Microlocal WKB expansions

André Martinez and Vania Sordoni
Dipartimento di Matematica
Università di Bologna
40127 Bologna, Italia

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1 Introduction

In this paper, we investigate the existence of microlocal WKB expansions for the eigenfunctions of pseudodifferential operators whose symbol admits a non degenerate minimum at some point (x_0, ξ_0) of $\mathbf{R}^{2n}$. For example, this is the case of an electromagnetic Schrödinger operator

$$P_A(x, hD_x) = \sum_{j=1}^{n} \left(hD_{x_j} - A_j(x)\right)^2 + V(x)$$

when V admits a non degenerate minimum at some point x_0. In the case where the A_j's can be taken small enough (that is when the magnetic field is small enough) and everything is analytic, it has been shown by Helffer and Sjöstrand [4] that the first eigenfunction u of P_A admits near x_0 a WKB expansion of the form

$$u(x, h) \sim e^{-\phi_A(x)/h} \sum_{k \geq 0} h^k a_k(x)$$

where ϕ_A and the a_k's are smooth functions. Moreover, the set of x's where such an expansion is valid can be estimated geometrically by means of the minimal geodesics starting from x_0, relatively to the

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so-called Agmon distance (i.e. the distance associated to the degenerate metric $(V(x)-V(x_0))dx^2$). But the problem remains entirely open for greater magnetic fields. Here, we are going to show that in any case (but still under assumptions of analyticity), a similar WKB expansion exists near (x_0,ξ_0) for the FBI-transform Tu of u, at least if one choose convenient symplectic coordinates in $\mathbf{R}^{2n}$. Moreover, the set of (x,ξ)'s where the expansion is valid will be estimated by means of simple constants attached to the symbol of P, and a more general notion of "admissible open set" will be given, in terms of deformation properties.

As we shall see, $|Tu(x,\xi;h)|$ behaves like $e^{-\psi(x,\xi)/h}$ where $\psi\geq 0$ has a non degenerate minimum at (x_0,ξ_0). This analogy with the behavior of the eigenfunctions in the case without magnetic field make us think that our construction can be useful, in future, to estimate various types of tunnelling in phase space.
Let us specify our main assumptions.
For $k\in\mathbf{N}$, let us consider a symbol p belonging to

$$\mathcal{S}_n(\langle x,\xi\rangle^k) =: \{p\in C^\infty(\mathbf{R}^{2n})\ ;\ \forall\,\alpha\in\mathbf{N}^{2n},\ \partial^\alpha p=\mathcal{O}(\langle x,\xi\rangle^k)\text{ uniformly}\} \tag{1.1}$$

and, for $p\in\mathcal{S}_n(\langle x,\xi\rangle^k)$, we consider the Weyl-semiclassical quantization $P(x,hD_x)=\mathrm{Op}_{h,\frac{1}{2}}(p)$ defined by:

$$P(x,hD_x)u(x)=\frac{1}{(2\pi h)^n}\int e^{i(x-y)\xi/h}p((x+y)/2,\xi)u(y)dyd\xi. \tag{1.2}$$

On our symbol $p\in\mathcal{S}_n(\langle x,\xi\rangle^k)$ we make the following assumptions

(H_1) $p(x,\xi)$ is real and non negative for real (x,ξ), $p^{-1}(0)=(0,0)$, $\mathrm{Hess}p(0,0)$ is positive definite, and there exist $\delta_1,\delta_2>0$ such that $p(x,\xi)\geq\delta_1$ if $|x|+|\xi|\geq\delta_2$.

(H_2) There exist $a,b>0$ such that p extends holomorphically to

$$S(a,b)=\{(x,\xi)\in\mathbb{C}^{2n}\ ;\ |\mathrm{Im}x|<a\ ,\ |\mathrm{Im}\xi|<b\},$$

and satisfies:

$$\forall \alpha \in \mathbf{N}^{2n} \ , \ \partial^\alpha p = \mathcal{O}(\langle \mathrm{Re}x, \mathrm{Re}\xi \rangle^k) \text{ uniformly in } S(a,b).$$

Assumptions $(H_1), (H_2)$ implies that the spectrum of P is discrete near 0. Actually, let us consider $\chi \in C_0^\infty(\mathbf{R}^{2n})$ with $\chi \geq 0$, $\chi(0) > 0$ and let us write $p = \hat{p} - \chi$ with $\hat{p} = p + \chi$ is such that $\mathrm{Inf}\hat{p} > 0$. Then P can be written as $P = \hat{P} + K$ where $\hat{P} = \mathrm{Op}_{h,\frac{1}{2}}(\hat{p})$ and $K = \mathrm{Op}_{h,\frac{1}{2}}(-\chi)$ is a compact operator If $z \in \mathbb{C}$ with $|z|$ sufficiently small, then $\hat{P} - z$ is invertible and

$$P - z = (\hat{P} - z)(I + (\hat{P} - z)^{-1} K).$$

Hence, by Fredholm theory, we can conclude that P has discrete spectrum near 0.

Moreover, assumption (H_1) implies (see, for example, [5]) that we can perform a linear symplectic change of variables such that, in the new coordinates, p can be written as

$$p(x,\xi) = \sum_{j=1}^{n} \mu_j (x_j^2 + \xi_j^2) + \mathcal{O}(|x,\xi|^3) \tag{1.3}$$

where $0 < \mu_1 \leq ... \leq \mu_n$.

For $h > 0$ small, $(x,\xi) \in \mathbf{R}^{2n}$, and u (possibly h-dependent) in $L^2(\mathbf{R}^n)$, we define the Fourier-Bros-Iagolnitzer transform (or global FBI transform) (see: [14] and [15]) of u as:

$$Tu(x,\xi;h) = c(n,h) \int e^{i(x-y)\xi/h - (x-y)^2/2h} u(y) dy \tag{1.4}$$

where $c(n,h) = 2^{-n/2}(\pi h)^{-3n/4}$ is chosen in such a way that:

$$\|Tu\|_{L^2(\mathbf{R}^{2n})} = \|u\|_{L^2(\mathbf{R}^n)}. \tag{1.5}$$

Our purpose is to show that the FBI transform Tu of the first eigenfunction u of $P(x, hD_x)$ admits, in a suitable neighborhood of $(0,0)$, a WKB asymptotics of the form:

$$Tu(x,\xi;h) \sim h^{-m_0} e^{-\xi^2/2h - \varphi(x - i\xi)/h} \sum_{j \geq 0} h^j a_j(x - i\xi) \tag{1.6}$$

where φ and the a_j's are holomorphic near $0 \in \mathbb{C}^n$. Incidentally, we also get an asymptotic expansion

$$E \sim h \sum_{k \geq 0} e_k h^k$$

as h tends to 0, for the first eigenvalue E of P. Moreover, by standard arguments, it should be possible to obtain asymptotic expansions for the eigenfunctions of a multiple wells problem and for the excited states. This would be a full generalization of the results of [12].

2 Construction of the WKB solution near a strict minima

Let us consider a symbol $p \in \mathcal{S}_n(\langle x, \xi \rangle^k)$ and assume that $(H_1), (H_2)$ hold. In [10] is studied in detail the action of the FBI transform T

on a pseudodifferential operator $P(x, hD_x) = \mathrm{Op}_{h,\frac{1}{2}}(p)$. In particular, it is shown that (see also [9]):

$$T(Pu)(x, \xi; h) = \tilde{P}(x, \xi, hD_x, hD_\xi) Tu(x, \xi; h) \tag{2.1}$$

with

$$\tilde{P}(x, \xi, hD_x, hD_\xi) = \mathrm{Op}_{h,\frac{1}{2}}(p(x - \xi^*, x^*)) \tag{2.2}$$

(Here and in the following, x^* and ξ^* denotes the dual variable of x and ξ respectively). Notice that, if we set $z = x - i\xi$, then $e^{\xi^2/2h} Tu(x, \xi; h)$ is

an holomorphic function of the variable z. For this reason, it is natural to look for a solution $v = Tu$ of $\tilde{P}(x, \xi, hD_x, hD_\xi)v = Ev$ of the form

$$v(z, h) \sim e^{-\xi^2/2h - \varphi(z)/h} a(z, h) \tag{2.3}$$

where $a(z, h) = \sum_{j \geq 0} h^j a_j(z)$ and $\varphi(z)$ and $a_j(z)$ are holomorphic near $0 \in \mathbb{C}^n$. The action of $\tilde{P}(x, \xi, hD_x, hD_\xi)$ on such a function v can be defined as following:

Let $\varphi = \varphi(z)$, $z = x - i\xi$, an holomorphic function defined on a complex neighborhood Ω of 0 in $\mathbb{C}^n$ and such that $|\mathrm{Im}(z - \partial_z \varphi(z))| < a$ and $|\mathrm{Re}\partial_z \varphi(z)| < b$ for $z \in \Omega$. If $a(z, h) = \sum_{j \geq 0} h^j a_j(z)$, with $a_j(z)$ holomorphic functions of $z \in \Omega$, is a formal power series, using a formal stationary phase expansion, we have:

$$
\begin{aligned}
& e^{\xi^2/2h+\varphi(z)/h}\tilde{P}(x,\xi,hD_x,hD_\xi)(e^{-\xi^2/2h-\varphi(z)/h})a(z,h) \\
& \quad \sim e^{\varphi(z)/h}\mathrm{Op}_{h,\frac{1}{2}}\left(p(z-\xi^*,x^*)\right)e^{-\varphi(z)/h}a(z,h) \\
& \quad \sim \sum_{\beta+\gamma=\alpha\in\mathbf{N}^{2n}} \frac{h^{|\alpha|}}{i^{|\alpha|}\beta!\gamma!}\partial^\beta_{y,\eta}\partial^\alpha_{x^*,\xi^*}p(\frac{z+z'}{2}-\xi^*-\partial_z\varphi, x^*+i\partial_z\varphi)\Big|_{\substack{y=x,\eta=\xi \\ x^*=\xi^*=0}} \\
& \qquad\qquad\qquad\qquad \times\partial^\gamma_{y,\eta}(e^{-\Phi(z,z')/h}a(z',h))\,|_{z=z'}
\end{aligned}
\tag{2.4}
$$

in the sense of the formal power series expansion. Here we have set $z' = y - i\eta$ and $\Phi(z,z') = \varphi(z) - \varphi(z') - \langle z - z', \nabla_z\varphi(z)\rangle = \mathcal{O}(|z-z'|^2)$. Hence, to find a solution v of the equation $(\tilde{P} - E)v = 0$ of the form $v(z,h) \sim e^{-\xi^2/2h-\varphi(z)/h}a(z,h)$, we are lead to solve the eikonal equation

$$p(z - \partial_z\varphi, i\partial_z\varphi) = 0 \tag{2.5}$$

where $z = x - i\xi$. To solve it, we proceed in an way analogous to [2] . We define

$$q(z,\zeta) = -p(z-\zeta, i\zeta)$$

so that (2.5) becomes $q(z,\partial_z\varphi) = 0$. Near $(0,0) \in \mathbb{C}^{2n}$ we have:

$$q(z,\zeta) = \sum_{j=1}^n \mu_j(2z_j\zeta_j - z_j^2) + \mathcal{O}(|z,\zeta|^3)$$

and therefore the fundamental matrix of q at $(0,0)$ is

$$F_q = \begin{pmatrix} 2\mu & 0 \\ 2\mu & -2\mu \end{pmatrix} \tag{2.6}$$

where $\mu = \mathrm{diag}(\mu_1,...,\mu_n)$. The spectrum of F_q is $\{\pm 2\mu_j\ ;\ j = 1,...,n\}$, and the direct sum of the eigenspaces

associated to $\{+2\mu_j\ ;\ j = 1,...,n\}$ (resp. to $\{-2\mu_j\ ;\ j = 1,...,n\}$) is the Lagrangian space $\mathcal{E}_+ = \{\zeta = z/2\}$ (resp. $\mathcal{E}_- = \{z = 0\}$). If we perform a canonical change of variables $z_j \to z_j/\sqrt{2}, \zeta_j \to \sqrt{2}\,\zeta_j - z_j/\sqrt{2}$, then, in this new coordinates,

$$q(z,\zeta) = \sum_{j=1}^{n} 2\mu_j z_j \zeta_j + \mathcal{O}(|z,\zeta|^3).$$

Then, we are exactly in the situation described in [15] Appendix [2]. By applying this analytic version of the 'stable-unstable manifold theorem' one can show that there exist two holomorphic complex Lagrangian manifolds $\Lambda_\pm$ containing $(0,0)$, stable under the action of H_q, and such that $T_{(0,0)}\Lambda_\pm = \mathcal{E}_\pm$. In particular, Λ_+ projects bijectively on the base $\{\zeta = 0\}$, and therefore there exists a holomorphic function φ such that in a complex neighborhood of 0, Λ_+ is given by:

$$\Lambda_+ = \{\zeta = \partial_z \varphi(z)\}.$$

Since $q(0,0) = 0$ and q is constant on Λ_+, we see in particular that φ solves (2.5). Notice that if we normalize φ by setting $\varphi(0) = 0$, we also have:

$$\varphi(z) = \frac{1}{4}z^2 + \mathcal{O}(|z|^3)$$

and therefore

$$\frac{\xi^2}{2} + \mathrm{Re}\varphi(x - i\xi) = \frac{x^2+\xi^2}{4} + \mathcal{O}(|x,\xi|^3).$$

Now, taking into account (2.4) and first working with z real, one can construct as in [1] a symbol $a(z,h) \sim \sum_{k\geq 0} h^k a_k(z)$ defined near 0, such that formally

$$\mathrm{Op}_{h,\frac{1}{2}}(p(z+i\zeta,\zeta) - E)(a(z,h)e^{-\varphi(z)/h}) \sim 0$$

with $E \sim h\sum_{k\geq 0} e_k h^k$, $e_0 = \sum_{j=1}^{n} \mu_j = \frac{1}{2}Tr^+ F_q$. Moreover, one can see, as in [14], that actually $a(z,h)$ is an analytic symbol in the sense that one has, for

[2]Actually, in [15], q is required to be real for (z,ζ) real, and here we are not in such a situation. Nevertheless, this assumption is used along the proof only to show that there exist analytic coordinates such that H_q can be written in the form

$$H_q = Az\partial_z - A^*\zeta\partial_\zeta + \mathcal{O}(|z,\zeta|)(\partial_z, \partial_\zeta)$$

for some $n \times n$ matrix A and A^* with $\mathrm{sp}(A), \mathrm{sp}(A^*) \subset \{\mathrm{Re}(\lambda) > 0\}$. Indeed, this is our case, with $A = A^* = 2\mu$

some $C > 0$ and for any $k \in \mathbf{N}$

$$|a_k(z)| \leq C^{k+1}\, k!$$

uniformly near zero. After resummation, this means that, if we set

$$v(x,\xi;h) = a(x-i\xi,h)e^{-\xi^2/2h-\varphi(x-i\xi)/h},$$

there exists $\varepsilon > 0$ such that, for (x,ξ) small enough:

$$(\tilde{P}-E)v = \mathcal{O}\left(e^{-\xi^2/2h-\mathrm{Re}\varphi(x-i\xi)/h-\varepsilon/h}\right). \tag{2.7}$$

where the action of the pseudodifferential operator $\tilde{P}$ on $v(x,\xi,h)$(which is defined only near $(0,0)$) is given by (2.4).
Let Ω_0 be the maximal connected open set where both φ and the a_k's extend holomorphically and such that $|\mathrm{Im}(z-\partial_z\varphi)| < a$, $|\mathrm{Re}\ \partial_z\varphi| < b$. Let $\chi, \tilde{\chi} \in C_0^\infty(\Omega_0)$ such that $\tilde{\chi} = 1$ on a neighborhood V_1 of suppχ. Then we claim that there exists $\varepsilon_1 > 0$ such that

$$\chi(\tilde{P}-E)\tilde{\chi}v = \mathcal{O}(e^{-\xi^2/2h-\mathrm{Re}\varphi(x-i\xi)/h-\varepsilon_1/h}) \quad \text{uniformly.} \tag{2.8}$$

To prove (2.8), let us set $X = (x,\xi)$, $Y = (y,\eta)$, $X^* = (x^*,\xi^*)$ and $\psi(X) = \xi^2/2 + \varphi(x-i\xi)$ and let us write

$$\begin{aligned}&(2\pi h)^{2n}e^{\psi(X)/h}\chi(X)((\tilde{P}-E)\tilde{\chi}v)(X) \\ &= \int e^{i(X-Y)X^*/h}\tilde{p}(\frac{X+Y}{2},X^*)\tilde{\chi}(Y)\chi(X)e^{(\psi(X)-\psi(Y))/h}a(Y,h)dYdX^*\end{aligned} \tag{2.9}$$

Then we split the integral in (2.10) in two parts. The first term is given by

$$I =: \int_{Y\in V_1} e^{i(X-Y)X^*/h}\tilde{p}(\frac{X+Y}{2},X^*)\tilde{\chi}(Y)\chi(X)e^{(\psi(X)-\psi(Y))/h}a(Y,h)dYdX^*$$

and, by making the change of contour of integration:

$$\mathbf{R}^{2n} \ni X^* \mapsto X^* - i\nabla\psi(X) \tag{2.10}$$

and by applying the analytic version of the Stationary Phase Theorem, we obtain

$$I = \mathcal{O}(e^{-\varepsilon_1/h})$$

for some $\varepsilon_1 > 0$. For the second term

$$II =: \int_{Y \notin V_1} e^{i(X-Y)X^*/h} \tilde{p}(\frac{X+Y}{2}, X^*) \tilde{\chi}(Y) \chi(X) e^{(\psi(X)-\psi(Y))/h} a(Y,h) dY dX^*,$$

by making the change of contour of integration:

$$\mathbf{R}^{2n} \ni (x^*, \xi^*) \mapsto (x^* + ib' \frac{x-y}{|x-y|}, \xi^* + ia' \frac{\xi - \eta}{|\xi - \eta|}) \tag{2.11}$$

with Sup $|\mathrm{Im}(z - \partial_z \varphi)| < a' < a$ and Sup $|\mathrm{Re}\, \partial_z \varphi| < b' < b$, we obtain

$$II = \mathcal{O}\left(\operatorname*{Sup}_{\substack{X \in \mathrm{supp}\chi \\ Y \in \mathrm{supp}\tilde{\chi} \setminus V_1}} e^{-\delta |X-Y|/h} \right) \tag{2.12}$$

with $0 < \delta < \mathrm{Inf}\{a - a', b - b'\}$. Since

$$\operatorname*{Inf}_{\substack{X \in \mathrm{supp}\chi \\ Y \in \mathrm{supp}\tilde{\chi} \setminus V_1}} |X - Y| > 0,$$

by possibly shrinking ε_1, we get

$$I = \mathcal{O}(e^{-\varepsilon_1/h})$$

and this ends the proof of (2.8).

3 WKB expansion of the first eigenvalue

In this section we show that the first eigenvalue of $P(x, hD_x)$ is simple and has the asymptotic expansions found in the previous section.
Let us start with the following:

Lemma 3.1 *Let $E \in \mathrm{Sp}_d(P)$ and let $u \in L^2(\mathbf{R}^n)$ be such that $Pu = Eu$. For $\delta > 0$, let $\chi_\delta \in C_0^\infty(\mathbf{R}^{2n})$ be such that $\chi_\delta(x, \xi) = 1$ on $B(0, \delta/2)$ and* $\mathrm{supp}(\chi_\delta) \subset B(0, \delta)$. *Then, for any $M \geq 0$ and for any $\delta > 0$, there exists $\varepsilon > 0$ and $C > 0$ such that*

$$|||(1 - \chi_\delta) Tu|||_M \leq C e^{-\varepsilon/h} |||Tu|||_0 \tag{3.1}$$

where $\|Tu\|_M$ *is defined by*

$$\|Tu\|_M = \sum_{j=1}^{M} \|Tu\|_j \tag{3.2}$$

and

$$\|Tu\|_j = \sum_{|\alpha|+|\beta|=j} \| \left(\frac{x}{\sqrt{h}}\right)^{\alpha} \left(\frac{\xi}{\sqrt{h}}\right)^{\beta} Tu\|_{L^2(\mathbf{R}^{2n})} \tag{3.3}$$

Proof - It is a consequence of the results of [10], [9] (see also [13]). $\diamond$

Let $P_0 = \sum_{j=1}^{n} \mu_j(h^2 D_{x_j}^2 + x_j^2)$ be the harmonic oscillator associated to P and let us fix $C_0 > 0$ such that $hC_0 \notin \mathrm{Sp}(P_0)$ and $[0, C_0 h] \cap \mathrm{Sp}_d(P) \neq \emptyset$. As in Section 5 of [2], we have the following:

Lemma 3.2 *Let* $E \in [0, C_0 h] \cap \mathrm{Sp}_d(P)$ *and let* $u \in L^2(\mathbf{R}^n)$ *be such that* $Pu = Eu$. *Then, for any* $M \geq 0$, *there exists* $C > 0$ *such that:*

$$\|Tu\|_{M+2} \leq C\|Tu\|_0. \tag{3.4}$$

Proof - For $M = 0$ we have

$$0 = \frac{1}{h}\|T(P-E)u\|_0 \geq \frac{1}{h}\|\tilde{P}_0 Tu\|_0 - \frac{1}{h}\|(\tilde{P} - \tilde{P}_0)Tu\|_0 - C_0\|Tu\|_0 \tag{3.5}$$

where $\tilde{P}_0 = \sum_{j=1}^{n} \mu_j \left(h^2 D_{x_j}^2 + (x_j - hD_{\xi_j})^2\right)$. It is clear that (see Theorem 3.1 in [9])

$$\begin{aligned} \|\tilde{P}_0 Tu\|_0 \quad &\geq \|\sum_{j=1}^{n} \mu_j(x_j^2 + \xi_j^2)Tu\|_0 - Ch\|Tu\|_0 \\ &\geq C_1 h \left(\|Tu\|_2 - \|Tu\|_0\right). \end{aligned} \tag{3.6}$$

On the other hand,

$$\|(\tilde{P} - \tilde{P}_0)Tu\|_0 \leq \sum_{j=0}^{2} h^j \|q_j(x,\xi)Tu\|_0 \tag{3.7}$$

with

$$|q_j(x,\xi)| \le c_j \langle x,\xi\rangle^k \left(\frac{|(x,\xi)|}{\langle x,\xi\rangle}\right)^{(3-2j)_+} \qquad j=1,2. \tag{3.8}$$

Hence, if we take $\chi_\delta \in C_0^\infty(\mathbf{R}^{2n})$ with $\chi_\delta(x,\xi)=1$ on $B(0,\delta/2)$ and with $\mathrm{supp}(\chi_\delta)\subset B(0,\delta)$ we have:

$$\sum_{j=0}^{2} h^j \|\chi_\delta(x,\xi) q_j(x,\xi) Tu\|_0 \le C_2 \sum_{j=0}^{2} \sum_{|\alpha|+|\beta|=(3-2j)_+} h^j \|$$

$\mathrm{x}^\alpha \xi^\beta Tu\|_{L^2(B(0,\delta))}$

$\le C_3 h\,(\delta\|Tu\|_2 + h\|Tu\|_0)$ (3.9)since, on $\mathrm{supp}(\chi_\delta)$, $|x|,|\xi|\le\delta$. On the other hand, by Lemma 3.1

$$\begin{aligned}\sum_{j=0}^{2} h^j \|(1-\chi_\delta(x,\xi))q_j(x,\xi)Tu\|_0 &\le C_4\|(1-\chi_\delta(x,\xi))Tu\|_k \\ &\le C_5 e^{-\varepsilon/h}\|Tu\|_0. \end{aligned} \tag{3.10}$$

Hence, from (3.9) and (3.10) we obtain

$$\|(\tilde P - \tilde P_0)Tu\|_0 \le C_6 h\,(\delta\|Tu\|_2 + h\|Tu\|_0) \tag{3.11}$$

Taking δ sufficiently small, (3.5), (3.6), (3.11) give (3.4) for $M=0$.
The proof in the case $M\ge 1$, can be done in the same way. $\diamond$

Proposition 3.3 *There exists a bijection $b : \mathrm{Sp}(P)\cap[0,C_0h] \to \mathrm{Sp}(P_0)\cap[0,C_0h]$ such that*

$$|b(E)-E| \le Ch^{3/2}$$

for h sufficiently small.

Proof - Let u be a normalized eigenfunction associated to an eigenvalue $E\in[0,C_0h]\cap\mathrm{Sp}_d(P)$ and let us write $\tilde P_0 Tu = ETu + (\tilde P - \tilde P_0)Tu$. By (3.7), (3.8) and Lemma 3.1, we have

$$\|(\tilde P - \tilde P_0)Tu\|_0 \le Ch^{3/2}\|Tu\|_3.$$

Applying Lemma 3.2 with $M=1$, we have that $\|Tu\|_3$ is uniformly bounded with respect to h. Hence,

$$\tilde P_0 Tu = ETu + \mathcal{O}(h^{3/2}), \qquad \text{in } L^2(\mathbf{R}^{2n})$$

and then

$$P_0 u = Eu + \mathcal{O}(h^{3/2}), \qquad \text{in } L^2(\mathbf{R}^n)$$

Then,

$$\mathrm{Sp}(P) \cap [0, C_0 h] \subset \mathrm{Sp}(P_0) + B(0, Ch^{3/2}). \tag{3.12}$$

As it was observed in [2], Section 5, equation (3.12) does not give the number of eigenvalues of P contained in $[0, C_0 h] + B(0, Ch^{3/2})$. On the other hand, setting $P_t = tP_0 + (1-t)P$, $t \in [0,1]$, it can be shown that the range of spectral projection Π_t on $\mathrm{Sp}(P_t) \cap [0, C_0 h]$ is constant for $t \in [0,1]$. Hence, we can conclude that the number of eigenvalue of P and P_0 in $[0, C_0 h]$ are the same and end the proof of the proposition. ◇

Proposition 3.3 shows, in particular, that the first eigenvalue E of P is simple and then, by applying Proposition 2.5 of [2], we have the following:

Theorem 3.4 *Under assumptions $(H_1), (H_2)$, if u is the normalized eigenvector associated to the first eigenvalue E of $P(x, hD_x)$ then E admits the following asymptotic expansions in powers of h*

$$E \sim h \sum_{k \geq 0} e_k h^k$$

Moreover, there exist constants m_0 and α_0, such that the FBI transform of u near 0 is given by :

$$Tu(x, \xi; h) = \alpha_0 h^{-m_0} a(x - i\xi, h) e^{-\xi^2/2h - \varphi(x-i\xi)/h} + \mathcal{O}(e^{-\varepsilon/h})$$

with some $\varepsilon > 0$.

In particular, denoting

$$v(x, \xi; h) = \alpha_0 h^{-m_0} a(x - i\xi, h) e^{-\xi^2/2h - \varphi(x-i\xi)/h}$$

we have

$$e^{\xi^2/2h + \mathrm{Re}\varphi(x-i\xi)/h}(Tu - v) = \mathcal{O}(e^{-\varepsilon'/2h}) \tag{3.13}$$

for $|(x, \xi)|$ sufficiently small compared with ε. Since not much is known about this ε', the problem is now to extend (3.13) in a neighborhood of $(0,0)$ that one can control more easily.

4 Admissible open sets

In order to extend (3.13) in a larger open set Ω containing $(0,0)$, we need to perform suitable Agmon-type estimates. Let us start with the following:

Proposition 4.1 *Under the previous assumptions, there exist an open set Ω containing $(0,0)$ and, for any $t \geq 0$, a real C^∞ function ϕ_t on Ω such that:*

1. $\phi_0 = -(\mathrm{Im} z)^2$ *and, for all $t \geq 0$,*

$$\phi_t(z, \overline{z}) = \frac{1}{2} \sum_{j=1}^{n} \Big((1 - e^{-4t\mu_j})(\mathrm{Re} z_j)^2 - (1 + e^{-4t\mu_j})(\mathrm{Im} z_j)^2 \Big) + \mathcal{O}(|z|^3) \tag{4.1}$$

where the $\mathcal{O}(|z|^3)$ is uniform with respect to t.

2.

$$\phi_t(z, \overline{z}) \to 2\mathrm{Re}\varphi(z) \ \text{ as } \ t \to +\infty \tag{4.2}$$

in $C^\infty(\Omega)$.

3. *There exists a constant $C_t > 0$ such that:*

$$-q\left(z, \partial_z \phi_t(z, \overline{z})\right) \geq \frac{1}{C_t} |z|^2, \qquad \forall z \in \Omega. \tag{4.3}$$

4. *On Ω, ϕ_t satisfies the evolution equation:*

$$\partial_t \phi_t(z, \bar{z}) = -2q(z, \partial_z \phi_t(z, \bar{z})) \tag{4.4}$$

5. *$\phi_t(z)$ is an increasing function of t for $z \neq 0$. and, for all $t > 0$ and $z \neq 0$,*

$$\phi_0(z, \overline{z}) < \phi_t(z, \overline{z}) < 2\mathrm{Re}\varphi(z) \tag{4.5}$$

6. *For $t \geq 0$, $(-\phi_t)$ is strictly plurisubharmonic on Ω.*

Proof - Let us set

$$\Lambda_0 = \{\zeta = i \mathrm{Im} z\}$$

(so that $(z, \zeta) \in \Lambda_0$ iff $(z - \zeta, i\zeta)$ is real), and for $t \geq 0$:

$$\Lambda_t = \exp tH_q\,(\Lambda_0).$$

Since Λ_0 is $\mathbf{R}$-Lagrangian (that is Lagrangian for the real symplectic form $\mathrm{Re}(d\zeta \wedge dz)$ on $\mathbb{C}^{2n} \approx \mathbf{R}^{4n}$), and the map $\exp tH_q$ is a complex canonical transformation, we have that Λ_t is $\mathbf{R}$-Lagrangian for all t. Moreover, on Λ_0, $-q(z,\zeta) = p(\mathrm{Re}z, -\mathrm{Im}z) \geq \frac{1}{C}|z|^2$ and then, for any t, $-q\,|_{\Lambda_t}$ is real and non negative and there exists $C_t > 0$ such that:

$$-q(z,\zeta) \geq \frac{1}{C_t}\left(|z|^2 + |\zeta|^2\right) \tag{4.6}$$

Let us show that Λ_t is transversal to $\{z = 0\}$ at $(0,0)$. Since $T_0\Lambda_t = \exp(tF_q(0))(\Lambda_0)$ and $F_q(0)$ is given by

(2.6) an easy calculus shows that the integral curves $(z_t,\zeta_t) = \exp(tF_q(0))(z,\zeta)$ with $(z,\zeta) \in \Lambda_0$ are given by

$$\begin{cases} z_t = e^{2t\mu} z \\ \zeta_t = \frac{1}{2}e^{2t\mu} z - \frac{1}{2}e^{-2t\mu}\overline{z}. \end{cases}$$

Hence

$$T_0\Lambda_t = \{\zeta = \frac{1}{2}z - \frac{1}{2}e^{-4t\mu}\overline{z}\}$$

and, in particular, it is transversal to $\{z = 0\}$. As a consequence, Λ_t admits near $(0,0)$ an equation of the form:

$$\Lambda_t \ : \qquad \zeta = \frac{\partial \phi_t(z,\overline{z})}{\partial z}$$

where ϕ_t is a real C^∞ function defined in a neighborhood of 0 and vanishing at 0. The quadratic approximation of ϕ_t is given by (4.1).

Looking carefully at the proof of [15], Appendix, one can also see that there exists a fix neighborhood Ω of 0 in $\mathbb{C}^n$ such that for all $t \geq 0$, ϕ_t is smooth in Ω, and that

$$\phi_t(z,\overline{z}) \to 2\mathrm{Re}\varphi(z) \ \text{ as } \ t \to +\infty$$

in $C^\infty(\Omega)$. Using (4.6), we can conclude that, for $z \in \Omega$, $-q\left(z, \partial_z\phi_t(z,\overline{z})\right)$ is real and non negative and that (4.3) holds.
Let us show that ϕ_t satisfies the evolution equation (4.4). If we denote by $(z_t,\zeta_t) = \exp tH_q(z,\zeta)$, $(z,\zeta) \in \Lambda_0$, we have

$$\begin{cases} \dot{z}_t = \partial_\zeta q(z_t,\zeta_t) \\ \dot{\zeta}_t = -\partial_z q(z_t,\zeta_t). \end{cases} \tag{4.7}$$

and

$$\zeta_t = \partial_z \phi_t(z_t, \overline{z}_t). \tag{4.8}$$

Hence

$$\begin{aligned} -\left(\partial_z q\right)(z_t, \zeta_t) = \dot{\zeta}_t &= \left(\partial_z \dot{\phi}_t\right)(z_t, \overline{z}_t) + \left(\partial_z^2 \phi_t\right)(z_t, \overline{z}_t)\dot{z}_t \\ &+ \left(\partial_z \partial_{\overline{z}} \phi_t\right)(z_t, \overline{z}_t)\dot{\overline{z}}_t. \end{aligned} \tag{4.9}$$

On the other hand, since $q(z, \zeta)$ is holomorphic in the variable z and real on Λ_t, one has, for any $z \in \Omega$,

$$\begin{aligned} \partial_z \left(q\left(z, \partial_z \phi_t(z, \overline{z})\right)\right) &= \left(\partial_z q\right)\left(z, \partial_z \phi_t(z, \overline{z})\right) \\ &+ \left(\partial_z^2 \phi_t\right)(z, \overline{z})\left(\partial_\zeta q\right)\left(z, \partial_z \phi_t(z, \overline{z})\right) \end{aligned}$$

and

$$\partial_{\overline{z}}\left(q(z, \partial_z \phi_t(z, \overline{z}))\right) = \left(\partial_{\overline{z}} \partial_z \phi_t\right)(z, \overline{z})(\partial_\zeta q)\left(z, \partial_z \phi_t(z, \overline{z})\right)$$

i.e.

$$\partial_z \left(\overline{q(z, \partial_z \phi_t(z, \overline{z}))}\right) = \left(\partial_z \partial_{\overline{z}} \phi_t\right)(z, \overline{z})\overline{\left(\partial_\zeta q\right)\left(z, \partial_z \phi_t(z, \overline{z})\right)}$$

Hence

$$\partial_z \left(\dot{\phi}_t(z, \overline{z}) + 2q(z, \partial_z \phi_t(z, \overline{z}))\right)|_{z=z_t} = 0. \tag{4.10}$$

Since, when z describes $\mathbb{C}^n$, z_t describes the neighborhood Ω of 0 where ϕ and ϕ_t are defined, we have that, for any $z \in \Omega$,

$$\dot{\phi}_t(z, \overline{z}) + 2q(z, \partial_z \phi_t(z, \overline{z}))$$

is constant. Moreover, for $z = 0$, $\dot{\phi}_t(0, 0) = 2q(0, \partial_z \phi_t(0, 0)) = 0$ and (4.4) follows. This implies that $\phi_t(z, \overline{z})$ is an increasing function of t for $z \neq 0$. As a consequence, we have for all $t > 0$ and $z \neq 0$:

$$\phi_0(z, \overline{z}) < \phi_t(z, \overline{z}) < 2\mathrm{Re}\varphi(z). \tag{4.11}$$

It remains to prove that $(-\phi_t)$ is plurisubharmonic. If we take the derivatives with respect to z of (4.4), we evaluate it for $z = z_t$ and we take into account (4.7), we get

$$\partial_z \dot{\phi}_t(z_t, \overline{z}_t) = -2\left(\partial_z q\right)(z_t, \zeta_t) - 2\partial_z^2 \phi_t(z_t, \overline{z}_t)\dot{z}_t. \tag{4.12}$$

Comparing (4.12) with (4.9) we get

$$\begin{aligned} - (\partial_z q)(z_t, \zeta_t) &= -2 (\partial_z q)(z_t, \zeta_t) - 2\partial_z^2 \phi_t(z_t, \overline{z}_t) \dot{z}_t \\ &+ \partial_z^2 \phi_t(z_t, \overline{z}_t) \dot{z}_t + \partial_z \partial_{\overline{z}} \phi_t(z_t, \overline{z}_t) \dot{\overline{z}}_t \end{aligned}$$

i.e.

$$(\partial_z q)(z_t, \zeta_t) = \partial_z \partial_{\overline{z}} \phi_t(z_t, \overline{z}_t) \dot{\overline{z}}_t - \partial_z^2 \phi_t(z_t, \overline{z}_t) \dot{z}_t. \tag{4.13}$$

On the other hand, if we take the derivative with respect to t of (4.4) we get:

$$\ddot{\phi}_t(z, \overline{z}) = -2 (\partial_\zeta q)(z, \partial_z \phi_t(z, \overline{z})) \partial_z \dot{\phi}_t(z, \overline{z}) \tag{4.14}$$

and if we evaluate (4.14) at $z = z_t$ and use (4.12), we obtain

$$\begin{aligned} \ddot{\phi}_t(z_t, \overline{z}_t) &= -2 (\partial_\zeta q)(z_t, \zeta_t) \Big(-2(\partial_z q)(z_t, \zeta_t) - 2\partial_z^2 \phi_t(z_t, \overline{z}_t) \dot{z}_t \Big) \\ &= 4\dot{z}_t \Big((\partial_z q)(z_t, \zeta_t) + \partial_z^2 \phi_t(z_t, \overline{z}_t) \dot{z}_t \Big). \end{aligned} \tag{4.15}$$

Comparing (4.15) with (4.13) we get

$$\ddot{\phi}_t(z_t, \overline{z}_t) = 4\dot{z}_t \partial_z \partial_{\overline{z}} \phi_t(z_t, \overline{z}_t) \dot{\overline{z}}_t \tag{4.16}$$

For $z \in \Omega$, let us set $M(t, z, \overline{z}) = \partial_z \partial_{\overline{z}} \phi_t(z, \overline{z})$. Taking the derivatives with respect to z and $\overline{z}$ of (4.4), it is easy to check that the matrix M satisfy the follow first order equation

$$X(t, z, \overline{z}, \partial_t, \partial_z, \partial_{\overline{z}}) M = -A(t, z, \overline{z}) M - M^t \overline{A}(t, z, \overline{z})$$

where

$$X(t, z, \overline{z}, \partial_t, \partial_z, \partial_{\overline{z}}) = \partial_t + \langle \partial_\zeta q(z, \phi_t(z, \overline{z})) \ ,$$

$\partial_z \rangle + \langle \overline{\partial_\zeta q(z, \partial_z \phi_t(z, \overline{z}))}, \partial_{\overline{z}} \rangle and A(t, z, \overline{z}) = (\partial_z \partial_\zeta q)(z, \phi_t(z, \overline{z})) + \partial_z^2 \phi_t(z, \overline{z}) \left(\partial_\zeta^2 q \right)(z, \phi_t(z, \overline{z})). If (z$
$\exp(tH_q)(z, \zeta)$ denotes, as before, the integral curves of q starting from points $(z, \zeta) \in \Lambda_0$, and we set $M_1(t) = M(t, z_t, \overline{z}_t)$ and $A_1(t) = A(t, z_t, \overline{z}_t)$,

we have

$$\frac{d}{dt} M_1(t) = -\Big(A_1(t) M_1(t) + M_1(t)^t \overline{A}_1(t) \Big)$$

Hence, by Liouville theorem

$$\det M_1(t) = \det M_1(0) \exp \left(-\int_0^t \mathrm{tr}(2A_1(s)) ds \right)$$

Since $M_1(t)$ is selfadjoint on $\mathbb{C}^n$ for any $t \geq 0$ and $M_1(0) = -\frac{1}{2}I$ then $M_1(t)$ is negative definite for any $t \geq 0$. This implies that

$$\ddot{\phi}_t(z_t, \overline{z}_t) = 4\dot{z}_t \partial_z \partial_{\overline{z}} \phi_t(z_t, \overline{z}_t) \dot{\overline{z}}_t \leq -C|z_t|^2 \tag{4.17}$$

and then

$$\ddot{\phi}_t(z, \overline{z}) \leq -C|z|^2 \tag{4.18}$$

for $z \in \Omega$ and $t \leq 0$. This ends the proof of Proposition 4.1. $\diamond$

Then we introduce the following notion of "admissible open set":

Definition 4.2 *Let Ω_1 be an open subset of Ω containing 0. Then Ω_1 is called "admissible" if for any compact subset K of Ω_1 there exist $\varepsilon_K > 0$ and a neighborhood $\mathcal{V}_K$*

of $\partial\Omega$ with the following property:
For every $t > 0$ large enough, there exists $\psi_t \in C^\infty(\mathbb{C}^n)$

real such that

$$\begin{aligned} &\psi_t = \phi_t \ \text{ on } K \ ; \\ &\psi_t - \phi_0 \ \text{ is constant outside } \Omega \ ; \\ &\psi_t \leq 2\mathrm{Re}\varphi - \varepsilon_K \ \text{ on } \mathcal{V}_K \ ; \\ &\mathrm{Sup}\ |\mathrm{Im}(z - \partial_z \psi_t)| < a \ \text{ and } \ \mathrm{Sup}\ |\mathrm{Re}\ \partial_z \psi_t| \end{aligned}$$

¡b ;
$\exists\ C'_t > 0$ such that $|q(z, \partial_z \psi_t)| \geq \frac{1}{C'_t}|z|^2$ on Ω.

In terms of deformation of **R**-Lagrangian manifolds, this means that one can deform Λ_t into Λ_0 within an arbitrarily small neighborhood of

$\Omega \backslash \Omega_1$, in such a way that q remains elliptic along the deformation. Moreover, the deformed weight has to be smaller than $2\mathrm{Re}\varphi$ near $\partial\Omega$ (this is imposed since the WKB constructions are ceasing to exist there).
Before proving that the estimate (3.13) remains valid locally uniformly in any admissible open set (which will be a relatively easy consequence of Theorem 3.1 in [9]), let us exhibit such a set in terms of some constants attached to q and easy to compute.
Noticing that $2\mathrm{Re}\varphi(z) - \phi_0(z, \overline{z}) = |z|^2/2 + \mathcal{O}(|z|^3)$, we set

$$|z|_\varphi = (2\mathrm{Re}\varphi(z) - \phi_0(z, \overline{z}))^{\frac{1}{2}} . \tag{4.19}$$

Taking into account the quadratic approximation of ϕ_t given by (4.1), we define the five constants γ_0', $\gamma_j > 0$ $(j = 0, 1, 2, 3)$ in the following way:

$$\gamma_0' = \operatorname*{Sup}_{z \in \Omega} \frac{|z|}{|z|_\varphi}$$

$$\gamma_0 = \operatorname*{Sup}_{z \in \Omega} \frac{|z|_\varphi}{|z|}$$

$$\gamma_1 = \operatorname*{Sup}_{\substack{t \geq 0 \\ z \in \Omega}}$$

(z,$\overline{z}$))$||z|_\varphi \gamma_2 = \operatorname*{Sup}_{\substack{t \geq 0 \\ z \in \Omega}} \frac{|\partial_z \phi_t(z,\overline{z}) - (z - e^{-4t\mu}\overline{z})/2|}{|z|_\varphi^2} \gamma_3 = \operatorname*{Sup}_{\substack{z \in \Omega \\ |\zeta| \leq \gamma_1 |z|_\varphi}} \frac{|\partial_\zeta q(z,\zeta) - 2\mu z|}{|z|_\varphi^2} .Thenwehave:$

Proposition 4.3 *If $r > 0$ satisfies:*

$$\gamma_0 (2\gamma_2 \mu_n + \frac{1}{2}\gamma_3)\sqrt{r} + \gamma_0^2 \gamma_2 \gamma_3 r \leq \mu_1 \tag{4.20}$$

and

$$\gamma_1 \sqrt{r} \leq b \qquad ; \qquad (\gamma_2 + \frac{\gamma_0'}{2})\sqrt{r} \leq a \tag{4.21}$$

then the set

$$\mathcal{B}_r = \{z \in \mathbb{C}^n \; ; \; (\mathrm{Im} z)^2 + 2\mathrm{Re}\varphi(z) < r\}$$

$=\{\ |z|_\varphi^2 < r\} is an admissible open set in the sense of definition 4.2.$ **Proof** - Fix $K \subset\subset \mathcal{B}_r$ and $0 < r_0 < r_1 < r$ such that $K \subset\subset \mathcal{B}_{r_0}$. Let $\chi \in C^\infty(\mathbf{R}_+)$ be such that $\chi(\theta) = \theta$ for $\theta \in [0, r_0]$, $\chi = \alpha_1$ is constant on $[r_1, +\infty)$, and $0 \leq \chi' \leq 1$ everywhere (so that necessarily $\alpha_1 < r_1$). Because of (4.2) and (4.5), we see that if $t > 0$ is large enough, then

$$\{\phi_t(z,\overline{z}) - \phi_0(z,\overline{z}) \leq r_1\} \subset\subset \mathcal{B}_r. \tag{4.22}$$

and for such t's, we set

$$\psi_t(z,\overline{z}) = \phi_0(z,\overline{z}) + \chi\left(\phi_t(z,\overline{z}) - \phi_0(z,\overline{z})\right). \tag{4.23}$$

Because of (4.5), we have

$$K \subset \{\phi_t(z,\overline{z}) - \phi_0(z,\overline{z}) \leq r_0\}.$$

and therefore

$$\psi_t = \phi_t \text{ on } K.$$

We also have $\psi_t = \phi_0 + \alpha_1$ on $\{\phi_t(z,\overline{z}) - \phi_0(z,\overline{z}) \geq r_1\}$, and thus by (4.22) and (4.2), we get on a t-independent neighborhood of $\partial\mathcal{B}_r$:

$$2\mathrm{Re}\varphi - \psi_t = 2\mathrm{Re}\varphi - \phi_0 + \alpha_1 \geq r_1 - \alpha_1 > 0.$$

Moreover, the fact that Sup $|\mathrm{Im}(z - \partial_z\psi_t)| < a$ and Sup $|\mathrm{Re}\ \partial_z\psi_t| < b$ is an easy consequence of (4.21).
It remains to show that $q(z, \partial_z\psi_t)$ is elliptic on $\{r_0 \leq \phi_t(z,\overline{z}) - \phi_0(z,\overline{z}) \leq r_1\}$.
For $0 \leq s \leq 1$, denote

$$g(s,z) = q(z, \partial_z\phi_0 + s\partial_z(\phi_t - \phi_0))$$

so that :

$$q(z, \partial_z\psi_t) = g(\chi'(\phi_t - \phi_0), z). \tag{4.24}$$

We have:

$$\frac{\partial g}{\partial s} = \langle \partial_z(\phi_t - \phi_0), (\partial_\zeta q)(z, \partial_z\phi_0 + s\partial_z(\phi_t - \phi_0))\rangle$$

and thus, using the definition of γ_j $(j = 0,1,2,3)$:

$$\frac{\partial g}{\partial s} = \langle \frac{1}{2}(1 - e^{-4t\mu})\overline{z} + R_1, 2\mu z + R_2\rangle$$

with

$$|R_1| \leq \gamma_2 |z|_\varphi^2$$
$$|R_2| \leq \gamma_3 |z|_\varphi^2.$$

As a consequence,

$$\frac{\partial g}{\partial s} = \langle \mu z, (1 - e^{-4t\mu})\overline{z}\rangle + R_3$$

with

$$|R_3| \leq \gamma_0(2\gamma_2\mu_n + \frac{1}{2}\gamma_3)|z|_\varphi |z|^2 + \gamma_0^2\gamma_2\gamma_3|z|_\varphi^2|z|^2.$$

In particular, if r satisfies (4.20) and t is large enough, we get

$$\mathrm{Re}\frac{\partial g}{\partial s} \geq \left(\mu_1(1 - e^{-4t\mu_1}) - \gamma_0(2\gamma_2\mu_n + \frac{1}{2}\gamma_3)|z|_\varphi - \gamma_0^2\gamma_2\gamma_3|z|_\varphi^2\right)|z|^2 \geq 0$$

on $\{r_0 \leq \phi_t(z,\overline{z}) - \phi_0(z,\overline{z}) \leq r_1\}$. Therefore, for any $s \in [0,1]$, we have

$$\mathrm{Re}\ g(s,z) \leq \mathrm{Re}\ g(1,z)$$

and since by (4.3) we have

$$g(1,z) = q(z, \partial_z \phi_t) \leq -\frac{1}{C_t}|z|^2,$$

we get in particular

$$\text{Re } q(z, \partial_z \psi_t) \leq -\frac{1}{C_t}|z|^2$$

which completes the proof of proposition 4. ◇

5 Extension of the WKB construction on admissible open sets

Theorem 5.1 *Let Ω_1 be any admissible open set in the sense of definition 4.2. Then for any compact set $K \subset \Omega_1$, there exists $\varepsilon > 0$ such that*

$$e^{\xi^2/2h + \text{Re}\varphi(x-i\xi)/h}(Tu(x,\xi;h) - v(x,\xi;h)) = \mathcal{O}(e^{-\varepsilon/h})$$

uniformly for $x - i\xi \in K$ and $h > 0$ small enough. Here u is the first normalized eigenfunction of $P = \text{Op}_{h,\frac{1}{2}}(p)$, and

$$v(x,\xi;h) = \alpha_0 h^{-m_0} a(x - i\xi, h) e^{-\xi^2/2h - \varphi(x-i\xi)/h}$$

is the WKB solution constructed in section 2.

Proof - Using (3.13), let $\varepsilon_0 > 0$ and V_0 be a neighborhood of $0 \in \mathbf{R}^{2n}$ such that

$$\|e^{\xi^2/2h + \text{Re}\varphi(x-i\xi)/h}(Tu - v)\|^2_{L^2(V_0)}$$

$=\text{O}(\text{e}^{-\varepsilon_0/h})$.(5.1)Then, fix $K \subset\subset \Omega_1$, and let $\varepsilon_K > 0$ and ψ_t ($t > 0$ large enough) be given by definition 4.2. By (4.2), we can fix t_0 sufficiently large so that:

$$|\psi_{t_0} - 2\text{Re}\varphi| \leq \frac{\varepsilon'_K}{4} \quad \text{on } K \tag{5.2}$$

with $\varepsilon'_K = \text{Min}(\varepsilon_0, \varepsilon_K)$. Let also $\chi \in C_0^\infty(\Omega)$ be such that $\chi = 1$ on K and $\text{Supp}\nabla\chi$ is included in the interior of the neighborhood $\mathcal{V}_K$ of $\partial\Omega$ where $\psi_{t_0} \leq 2\text{Re}\varphi - \varepsilon_K$, and define

$$w = \chi v - Tu.$$

Then by construction, we have that there exists $\varepsilon > 0$ such that (by eventually shrinking ε_K)

$$\begin{aligned}(hD_x - \xi - ihD_\xi)w &= h(D_x\chi - iD_\xi\chi)\, v + \mathcal{O}(e^{-(\xi^2/2+\mathrm{Re}\varphi+\varepsilon)/h}) \\ &= \mathcal{O}(e^{-(\xi^2+\psi_{t_0}+\varepsilon_K)/2h})\end{aligned} \tag{5.3}$$

since $D_x\chi - iD_\xi\chi$ is supported di $\mathcal{V}_K$. Moreover, if we set $\tilde{v} = \tilde{\chi}v$ with $\tilde{\chi} \in C_0^\infty(\Omega)$ such that $\tilde{\chi} = 1$ on a neighborhood of suppχ, (2.8) gives

$$\begin{aligned}(\tilde{P} - E)w &= (\tilde{P} - E)\chi v \\ &= [\tilde{P}, \chi]\tilde{v} + \mathcal{O}(e^{-\xi^2/2h-\mathrm{Re}\varphi(x-i\xi)/h-\varepsilon_1/h})\end{aligned} \tag{5.4}$$

with $\varepsilon_1 > 0$. We set

$$\psi = \frac{1}{2}\left(\xi^2 + \psi_{t_0} + \frac{\varepsilon_K'}{2}\right)$$

(which is constant outside Ω), and we plan to apply Theorem 3.5.1 in [10] with this ψ, but with Tu replaced by w. Actually, since $(hD_x - \xi - ihD_\xi)w$ is not zero but we have only (5.3), following the proof of Theorem 3.1 in [9] we see that an extra term appears in the error, namely:

$$\begin{aligned}\|e^{\psi/h}\langle x,\xi\rangle^{-k}(\tilde{P} - E)w\|^2 &= \|\langle x,\xi\rangle^{-k}p(x - 2\partial_z\psi, \xi + i\partial_z\psi)e^{\psi/h}w\|^2 \\ &+ \mathcal{O}(h)\|e^{\psi/h}w\|^2 \\ &+ \mathcal{O}\left(h\|e^{\psi/h}v\|_{L^2(\mathcal{V}_K)} + e^{-\varepsilon_K/4h}\right)\|e^{\psi/h}w\|.\end{aligned} \tag{5.5}$$

However, since $\psi \leq \dfrac{\xi^2}{2} + \mathrm{Re}\varphi - \dfrac{\varepsilon_K}{4}$ on $\mathcal{V}_K$, we have

$$\|e^{\psi/h}v\|_{L^2(\mathcal{V}_K)} = \mathcal{O}(e^{-\varepsilon_K/8h})$$

and therefore, using also the fact that $p(x - 2\partial_z\psi, \xi + 2i\partial_z\psi) = -q(z, \partial_z\psi_{t_0})$ is elliptic outside 0, we get from (5.4) and (5.5) :

$$\|e^{\psi/h}w\|^2 = \mathcal{O}\left(1 + \|e^{\psi/h}w\|^2_{L^2(V_0)} + \|e^{\psi/h}\langle x,\xi\rangle^{-k}[\tilde{P},\chi]\tilde{v}\|^2\right). \tag{5.6}$$

Using (5.2), we also have

$$|\psi - \frac{\xi^2}{2} - \mathrm{Re}\varphi| \leq \frac{\varepsilon_K'}{2} \tag{5.7}$$

and thus, by (5.1):

$$\|e^{\psi/h}w\|^2_{L^2(V_0)} = \mathcal{O}(e^{-\varepsilon'_K/2h}). \tag{5.8}$$

In view of (5.6), it remains to study the term $\|e^{\psi/h}\langle x,\xi\rangle^{-k}[\tilde{P},\chi]\tilde{v}\|^2$. We write

$$\begin{aligned}&\langle X\rangle^{-k}[\tilde{P},\chi]\tilde{v}(X)\\ &\quad= \frac{\langle X\rangle^{-k}}{(2\pi h)^{2n}}\int e^{i(X-Y)X^*/h}\tilde{p}(\frac{X+Y}{2},X^*)(\chi(Y)-\chi(X))\tilde{v}(Y)dYdX^*\end{aligned}$$

and, denoting $X=(x,\xi)$, $Y=(y,\eta)$ and $X^*=(x^*,\xi^*)$, we make in (5.9) the change of contour of integration:

$$\mathbf{R}^{2n}\ni(x^*,\xi^*)\mapsto(x^*+ib'\frac{x-y}{|x-y|},\xi^*+ia'\frac{\xi-\eta}{|\xi-\eta|}) \tag{5.9}$$

with Sup $|\mathrm{Im}(z-\partial_z\varphi)|<a'<a$ and Sup $|\mathrm{Re}\;\partial_z\varphi|<b'<b$. Then

$$\|e^{\psi/h}\langle x,\xi\rangle^{-k}[\tilde{P},\chi]\tilde{v}\|^2 = \mathcal{O}\left(1+\sup_{\substack{Y\notin\mathcal{V}_K\\ \chi(X)\neq\chi(Y)}} e^{-\delta|X-Y|/h+\varepsilon'_K/2h}\right) \tag{5.10}$$

with some $\delta>0$ depending only on K. Now we have

$$\inf_{\substack{Y\notin\mathcal{V}_K\\ \chi(X)\neq\chi(Y)}}|X-Y|>0$$

and we see that we can shrink ε_K without modifying the set $\{Y\notin\mathcal{V}_K\;;\;\chi(X)\neq\chi(Y)\}$. Then we deduce from (5.10) that $\|e^{\psi/h}\langle x,\xi\rangle^{-k}[\tilde{P},\chi]\tilde{v}\|^2$ is exponentially small, and in view of (5.6) and (5.8), we get finally get

$$\|e^{\psi/h}w\|^2 = \mathcal{O}(1).$$

Since $\psi(x,\xi)\geq\frac{1}{2}\xi^2+\mathrm{Re}\varphi(x-i\xi)+\frac{1}{8}\varepsilon'_K$ on K, this completes the proof of theorem 5.1. $\diamond$

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