Degenerate Homoclinic Crossings

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Abstract

Let $F : M \to M$ denote a diffeomorphism of a C^∞-manifold M. Let $p \in M$ be a hyperbolic fixed point with stable and unstable manifolds W_S and W_U respectively. Let q be a point other than p at which these manifolds meet. Suppose W_U and W_S meet non-transversally, i.e., they have a *degenerate homoclinic contact.*

It is shown that, subject to C^1-linearizability and diagonalizability of the linear part of a map, a transverse crossing will arise and imply a horseshoe structure, if the dimension of one of the invariant manifolds is 1. A more general manifolds are considered. Also, it is shown that in the planar case the Hirsch's linearizability assumption is unnecessary.

1 Introduction

A celebrated Theorem of Birkhoff-Smale [Sm] gives one of the few methods to rigorously conclude the existence of a horseshoe structure. The basic assumption of this theorem is the presence of a transverse homoclinic point.
The work in this paper was inspired by the idea of Hirsch [Hi] and Churchill and Rod [CR], that a transverse crossing arises necessarily for a wide class of diffeomorphisms on $\mathbf{R^2}$.
Let $F : M \to M$ denote a diffeomorphism of a C^∞-manifold M. Let $p \in M$ be a hyperbolic fixed point with stable and unstable manifolds W_S and W_U respectively. Let q be a point other then p at which these manifolds meet. If they meet transversely, q is called *transverse homoclinic point.* In this case the Birkhoff-Smale Theorem applies to give a horseshoe.
Suppose W_S and W_U meet tangentially, i.e. they have a *degenerate homoclinic contact.* We say, that they have *order of contact* $l < \infty$ ($l \in \mathbf{R}$, $l > 1$), provided part of unstable (stable) manifold can be approximated near q by an l^{th}-order parabolic surface. Otherwise they have *infinite order of contact.* (See Section 2 for a precise definition).

The object of the papers of Hirsch and Churchill-Rod was to show that in the planar case ($M = \mathbf{R^2}$) the manifolds W_S and W_U cross transversely at some other point, if the order of contact is finite.
Churchill and Rod in their paper made the additional assumptions that F is *analytic* and *area-preserving.* In that case, by Moser's Theorem [M], [SM],this diffeomorphism can be assumed to be in the Birkhoff normal form, and any branch of the stable (unstable) manifold can be reparameterized, so that $F^n(W_U) \cap W_S$ contains a positive orbit $q', Fq', F^2q', ..., F^kq'$, that lies on a hyperbola. Then, direct calculations imply that q' is a transverse homoclinic point. But area preservation is a strong restriction. Also, analyticity permits one to consider only a finite order of contact in the tangential contact.
Hirsch in his paper replaced the above assumptions by the smooth linearizability assumption (near p). If we substitute L, the linear part of F, for F, the order of contact will not be changed, and we can calculate the "slope" between two manifolds at the point q', where $L^n(W_U)$ meets W_S. If the crossing at q' is transverse, then $F^n(W_U) = \Phi L^n \Phi^{-1}(W_U)$ also meets W_S non-degenerately, where Φ is the C^1 linearizing map.

There arises the question: *"Does the Hirsch's linearizability assumption give a serious restriction on the diffeomorphism F?"*
For dimension 2, the answer is "No". A C^2-diffeomorphism in $\mathbf{R^2}$ can be C^1-linearized near a hyperbolic fixed point (Hartman [H1]). Although we might hope to get the same answer in higher dimensions, the well-known example of Hartman [H2] introduces the effect of *resonance.* It shows that a C^∞ diffeomorphism on $\mathbf{R^3}$ may not be C^1-linearizable.
In section 2 we will discuss the 2-dimensional case and show that it is unnecessary to make resonance restrictions in the theorem proved by Hirsch. Section 3 contains the proof that any C^1-linearizable diffeomorphism on $\mathbf{R^n}$ (with diagonalizable linear part) that has a tangential homoclinic point described above, admits a horseshoe, if the dimension of one of the invariant manifolds is 1. A more general manifolds are considered (Theorem 5). We close this topic with a Conjecture 1 that any assumption on the form of the invariant manifolds is redundant. This statement permits to define the concept of order of contact for manifolds of dimensions higher than one. The above definition suggests the Conjecture 2 on classification of singularities.

2 2-Dimensional Dynamics

Consider a smooth manifold M without boundary and a C^2 map $F : M \to M$ with a hyperbolic fixed point p and stable and unstable manifolds W_S, W_U respectively. A point other than p that is in $W_S \cap W_U$ is said to be *homoclinic to p.*
If W_U and W_S intersect transversely in a point, then the point is called a

transverse homoclinic point. If W_U and W_S intersect non-transversely we will say that they have a *degenerate homoclinic crossing (point).*
For the case of a transverse homoclinic crossing (see Figure 1, left) horseshoe can be established via the Birkhoff-Smale Theorem:

Theorem 1 (Birkhoff-Smale) *Let $f : \mathbf{R^n} \to \mathbf{R^n}$ be a diffeomorphism such that p is a hyperbolic fixed point. If $q \neq p$ is a transverse homoclinic point, then there exists a hyperbolic invariant set Λ, a set of bi-infinite sequences Σ and a shift map σ such that for some m the following diagram commutes:*

$$\begin{array}{ccc} \Lambda & \stackrel{f^m}{\to} & \Lambda \\ h\downarrow & & \downarrow h \\ \Sigma & \stackrel{\sigma}{\to} & \Sigma \end{array}$$

*Here $\Sigma = \{...A_{-2}A_{-1}.A_0A_1A_2...\}$, $A_i \in \{0, ..., N\}$,
$\sigma\{...A_{-2}A_{-1}.A_0A_1A_2...\} = \{...A_{-2}A_{-1}A_0.A_1A_2...\}$ and h is a homeomorphism mapping Λ onto Σ.*

It is not difficult to prove that the shift map σ has :
(i) a countable infinity of periodic orbits of all periods;
(ii) an uncountable infinity of non-periodic orbits; and
(iii) a dense orbit.
Using this theorem and employing the Poincaré first return map, one can show that the flow corresponding to this Poincaré map has similar properties.

In practice, it is very difficult to find what kind of homoclinic crossing of stable and unstable manifolds we have. There were many attempts to remove the assumption on transversality of the above crossing (see [BW], [CR], [Hi], [Pa]). In this chapter we are going to consider 2-dimensional dynamics with a degenerate homoclinic point (see Figure 1, right) and to prove that in this situation we can make the same conclusion as in the Birkhoff-Smale Theorem. In this 2-dimensional case, we can define the *order of contact* of a degenerate homoclinic crossing in the following way:

Definition 1 *We say that the parameterized curves $\gamma_1(t)$ and $\gamma_2(t)$, immersed in $\mathbf{R^2}$, which meet tangentially at a point $q = \gamma_1(0) = \gamma_2(0)$ have the order of contact l if the following condition holds: for some m and M positive numbers and all sufficiently small $t \neq 0$*

$$m \leq \frac{|\gamma_1(t) - \gamma_2(t)|}{|t^l|} \leq M$$

Remark 1 *Such an l is unique (does not depend on parameterization), if it is exists.*

Hirsch in his paper ([Hi]) proved the following theorem:

Theorem 2 (Hirsch) *Assume $F : M \to M$ is C^r diffeomorphism, $r \geq 2$. Denote by $p \in M$ fixed point of F whose invariant curves have a degenerate homoclinic crossing $q \in M$. If*
(a) The invariant curves (γ_1, γ_2) at p have contact of order l at a degenerate homoclinic crossing $q \in M$, $l \in (1, r]$.
(b) There is a C^r local diffeomorphism from a neighborhood of the origin in the plane which locally conjugates F to a linear map.
Then, the invariant curves also have a point of transverse intersection. Therefore F admits a horseshoe by the Birkhoff-Smale theorem.

Remark 2 *Hirsch in his paper defines the order of contact as having $\gamma_1^{(i)}(0) = \gamma_2^{(i)}(0)$ for $i = 1, ..., l-1$. If F is C^r, then the invariant curves $\gamma_1(t)$ and $\gamma_2(t)$ are also C^r. Hirsch in his proof assumes that the smoothness of F is at least l (here l is the order of contact of $\gamma_1(t)$ and $\gamma_2(t)$) only because of the way he defines the order of contact. Therefore, if we use Definition 1 to define l, the order of contact can exceed the smoothness of $\gamma_1(t)$ and $\gamma_2(t)$, and the assumption F is C^r, where $r \geq l$ can be changed to the assumption F is C^2.*

In this chapter we will show, that it is unnecessary to make an assumption on linearizability in order to establish the conclusion of Hirsch's theorem. In other words, the theorem can be stated in the following way:

Theorem 3 *Assume $F : M \to M$ is a C^2 diffeomorphism on $\mathbf{R}^2$. If the invariant curves of a hyperbolic fixed point p have a contact of finite order l at a homoclinic point, then F admits a horseshoe.*

To prove Theorem 3 we need the following lemma.

Lemma 1 *Let M be a C^∞ surface without boundary and let ϕ be a C^1 diffeomorphism that maps a neighborhood $N' \subset M$ onto some neighborhood $N \subset \mathbf{R^2}$. Assume that $u(t), v(t)$ are C^1 curves, such that $u(0) = v(0)$, $u'(0) \neq 0$ and $v'(0) \neq 0$.*
Then, ϕ preserves the order of contact of these curves.

Proof:
It is no loss of generality that we assumed that u(0)=v(0)=0 .
We have curves

$$\phi \circ u(t), \phi \circ v(t),$$

transformed by the diffeomorphism ϕ.
There are positive constants m and M such that for all small $|t|$

$$m \leq \frac{|u(t) - v(t)|}{t^l} \leq M.$$

By the C^1 *Mean Value Theorem*,

$$\phi(x) - \phi(y) = \left[\int_0^1 (D\phi)_{\sigma(s)} ds\right](x-y),$$

where $\sigma(s) = (1-s)x + sy$.
Then

$$(\phi \circ u)(t) - (\phi \circ v)(t) = \left[\int_0^1 (D\phi)_{\sigma(s)} ds\right](u(t) - v(t)),$$

where $\sigma(s) = (1-s)u(t) + sv(t)$.
Therefore

$$\frac{(\phi \circ u)(t) - (\phi \circ v)(t)}{t^l} = \left[\int_0^1 (D\phi)_{\sigma(s)} ds\right]\left(\frac{u(t) - v(t)}{t^l}\right)$$

As $t \to 0$, $\sigma(s) \to u(0)$ and the matrix $\int_0^1 (D\phi)_{\sigma(s)} ds$ tends to the invertible matrix $(D\phi)_{u(0)}$.
The ratio $\frac{u(t)-v(t)}{t^l}$ is a vector whose norm stays bounded between m and M, $0 < m < M < \infty$. Hence the vector

$$\left[\int_0^1 (D\phi)_{\sigma(s)} ds\right]\left(\frac{u(t) - v(t)}{t^l}\right)$$

stays bounded and bounded away from 0.

Q.E.D.

Thus, to complete this proof, it is suffices to find a C^1-linearization. Hartman in his paper [H1] proves an even more general theorem, which is stated below. Also, for more details on this theorem see [R].

Theorem 4 (Hartman) *Let the map T be given by*

$$T : \tilde{x} = Ax + X(x,z), \qquad \tilde{z} = Cz + Z(x,z),$$

where x is an I-vector and z is a K-vector and where A,C are constant square matrices with eigenvalues $a_1, ..., a_I$, $c_1, ..., c_K$ satisfying the bunching conditions

$$0 < |a_1| \leq ... \leq |a_I| < 1 < |c_1| \leq ... \leq |c_K|.$$

We assume X, Z are functions of class C^1 for small $|x|$ and $|z|$, are $o(|x| + |z|)$ as $(x,z) \to 0$, and have uniformly Lipschitz continuous partial derivatives. Let

$\lambda = \lambda(|a_1|, ..., |a_I|)$, $\qquad \mu = \lambda(1/|c_K|, ..., 1/|c_1|)$ $\quad$ *for some positive λ and μ,* *and let the eigenvalues of* A, C *satisfy*

$$|a_I|^\lambda |c_K|/|c_1| < 1 \textit{ and } |c_1|^\mu |a_1|/|a_I| > 1.$$

Then there exists a C^1 *map* R *of the form*

$$R : u = x - \phi(x,z), \; w = z - \chi(x,z)$$

for small $|x|$, $|z|$ *in which*

$$\phi, \chi = o(|x| + |z|) \qquad\qquad as\ (x,z) \to 0$$

that linearizes T,

$$RTR^{-1} : \tilde{u} = Au, \qquad \tilde{w} = Cw.$$

It is clear that for the case $I = K = 1$ there is a simple corollary to the above theorem.

Corollary 1 *Let* T *be a* C^2 *map with a hyperbolic fixed point* 0. *Then there exists a* C^1 *map that linearizes* T.

Thus, the proof of Theorem 3 follows from the Hirsch's Theorem, Corollary 1, Lemma 1 and Remark 2.

3 Degenerate Homoclinic Crossings in Higher Dimensions

One of the assumptions of the Birkhoff-Smale Theorem [Sm] that describes chaotic dynamics is the presence of a transverse homoclinic crossing. This is a serious restriction and there have been many attempts to remove it. (See [BW], [CR], [Hi], [Pa]). In particular, the results were obtained for planar dynamics. ([CR], [Hi]).
In this chapter we will consider a C^1-linearizable diffeomorphism on $\mathbf{R^n}$ (with diagonalizable linear part) that has a homoclinic point. In our theorem we do not make an assumption that the homoclinic point is transverse.
Thus we are going to prove the following:

Theorem 5 *Assume that* M *is a smooth manifold,* $F : M \to M$ *is a locally* C^1*-linearizable map in the neighborhood of* $o \in M$, *a saddle point of* F *whose invariant manifolds* W_U *(unstable) and* W_S *(stable) have a degenerate homoclinic crossing at an isolated point* $q \in M$ $(q \neq o)$.
Assume that either condition **1** *or* **2** *is true:*
1. *The dimension of* <u>*one*</u> *of the invariant manifolds is* 1.

2. *There exists a parameterization such that one of the invariant manifolds can be written (after linearization of a map) in the special diagonal form (1). (See page 11.)*
If the linear part of F (call it T) is diagonalisable, then the invariant manifolds also have a point of transverse intersection. Therefore F admits a horseshoe by the Birkhoff-Smale Theorem.

Remark 3 *The saddle point p from the last section will be denoted by o in this section.*

Proof:
Consider a linearization $\phi : N' \mapsto N$, from a neighborhood $N' \subset M$ of o onto a neighborhood $N \subset \mathbf{R}^{p+m}$ of the origin which is at least of class C^1. We may assume ϕ takes the local stable manifold at o to a neighborhood in the Y-hyperplane, and the local unstable manifold to a neighborhood in the X-hyperplane. In N' there are images of q, under forward and backward iterates of F, on both the local stable and local unstable manifolds of o.
If Γ is in the first orthant and T is diagonal, then forward and backward iterates $T^n\Gamma$ (here n is some subsequence of whole numbers) are in the first orthant too. Therefore there are compact manifolds $\Lambda, \Gamma \subset N$ (see Figure 2) with the following properties:
(i) Γ and Λ are images under F of stable and unstable manifolds.
(ii) Λ is tangent to the Y-hyperplane at the point (0,A), $A_1, ..., A_p > 0$; Γ is tangent to the X-hyperplane at the point (B,0), $B_1, ..., B_m > 0$. These manifolds are in the first orthant.
Any smooth p-manifold Γ immersed in $\mathbf{R^n}$, that has a crossing with the X-hyperplane at the point $B = (B_1, ..., B_m, 0, ..., 0)$ locally can be written in the form:

$$\Gamma \sim \left\{ \begin{array}{lcl} u^1 & = & \phi_1(\tau_1, ..., \tau_p) + B_1 \\ . & . & . \\ . & . & . \\ . & . & . \\ u^m & = & \phi_m(\tau_1, ..., \tau_p) + B_m \\ v^1 & = & \phi_{m+1}(\tau_1, ..., \tau_p) \\ . & . & . \\ . & . & . \\ . & . & . \\ v^p & = & \phi_{m+p}(\tau_1, ..., \tau_p). \end{array} \right.$$

for some positive constants B_j ($j = 1, ..., m$), constant $l_p > 1$ and smooth functions ϕ_j ($j = 1, ..., m+p$) such that $\phi_j(0) = 0$ and $\frac{\partial}{\partial \tau_k}(\phi_j)$ ($k = 1, ..., p$) are bounded in a small neighborhood of 0.

Let Λ be written as

$$\Lambda \sim \begin{cases} x^1 & = & t_1 \\ . & . & . \\ . & . & . \\ . & . & . \\ x^m & = & t_m \\ y^1 & = & f_1(t_1,...,t_m)+A_1 \\ . & . & . \\ . & . & . \\ . & . & . \\ y^p & = & f_p(t_1,...,t_m)+A_p \end{cases}$$

for some positive constants A_i, continuously differentiable functions f_i (by assumption our linearization is at least of class C^1) and positive parameters t_j. Also, we assume that $f_i(0)=0$.
It suffices to prove the following lemma, in order to complete the proof of this theorem.

Lemma 2 *Suppose*

$$T:(x,y)\rightarrow(\beta_1x^1,...,\beta_mx^m,\alpha_1y^1,...,\alpha_py^p),$$

$$0<\alpha_k<1<\beta_j(k=1,...,p,j=1,...,m).$$

Then there exists a natural number n_* such that $T^n\Lambda$ meets Γ transversely if $n>n_*$.

Proof:
We can find sequences $(x_i,y_i)\in\Lambda$ and $n_i\rightarrow\infty$ such that $(x_i,y_i)\rightarrow(0,A)$ and $T^{n_i}(x_i,y_i)\rightarrow(B,0)$. This implies that $T^{n_i}\Lambda$ and Γ will meet at some point different from $(0,A)$ and $(B,0)$. (See Figure 2.)

$$T^{n_i}\Lambda \sim \begin{cases} u^1 & = & \beta_1^{n_i}t_1 \\ . & . & . \\ . & . & . \\ . & . & . \\ u^m & = & \beta_m^{n_i}t_m \\ v^1 & = & \alpha_1^{n_i}\left(f_1(t_1,...,t_m)+A_1\right) \\ . & . & . \\ . & . & . \\ . & . & . \\ v^p & = & \alpha_p^{n_i}\left(f_p(t_1,...,t_m)+A_p\right) \end{cases}$$

We can find a subsequence of $\{n_i\}_{i=1}^{\infty}$ such that $\alpha_k^{n_i}\frac{\partial}{\partial t_j}(f_k)_i\rightarrow 0$ as $i\rightarrow\infty$

Consider the matrix, constructed in the following way.
The first m rows represent the linear space of the tangent bundle of the m-manifold $T^{n_i}\Lambda$.
The second p rows represent the linear space of the tangent bundle of the p-manifold Γ.
Recall that $p+m = n$, where n denotes the dimension of the space, these manifolds embedded into. Therefore Γ and $T^{n_i}\Lambda$ meet transversely if, and only if, the described tangent bundles span the $\mathbf{R^n}$. Thus, in order to prove that $T^{n_i}\Lambda$ meets Γ transversely for some n at the point $(u^1,...,u^m,v^1,...,v^p)$ we will show that the determinant of the described matrix is never 0 for n_i sufficiently large. We will consider the following determinant D_i for i sufficiently large.

$$D_i = \begin{vmatrix} \beta_1^{n_i} & & 0 & \alpha_1^{n_i}\frac{\partial}{\partial t_1}(f_1)_i & \dots & \alpha_p^{n_i}\frac{\partial}{\partial t_1}(f_p)_i \\ & \ddots & & \vdots & \ddots & \vdots \\ 0 & & \beta_m^{n_i} & \alpha_1^{n_i}\frac{\partial}{\partial t_m}(f_1)_i & \dots & \alpha_p^{n_i}\frac{\partial}{\partial t_m}(f_p)_i \\ \frac{\partial}{\partial \tau_1}(\phi_1)_i & \dots & \frac{\partial}{\partial \tau_1}(\phi_m)_i & \frac{\partial}{\partial \tau_1}(\phi_{m+1})_i & & \frac{\partial}{\partial \tau_1}(\phi_{m+p})_i \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial}{\partial \tau_p}(\phi_1)_i & \dots & \frac{\partial}{\partial \tau_p}(\phi_m)_i & \frac{\partial}{\partial \tau_p}(\phi_{m+1})_i & & \frac{\partial}{\partial \tau_p}(\phi_{m+p})_i \end{vmatrix}$$

We will prove that there are some real constants $l_1,...,l_p \geq 1$ (not all equal to 1) such that

$$D_i = (C+o(1))\cdot \alpha_1^{n_i\frac{l_1-1}{l_1}} \cdot ... \cdot \alpha_p^{n_i\frac{l_p-1}{l_p}} \cdot \beta_1^{n_i} \cdot ... \cdot \beta_m^{n_i}\ , \quad \text{as } i \to \infty.$$

(Here C is some non-zero constant).

Remark 4 *Without loss of generality we can assume that $m \geq p$ (otherwise we can substitute T^{-1} for T).*

1. Now we will prove Lemma 2 assuming that $p=1$ and $m \geq 1$. I. e., Γ is going to be a smooth curve immersed in $m+1$-dimensional space and having an intersection with X hyperplane at the point $A_1,...,A_m$. One can always find a parameterization such that locally this Γ is in the form

$$\Gamma \sim \begin{cases} u^1 & = & \phi_1(\tau_1) + B_1 \\ . & . & . \\ . & . & . \\ . & . & . \\ u^m & = & \phi_m(\tau_1) + B_m \\ v^1 & = & \tau_1^{l_1}, \end{cases}$$

where $\phi_1, ..., \phi_m$ are smooth functions and $\phi_1'(0), ..., \phi_m'(0)$ are bounded. The m-manifold Λ we can write in the form

$$\Lambda \sim \begin{cases} x^1 & = & t_1 \\ . & . & . \\ . & . & . \\ . & . & . \\ x^m & = & t_m \\ y^1 & = & f_1(t_1, ..., t_m) + A_1 \end{cases}$$

The determinant $D_i^{m,1}$ is

$$D_i^{m,1} = \begin{vmatrix} \beta_1^{n_i} & & 0 & \alpha_1^{n_i} \frac{\partial}{\partial t_1}(f_1)_i \\ & \ddots & & \vdots \\ 0 & & \beta_m^{n_i} & \alpha_1^{n_i} \frac{\partial}{\partial t_m}(f_1)_i \\ (\phi_1')_i & \dots & (\phi_m')_i & l_1(\tau_1^{l_1-1})_i \end{vmatrix}$$

Assume that the n_i^{th} iteration of Λ meets Γ at the point $u^1, ..., u^m, v^1$. Then at this point $T^{n_i}\Lambda$ can be written

$$T^{n_i}\Lambda \sim \begin{cases} u^1 & = & \beta_1^{n_i} t_1 \\ . & . & . \\ . & . & . \\ . & . & . \\ u^m & = & \beta_m^{n_i} t_m \\ v^1 & = & \alpha_1^{n_i}\left(f_1(t_1, ..., t_m) + A_1\right) \end{cases}$$

Then

$$l_1 \tau_1^{l_1-1} = l_1 \left(\left(\tau_1^{l_1} \right)^{\frac{l_1-1}{l_1}} \right)_i = l_1 \left(v_i^1 \right)^{\frac{l_1-1}{l_1}} = l_1 \left(\alpha_1^{n_i} y_i^1 \right)^{\frac{l_1-1}{l_1}} =$$

$$l_1\left(\alpha_1^{n_i}\right)^{\frac{l_1-1}{l_1}}\cdot\left(y_i^1\right)^{\frac{l_1-1}{l_1}}$$

Thus, the determinant $D_i^{m,1}$ takes the form

$$D_i^{m,1}=\begin{vmatrix} \beta_1^{n_i} & 0 & \alpha_1^{n_i}\frac{\partial}{\partial t_1}(f_1)_i \\ & \ddots & \vdots \\ 0 & \beta_m^{n_i} & \alpha_1^{n_i}\frac{\partial}{\partial t_m}(f_1)_i \\ (\phi_1')_i \quad \dots & (\phi_m')_i & l_1\left(\alpha_1^{n_i}\right)^{\frac{l_1-1}{l_1}}\left(y_i^1\right)^{\frac{l_1-1}{l_1}} \end{vmatrix}$$

There exists a subsequence of $\{n_i\}_{i=1}^{\infty}$ such that $\alpha_1^{\frac{n_i}{l_1}}\frac{\partial}{\partial t_j}(f_1)_i\to 0$. Since $(\phi_j')_i$ are bounded, $\alpha_1^{\frac{n_i}{l_1}}\frac{\partial}{\partial t_j}(f_1)_i$ are arbitrarily small for i big enough ($j=1,...,m$) and $\left(y_i^1\right)^{\frac{l_1-1}{l_1}}\to(A_1)^{\frac{l_1-1}{l_1}}\neq 0$,

$$D_i^{m,1}=(C+o(1))\cdot\alpha_1^{n_i\frac{l_1-1}{l_1}}\cdot\beta_1^{n_i}\cdot\ldots\cdot\beta_m^{n_i}\ ,\quad \text{as } i\to\infty.$$

The constant C is different from zero because A_1 is a non-zero constant.

Next we are going to prove Lemma 2 assuming that after a linearization of a map T, Γ has the following diagonal form:

$$\Gamma\sim\begin{cases} u^1 & = & \phi_1(\tau_1,...,\tau_p)+B_1 \\ . & . & . \\ . & . & . \\ . & . & . \\ u^m & = & \phi_m(\tau_1,...,\tau_p)+B_m \\ v^1 & = & \tau_1^{l_1} \\ . & . & . \\ . & . & . \\ . & . & . \\ v^p & = & \tau_p^{l_p}, \end{cases} \qquad (1)$$

where ϕ_k are smooth functions and $\frac{\partial}{\partial\tau_j}\phi_k(0)$ are bounded ($k=1,...,m,\ j=1,...,p$).

Then the determinant $D_i^{m,p}$ is

$$D_i^{m,p} = \begin{vmatrix} \beta_1^{n_i} & & 0 & \alpha_1^{n_i}\frac{\partial}{\partial t_1}(f_1)_i & \dots & \alpha_p^{n_i}\frac{\partial}{\partial t_1}(f_p)_i \\ & \ddots & & \vdots & \ddots & \vdots \\ 0 & & \beta_m^{n_i} & \alpha_1^{n_i}\frac{\partial}{\partial t_m}(f_1)_i & \dots & \alpha_p^{n_i}\frac{\partial}{\partial t_m}(f_p)_i \\ \frac{\partial}{\partial \tau_1}(\phi_1)_i & \dots & \frac{\partial}{\partial \tau_1}(\phi_m)_i & l_1\tau_1^{l_1-1} & & 0 \\ \vdots & \ddots & \vdots & & \ddots & \\ \frac{\partial}{\partial \tau_p}(\phi_1)_i & \dots & \frac{\partial}{\partial \tau_p}(\phi_m)_i & 0 & & l_p\tau_p^{l_p-1} \end{vmatrix}$$

Assume that the n_i^{th} iteration of Λ meets Γ at the point $u^1, ..., u^m, v^1, ..., v^p$. Then at this point

$$l_j\tau_j^{l_j-1} = l_j\left(\left(\tau_j^{l_j}\right)^{\frac{l_j-1}{l_j}}\right)_i = l_j\left(v_i^j\right)^{\frac{l_j-1}{l_j}} = l_j\left(\alpha_j^{n_i}y_i^j\right)^{\frac{l_j-1}{l_j}} =$$

$$l_j\left(\alpha_j^{n_i}\right)^{\frac{l_j-1}{l_j}} \cdot \left(y_i^j\right)^{\frac{l_j-1}{l_j}}$$

where $j = 1, ..., p$. Thus, the determinant $D_i^{m,p}$ takes the form

$$D_i^{m,p} = \begin{vmatrix} \beta_1^{n_i} & & 0 & \alpha_1^{n_i}\frac{\partial}{\partial t_1}(f_1)_i & \dots & \alpha_p^{n_i}\frac{\partial}{\partial t_1}(f_p)_i \\ & \ddots & & \vdots & \ddots & \vdots \\ 0 & & \beta_m^{n_i} & \alpha_1^{n_i}\frac{\partial}{\partial t_m}(f_1)_i & \dots & \alpha_p^{n_i}\frac{\partial}{\partial t_m}(f_p)_i \\ \frac{\partial}{\partial \tau_1}(\phi_1)_i & \dots & \frac{\partial}{\partial \tau_1}(\phi_m)_i & l_1\left(\alpha_1^{n_i}y_i^1\right)^{\frac{l_1-1}{l_1}} & & 0 \\ \vdots & \ddots & \vdots & & \ddots & \\ \frac{\partial}{\partial \tau_p}(\phi_1)_i & \dots & \frac{\partial}{\partial \tau_p}(\phi_m)_i & 0 & & l_p\left(\alpha_p^{n_i}y_i^p\right)^{\frac{l_p-1}{l_p}} \end{vmatrix}$$

There exists a subsequence of $\{n_i\}_{i=1}^{\infty}$ such that $\alpha_k^{\frac{n_i}{l_k}} \frac{\partial}{\partial t_j}(f_k)_i \to 0$ $(k = 1, ..., p,$ $j = 1, ..., m)$. Since $\frac{\partial}{\partial \tau_j}(\phi_k)_i$ are bounded, $\alpha_k^{\frac{n_i}{l_k}} \frac{\partial}{\partial t_j}(f_k)_i$ are arbitrarily small for i big enough $(j = 1, ..., m)$ and $\left(y_i^1\right)^{\frac{l_1-1}{l_1}} \to (A_1)^{\frac{l_1-1}{l_1}} \neq 0$,

$$D_i^{m,1} = (C + o(1)) \cdot \alpha_1^{n_i \frac{l_1-1}{l_1}} \alpha_p^{n_i \frac{l_p-1}{l_p}} \cdot \beta_1^{n_i} \cdot ... \cdot \beta_m^{n_i}\ , \quad \text{as } i \to \infty.$$

Here C is a non-zero constant.

Thus, $D_i \neq 0$ for i big enough. This implies that for some n, iteration T^n applied to the manifold Λ near the origin will produce a transverse crossing with Γ.

Q.E.D.

This completes the proof of Theorem 5
Q.E.D.

Conjecture 1 *Assumptions* **1** *and* **2** *in Theorem 5 are redundant. The vector* $(l_1, ..., l_p)$ *defined via the determinant* $D^{m,p}$ *can be called the order of contact. This vector does not depend on parameterization.*

Conjecture 2 *For any two smooth manifolds having the order of contact* $l_1, ..., l_p$ *there exists a smooth change of coordinates such that locally one of the manifolds is an* $m-$*hyperplane* $u^1 = ... = u^p = 0$, $v^1 = \tau_1, ..., v^p = \tau_p, u^{p+1} = \tau_{p+1}, ..., u^{m-p} = \tau_m$ *and the other manifold locally is*

$$\left\{\begin{array}{lll} u^1 & = & a_1^1\tau_1 + ... + a_p^1\tau_p + B_1 \\ . & . & . \\ . & . & . \\ . & . & . \\ u^m & = & a_1^m\tau_1 + ... + a_p^m\tau_p + B_m \\ v^1 & = & a_1^{m+1}\tau_1^{l_1} + ... + a_p^{m+1}\tau_p^{l_1} \\ . & . & . \\ . & . & . \\ . & . & . \\ v^p & = & a_1^{m+p}\tau_1^{l_p} + ... + a_p^{m+p}\tau_p^{l_p}. \end{array}\right.$$

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Figure 1: Transverse and degenerate homoclinic crossings near the fixed point p.

Figure 2: The degenerate homoclinic crossing near the fixed point o.