RELATIVITY ON 3-MANIFOLDS

An alternative approach to Relativity
And a new solution of the vacuum Einstein field equations

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ABSTRACT

We present a new formulation of General Relativity in an entirely 3-dimensional pseudo–Riemannian space. Here the basic idea is to represent local physical observers by nonsingular 3-dimensional vector fields of bounded length and the dynamics of physical fields by the flows of vector fields. The flow parameter plays the role of local time for each physical observer, and the Lie derivative replaces the time derivative. Some wellknown solutions and a new solution of the vacuum Einstein field equations are formulated in this framework.

Introduction: The original 1905 paper of Einstein [1] , which laid the foundation of Special theory of Relativity, did not use the ubiquitous 4-dimensional geometry of space-time. It was Minkowski [2] who showed how Einstein's theory could be formulated elegantly in a 4-dimensional geometry , and ever since relativists have become accustomed to think always in the framework of a 4-dimensional continuum. At about the same time as Minkowski, Poincaré [3] made a careful analysis of the role of geometry in physical theories and concluded: *"One geometry cannot be more true than another; it can only be more convenient."* It is therefore instructive to enquire whether the theory of Relativity can be formulated *equivalently* in another geometrical formalism.

In this note we present the outline of a new formulation of both relatvistic kinematics and field dynamics in an entirely 3-dimensional space. For details we refer the reader to [4],[5].

The 3-dimensional formalism and generalized Lorentz matrices: Let (M^3, g) be a 3-dimensional space M^3 with a pseudo–Riemannian metric g . By an observer (or a particle path) we usually mean a regular smooth curve in M^3 . Since to every such curve c one can associate a *local* vector field C such that c is an integral curve of C , [6], we shall regard the set of all *nonsingular vector fields* on M^3 as the set of all *observers* in M^3 . By a physical observer we shall mean a nonsingular vectr field X such that

$$0 < g(X,X) < 1 \tag{1}$$

The principle of Relativity, in our formalism, is contained in (1).

The *relative velocity* between a pair of physical observers X, Y is assumed to be given by:

$$\left.\begin{aligned} V(X,Y) &= \left[1 - \frac{1}{[F(X,Y)]^2}\right]^{1/2} \\ F(X,Y) &= \frac{1-g(X,Y)}{[1-g(X,X)]^{1/2}[1-g(Y,Y)]^{1/2}} \end{aligned}\right\} \tag{2}$$

Note that $V(X,Y)$ is a funcion on M^3 . Two observers X, Y are said to be (*inertially*) equivalent if $V(X,Y)$ is a constant function on M^3 .

Let $c : I \to M^3$, $\lambda \mapsto c(\lambda)$ be a particle path and C any representative corresponding vector field associated with it such that c is an integral curve of C . For a physical particle path $0 < g(C,C) < 1$ on $c(I)$.

Definition. The *time interval of* c *relative to an observer* X , denoted by Δt_X , is given by the following integral over I :

$$\Delta t_X = \int_I F(X,C) d\lambda \tag{3}$$

Consider now two *inertially equivalent* observers X and Y , such that $V(X,Y) = v =$ const . Then $F(X,Y) = (1 - v^2)^{-1/2} \equiv \gamma$. Let c be an integral curve of

X , i.e., $X = C$. Then from (2), $F(X,C) = F(C,C) = 1$ and $V(C,C) = 0$, so that $\Delta t_X = \int_I d\lambda \equiv \Delta\lambda$.

Relative to the observer Y , however, the time interval of c is given by

$$\Delta t_Y \;=\; \int_I F(Y,C)d\lambda \;=\; \int_I F(Y,X)d\lambda \;=\; \int_I \gamma d\lambda \;=\; \gamma\Delta\lambda \tag{4}$$

so that $\Delta t_Y = \gamma\Delta t_X$, which is the Lorentz time-dilation formula.

We now consider the problem of constructing a class of inertial observers starting from a representative physical observer; in other words, given a physical observer u , the problem of constructing an observer w such that u and w are inertially equivalent. In some local coordinate system (x^i) , since u is a physical observer, we have $g(u,u) = g_{ik}u^i u^k = u_i u^i < 1$, where $u_i = g_{ik}u^k$.

Let

$$\left.\begin{array}{rcl} \gamma_u &=& [1 - g(u,u)]^{-1/2} \\ A_u &=& (\gamma_u - 1)/g(u,u) \end{array}\right\} \tag{5}$$

and let us define the 4×4 matrix functions $\tilde{L}^\mu_\lambda(u)$, $\mu,\lambda = 0,1,2,3$ as follows: 1)

$$\left.\begin{array}{rclcrcl} \tilde{L}^0_0(u) &=& \gamma_u &,& \tilde{L}^0_k(u) &=& \gamma_u u_k \\ \tilde{L}^k_0(u) &=& \gamma_u u^k &,& \tilde{L}^k_j(u) &=& \delta^k_j + A_u u^k u_j \ . \end{array}\right\} \tag{6}$$

Note that in (6) u_j differ from u^k by the lowering of indices by g_{jk} and also that $\tilde{L}^\mu_\lambda$ are *functions* on M . These "generalized Lorentz matrices" $\tilde{L}^\mu_\lambda(u)$ are easily seen to satisfy the following "generalized" properties

$$\left.\begin{array}{rcl} g_{ik}\tilde{L}^i_0\tilde{L}^k_0 &=& (\tilde{L}^0_0)^2 - 1 \\ g_{mn}\tilde{L}^m_0\tilde{L}^n_i &=& \tilde{L}^0_0\tilde{L}^0_i \\ g_{mn}\tilde{L}^m_i\tilde{L}^n_j &=& \tilde{L}^0_i\tilde{L}^0_j + g_{ij} \end{array}\right\} \tag{7}$$

and when $(M^3,g) = (^3,\delta)$, that is, when (M^3,g) is the Euclidean space ^{3} with the Euclidean metric $g_{ik} = \delta_{ik}$, $\tilde{L}^\mu_\lambda(u)$ reduces to the usual Lorentz matrix $L^\mu_\lambda(\mathbf{u})$ with the pure boost given by $\mathbf{u} = (u^1,u^2,u^3)$.

Let v be another vector field where $g(v,v) < 1$. We now define a vector field w by

$$w^i \;=\; \frac{\tilde{L}^i_0(u) + \tilde{L}^i_k(u)v^k}{\tilde{L}^0_0(u) + \tilde{L}^0_k(u)v^k} \tag{8}$$

Then it follows that $g(w,w) < 1$, so that w represents also a physical observer.

Using the "generalized" properties (7) one now shows that the relative velocity function $V(u,w)$ as given by (2) satisfies: $\big[V(u,w)\big]^2 \;=\; g(v,v)$.

1) In what follows Latin indices $i,j,k,\ldots=1,2,3$ and the Greek indices $\alpha,\beta,\gamma,\ldots=0,1,2,3$.

4-metric from a 3-metric and a fundamental vector field: In classical field theory the dynamics of a field is usually described by a set of evolution equations (together possibly with some constraint equations) in some space-time manifold. We shall now demonstrate that in many cases the dynamics can also be described in a purely 3-dimensional setting without the necessity of introducing explicitly a time-coordinate. Instead of the time coordinate and time-dependent quantities, the temporal evolution is provided by the "flow" of a 3-dimensional vector field representing a physical observer.

In conventional General Relativity the space-time metric $g_{\alpha\beta}$ satisfies the Einstein field equations, which are also basically evolution equations with certain constraints. These equations (for a dust like matter distribution with density ϱ, no pressure, and a cosmological constant Λ) when expressed in a Gaussian normal coordinate system, are:

$$G_i^0 \equiv \tfrac{1}{2}(g^{k\ell}g_{ik,0})_{;\ell} - \tfrac{1}{2}(g^{\ell k}g_{\ell k,0})_{,i} \;=\; 0 \tag{9}$$

$$G_0^0 \equiv -\tfrac{1}{2}\overline{R} + \tfrac{1}{8}g^{\ell m}g_{\ell m,0}g^{ik}g_{ik,0} - \tfrac{1}{8}g^{i\ell}g^{km}g_{ik,0}g_{\ell m,0} \;=\; \varrho - \Lambda \tag{10}$$

$$R_{ik} \equiv \overline{R}_{ik} - \tfrac{1}{2}g_{ik,00} - \tfrac{1}{4}g^{\ell m}g_{\ell m,0}g_{ik,0} + \tfrac{1}{2}g^{\ell m}g_{i\ell,0}g_{km,0} = (-\tfrac{1}{2}\varrho + \Lambda)g_{ik} \tag{11}$$

where

$$g_{ik,0} \;\equiv\; \partial g_{ik}/\partial x^0 = \partial g_{ik}/\partial\tau \;; \qquad g_{ik,00} \equiv \partial^2 g_{ik}/\partial\tau^2$$

$$\overline{R}_{ik} \;\equiv\; \text{Ricci tensor of the 3-metric } \; g_{ik}$$

$$\overline{R} \;\equiv\; \text{Ricci-scalar of the 3-metric } \; g_{ik}$$

$$A_{;\ell} \;\equiv\; \text{covariant derivative of } \; A \; \text{ with respect to the 3-metric } \; g_{ik}$$

Here the metric is: $ds^2 = -d\tau^2 + g_{ik}(x)dx^i dx^k$. Note that we are not assuming any condition on the *signature* of the 3-metric g_{ik} . Thus, for example, *the 3-metric g_{ik} need not be +ve definite, only that it be non-singular.* The -ve sign in front of $d\tau^2$ is a matter of convention. Thus $x^0 = \tau$ need not represent in general the physical time coordinate and the signature of the 4-metric $g_{\alpha\beta}$ can be arbitrary.

We now reformulate the equations in a purely 3-dimensional setting. Consider a 3-dimensional pseudo-Riemannian manifold (M^3, g) where g is a non singular (but possibly indefinite) 3-metric on M^3 . Let $X = X^i(x)\partial/\partial x^i$ be a vector field on M^3 , and $h = \mathcal{L}_X g$, the Lie-derivative of g with respect to X . So that

$$h \;=\; h_{ik}dx^i \otimes dx^k \qquad \text{where} \quad h_{ik} \;=\; (\mathcal{L}_X g)_{ik} \;.$$

Consider now the equations (9)–(11) and replace

$$g_{ik,0} \quad \textit{by} \quad h_{ik} \quad \textit{and} \quad g_{ik,00} \quad \textit{by} \quad (\mathcal{L}_X h)_{ik} \;,$$

We then obtain

$$h^{\ell}_{i;\ell} - (g^{\ell k}h_{\ell k})_{,i} \;=\; 0 \tag{12}$$

$$-\frac{1}{2}\overline{R} + \frac{1}{8}(g^{\ell m}h_{\ell m})(g^{ik}h_{ik}) - \frac{1}{8}(g^{i\ell}h_{ik})(g^{km}h_{\ell m}) \;=\; \varrho - \Lambda \tag{13}$$

$$\overline{R}_{ik} - \frac{1}{2}(\mathcal{L}_X h)_{ik} - \frac{1}{4}(g^{\ell m}h_{\ell m})h_{ik} + \frac{1}{2}(g^{\ell m}h_{i\ell})h_{km} \;=\; (-\tfrac{1}{2}\varrho + \Lambda)g_{ik} \tag{14}$$

We regard equations (12)–(14) as *a set of differential equations for both the 3-metric* g *and the 3-dimensional vector field* X *on* (M^3, g). *Every solution* (g, X) *of* (12)–(14) *determines uniquely a solution of the Einstein field equations (but not vice-versa)in Gaussian normal coordinates* as follows.

The integral curves of X determines a flow χ_τ, which are local 1-parameter group of local diffeomorphisms [7] of M^3. Thus, for each value of the flow parameter τ, χ_τ defines a point transformation: $x \mapsto \tilde{x} = \chi_\tau(x)$ with $\chi_0(x) = x$. The transformed metric $\tilde{g} = \tilde{\chi}_\tau(g)$ is given by $\tilde{g}_{ik}(\tau, \tilde{x}) = \frac{\partial x^\ell}{\partial \tilde{x}^i}\frac{\partial x^m}{\partial \tilde{x}^k} g_{\ell m}(x)$, and is thus τ-dependent.

Then the following 4-dimensional space-time metric, in Gaussian normal coordinates $(\tau, \tilde{x}^i)$

$$ds^2 \;=\; -d\tau^2 + \tilde{g}_{ik}(\tau, \tilde{x}) d\tilde{x}^i d\tilde{x}^k$$

is a solution of the Einstein equations. Whether τ is the time coordinate depends on the signature of the 3-metric.

Suppose $\varrho = \Lambda = 0$. If $X = 0$ or if X *is a Killing vector field* of g then $h = \mathcal{L}_X g = 0$. So that (12) is identically satisfied, and (13) implies that $\overline{R}_{ik} = 0$, which in turn implies that $\overline{R}_{ijk\ell} = 0$ (since M^3 is 3-dimensional), that is, M^3 is flat with metric $g = \delta$ .So ($X = 0, g = \delta$) and (X Killing, $g = \delta$) are both solutions of (12)-(14). These are *trivial* flat-space solutions. We shall now indicate some *non-trivial* flat space solutions. These turn out to correspond to some well known 4-metric solutions and a new solution of the vacuum Einstein equations.

The Schwarzschild solution($\varrho = \Lambda = 0$):

$$\left.\begin{aligned} \text{3-metric:} \quad ds^2 &= d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\phi^2) \\ \text{vector field:} \quad X &= k\rho^{-\frac{1}{2}}\frac{\partial}{\partial\rho} \end{aligned}\right\} \tag{15}$$

The transformation $\rho = \left(\frac{3}{2}kR\right)^{\frac{2}{3}}$ transforms the metric as well as the vector field in (15), into

$$\left.\begin{aligned} ds^2 &= k^2(\frac{3}{2}kR)^{-\frac{2}{3}}dR^2 + (\frac{3}{2}kR)^{\frac{4}{3}}(d\theta^2 + \sin^2\theta d\phi^2) \\ X &= \frac{\partial}{\partial R} \end{aligned}\right\} \tag{16}$$

The flow generated by X is given by: $R \mapsto \widetilde{R} = R + \tau, \quad \theta \mapsto \tilde{\theta} = \theta, \quad \phi \mapsto \tilde{\phi} = \phi$ .The corresponding *space-time 4-metric (in Gaussian normal coordinates* $(\tau, \widetilde{R}, \tilde{\theta}, \tilde{\phi})$) is:

$$ds^2 \;=\; -d\tau^2 + k^2\left[\frac{3}{2}k(\widetilde{R} - \tau)\right]^{-\frac{2}{3}} d\widetilde{R}^2 + \left[\frac{3}{2}k(\widetilde{R} - \tau)\right]^{\frac{4}{3}} (d\tilde{\theta}^2 + \sin^2\tilde{\theta} d\tilde{\phi}^2). \tag{17}$$

Eq.(17) is the *Schwarzschild* solution in so called *Lemaitre coordinates* [8], [9] if we take $k = (2m)^{\frac{1}{2}}$, as can be seen by considering the following transformation: $(\tau, \widetilde{R}, \tilde{\theta}, \tilde{\phi}) \mapsto$

(t, r, θ, ϕ), where

$$\left.\begin{aligned}
\tau &= 2\left(\frac{r}{2m}\right)^{\frac{1}{2}} + 2m \log\left|\frac{\sqrt{r}-\sqrt{2m}}{\sqrt{r}+\sqrt{2m}}\right| - t \\
\widetilde{R} &= \frac{2}{3} r^{\frac{3}{2}} (2m)^{-\frac{1}{2}} + \tau \\
\tilde{\theta} &= \theta, \quad \tilde{\phi} = \phi
\end{aligned}\right\} \tag{18}$$

The vector field $X = \left[\frac{2m}{\rho}\right]^{\frac{1}{2}} \frac{\partial}{\partial\rho}$ in (15) ceases to be *physical* when $\rho \leq 2m$ or when $R \leq \frac{4}{3}m$, even though X has a *mathematical* singularity only at $\rho = 0$. $\rho = 2m$ is of course the Schwarzschild horizon corresponding to $r = 2m$ in Schwarzschild coordinates.

One might think that if (X, g) is a solution of (12)-(14) then adding a Killing vector field of g to X would again give us another solution. In fact, one has the following proposition.

Proposition 1: *Let (X, g) be a solution and X_0 a Killing vector field of g. Then $(X + X_0, g)$ is also a solution provided $[X, X_0] = 0$.*

Proof: Let $\tilde{X} = X + X_0$. Then $\mathcal{L}_{\tilde{X}} g = \mathcal{L}_X g$ and $\mathcal{L}_{\tilde{X}}(\mathcal{L}_{\tilde{X}} g) = \mathcal{L}_X(\mathcal{L}_X g) + \mathcal{L}_{X_0}(\mathcal{L}_X g)$. Now $[\mathcal{L}_{X_0}, \mathcal{L}_X] = \mathcal{L}_{[X_0, X]}$ .Hence $[X_0, X] = 0$ implies that $\mathcal{L}_{X_0}(\mathcal{L}_X g) = \mathcal{L}_X(\mathcal{L}_{X_0} g) = 0$, and therefore, we have also $\mathcal{L}_{\tilde{X}}(\mathcal{L}_{\tilde{X}} g) = \mathcal{L}_X(\mathcal{L}_X g)$.

However, this does not generate a new solution in view of the following Corollary.

Proposition 2: *(X, g) and $(X + X_0, g)$, where $[X, X_0] = 0$, generate equivalent space-time 4-metrics.*

For example, consider the solution (15) together with the two Killing vectors X_0 and X_1 of the 3-metric:

$$\begin{aligned}
g:\quad ds^2 &= d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\phi^2) \\
X &= (2m/\rho)^{\frac{1}{2}} \frac{\partial}{\partial\rho} \\
X_0 &= sin\phi \frac{\partial}{\partial\theta} + cot\theta cos\phi \frac{\partial}{\partial\phi} \\
X_1 &= sin\theta cos\phi \frac{\partial}{\partial\rho} + (cos\theta cos\phi/\rho) \frac{\partial}{\partial\theta} \\
&\quad - (cosec\theta sin\phi/\rho) \frac{\partial}{\partial\phi}
\end{aligned}$$

Here the Killing vector field X_0 corresponds to rotational isometry whereas X_1 corresponds to translational isometry. $[X, X_0] = 0$, but $[X, X_1] \neq 0$. $(X + X_0, g)$ is again a solution, but $(X + X_1, g)$ is not. The 4-metrics corresponding to (X, g) and $(X + X_0, g)$ are equivalent.

The de Sitter solution($\varrho = 0, \Lambda \neq 0$):

3-metric: $ds^2 = \delta_{ik} dx^i dx^k$

Vector field: $X = -\lambda x^i \frac{\partial}{\partial x^i}$

with $\lambda = \sqrt{\frac{-\Lambda}{3}}$. The flow generated by X is simply $x^i \mapsto \tilde{x}^i = e^{-\lambda\tau}x^i$, and the corresponding 4-metric:

$$ds^2 = -d\tau^2 + e^{2\lambda\tau}\delta_{ik}d\tilde{x}^i d\tilde{x}^k$$

with $\lambda = \sqrt{\frac{-\Lambda}{3}}$ is the well-known de Sitter solution.

A new solution of the vacuum Einstein field equations: Let $\varrho = \Lambda = 0$. Then the flat 3-metric $ds^2 = \delta_{ik}dx^i dx^k$ and the vector field $X = -\lambda_i x^i \frac{\partial}{\partial x^i}$ is a solution of (12)–(14), provided

$$\left.\begin{aligned} \lambda_1 + \lambda_2 + \lambda_3 &= 0 \\ \lambda_1^2 + \lambda_2^2 + \lambda_3^2 &= 0 \end{aligned}\right\} \tag{19}$$

which is possible only if, either all λ_i are zero or some of the λ_i are *complex*. Let us entertain the possibility that some of the λ_i are *complex*. An obvious solution of (19) is $\lambda_1 = p + iq, \lambda_2 = p - iq, \lambda_3 = r$ with $p = -\frac{r}{2}$, $q = r\frac{\sqrt{3}}{2}$ where r is real number. The flow generated by the vector field X is now $x^i \mapsto \tilde{x}^i = e^{-\lambda_i\tau}x^i$ (no summation), The corresponding *complex* 4-metric is then

$$ds^2 = -d\tau^2 + e^{2\lambda_1\tau}(d\tilde{x}^1)^2 + e^{2\lambda_2\tau}(d\tilde{x}^2)^2 + e^{2\lambda_3\tau}(d\tilde{x}^3)^2 \tag{20}$$

which is related to the socalled (*real*) Kasner [10] solution.

We now make a *complex* coordinate transformation: $\tilde{x}^i \mapsto \bar{x}^i$ to obtain a *real* metric as follows:

$$\begin{aligned} \tilde{x}^1 &= (a+ib)^{\frac{1}{2}}\bar{x}^1 + (c+id)^{\frac{1}{2}}\bar{x}^2 \\ \tilde{x}^2 &= (a-ib)^{\frac{1}{2}}\bar{x}^1 + (c-id)^{\frac{1}{2}}\bar{x}^2 \\ \tilde{x}^3 &= i\bar{x}^3 \end{aligned} \tag{21}$$

Here a, b, c, d are any *real* numbers subject to the condition $D = \mathrm{Im}\left[(a+ib)^{\frac{1}{2}}(c-id)^{\frac{1}{2}}\right] \neq 0$.

We now have a *real* 4-metric $\bar{g}_{\alpha\beta}$ containing 5 *real* parameters a, b, c, d, r

$$\left.\begin{aligned} \bar{g}_{11} &= 2e^{-r\tau}\big(a\cos(r\sqrt{3}\tau) - b\sin(r\sqrt{3}\tau)\big) \\ \bar{g}_{22} &= 2e^{-r\tau}\big(c\cos(r\sqrt{3}\tau) - d\sin(r\sqrt{3}\tau)\big) \\ \bar{g}_{12} &= 2e^{-r\tau}\big(A\cos(r\sqrt{3}\tau) - B\sin(r\sqrt{3}\tau)\big) \\ \bar{g}_{33} &= -e^{2r\tau} \\ \bar{g}_{00} &= -1 \,. \end{aligned}\right\} \tag{22}$$

where $A = \mathrm{Re}\left[(a+ib)^{\frac{1}{2}}(c+id^{\frac{1}{2}}\right], B = \mathrm{Im}\left[(a+ib)^{\frac{1}{2}}(c+id)^{\frac{1}{2}}\right]$.

The signature of (22) is Lorentzian: $(--+-)$ or $(-+--)$; but $\bar{x}^0 = \tau$ here is *not* the physical time coordinate. Incidentally, by changing the sign of $\bar{g}_{33}$ (which corresponds to taking $\tilde{x}^3 = \bar{x}^3$ in (21)) we obtain a non-Lorentzian solution of the vacuum Einstein equations.

A special case of (22) is given by $a=1, b=c=0, d=1$ with $A=B=1/\sqrt{2}, D=-1/\sqrt{2}$

$$\left.\begin{aligned} \bar{g}_{11} &= 2e^{-r\tau}\cos(r\sqrt{3}\tau) \\ \bar{g}_{22} &= -2e^{-r\tau}\sin(r\sqrt{3}\tau) \\ \bar{g}_{12} &= \sqrt{2}e^{-r\tau}\big(\cos(r\sqrt{3}\tau) - \sin(r\sqrt{3}\tau)\big) \\ \bar{g}_{33} &= -e^{2r\tau} \\ \bar{g}_{00} &= -1\ . \end{aligned}\right\} \tag{23}$$

That this is a solution can also be checked by direct calculaton. The metric (23) has three Killing vector fields $\frac{\partial}{\partial \bar{x}^i}$ ($i=1,2,3$). However,since both $\bar{g}_{11}, \bar{g}_{22}$ change sign with τ (as well as r), the metric is static for certain value of τ and non-static for other values.
The metric (22) is generated by:
3-metric: $ds^2 = 2a(d\bar{x}^1)^2 + 2c(d\bar{x}^2)^2 + 4Ad\bar{x}^1 d\bar{x}^2 - (d\bar{x}^3)^2$
Vector field:

$$\begin{aligned} \bar{X} = &\left(\frac{(-qC-pD)\bar{x}^1 - q(c^2+d^2)^{\frac{1}{2}}\bar{x}^2}{D}\right)\frac{\partial}{\partial \bar{x}^1} \\ &+ \left(\frac{q(a^2+b^2)^{\frac{1}{2}}\bar{x}^1 + (qC-pD)\bar{x}^2}{D}\right)\frac{\partial}{\partial \bar{x}^2} - ir\bar{x}^3\frac{\partial}{\partial \bar{x}^3} \end{aligned} \tag{24}$$

where $C = \mathrm{Re}\left[(a+ib)^{\frac{1}{2}}(c-id)^{\frac{1}{2}}\right]$.
Similarly, (23) is generated by:
3-metric: $ds^2 = 2(d\bar{x}^1)^2 + 2\sqrt{2}d\bar{x}^1 d\bar{x}^2 - (d\bar{x}^3)^2$

Vector field:

$$\bar{X} = r\big(\frac{1}{2}(\sqrt{3}+1)\bar{x}^1 + \sqrt{\frac{3}{2}}\bar{x}^2\big)\frac{\partial}{\partial \bar{x}^1} - r\big(\sqrt{\frac{3}{2}}\bar{x}^1 + \frac{1}{2}(\sqrt{3}-1)\bar{x}^2\big)\frac{\partial}{\partial \bar{x}^2} - ir\bar{x}^3\frac{\partial}{\partial \bar{x}^3} \tag{25}$$

It should be noted that the 3-metrics must have *indefinite* signatures and the vector fields have to be *complex* if the corresponding 4-metrics are to be *Lorentzian.* The *non-Lorentzian* solutions can however be generated by a *real* vector field acting on a 3-metric, also with an indefinite signature.

Conclusion: The above examples show that the *geometry* of certain space-times can also be formulated in a purely 3-dimensional setup by the triple (M^3, g, X) satisfying eqs.(12)–(14). Every solution (M^3, g, X) of (12)–(14) describes in some sense a physical space-time as 'perceived' by a *fundamental* observer X , living in a 3-dimensional space; (M^3, g) 'appears' to X as a 4-dimesional space-time.

It should be clear that the present formalism is not a $(3+1)$ formalism, such as the well known ADM theory. Since the space of vector fields on a manifold is in general infinite dimensional, our formalism may be considered as a $(3+\infty)$ theory instead of a $(3+1)$ theory.

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