to

$$-\beta\sum_k |k|^3|\hat{f}_k|^2. \qquad (3.14)$$

(iii) Third term. As before we use partial integration,

$$\begin{aligned}
&\mathbb{E}_N(\gamma^N(f)^2) \\
= &\sum_{i,j=1}^N \left\{\mathbb{E}_N[a_i^N(x)f'(x_i)a_j^N(x)f'(x_j)] + \frac{2}{N}\mathbb{E}_N[a_i^N(x)f'(x_i)f''(x_j)]\right. \\
&\left. + \frac{1}{N^2}\mathbb{E}_N[f''(x_i)f''(x_j)]\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{N^2}\sum_{i,j=1}^{N}\{\mathbb{E}_N[\partial_i\partial_j(f'(x_i)f'(x_j)) - N(\partial_j a_i^N)(x)f'(x_i)f'(x_j)] \\
&\quad -2\mathbb{E}_N[\partial_i(f'(x_i)f''(x_j))] + \mathbb{E}_N[f''(x_i)f''(x_j)]\} \qquad (3.15)\\
&= \frac{1}{N^2}\mathbb{E}_N[\sum_{j=1}^{N} f''(x_j)^2] + \frac{\beta}{8N^2}\mathbb{E}_N[\sum_{i,j=1}^{N}(\sin((x_i-x_j)/2))^{-2}(f'(x_i)-f'(x_j))^2].
\end{aligned}
$$

The first summand vanishes as $N \to \infty$ by the law of large numbers (2.4) and since $f \in \mathcal{H}_4$. Again by the law of large numbers the second summand converges to

$$
\begin{aligned}
&\frac{\beta}{8}(2\pi)^{-2}\int_0^{2\pi}\int_0^{2\pi}(\sin((x-y)/2))^{-2}(f'(x)-f'(y))^2 dxdy \\
&= \frac{\beta}{2}\sum_k |k|^3|\hat{f}_k|^2. \qquad (3.16)
\end{aligned}
$$

We conclude that the limit in (3.10) vanishes. □

Next we have to establish support and tightness of Q^N. We denote by $||f||_\alpha$ the norm in $\mathcal{H}_\alpha$, cf. definition above (1.12).

Lemma 5. *Let $f \in \mathcal{H}_2$. Then*

$$
Q^N(\sup_{0\le t\le T}|\xi^N(f,t)|^2) \le c_1 T\sum_k |k|^2|\hat{f}_k|^2. \qquad (3.17)
$$

Proof: For any stationary Markov process, X_t, with generator L and invariant measure μ we have

$$
\mathbb{E}(\sup_{0\le t\le T}|G(X_t)|^2) \le 3\mathbb{E}(|G|^2) - 56T\mathbb{E}(GLG) \qquad (3.18)
$$

for $G \in L^2(\mu)\cap H_1(\mu)$ [23]. The assertion follows from Lemma 1 and from $||f||_1 \le ||f||_2$. □

Lemma 6. *Q^N and Q are supported by $C(\mathbb{R}, \mathcal{H}_{-3-\kappa})$ for some $\kappa > 0$.*

Proof: We have as distribution

$$
\xi^N(x,t) = \sum_{j=1}^{N}\delta(x_j(t)-x) - \frac{N}{2\pi}. \qquad (3.19)
$$

Therefore

$$\hat{\xi}_k^N(t) = \frac{1}{2\pi}\sum_{j=1}^{N} \exp[-ikx_j(t)],\ k \neq 0,\ \hat{\xi}_0^N(t) = 0, \tag{3.20}$$

and

$$\mathbb{E}_N\big(\sup_{0\le t\le T} ||\xi(\cdot,t)||^2_{-3-\kappa}\big) \le \sum_{k\neq 0} |k|^{-3-\kappa}\mathbb{E}_N\big(\sup_{0\le t\le T} |\hat{\xi}_k(t)|^2\big)$$
$$\le\ c'\sum_{k\neq 0} |k|^{-3-\kappa}|k|^2 < \infty \tag{3.21}$$

by Lemma 5. The continuity of $t \mapsto \xi^N(f,t)$ and $t \mapsto \xi(f,t)$ can be checked directly, since $f \in \mathcal{H}_{3+\kappa}$. □

Lemma 7. *The family of measures* $\{Q^N, N = 1, 2, \ldots\}$ *is tight.*

Proof: The proof is identical to [24], page 207. □

Proof of Theorem 1: Since the family $\{Q^N, N = 1, 2, \ldots\}$ is tight, we can choose a convergent subsequence with limit point Q. By Lemma 4 we conclude that for $f \in \mathcal{H}_4$

$$M(f,t) = \xi(f,t) - \xi(f,0) - \int_0^t \xi(-(\beta/2)\sqrt{-(\partial^2/\partial x^2)}f, s)ds \tag{3.22}$$

is a martingale under Q. For $f \in \mathcal{H}_4 \subset \mathcal{H}_3$ by Lemma 3

$$M_1(f,t) = M(f,t)^2 - 2t\frac{1}{2\pi}\int_0^{2\pi} f'(x)^2 dx \tag{3.23}$$

is also a martingale under Q. The martingale problem (3.22), (3.23) with $f \in \mathcal{H}_4$ has as unique solution the infinite dimensional Ornstein-Uhlenbeck process (2.6) [25]. Its inital measure is fixed by (3.7). Alternatively we can use that the Ornstein-Uhlenbeck process (2.6) must be stationary as a limit of stationary processes. This determines (3.7) uniquely as its single time distribution. Since the limit measure is unique, any subsequence has the same limit. □

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