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Exact Solution of the Bogoliubov Hamiltonian for Weakly Imperfect Bose-Gas

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Abstract

We show that the pressure for the Bogoliubov Weakly Imperfect Bose-Gas can be calculated exactly in the thermodynamic limit. We point out sufficient and necessary conditions that it does not coincide with the pressure for the Ideal Bose-Gas.

Keywords : Bogoliubov Weakly Imperfect Gas, Bose-condensation, Exact solution

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1. Introduction and set up of the problem

A pragmatic procedure for the description of the properties of superfluids, e.g. derivation of the experimentally observed spectra, was initiated in Bogoliubov's classical paper [1, 2], where he considered a Hamiltonian with truncated interaction, giving rise to what will be called the Bogoliubov Hamiltonian for a Weakly Imperfect Bose-Gas (WIBG).

Consider a system of bosons of mass m in a cubic box $\Lambda \subset \mathbb{R}^3$ of the volume $V = L^3$, with periodic boundary conditions. If $\varphi(x)$ denotes an integrable two-body interaction potential and

$$v(q) = \int_{\mathbb{R}^3} d^3x \varphi(x) e^{-iqx}, \ q \in \mathbb{R}^3, \tag{1.1}$$

then its second-quantized Hamiltonian acting in the boson Fock space $\mathcal{F}_\Lambda$ can be written as

$$H_\Lambda = \sum_k \varepsilon_k a_k^* a_k + \frac{1}{2V} \sum_{k_1,k_2,q} v(q)\, a_{k_1+q}^* a_{k_2-q}^* a_{k_1} a_{k_2} \tag{1.2}$$

where all sums run over the set Λ^* defined by

$$\Lambda^* = \left\{ k \in \mathbb{R}^3 : \alpha = 1,2,3,\ k_\alpha = \frac{2\pi n_\alpha}{L} \text{ et } n_\alpha = 0, \pm 1, \pm 2, ... \right\} \tag{1.3}$$

Here $\varepsilon_k = \hbar^2 k^2/2m$ is the kinetic energy, and $a_k^\# = \{a_k^*, a_k\}$ are the usual boson creation and annihilation operator in the one particle state $\psi_k(x) = V^{-\frac{1}{2}} e^{ikx}, k \in \Lambda^*,\ x \in \Lambda$; $a_k^* \equiv a^*(\psi_k) = \int_\Lambda dx \psi_k(x) a^*(x)$; $a^\#(x)$ are the basic boson operators in the Fock space over $L^2(\mathbb{R}^3)$.

If one expects that Bose-Einstein condensation, which occurs in the ideal Bose-gas for $k = 0$, persists for a weak interaction $\varphi(x)$ then, according to Bogoliubov, the most important terms in (1.2) should be those in which at least two operators a_0^*, a_0 appear. We are thus led to consider the following truncated Hamiltonian (the Bogoliubov Hamiltonian for WIBG, see [2], Part 3, eq. (3.81)) :

$$H_\Lambda^B = T_\Lambda + U_\Lambda^D + U_\Lambda \tag{1.4}$$

where

$$T_\Lambda = \sum_{k \in \Lambda^*} \varepsilon_k a_k^* a_k, \tag{1.5}$$

$$U_\Lambda^D = \frac{v(0)}{V} a_0^* a_0 \sum_{k \in \Lambda^*, k \neq 0} a_k^* a_k + \frac{1}{2V} \sum_{k \in \Lambda^*, k \neq 0} v(k)\, a_0^* a_0 \left(a_k^* a_k + a_{-k}^* a_{-k} \right) + \frac{v(0)}{2V} a_0^{*^2} a_0^2, \tag{1.6}$$

$$U_\Lambda = \frac{1}{2V} \sum_{k \in \Lambda^*, k \neq 0} v(k) \left(a_k^* a_{-k}^* a_0^2 + a_0^{*^2} a_k a_{-k} \right). \tag{1.7}$$

$H_\Lambda^{BD} \equiv \left(T_\Lambda + U_\Lambda^D \right)$ represents the diagonal part of the Bogoliubov Hamiltonian H_Λ^B while U_Λ represents the non-diagonal part.

Remark 1.1. *Below the following assumptions on the interaction potential* φ *are imposed :*
(A) $\varphi \in L^1(\mathbb{R}^3)$;
(B) $v(k)$ *is a real continuous function, satisfying* $v(0) > 0$ *and* $0 \leq v(k) = v(-k) \leq v(0)$ *for* $k \in \mathbb{R}^3$.
Under these (in fact, even under weaker) conditions it is known [3, 4] that φ *is superstable and hence that the grand-canonical partition function associated with the full Hamiltonian (1.2)*

$$\Xi_\Lambda(\beta, \mu) = Tr_{\mathcal{F}_\Lambda}\left(e^{-\beta(H_\Lambda - \mu N_\Lambda)}\right) \tag{1.8}$$

and the finite-volume pressure

$$p_\Lambda[H_\Lambda] \equiv p_\Lambda(\beta, \mu) = \frac{1}{\beta V} \ln \Xi_\Lambda(\beta, \mu) \tag{1.9}$$

are finite for all real μ *and all* $\beta > 0$.

However, it is not true for the Bogoliubov Hamiltonian (1.4) :

Proposition 1.2. *[5] Let* $\Xi_\Lambda^B(\beta, \mu)$ *be the grand-canonical partition function associated for the Hamiltonian (1.4). Then :*
(a) *the Bogoliubov model of WIBG is stable (* $\Xi_\Lambda^B(\beta, \mu) < +\infty$ *) for* $\mu \leq 0$ *and unstable for* $\mu > 0$.
(b) *for* $\mu \leq \mu^* = -\frac{1}{2}\varphi(0)$ *the pressure*

$$p^B(\beta, \mu) = \lim_\Lambda p_\Lambda\left[H_\Lambda^B\right] \tag{1.10}$$

coincides with the pressure of the Ideal Bose-Gas (IBG)

$$p^I(\beta, \mu) = \lim_\Lambda p_\Lambda[T_\Lambda]. \tag{1.11}$$

Moreover, the following conjecture was formulated in [5] :

Conjecture 1.3. *The Bogoliubov Hamiltonian* H_Λ^B *is exactly soluble in the sense that it is thermodynamically equivalent, in the grand-canonical ensemble, to the IBG for all chemical potential* $\mu \leq 0$*; which means precisely that*

$$p^B(\beta, \mu \leq 0) = p^I(\beta, \mu \leq 0). \tag{1.12}$$

The aim of the present paper is twofold :
- first, to show that the phase diagram of the Bogoliubov model (1.4) is less trivial than it is expressed by Conjecture 1.3;
- second, to calculate exactly $p^B(\beta, \mu)$ in domain where it does not coincide with $p^I(\beta, \mu)$.

Our results are organized as follows :
In the next Section 2, we show that

$$p^{BD}(\beta, \mu \leq 0) = \lim_\Lambda p_\Lambda\left[H_\Lambda^{BD}\right] = p^I(\beta, \mu \leq 0), \tag{1.13}$$

i.e. that thermodynamics of the diagonal part of the Bogoliubov Hamiltonian and that of the ideal gas coincide including Bose-condensation which occurs at $k=0$. This means in particular that the thermodynamic non-equivalence between Bogoliubov Hamiltonian and IBG is due to non-diagonal terms of interaction (1.7). Moreover, in this section, we study the Conjecture 1.3. First, we show that for any interaction which satisfies (A) and (B) there is a domain Γ of the phase diagram (plane $Q = \left(\mu \leq 0, \theta = \beta^{-1} \geq 0\right)$) where indeed

$$p^{B}\left(\beta, \mu \leq 0\right) = p^{I}\left(\beta, \mu \leq 0\right). \tag{1.14}$$

Then we formulate a sufficient condition on the interaction $v\left(k\right)$ to ensure the existence of domain $D_0 \subset Q$ where

$$p^{B}\left(\beta, \mu\right) \neq p^{I}\left(\beta, \mu\right). \tag{1.15}$$

In fact we show that this is equivalent to the statement that the system H_{Λ}^{B} manifests in this domain a Bose-condensation.

Thermodynamic limit of the pressure (1.10) of the system H_{Λ}^{B} in domain $D \supseteq D_0$ defined by

$$p^{B}\left(\beta, \mu\right) \neq p^{I}\left(\beta, \mu\right). \tag{1.16}$$

is studied on Section 3. We give an exact formula for $p^{B}\left(\beta, \mu\right)$ showing its relation to the concept of the Bogoliubov approximation à la Ginibre [4]. As a corollary we get that $D = D_0$. In Section 4 we study the breaking of the gauge symmetry and the behaviour of the Bose-condensate, i.e., the phase diagram of the WIBG. We reserve Section 5 for discussions and concluding remarks.

2. Bogoliubov weakly imperfect gas versus ideal Bose-gas

2.1. Diagonal part of the Bogoliubov Hamiltonian

The diagonal part of the Bogoliubov Hamiltonian $H_{\Lambda}^{BD} = \left(T_{\Lambda} + U_{\Lambda}^{D}\right)$ (1.5)-(1.6) can be rewritten using the occupation-number operators for modes $k \in \Lambda^{*}, n_k = a_k^{*}a_k$. So, the Hamiltonian $H_{\Lambda}^{BD}\left(\mu\right) \equiv H_{\Lambda}^{BD} - \mu N_{\Lambda}$ has the form

$$\begin{aligned} H_{\Lambda}^{BD}\left(\mu\right) &= \sum_{k\in\Lambda^{*}} \left(\varepsilon_k - \mu\right) a_k^{*}a_k + \frac{v\left(0\right)}{V} a_0^{*}a_0 \sum_{k\in\Lambda^{*}, k\neq 0} a_k^{*}a_k \\ &\quad + \frac{1}{2V} \sum_{k\in\Lambda^{*}, k\neq 0} v\left(k\right) a_0^{*}a_0 \left(a_k^{*}a_k + a_{-k}^{*}a_{-k}\right) + \frac{v\left(0\right)}{2V} a_0^{*^{2}} a_0^{2}, \end{aligned} \tag{2.1}$$

where $N_{\Lambda} = \sum_{k\in\Lambda^{*}} a_k^{*}a_k$. If $v\left(k\right)$ satisfies (B) then one obviously gets :

$$\begin{aligned} H_{\Lambda}^{BD}\left(\mu\right) &= \sum_{k\in\Lambda^{*}} \left(\varepsilon_k - \mu\right) n_k + \frac{v\left(0\right)}{V} n_0 N_{\Lambda} - \frac{v\left(0\right)}{2V}\left(n_0^{2} + n_0\right) \\ &\quad + \frac{1}{V} \sum_{k\in\Lambda^{*}, k\neq 0} v\left(k\right) n_0 n_k \qquad (2.2) \\ &\geq T_{\Lambda}\left(\mu\right) \equiv T_{\Lambda} - \mu N_{\Lambda}. \qquad (2.3) \end{aligned}$$

Theorem 2.1. *Let $v(k)$ satisfy (A) and (B). Then*
(a) *for any $\mu \leq 0$ and $\beta > 0$ one has*

$$p^{BD}(\beta,\mu) \equiv \lim_{\Lambda} p_\Lambda \left[H_\Lambda^{BD}\right] = p^I(\beta,\mu) \ , \tag{2.4}$$

(b) *for any $\beta > 0$ one has*

$$p^{BD}(\beta, \mu > 0) = +\infty$$

Proof :
(a) In virtue of representation (2.2) and inequality (2.3) we get that partition function

$$\Xi_\Lambda^{BD}(\beta,\mu) = Tr_{\mathcal{F}_\Lambda} e^{-\beta H_\Lambda^{BD}(\mu)} \leq Tr_{\mathcal{F}_\Lambda} e^{-\beta T_\Lambda(\mu)} = \Xi_\Lambda^I(\beta,\mu) \,.$$

Hence, for any $\mu < 0$

$$p_\Lambda\left[H_\Lambda^{BD}\right] \leq p_\Lambda\left[T_\Lambda\right]. \tag{2.5}$$

By (2.2), we can calculate $Tr_{\mathcal{F}_\Lambda}$ in the basis of occupation-number operators :

$$\Xi_\Lambda^{BD}(\beta,\mu) = \sum_{n_0=0}^{\infty} \left\{ e^{\left[-\beta\left(\frac{v(0)}{2V}\left(n_0^2-n_0\right)-\mu n_0\right)\right]} \prod_{k\in\Lambda^*, k\neq 0} \left(1 - e^{\left[-\beta\left(\varepsilon_k - \mu + \frac{[v(0)+v(k)]}{V} n_0\right)\right]}\right)^{-1} \right\},$$

which gives estimate

$$\Xi_\Lambda^{BD}(\beta,\mu) \geq \prod_{k\in\Lambda^*, k\neq 0} \left(1 - e^{[-\beta(\varepsilon_k-\mu)]}\right)^{-1}.$$

Therefore,

$$p_\Lambda\left[H_\Lambda^{BD}\right] \geq \widetilde{p}_\Lambda^I(\beta,\mu) \equiv \frac{1}{\beta V} \sum_{k\in\Lambda^*, k\neq 0} \ln\left[\left(1 - e^{[-\beta(\varepsilon_k-\mu)]}\right)^{-1}\right] \tag{2.6}$$

Note that $\widetilde{p}_\Lambda^I(\beta,\mu)$ is the pressure of an ideal Bose-gas with excluded mode $k = 0$ and $\widetilde{p}_\Lambda^I(\beta,\mu) < +\infty$ for $\mu < \inf_{k\neq 0} \varepsilon_k$. Hence, for any $\mu < 0$ one gets

$$\lim_{\Lambda} \widetilde{p}_\Lambda^I(\beta,\mu) = \lim_{\Lambda} p_\Lambda\left[T_\Lambda\right] = p^I(\beta,\mu) \tag{2.7}$$

Therefore, taking thermodynamic limit in (2.5), (2.6), by (2.7) we obtain (2.4) for $\mu < 0$. Then taking limit $\mu \to 0^-$ one gets (2.4) for $\mu = 0$.
(b) Follows directly from the estimate (2.6). ■

Corollary 2.2. *Since functions $\left\{p_\Lambda^{BD}(\beta,\mu) \equiv p_\Lambda\left[H_\Lambda^{BD}\right]\right\}_{\Lambda\subset\mathbb{R}^d}$ are convex for $\mu \leq 0$ and the limit $p^I(\beta,\mu)$ is differentiable for $\mu < 0$, by the Griffiths Lemma (see e.g. [6] and Appendix B)*

$$\lim_{\Lambda} \partial_\mu p_\Lambda^{BD}(\beta,\mu) = \partial_\mu p^I(\beta,\mu)\,,$$

i.e, the particle-density for the system (2.1) coincides with that for the ideal gas :

$$\rho^{BD}(\beta,\mu) \equiv \lim_{\Lambda} \left\langle \frac{N}{V} \right\rangle_{H_\Lambda^{BD}} (\beta,\mu) = \partial_\mu p^I(\beta,\mu) \equiv \rho^I(\beta,\mu) \tag{2.8}$$

Here $\langle - \rangle_{H_\Lambda}(\beta,\mu)$ corresponds to the grand-canonical Gibbs state for Hamiltonian H_Λ. Taking in (2.8) the limit $\mu \to 0^-$ we extend this equality to $\mu \in (-\infty, 0]$.

Resuming (2.4) and (2.8) we see that diagonal part of the Bogoliubov Hamiltonian H_Λ^{BD} is thermodynamically equivalent to T_Λ. The Bose-condensate in the system (2.1) has the same properties as in the IBG. Below we show that it is a non-diagonal interaction (1.7) that makes an essential difference between H_Λ^B and T_Λ.

2.2. Domain Γ : $p^B(\beta,\mu) = p^I(\beta,\mu)$

Similar to IBG the Bogoliubov WIBG exists only for $\mu \le 0$, see Proposition 1.2. In fact we can say more (cf. [5]).

Lemma 2.3. *For any $\mu \le 0$, one has*

$$p^I(\beta,\mu) \le p^B(\beta,\mu). \tag{2.9}$$

Proof :
By the Bogoliubov inequality (see e.g. [3, 7]), one gets :

$$\frac{1}{V}\langle U_\Lambda\rangle_{H_\Lambda^B} \le p_\Lambda\left[H_\Lambda^{BD}\right] - p_\Lambda\left[H_\Lambda^B\right] \le \frac{1}{V}\langle U_\Lambda\rangle_{H_\Lambda^{BD}}. \tag{2.10}$$

Since $\frac{1}{V}\langle U_\Lambda\rangle_{H_\Lambda^{BD}} = 0$, combining (2.6) and (2.10) we obtain in the thermodynamic limit (2.9). ■

Remark 2.4. *Let $v(k)$ satisfy (A) and (B). Then regrouping terms in (1.6), (1.7) one gets*

$$H_\Lambda^B = \widetilde{H}_\Lambda + \frac{1}{2V}\sum_{k\in\Lambda^*,k\neq 0} v(k)\left(a_0^*a_k + a_{-k}^*a_0\right)^*\left(a_0^*a_k + a_{-k}^*a_0\right) \ge \widetilde{H}_\Lambda\ , \tag{2.11}$$

where

$$\widetilde{H}_\Lambda = \sum_{k\in\Lambda^*,k\neq 0}\left(\varepsilon_k - \frac{v(k)}{2V} + \frac{v(0)}{V}n_0\right)n_k + \frac{v(0)}{2V}n_0^2 - \frac{1}{2}\varphi(0)\,n_0. \tag{2.12}$$

Hence, by the Bogoliubov inequality for (2.11) and for its diagonal part (2.12) we obtain in the thermodynamic limit for $\mu \le 0$

$$p^B(\beta,\mu) \le \lim_\Lambda p_\Lambda\left[\widetilde{H}_\Lambda\right] = \sup_{\rho_0\ge 0} G(\beta,\mu;\rho_0). \tag{2.13}$$

Here

$$G(\beta,\mu;\rho_0) \equiv \left[-\frac{v(0)}{2}\rho_0^2 + \left(\mu + \frac{1}{2}\varphi(0)\right)\rho_0 + p^I(\beta,\mu - v(0)\rho_0)\right]. \tag{2.14}$$

Corollary 2.5. *[5] If $\mu \le -\frac{1}{2}\varphi(0)$, then $\sup_{\rho_0\ge 0} G(\beta,\mu;\rho_0) = p^I(\beta,\mu)$.Therefore, by Lemma 2.3 and inequality (2.13) we get*

$$p^B(\beta,\mu) = p^I(\beta,\mu)\,,\ \textit{for}\ \Gamma_{\mu_*} = \left\{\theta \ge 0,\ \mu \le -\frac{1}{2}\varphi(0) \equiv \mu_*\right\}. \tag{2.15}$$

The next statement extends the domain Γ_{μ_*}, see Fig. 1.

Theorem 2.6. *Let $v(k)$ satisfy (A) and (B) and let*

$$h(z,\beta) \equiv z + \frac{v(0)}{(2\pi)^3} \int_{\mathbb{R}^3} d^3k \left(e^{[\beta(\varepsilon_k+z)]} - 1\right)^{-1}. \tag{2.16}$$

Then we have

$$p^B(\beta,\mu) = p^I(\beta,\mu), \text{ for } (\theta,\mu) \in \Gamma, \tag{2.17}$$

where

$$\Gamma = \left\{(\theta,\mu) : \frac{1}{2}\varphi(0) \le \inf_{z \ge -\mu} h(z,\beta)\right\} \subset Q. \tag{2.18}$$

Proof :
In virtue of (2.9) and (2.13), (2.14), the equality (2.17) is insured by

$$\sup_{\rho_0 \ge 0} G(\beta,\mu;\rho_0) = G(\beta,\mu;\rho_0 = 0). \tag{2.19}$$

If $\partial_{\rho_0} G(\beta,\mu;\rho_0) \le 0$ or equivalently $\frac{1}{2}\varphi(0) \le h(v(0)\rho_0 - \mu, \beta)$ for $\rho_0 \ge 0$, then sufficient condition (2.18) guarantees (2.19) and hence, (2.17). ■

Corollary 2.7. *Since $h(z,\beta)$ is a convex function of $z \ge 0$ and $h(z,\beta) \ge z$, then by (2.16) we get*

$$\lambda(\theta) \le \inf_{z \ge -\mu} h(z,\beta), \tag{2.20}$$

where

$$\lambda(\theta) \equiv \inf_{z \ge 0} h(z,\beta). \tag{2.21}$$

Therefore, by (2.20) we get a sufficient condition independent of $\mu \le 0$ (high-temperature domain, see Fig. 1) :

$$\Gamma_{\theta_*} = \left\{(\theta, \mu \le 0) : \frac{1}{2}\varphi(0) \le \lambda(\theta)\right\}, \tag{2.22}$$

which insures (2.17).

Remark 2.8. *Note that the inequality $h(z,\beta) \ge z$ and (2.18), for $-\mu \ge \frac{1}{2}\varphi(0)$ implies (2.15), i.e., $\Gamma_{\mu_*} \subset \Gamma$. On the other hand, (2.18) for $\mu = 0$ implies (2.22), i.e., $\Gamma_{\theta_*} \subset \Gamma$. See Fig. 1.*

Remark 2.9. *Since $\partial_{v(0)}\lambda(\theta) \ge 0$, one can always insure (2.22) for a fixed temperature θ, by increasing $v(0)$ without changing $\varphi(0)$.*

Remark 2.10. *Note that $p^I(\beta = +\infty, \mu) = 0$ and $\lambda(\theta = 0) = 0$. Therefore, at zero temperature the sufficient condition (2.18) reduces to (2.15), see Fig. 1. In fact this part of Γ is known since [5]. Theorem 2.6 shows that Conjecture 1.3 formulated there can be extended at least to domain Γ (2.18).*

Below we show that this conjecture is not valid in the complement $Q \backslash \Gamma$, see Fig. 1.

2.3. Domain D : $p^B(\beta,\mu) \neq p^I(\beta,\mu)$

Let $\mathcal{H}_{0\Lambda} \subset L^2(\Lambda)$ be the one-dimensional subspace generated by $\psi_{k=0}$, see Section 1. Then $\mathcal{F}_\Lambda \approx \mathcal{F}_{0\Lambda} \otimes \mathcal{F}_\Lambda'$ where $\mathcal{F}_{0\Lambda}$ and $\mathcal{F}_\Lambda'$ are the boson Fock spaces constructed out of $\mathcal{H}_{0\Lambda}$ and of its orthogonal complement $\mathcal{H}_{0\Lambda}^\perp$ respectively. For any complex $c \in \mathbb{C}$, we can define in $\mathcal{F}_{0\Lambda}$ a coherent vector

$$\psi_{0\Lambda}(c) = e^{-V|c|^2/2} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sqrt{V} c\right)^k (a_0^*)^k \Omega_0, \tag{2.23}$$

where Ω_0 is the vacuum of $\mathcal{F}_\Lambda$. Then $a_0 \psi_{0\Lambda}(c) = c\psi_{0\Lambda}(c)$.

Definition 2.11. *[4] The Bogoliubov approximation to a Hamiltonian $H_\Lambda(\mu) \equiv H_\Lambda - \mu N_\Lambda$ in $\mathcal{F}_\Lambda$ is the operator $H_\Lambda\left(c^\#, \mu\right)$ defined in $\mathcal{F}_\Lambda'$ by its quadratic form*

$$\left(\psi_1', H_\Lambda\left(c^\#\right) \psi_2'\right)_{\mathcal{F}_\Lambda'} = \left(\psi_{0\Lambda}(c) \otimes \psi_1', H_\Lambda \psi_{0\Lambda}(c) \otimes \psi_2'\right)_{\mathcal{F}_\Lambda} \tag{2.24}$$

for $\psi_{0\Lambda}(c) \otimes \psi_{1,2}'$ in the form domain of $H_\Lambda(\mu)$, where $c^\# = (c, \overline{c})$.

This formulation of the Bogoliubov approximation [1, 2] provides an estimate for the pressure $p_\Lambda\left[H_\Lambda^B\right]$ from below which allows to refine (2.9).

Proposition 2.12. *[5] For any $(\theta,\mu) \in Q$ one has*

$$\sup_{c\in\mathbb{C}} \widetilde{p}_\Lambda^B\left(\beta,\mu;c^\#\right) \leq p_\Lambda\left[H_\Lambda^B\right], \tag{2.25}$$

where

$$\widetilde{p}_\Lambda^B\left(\beta,\mu;c^\#\right) \equiv \frac{1}{\beta V} \ln Tr_{\mathcal{F}_\Lambda'} e^{-\beta H_\Lambda^B\left(c^\#,\mu\right)}. \tag{2.26}$$

Remark 2.13. *By Definition 2.11 we get from (1.4)-(1.7) that*

$$\begin{aligned} H_\Lambda^B\left(c^\#,\mu\right) &= \sum_{k\in\Lambda^*,k\neq 0} \left[\varepsilon_k - \mu + v(0)|c|^2\right] a_k^* a_k \\ &+ \frac{1}{2}\sum_{k\in\Lambda^*,k\neq 0} v(k)|c|^2 \left[a_k^* a_k + a_{-k}^* a_{-k}\right] \\ &+ \frac{1}{2}\sum_{k\in\Lambda^*,k\neq 0} v(k)\left[c^2 a_k^* a_{-k}^* + \overline{c}^2 a_k a_{-k}\right] \\ &-\mu|c|^2 V + \frac{1}{2} v(0)|c|^4 V. \end{aligned} \tag{2.27}$$

Therefore, after diagonalization one can calculate (2.26) in the explicit form :

$$\begin{aligned} &\widetilde{p}_\Lambda^B\left(\beta,\mu;c^\#\right) = \xi_\Lambda(\beta,\mu;x) + \eta_\Lambda(\mu;x), \\ &\xi_\Lambda(\beta,\mu;x) = \tfrac{1}{\beta V} \sum_{k\in\Lambda^*,k\neq 0} \ln\left(1 - e^{-\beta E_k}\right)^{-1}, \\ &\eta_\Lambda(\mu;x) = -\tfrac{1}{2V} \sum_{k\in\Lambda^*,k\neq 0} (E_k - f_k) + \mu x - \tfrac{1}{2} v(0) x^2, \end{aligned} \tag{2.28}$$

where E_k *and* f_k *are functions of* $x = |c|^2 \geq 0$ *and* $\mu \leq 0$ *:*

$$\begin{aligned} f_k &= \varepsilon_k - \mu + x\left[v(0) + v(k)\right], \\ h_k &= xv(k), \\ E_k &= \sqrt{f_k^2 - h_k^2}. \end{aligned} \tag{2.29}$$

Now the strategy of localization of domain D (see Fig. 1) gets clear : in virtue of (2.9) and (2.25) one has to $(\theta, \mu) \in Q$ such that

$$p^I(\beta, \mu) < \lim_{\Lambda} \left[\sup_{c \in \mathbb{C}} \widetilde{p}_{\Lambda}^B\left(\beta, \mu; c^{\#}\right)\right]. \tag{2.30}$$

Proposition 2.14. *[5] Let* $v(k)$ *satisfy (A), (B) and*

$$v(0) \geq \frac{1}{2(2\pi)^3} \int_{\mathbb{R}^3} d^3k \frac{[v(k)]^2}{\varepsilon_k}. \tag{2.31}$$

Then, cf. (2.6),

$$\sup_{c \in \mathbb{C}} \widetilde{p}_{\Lambda}^B\left(\beta, \mu; c^{\#}\right) = \widetilde{p}_{\Lambda}^B(\beta, \mu; 0) = \widetilde{p}_{\Lambda}^I(\beta, \mu).$$

Therefore, in the thermodynamic limit (see (2.7)) we get

$$\lim_{\Lambda} \left[\sup_{c \in \mathbb{C}} \widetilde{p}_{\Lambda}^B\left(\beta, \mu; c^{\#}\right)\right] = p^I(\beta, \mu). \tag{2.32}$$

Lemma 2.15. *Let* $v(k)$ *satisfy (A), (B) and (C) :*

$$v(0) < \frac{1}{2(2\pi)^3} \int_{\mathbb{R}^3} d^3k \frac{[v(k)]^2}{\varepsilon_k}. \tag{2.33}$$

Then, there is $\mu_0 < 0$ *such that*

$$\lim_{\Lambda} \left(\sup_{x \geq 0} \eta_{\Lambda}(\mu; x)\right) = \eta\left(\mu; \overline{x}(\mu) > 0\right) > 0 \text{ for } \mu \in (\mu_0, 0]. \tag{2.34}$$

Proof :
By the explicit formulae (2.28) and (2.29) we readily get that for $\mu \leq 0$:
(a) $\eta_{\Lambda}(\mu; x = 0) = 0$ and $\eta_{\Lambda}(\mu; x) \leq const - \frac{1}{2}v(0)\, x^2$;
(b) $\partial_x \eta_{\Lambda}(\mu; x = 0) = \mu$ and

$$\partial_x^2 \eta_{\Lambda}(\mu; x = 0) = \frac{1}{2V} \sum_{k \in \Lambda^*, k \neq 0} \frac{[v(k)]^2}{(\varepsilon_k - \mu)} - v(0).$$

Since

$$\lim_{\Lambda} \frac{1}{2V} \sum_{k \in \Lambda^*, k \neq 0} \frac{[v(k)]^2}{(\varepsilon_k - \mu)} = \frac{1}{2(2\pi)^3} \int_{\mathbb{R}^3} d^3k \frac{[v(k)]^2}{(\varepsilon_k - \mu)},$$

the condition (2.33) implies the existence of $\widetilde{\mu}_0 < 0$ such that

$$\lim_{\Lambda} \partial_x^2 \eta_\Lambda \left(\mu > \widetilde{\mu}_0; x = 0\right) > 0 \ .$$

In virtue of (a), (b), and $\lim_{\Lambda} \partial_x \eta_\Lambda \left(\mu = 0; x = 0\right) = 0$ this means that

$$\lim_{\Lambda} \left(\sup_{x \geq 0} \eta_\Lambda \left(\mu = 0; x\right) \right) = \eta \left(\mu = 0; \overline{x} \left(\mu = 0\right) > 0\right) > 0 \ . \tag{2.35}$$

Therefore, by continuity of (2.35) on the interval $(\widetilde{\mu}_0, 0]$ it follows the existence of $\mu_0 : \widetilde{\mu}_0 \leq \mu_0 < 0$, such that one has (2.34). ■

Theorem 2.16. *Let $v(k)$ satisfy (A), (B) and (C). Then, for any $\mu \in (\mu_0, 0]$, there is $\theta_0(\mu) > 0$ such that one has (see Fig. 1) :*

$$p^I(\beta, \mu) < p^B(\beta, \mu) \ , \ \textit{in } D_0 = \{(\theta, \mu) : \mu_0 < \mu \leq 0, 0 \leq \theta < \theta_0(\mu)\} \ , \tag{2.36}$$

where μ_0 is defined in Lemma 2.15 and domain D_0 coincides in fact with

$$D_0 \equiv \left\{ (\theta, \mu) : \ \lim_{\Lambda} \ \sup_{c \in \mathbb{C}} \widetilde{p}_\Lambda^B \left(\beta, \mu; c^{\#}\right) > p^I(\beta, \mu) \right\} .$$

Proof :
First we note that by (2.28)-(2.29) one has $\xi_\Lambda(\beta, \mu; x = 0) = \widetilde{p}_\Lambda^I(\beta, \mu)$ and

$$\begin{aligned} &(i) \ \ \partial_x \xi_\Lambda(\beta, \mu; x) \leq 0 \text{ and } \lim_{x \to +\infty} \xi_\Lambda(\beta, \mu; x) = 0, \text{ for any } \Lambda; \\ &(ii) \ \ \partial_\theta \xi_\Lambda(\beta, \mu; x) \geq 0 \text{ and } \lim_{\theta \to 0} \xi_\Lambda(\beta, \mu; x) = 0, \text{ for any } \Lambda. \end{aligned} \tag{2.37}$$

Next, by Lemma 2.15 for $\mu = \mu_0 < 0$ we have

$$\lim_{\Lambda} \left(\sup_{x \geq 0} \eta_\Lambda(\mu_0; x) \right) = \eta(\mu_0; 0) = \eta(\mu_0; \overline{x}(\mu_0) > 0) = 0 \ . \tag{2.38}$$

Hence, according to (2.37) and (2.38) one obtains :

$$\begin{aligned} (iii) \ \ \theta > 0 : \ \ \lim_{\Lambda} \left[\sup_{c \in \mathbb{C}} \widetilde{p}_\Lambda^B \left(\beta, \mu_0; c^{\#}\right) \right] &= \sup_{x \geq 0} \left[\xi(\beta, \mu_0; x) + \eta(\mu_0; x)\right] \\ &= \widetilde{p}^B \left(\beta, \mu_0; c^{\#} = 0\right) = p^I(\beta, \mu) \ , \end{aligned} \tag{2.39}$$

and by (2.37), (ii) and (2.38) we get :

$$\begin{aligned} &(i\nu) \ \ \theta = 0 : \ \ \lim_{\Lambda} \left[\sup_{c \in \mathbb{C}} \widetilde{p}_\Lambda^B \left(\beta = \infty, \mu_0; c^{\#}\right) \right] = \widetilde{p}^B \left(\beta = \infty, \mu_0; c^{\#} = 0\right) \\ &= \widetilde{p}^B \left(\beta = \infty, \mu_0; c^{\#}\right) \Big|_{|c|^2 = \overline{x}(\mu_0) > 0} = 0, \end{aligned}$$

see Fig. 1 and Fig. 2.

Now by (2.28), (2.37) and Lemma 2.15 one gets that for $\mu_0 < \mu \leq 0$

$$\lim_{\Lambda}\left[\sup_{c\in\mathbb{C}}\widetilde{p}_{\Lambda}^{B}\left(\beta,\mu>\mu_0;c^{\#}\right)\right]\geq\eta\left(\mu>\mu_0;\overline{x}\left(\mu\right)>0\right)>0. \tag{2.40}$$

Since by (2.37), (ii) the pressure $p^{I}\left(\beta,\mu\leq 0\right)$ is monotonously decreasing for $\theta\searrow 0$, there is a temperature $\theta_0\left(\mu\right)$ such that for $\theta<\theta_0\left(\mu>\mu_0\right)$ we get from (2.40)

$$\begin{aligned} p^{I}\left(\beta>\beta_0\left(\mu\right),\mu>\mu_0\right) &< \eta\left(\mu>\mu_0;\overline{x}\left(\mu\right)>0\right) \\ &< \lim_{\Lambda}\left[\sup_{c\in\mathbb{C}}\widetilde{p}_{\Lambda}^{B}\left(\beta>\beta_0\left(\mu\right),\mu>\mu_0;c^{\#}\right)\right]. \end{aligned} \tag{2.41}$$

Then (2.25) and (2.41) imply (2.36) for $(\theta,\mu)\in D_0$.■

Corollary 2.17. *Let*

$$D\equiv\left\{(\theta,\mu):p^{B}\left(\beta,\mu\right)>p^{I}\left(\beta,\mu\right)\right\}. \tag{2.42}$$

Then by (2.25) and (2.36) one obviously gets

$$D\supseteq D_0=\left\{(\theta,\mu):\mu_0<\mu\leq 0, 0\leq\theta<\theta_0\left(\mu\right)\right\}.$$

Here $\mu_0<0$ is defined in Lemma 2.15 and $\theta_0\left(\mu\right)$ in Theorem 2.16.

Remark 2.18. *The condition (2.33) is sufficient to guarantee that $\mu_0<0$, i.e., $D\supseteq D_0\neq\{\emptyset\}$. On the other hand, the contrary condition (2.31) implies only the triviality (2.32) of the lower bound (2.25) for $p^{B}\left(\beta,\mu\right)$ but not $D=\{\emptyset\}$, see Lemma 2.3.*

Therefore, for the moment we do not know whether condition (C) is necessary for $D\neq\{\emptyset\}$, we postpone the answer to this question to Section 3. Here we discuss a relation of the conditions (2.31),(2.33) which result from the rather restricted analysis of convexity and monotonicity of the $\widetilde{p}_{\Lambda}^{B}\left(\beta,\mu;c^{\#}\right)$ in the vicinity of $x=0$ and the condition (2.15) which gives triviality of the upper bound (2.13) for $p^{B}\left(\beta,\mu\right)$ for all temperatures, see Fig. 1.

Remark 2.19. *Let $v\left(k\right)$ satisfy (A), (B) and (C). Then there is $\widetilde{\mu}_0<0$ such that for $\mu\leq\widetilde{\mu}_0$ one has*

$$v\left(0\right)\geq\frac{1}{2\left(2\pi\right)^{3}}\int_{\mathbb{R}^{3}}\frac{\left[v\left(k\right)\right]^{2}}{\left(\varepsilon_{k}-\mu\right)}d^{3}k, \tag{2.43}$$

and by consequence $\partial_{x}^{2}\eta\left(\mu\leq\widetilde{\mu}_0;x=0\right)\leq 0$, see Lemma 2.15. One can represent the inequality (2.43) as

$$\int_{\mathbb{R}^{3}}\frac{d^{3}k}{\left(2\pi\right)^{3}}v\left(k\right)\left\{\frac{v\left(k\right)}{2\left(\varepsilon_{k}-\mu\right)}-\frac{v\left(0\right)}{\varphi\left(0\right)}\right\}\leq 0. \tag{2.44}$$

Since by (B) and $\mu\leq 0$

$$\frac{v\left(k\right)}{2\left(\varepsilon_{k}-\mu\right)}\leq\frac{v\left(0\right)}{\left(-2\mu\right)},$$

the condition $\mu<-\frac{1}{2}\varphi\left(0\right)\equiv\mu_{*}$ (2.15) implies (2.44), i.e., $\mu_{*}\leq\widetilde{\mu}_0$ see Figs. 1 and 2. Therefore, a local convexity condition (2.43) is intimately related to condition insuring $p^{B}\left(\beta,\mu\right)=p^{I}\left(\beta,\mu\right)$. In particular, one has to note that for condition (2.31) the inequality (2.43) is valid for any $\mu<0$.

We conclude this section by a new simple and important for below characterization of domain D (cf. (2.42)).

Theorem 2.20. *Let*

$$\rho_0^B(\beta,\mu) \equiv \lim_\Lambda \left\langle \frac{a_0^* a_0}{V} \right\rangle_{H_\Lambda^B} (\beta,\mu) \tag{2.45}$$

be density of the Bose-condensate for the Bogoliubov Hamiltonian (1.4). Then

$$D = \left\{ (\theta,\mu) \in Q : \ \rho_0^B(\beta,\mu) > 0 \right\}. \tag{2.46}$$

Proof :
Put

$$\widehat{H}_\Lambda^B \equiv H_\Lambda^B + \frac{1}{2}\varphi(0)\, a_0^* a_0. \tag{2.47}$$

Then by Remark 2.4 we get

$$\lim_\Lambda p_\Lambda\left[\widehat{H}_\Lambda^B\right] \leq \sup_{\rho_0 \geq 0} \left\{ G(\rho_0,\mu) - \frac{1}{2}\varphi(0)\,\rho_0 \right\} = p^I(\beta,\mu). \tag{2.48}$$

By the Bogoliubov inequality for H_Λ^B and $\widehat{H}_\Lambda^B$ one has

$$p_\Lambda\left[H_\Lambda^B\right] - \frac{\varphi(0)}{2} \left\langle \frac{a_0^* a_0}{V} \right\rangle_{H_\Lambda^B} \leq p_\Lambda\left[\widehat{H}_\Lambda^B\right]. \tag{2.49}$$

Now in virtue of (2.9), (2.48) and (2.49) we get in the thermodynamic limit that

$$p^I(\beta,\mu) - \frac{\varphi(0)}{2}\rho_0^B(\beta,\mu) \leq p^B(\beta,\mu) - \frac{\varphi(0)}{2}\rho_0^B(\beta,\mu) \leq p^I(\beta,\mu).$$

Therefore, $p^B(\beta,\mu) = p^I(\beta,\mu)$ if and only if $\rho_0^B(\beta,\mu) = 0$, which gives (2.46). ■

Remark 2.21. *The fact that $p^B(\beta,\mu) \neq p^I(\beta,\mu)$ only when $\rho_0^B(\beta,\mu) \neq 0$ is very similar to what is known since Bogoliubov model for superfluidity [1, 2]. An essential difference is that in the Bogoliubov model the gapless spectrum occurs for positive chemical potential $\mu = v(0)\rho$ where the system corresponding to the Bogoliubov Hamiltonian for WIBG is unstable, for further discussion see [5] and [8, 9].*

3. Exactness of the Bogoliubov approximation

Since the pressure $p^B(\beta,\mu) \neq p^I(\beta,\mu)$ only in domain D, where the Bose condensate $\rho_0^B(\beta,\mu) > 0$, the aim of this section is to identify $p^B(\beta,\mu)$ in this domain. Below we shall show that

$$\begin{aligned} p^B(\beta,\mu) &= \lim_\Lambda \left[\sup_{c\in\mathbb{C}} \widetilde{p}_\Lambda^B\left(\beta,\mu;c^\#\right) \right] \\ &= \sup_{c\in\mathbb{C}} \widetilde{p}^B\left(\beta,\mu;c^\#\right), \end{aligned} \tag{3.1}$$

and that in fact (cf. (2.36), (2.42)) one has

$$D = D_0. \tag{3.2}$$

Therefore, the condition (C) is necessary for $D \neq \{\emptyset\}$, cf. Remark 2.18. By definition of $\widetilde{p}^B\left(\beta,\mu;c^{\#}\right)$, see (2.25)-(2.28), the statement (3.1) means that the Bogoliubov approximation for the WIBG is exact. Since $\widetilde{p}^B_\Lambda\left(\beta,\mu;c^{\#}\right)$ is known explicitly the statement (3.1) gives the exact solution of thermodynamics of this model.
By results of the Section 2 it is non-diagonal part U_Λ (1.7) of the Bogoliubov Hamiltonian (1.4) that makes that $p^B\left(\beta,\mu\right) \neq p^I\left(\beta,\mu\right)$ in domain $D \neq \{\emptyset\}$ for (2.33). The interaction U_Λ is known to be effectively attractive [5]. Therefore, to prove (3.1) we use the Approximation Hamiltonian Method originally invented for quantum systems with attractive interactions, see e.g. [7].

Remark 3.1. *This method was adapted by Ginibre [4] to prove the exactness of the Bogoliubov approximation for non ideal Bose-gas (1.2) with superstable interaction, which is the case if $v(q)$ satisfies (B). But after truncation of (1.2) the Hamiltonian H^B_Λ (1.4) is not superstable. The system (1.4) is unstable for $\mu > 0$, Proposition 1.2. Below we follow the Approximation Hamiltonian Method à la Ginibre modified for the WIBG.*

Since in the approximating Hamiltonian $H^B_\Lambda\left(c^{\#},\mu\right)$ (2.27) the gauge symmetry is broken, we introduce

$$\begin{aligned} H^B_\Lambda\left(\nu^{\#}\right) &= H^B_\Lambda - \sqrt{V}\left(\overline{\nu}a_0 + \nu a_0^*\right), \\ H^B_\Lambda\left(\mu,\nu^{\#}\right) &= H^B_\Lambda\left(\nu^{\#}\right) - \mu N_\Lambda \end{aligned} \tag{3.3}$$

with sources $\nu \in \mathbb{C}$ breaking the symmetry of H^B_Λ, here $\nu^{\#} = (\nu,\overline{\nu})$. Then by Proposition 2.12 and the Bogoliubov inequality for $H^B_\Lambda\left(\mu,\nu^{\#}\right)$ and $H^B_\Lambda\left(c^{\#},\mu,\nu^{\#}\right)$ one gets :

$$\begin{aligned} 0 &\leq \Delta_\Lambda\left(\beta,\mu;c^{\#},\nu^{\#}\right) \equiv p_\Lambda\left[H^B_\Lambda\left(\nu^{\#}\right)\right] - \widetilde{p}^B_\Lambda\left(\beta,\mu;c^{\#},\nu^{\#}\right) \leq \\ &\leq \frac{1}{V}\left\langle H^B_\Lambda\left(c^{\#},\mu,\nu^{\#}\right) - H^B_\Lambda\left(\mu,\nu^{\#}\right)\right\rangle_{H^B_\Lambda\left(\nu^{\#}\right)}. \end{aligned} \tag{3.4}$$

Let $A \equiv a_0 - \sqrt{V}c, A^* \equiv a_0^* - \sqrt{V}\overline{c}$. Then a Taylor expansion around $a_0^{\#}$ gives :

$$\begin{aligned} H^B_\Lambda\left(c^{\#},\mu,\nu^{\#}\right) - H^B_\Lambda\left(\mu,\nu^{\#}\right) &= -A^*\left[a_0, H^B_\Lambda\left(\mu,\nu^{\#}\right)\right] + h.c. \\ &+\frac{1}{2}A^{*^2}\left[a_0,\left[a_0, H^B_\Lambda\left(\mu,\nu^{\#}\right)\right]\right] + h.c. \\ &+A^*\left[a_0,\left[H^B_\Lambda\left(\mu,\nu^{\#}\right), a_0^*\right]\right]A \\ &-\frac{1}{2}A^{*^2}\left[a_0,\left[a_0,\left[H^B_\Lambda\left(\mu,\nu^{\#}\right), a_0^*\right]\right]\right]A + h.c. \\ &+\frac{1}{4}A^{*^2}\left[a_0,\left[a_0,\left[\left[H^B_\Lambda\left(\mu,\nu^{\#}\right), a_0^*\right]a_0^*\right]\right]\right]A^2. \end{aligned} \tag{3.5}$$

Remark 3.2. *Explicit calculations show that the third and the fourth other terms in (3.5) are bounded from above :*

$$\begin{aligned} -\frac{v(0)}{\sqrt{V}}\left(\bar{c}A^*AA + cA^*A^*A\right) - \frac{v(0)}{2V}A^{*^2}A^2 &= 2v(0)\left|c\right|^2 A^*A - \\ -\frac{v(0)}{2V}\left(A^2 + 2\sqrt{V}cA\right)^*\left(A^2 + 2\sqrt{V}cA\right) &\leq 2v(0)\left|c\right|^2 A^*A. \end{aligned} \tag{3.6}$$

Remark 3.3. *After some manipulations the terms of the first and the second order in (3.5) can be combined in*

$$\begin{aligned} &-\frac{1}{2}\left[A^*A,\left[H_\Lambda^B\left(\mu,\nu^\#\right),A^*A\right]\right] + 2A^*\left[A,\left[H_\Lambda^B\left(\mu,\nu^\#\right),A^*\right]\right]A \\ &-\frac{3}{2}A^*\left[A,H_\Lambda^B\left(\mu,\nu^\#\right)\right] - \frac{3}{2}\left[H_\Lambda^B\left(\mu,\nu^\#\right),A^*\right]A. \end{aligned} \tag{3.7}$$

Lemma 3.4. *One has the following inequality :*

$$\left\langle\left[A^*A,\left[H_\Lambda^B\left(\mu,\nu^\#\right),A^*A\right]\right]\right\rangle_{H_\Lambda^B(\nu^\#)} \geq 0. \tag{3.8}$$

Proof :
Denote by $(.,.)_{H_\Lambda}$ a positive semi-definite scalar product with respect to a Hamiltonian H_Λ (see, e.g. [10]) :

$$(X,Y)_{H_\Lambda} \equiv \frac{1}{\beta\Xi_\Lambda(\beta,\mu)}\int_0^\beta d\tau Tr_{\mathcal{F}_\Lambda}\left(e^{-(\beta-\tau)H_\Lambda(\mu)}X^*e^{-\tau H_\Lambda(\mu)}Y\right). \tag{3.9}$$

Then $(\mathbf{1},Y)_{H_\Lambda} = \langle Y\rangle_{H_\Lambda}$ and

$$\beta\left(\left[X,H_\Lambda(\mu)\right],\left[X,H_\Lambda(\mu)\right]\right)_{H_\Lambda} = \left\langle\left[X,\left[H_\Lambda(\mu),X^*\right]\right]\right\rangle_{H_\Lambda}. \tag{3.10}$$

Applying (3.10) to $H_\Lambda(\mu) = H_\Lambda^B\left(\mu,\nu^\#\right)$ and $X = A^*A$ one gets (3.8). ■

Lemma 3.5. *One has the following estimate :*

$$\begin{aligned} -2\left\langle A^*\left[A,H_\Lambda^B\left(\mu,\nu^\#\right)\right]\right\rangle_{H_\Lambda^B(\nu^\#)} &\leq \left\langle\left[A^*,\left[H_\Lambda^B\left(\mu,\nu^\#\right),A\right]\right]\right\rangle_{H_\Lambda^B(\nu^\#)} \\ &+\left\langle\left[A^*,\left[H_\Lambda^B\left(\mu,\nu^\#\right),A\right]\right]^*\right\rangle_{H_\Lambda^B(\nu^\#)} \\ &+2\beta^{-1}\left\langle\{A,A^*\}\right\rangle_{H_\Lambda^B(\nu^\#)}, \end{aligned} \tag{3.11}$$

where $\{X,Y\} = XY + YX$.

Proof :
By the spectral decomposition of the Hamiltonian $\left(H_\Lambda^B\left(\mu,\nu^\#\right)\psi_n = E_n\psi_n\right)$ one gets

$$\begin{aligned} \left\langle\left\{A^*,\left[H_\Lambda^B\left(\mu,\nu^\#\right),A\right]\right\}\right\rangle_{H_\Lambda^B(\nu^\#)} &= \frac{1}{\Xi_\Lambda^B\left(\beta,\mu,\nu^\#\right)}\sum_{m,n}\left|\left(\psi_m,A\psi_n\right)\right|^2 \\ &\times\left(E_m - E_n\right)\left(e^{-\beta E_n} + e^{-\beta E_m}\right). \end{aligned} \tag{3.12}$$

Since

$$\frac{1}{2}\left(e^{x}+e^{y}\right)-\frac{1}{2}\left|e^{x}-e^{y}\right| \leq \frac{e^{x}-e^{y}}{x-y} \leq \frac{1}{2}\left(e^{x}+e^{y}\right), \tag{3.13}$$

one gets

$$\begin{aligned}
\beta\left(E_{m}-E_{n}\right)\left(e^{-\beta E_{n}}+e^{-\beta E_{m}}\right) &\leq 2\left(e^{-\beta E_{n}}-e^{-\beta E_{m}}\right) \\
&\quad +\beta\left(E_{m}-E_{n}\right)\left|e^{-\beta E_{n}}-e^{-\beta E_{m}}\right| \\
&\leq 2\left(e^{-\beta E_{n}}+e^{-\beta E_{m}}\right) \\
&\quad +\beta\left(E_{m}-E_{n}\right)\left(e^{-\beta E_{n}}-e^{-\beta E_{m}}\right). \qquad (3.14)
\end{aligned}$$

Inserting the estimate (3.14) into (3.12) we obtain

$$\begin{aligned}
\left\langle\left\{A^{*},\left[H_{\Lambda}^{B}\left(\mu, \nu^{\#}\right), A\right]\right\}\right\rangle_{H_{\Lambda}^{B}\left(\nu^{\#}\right)} &\leq 2 \beta^{-1}\left\langle A A^{*}+A^{*} A\right\rangle_{H_{\Lambda}^{B}\left(\nu^{\#}\right)} \\
&\quad +\left\langle\left[A^{*},\left[H_{\Lambda}^{B}\left(\mu, \nu^{\#}\right), A\right]\right]\right\rangle_{H_{\Lambda}^{B}\left(\nu^{\#}\right)}.
\end{aligned} \tag{3.15}$$

Note that

$$\begin{aligned}
-2\left\langle A^{*}\left[A, H_{\Lambda}^{B}\left(\mu, \nu^{\#}\right)\right]\right\rangle_{H_{\Lambda}^{B}\left(\nu^{\#}\right)} &= \left\langle\left[A^{*},\left[H_{\Lambda}^{B}\left(\mu, \nu^{\#}\right), A\right]\right]\right\rangle_{H_{\Lambda}^{B}\left(\nu^{\#}\right)} \\
&\quad +\left\langle\left\{A^{*},\left[H_{\Lambda}^{B}\left(\mu, \nu^{\#}\right), A\right]\right\}\right\rangle_{H_{\Lambda}^{B}\left(\nu^{\#}\right)}.
\end{aligned} \tag{3.16}$$

Then combining (3.15) and (3.16) one gets (3.11). ■

Corollary 3.6. *Since*

$$\left\langle A^{*}\left[A, H_{\Lambda}^{B}\left(\mu, \nu^{\#}\right)\right]\right\rangle_{H_{\Lambda}^{B}\left(\nu^{\#}\right)}=\left\langle\left[H_{\Lambda}^{B}\left(\mu, \nu^{\#}\right), A^{*}\right] A\right\rangle_{H_{\Lambda}^{B}\left(\nu^{\#}\right)},$$

by (3.11) the mean value of the last two terms of (3.7) is bounded from above :

$$\begin{aligned}
-3\left\langle A^{*}\left[A, H_{\Lambda}^{B}\left(\mu, \nu^{\#}\right)\right]\right\rangle_{H_{\Lambda}^{B}\left(\nu^{\#}\right)} &\leq \frac{3}{2}\left\langle\left[A^{*},\left[H_{\Lambda}^{B}\left(\mu, \nu^{\#}\right), A\right]\right]+h.c.\right\rangle_{H_{\Lambda}^{B}\left(\nu^{\#}\right)}+ \\
&\quad +3 \beta^{-1}\left\langle A A^{*}+A^{*} A\right\rangle_{H_{\Lambda}^{B}\left(\nu^{\#}\right)}. \qquad (3.17)
\end{aligned}$$

Since we are looking for the estimate of (3.5) (and consequently by of (3.7)) from above, inequalities (3.8) and (3.17) show that it rests to estimate the mean value only of the second term in (3.7). Here we formulate the result, the proof is postponed to Appendix A.

Theorem 3.7. *Let $(\theta, \mu) \in D$ Then in D there are non-negative functions*

$$\begin{aligned}
a&=a\left(\theta, \mu, \nu^{\#}\right), \\
b&=b\left(\theta, \mu, \nu^{\#}\right),
\end{aligned} \tag{3.18}$$

such that for $|\nu| \leq r_{0}$, $r_{0}>0$, one has :

$$\left\langle A^{*}\left[A,\left[H_{\Lambda}^{B}\left(\mu, \nu^{\#}\right), A^{*}\right]\right] A\right\rangle_{H_{\Lambda}^{B}\left(\nu^{\#}\right)} \leq a\left\langle A^{*} A\right\rangle_{H_{\Lambda}^{B}\left(\nu^{\#}\right)}+b. \tag{3.19}$$

To prove this statement we need the following lemata which have its own importance for below.

Lemma 3.8. *For $(\theta,\mu)\in Q$ and $\nu\in\mathbb{C}$ we have*

$$p_\Lambda\left[H_\Lambda^B\left(\nu^\#\right)\right]\le\widetilde{p}_\Lambda^I\left(\beta,\mu\right)+\left\{\frac{1}{\beta V}\sum_{n_0=0}^{\infty}e^{\frac{\beta}{2}\left[\left(\varphi(0)+2\right)n_0-v(0)n_0^2/V\right]}\right\}+\left|\nu\right|^2. \tag{3.20}$$

Proof :
By the inequality

$$-\sqrt{V}\left(\overline{\nu}a_0+\nu a_0^*\right)\ge -a_0^*a_0-\left|\nu\right|^2V,$$

it follows immediately from the estimate (cf. (2.11), (2.12))

$$H_\Lambda^B\left(\nu^\#\right)-\mu N_\Lambda\ge\sum_{k\in\Lambda^*,k\neq 0}\left(\varepsilon_k-\mu-\frac{v(k)}{2V}\right)n_k+\frac{v(0)}{2V}n_0^2-\left(\mu+\frac{1}{2}\varphi\left(0\right)+1\right)n_0-\left|\nu\right|^2V.$$

■

Corollary 3.9. *By (3.20), in the thermodynamic limit, one gets*

$$p^B\left(\beta,\mu;\nu^\#\right)\le p^I\left(\beta,\mu\right)+\frac{1}{2}\sup_{\rho\ge 0}\left[\left(\varphi\left(0\right)+2\right)\rho-v\left(0\right)\rho^2\right]+\left|\nu\right|^2 \tag{3.21}$$

for $(\theta,\mu)\in Q,\nu\in\mathbb{C}$.

Lemma 3.10. *For any $\mu<0$ and $\nu\in\mathbb{C}$ one has the estimate*

$$\left\langle\frac{N_\Lambda}{V}\right\rangle_{H_\Lambda^B\left(\nu^\#\right)}\le g_\Lambda\left(\beta,\mu;\nu^\#\right)<\infty. \tag{3.22}$$

Proof :
For any $\mu<0$ there is $\delta>0$ such that $\mu+\delta<0$. Then by the Bogoliubov inequality we obtain

$$\delta\left\langle\frac{N_\Lambda}{V}\right\rangle_{H_\Lambda^B\left(\nu^\#\right)}\le p_\Lambda\left[H_\Lambda^B\left(\nu^\#\right)-\delta N_\Lambda\right]-p_\Lambda\left[H_\Lambda^B\left(\nu^\#\right)\right]. \tag{3.23}$$

Therefore, by Lemma 3.8 one gets (3.22) for

$$g_\Lambda\left(\beta,\mu;\nu^\#\right)\equiv\frac{1}{\delta}\left(p_\Lambda^B\left(\beta,\mu+\delta;\nu^\#\right)-p_\Lambda^B\left(\beta,\mu;\nu^\#\right)\right). \tag{3.24}$$

■

Corollary 3.11. *In the thermodynamic limit (3.24) gives*

$$\begin{aligned}\rho^B\left(\beta,\mu;\nu^\#\right)&=\lim_\Lambda\left\langle\frac{N_\Lambda}{V}\right\rangle_{H_\Lambda^B\left(\nu^\#\right)}\le\frac{1}{\delta}\left(p^B\left(\beta,\mu+\delta;\nu^\#\right)-p^B\left(\beta,\mu;\nu^\#\right)\right)\\&\equiv g\left(\beta,\mu;\nu^\#\right).\end{aligned} \tag{3.25}$$

In fact, in domain D we have that

$$\rho^{B}\left(\beta,\mu;\nu^{\#}\right)=\partial_{\mu}p^{B}\left(\beta,\mu;\nu^{\#}\right);\mu<0,\nu\in\mathbb{C} \tag{3.26}$$

by the Griffiths lemma ([6] and Appendix B).

Corollary 3.12. *In virtue of (3.22) one obviously get :*

$$\begin{aligned}\left\langle\frac{a_0^*a_0}{V}\right\rangle_{H_\Lambda^B(\nu^\#)} &\leq g_\Lambda\left(\beta,\mu;\nu^{\#}\right),\\ \left|\left\langle\frac{a_0^*}{\sqrt{V}}\right\rangle_{H_\Lambda^B(\nu^\#)}\right| &= \left|\left\langle\frac{a_0^*}{\sqrt{V}}\right\rangle_{H_\Lambda^B(\nu^\#)}\right|\leq\sqrt{g_\Lambda\left(\beta,\mu;\nu^{\#}\right)}.\end{aligned} \tag{3.27}$$

Remark 3.13. *To optimize the estimate (3.4) we have to look for* $\sup_{c\in\mathbb{C}}\widetilde{p}_\Lambda^B\left(\beta,\mu;c^{\#},\nu^{\#}\right)$. *Since by Definition 2.11 and (3.3)*

$$H_\Lambda^B\left(c^{\#},\mu,\nu^{\#}\right)=H_\Lambda^B\left(c^{\#},\mu\right)-V\left(\nu\overline{c}+\overline{\nu}c\right)\geq H_\Lambda^B\left(c^{\#},\mu\right)-V\left(\left|\nu\right|^2\left|c\right|^2+1\right), \tag{3.28}$$

one gets by (2.28) that for any $(\theta,\mu)\in Q$ *and a fixed* $\nu^{\#}$ *there is* $A\geq 0$ *such that*

$$\widetilde{p}_\Lambda^B\left(\beta,\mu;c^{\#},\nu^{\#}\right)\leq A-\frac{1}{2}v\left(0\right)\left|c\right|^4. \tag{3.29}$$

Therefore, the optimal value of $|c|$ *is bounded by a positive constant* $M<\infty$.

Now we are in position to prove the main statement of this Section (see (3.1)) about exactness of the Bogoliubov approximation for the WIBG.

Theorem 3.14. *Let* $(\theta,\mu)\in D$. *Then*

$$\lim_{\Lambda}\left\{p_\Lambda^B\left(\beta,\mu,\nu^{\#}\right)-\sup_{c\in\mathbb{C}}\widetilde{p}_\Lambda^B\left(\beta,\mu;c^{\#},\nu^{\#}\right)\right\}=0 \tag{3.30}$$

for $|\nu|\leq r_0$, $r_0>0$.

Proof :
By inequality (3.4) one gets

$$\begin{aligned}0 &\leq \inf_{c\in\mathbb{C}}\Delta_\Lambda\left(\beta,\mu;c^{\#},\nu^{\#}\right)\equiv\Delta_\Lambda\left(\beta,\mu;\widehat{c}_\Lambda^{\#}\left(\beta,\mu,\nu^{\#}\right),\nu^{\#}\right)\\ &\leq \frac{1}{V}\left\langle H_\Lambda^B\left(c^{\#},\mu,\nu^{\#}\right)-H_\Lambda^B\left(\mu,\nu^{\#}\right)\right\rangle_{H_\Lambda^B(\nu^\#)}.\end{aligned} \tag{3.31}$$

In virtue of (3.5)-(3.7), estimates (3.6) (3.8), (3.11), (3.17), (3.19) and Remark 3.13, there are positive constants u and w independent of the volume V, such that

$$\begin{aligned}\frac{1}{V}\left\langle H_\Lambda^B\left(c^{\#},\mu,\nu^{\#}\right)-H_\Lambda^B\left(\mu,\nu^{\#}\right)\right\rangle_{H_\Lambda^B(\nu^\#)} &\leq u+\frac{w}{2}\times\\ &\left\langle\left\{\left(a_0^*-\sqrt{V}c^*\right),\left(a_0-\sqrt{V}c\right)\right\}\right\rangle_{H_\Lambda^B(\nu^\#)}.\end{aligned} \tag{3.32}$$

Put $c = \left\langle a_0/\sqrt{V}\right\rangle_{H_\Lambda^B(\nu^\#)}$ which is bounded, see (3.27). Then

$$\Delta_\Lambda\left(\beta,\mu;\widehat{c}_\Lambda^\#,\nu^\#\right) \leq \Delta_\Lambda\left(\beta,\mu;\left\langle a_0^\#/\sqrt{V}\right\rangle_{H_\Lambda^B(\nu^\#)},\nu^\#\right),$$

and estimates (3.31), (3.32) give

$$0 \leq \inf_{c\in\mathbb{C}} \Delta_\Lambda\left(\beta,\mu;c^\#,\nu^\#\right) \leq \frac{u}{V} + \frac{w}{2V}\left\langle\left\{\left(a_0^* - \langle a_0^*\rangle\right),\left(a_0 - \langle a_0\rangle\right)\right\}\right\rangle_{H_\Lambda^B(\nu^\#)}, \tag{3.33}$$

where for the shorthands $\left\langle a_0^\#\right\rangle \equiv \left\langle a_0^\#\right\rangle_{H_\Lambda^B(\nu^\#)}$. Let $\delta a_0^\# \equiv a_0^\# - \left\langle a_0^\#\right\rangle$. Then, by the Harris inequality [11] (see Appendix C) one gets

$$\frac{1}{2}\left\langle\left\{\delta a_0^*,\delta a_0\right\}\right\rangle_{H_\Lambda^B(\nu^\#)} \leq \left(\delta a_0^*,\delta a_0\right)_{H_\Lambda^B(\nu^\#)} + \frac{\beta}{12}\left\langle\left[\delta a_0^*,\left[H_\Lambda^B\left(\mu,\nu^\#\right),\delta a_0\right]\right]\right\rangle_{H_\Lambda^B(\nu^\#)}. \tag{3.34}$$

Since by (B) and Lemma 3.10 we have :

$$\begin{aligned}\left\langle\left[\delta a_0^*,\left[H_\Lambda^B\left(\mu,\nu^\#\right),\delta a_0\right]\right]\right\rangle_{H_\Lambda^B(\nu^\#)} &= \left\langle\frac{v(0)}{V}N_\Lambda - \mu + \frac{1}{V}\sum_{k\in\Lambda^*} v(k)\,a_k^*a_k\right\rangle_{H_\Lambda^B(\nu^\#)} \leq \\ &\leq 2v(0)\,g_\Lambda\left(\beta,\mu;\nu^\#\right) - \mu,\end{aligned} \tag{3.35}$$

by (3.25) and the uniform boundedness of $g_\Lambda\left(\beta,\mu;\nu^\#\right)$ on D for $|\nu| \leq r_0$ by g_0, the estimate (3.33) in this compact set gets the form :

$$0 \leq \inf_{c\in\mathbb{C}} \Delta_\Lambda\left(\beta,\mu;c^\#,\nu^\#\right) \leq \frac{1}{V}\left[\widetilde{u} + w\left(\delta a_0^*,\delta a_0\right)_{H_\Lambda^B(\nu^\#)}\right]. \tag{3.36}$$

Now we can proceed along the standard reasoning of the Approximation Hamiltonian Method, see [7].First we note that

$$\left(\delta a_0^*,\delta a_0\right)_{H_\Lambda^B(\nu^\#)} = \frac{1}{\beta}\partial_\nu\partial_{\overline{\nu}}p_\Lambda\left[H_\Lambda^B\left(\nu^\#\right)\right]. \tag{3.37}$$

By the (canonical) gauge transformation $a_0 \to a_0e^{i\varphi}$, $\varphi = \arg\nu$, one gets that in fact

$$p_\Lambda\left[H_\Lambda^B\left(\nu^\#\right)\right] = p_\Lambda^B\left(\beta,\mu;|\nu| \equiv r\right).$$

Then passing in (3.37) to polar coordinates (r,φ) we obtain :

$$\left(\delta a_0^*,\delta a_0\right)_{H_\Lambda^B(\nu^\#)} = \frac{1}{4\beta r}\partial_r\left(r\partial_r p_\Lambda^B\right). \tag{3.38}$$

Let $c = |c|\,e^{i\psi}$, $\psi = \arg c$. Then by (3.3), (3.4) one gets

$$\begin{aligned}\inf_{c\in\mathbb{C}} \Delta_\Lambda\left(\beta,\mu;c^\#,\nu^\#\right) &= \inf_{|c|,\psi} \Delta_\Lambda\left(\beta,\mu;|c|\,e^{\pm i\psi},re^{\pm i\varphi}\right) = \inf_{|c|} \widehat{\Delta}_\Lambda\left(\beta,\mu;|c|\,e^{\pm i\varphi},r\right) \\ &\equiv \inf_{|c|} \widetilde{\Delta}_\Lambda(r).\end{aligned} \tag{3.39}$$

Therefore, by (3.36)

$$\int\limits_{R}^{R+\varepsilon} r \inf_{|c|} \widetilde{\Delta}_\Lambda(r)\, dr \leq \frac{1}{V}\left\{\widetilde{u}\frac{(R+\varepsilon)^2-R^2}{2} + \frac{w}{4\beta}\left(r\partial_r p_\Lambda^B\right)\Big|_R^{R+\varepsilon}\right\}, \tag{3.40}$$

for $[R, R+\varepsilon] \subset [0, r_0]$. Note that by (3.27) we have

$$\partial_r p_\Lambda^B = 2\left|\left\langle a_0/\sqrt{V}\right\rangle_{H_\Lambda^B(\nu\#)}\right| \leq 2g_0^{\frac{1}{2}}, \ \ (\theta,\mu)\in D, |\nu| \leq r_0. \tag{3.41}$$

Therefore, (3.40) gets the form

$$\int\limits_{R}^{R+\varepsilon} r \inf_{|c|} \widetilde{\Delta}_\Lambda(r)\, dr \leq \frac{1}{V}\left\{\widetilde{u}\frac{(R+\varepsilon)^2-R^2}{2} + \frac{w}{2\beta} g_0^{\frac{1}{2}}(2R+\varepsilon)\right\}. \tag{3.42}$$

Since by Corollary 3.12 and Remark 3.13

$$\left|\partial_r \inf_{|c|} \widetilde{\Delta}_\Lambda(r)\right| \leq 2g_\Lambda^{\frac{1}{2}} + 2\left|\widehat{c}_\Lambda\right| \leq 2\left(g_0^{\frac{1}{2}} + M\right),$$

for $r \in [R, R+\varepsilon]$ we get :

$$\inf_{|c|} \widetilde{\Delta}_\Lambda(R) \leq \inf_{|c|} \widetilde{\Delta}_\Lambda(r) + 2(r-R)\left(g_0^{\frac{1}{2}} + M\right).$$

Hence,

$$\inf_{|c|} \widetilde{\Delta}_\Lambda(R)\frac{(R+\varepsilon)^2-R^2}{2} \leq \int\limits_{R}^{R+\varepsilon} r \inf_{|c|} \widetilde{\Delta}_\Lambda(r)\, dr + 2\left(g_0^{\frac{1}{2}} + M\right)\left(\frac{r^3}{3} - R\frac{r^2}{2}\right)\Big|_R^{R+\varepsilon}.$$

Then by (3.42) we obtain

$$\inf_{|c|} \widetilde{\Delta}_\Lambda(R) \leq \frac{1}{V}\left\{\widetilde{u} + \frac{w}{\beta} g_0^{\frac{1}{2}}\varepsilon^{-1}\right\} + \left(g_0^{\frac{1}{2}} + M\right)\varepsilon\frac{R+\frac{2}{3}\varepsilon}{R+\frac{1}{2}\varepsilon}. \tag{3.43}$$

Note that $\varepsilon > 0$ is still arbitrary. Minimizing the right-hand side of (3.43) one obtains that for large V the optimal value of $\varepsilon \sim 1/\sqrt{V}$. Hence, for $V \to \infty$ one gets from (3.43) the asymptotic estimate

$$0 \leq \inf_{c\in\mathbb{C}} \Delta_\Lambda\left(\beta,\mu; c^\#, \nu^\#\right) \leq \delta_\Lambda = const.\frac{1}{\sqrt{V}} \tag{3.44}$$

valid for $(\theta,\mu)\in D$ and $|\nu| \leq r_0$. ■

Corollary 3.15. *Let $(\theta,\mu)\in D$. Then, if one considers the Bogoliubov approximation for the density operator W_Λ*

$$W_\Lambda = e^{-\beta H_\Lambda^B(\mu,\nu^\#)},$$

we have

$$\lim_{\Lambda}\left\{p_{\Lambda}^{B}\left(\beta,\mu,\nu^{\#}\right)-\sup_{c\in\mathbb{C}}p_{\Lambda}^{B}\left(\beta,\mu;c^{\#},\nu^{\#}\right)\right\}=0, \tag{3.45}$$

where

$$p_{\Lambda}^{B}\left(\beta,\mu;c^{\#},\nu^{\#}\right)\equiv\frac{1}{\beta V}\ln Tr_{\mathcal{F}_{\Lambda}^{\prime}}W_{\Lambda}\left(c^{\#}\right), \tag{3.46}$$

for $|\nu|\leq r_0$, $r_0>0$.

Proof :
Using a product basis in $\mathcal{F}_{\Lambda}$ in calculating $Tr_{\mathcal{F}_{\Lambda}}(-)=Tr_{\mathcal{F}_{0\Lambda}\otimes\mathcal{F}_{\Lambda}^{\prime}}(-)$, one gets

$$Tr_{\mathcal{F}_{\Lambda}}(-)\geq\sup_{\{\psi_{n}^{\prime}\}_{n}}\sum_{n}\left(\psi_{0\Lambda}(c)\otimes\psi_{n}^{\prime},e^{\beta H_{\Lambda}^{B}\left(\mu,\nu^{\#}\right)}\psi_{0\Lambda}(c)\otimes\psi_{n}^{\prime}\right)\equiv Tr_{\mathcal{F}_{\Lambda}^{\prime}}W_{\Lambda}\left(c^{\#}\right),$$

where $\{\psi_{n}^{\prime}\}_{n}$ is an arbitrary orthonormal basis in $\mathcal{F}_{\Lambda}^{\prime}$. Now, whenever $\psi_{0\Lambda}(c)\otimes\psi_{n}^{\prime}$ are in the form domain of $H_{\Lambda}^{B}\left(\mu,\nu^{\#}\right)$, Peierl inequality [3] gives, by definition of $H_{\Lambda}^{B}\left(\mu,c^{\#},\nu^{\#}\right)$,

$$\left(\psi_{0\Lambda}(c)\otimes\psi_{n}^{\prime},e^{\beta H_{\Lambda}^{B}\left(\mu,\nu^{\#}\right)}\psi_{0\Lambda}(c)\otimes\psi_{n}^{\prime}\right)\geq e^{-\beta\left(\psi_{n}^{\prime},H_{\Lambda}^{B}\left(\mu,c^{\#},\nu^{\#}\right)\psi_{n}^{\prime}\right)}.$$

Therefore one can obtain

$$\widetilde{p}_{\Lambda}^{B}\left(\beta,\mu;c^{\#},\nu^{\#}\right)\leq p_{\Lambda}^{B}\left(\beta,\mu;c^{\#},\nu^{\#}\right)\leq p_{\Lambda}^{B}\left(\beta,\mu,\nu^{\#}\right), \tag{3.47}$$

from which one can deduce by Theorem 3.14 the thermodynamic limit (3.45). ∎

Corollary 3.16. *Since the variational pressure* $\widetilde{p}_{\Lambda}^{B}\left(\beta,\mu;c^{\#},\nu^{\#}\right)$ *is known in the explicit form (see (2.27)) :*

$$\widetilde{p}_{\Lambda}^{B}\left(\beta,\mu;c^{\#},\nu^{\#}\right)=\widetilde{p}_{\Lambda}^{B}\left(\beta,\mu;c^{\#}\right)+2|\nu||c|, \tag{3.48}$$

we get that the thermodynamic limits

$$\begin{aligned}\widetilde{p}^{B}\left(\beta,\mu;c^{\#},\nu^{\#}\right)&=\lim_{\Lambda}\widetilde{p}_{\Lambda}^{B}\left(\beta,\mu;c^{\#},\nu^{\#}\right),\\ \widetilde{p}^{B}\left(\beta,\mu;\widehat{c}^{\#}\left(\beta,\mu;\nu^{\#}\right),\nu^{\#}\right)&=\lim_{\Lambda}\left[\sup_{c\in\mathbb{C}}\widetilde{p}_{\Lambda}^{B}\left(\beta,\mu;c^{\#},\nu^{\#}\right)\right]\\ &=\sup_{c\in\mathbb{C}}\widetilde{p}^{B}\left(\beta,\mu;c^{\#},\nu^{\#}\right)\end{aligned} \tag{3.49}$$

exist. Then in virtue of the uniform estimate (3.44) we get

$$p^{B}\left(\beta,\mu;\nu^{\#}\right)=\lim_{\Lambda}p_{\Lambda}\left[H_{\Lambda}^{B}\left(\nu^{\#}\right)\right]=\sup_{c\in\mathbb{C}}\widetilde{p}^{B}\left(\beta,\mu;c^{\#},\nu^{\#}\right) \tag{3.50}$$

for $(\theta,\mu)\in D$, $|\nu|\leq r_0$ *and (cf. (3.1)) the limit* $|\nu|\rightarrow 0$ *:*

$$p^{B}(\beta,\mu)=\sup_{c\in\mathbb{C}}\widetilde{p}^{B}\left(\beta,\mu;c^{\#}\right). \tag{3.51}$$

Corollary 3.17. *Inequalities (2.25) and (2.30) give*

$$p^{I}\left(\beta,\mu\right)\leq\lim_{\Lambda}\left[\sup_{c\in\mathbb{C}}\widetilde{p}_{\Lambda}^{B}\left(\beta,\mu;c^{\#}\right)\right]\leq p^{B}\left(\beta,\mu\right).$$

Then definitions (2.36), (2.42) imply $D_0 \subseteq D$, whereas (3.30) implies that $D_0 = D$, which proves (3.2). Hence, we have

$$p^{B}\left(\beta,\mu\right)=\sup_{c\in\mathbb{C}}\widetilde{p}^{B}\left(\beta,\mu;c^{\#}\right)\ \text{ for }\left(\theta,\mu\right)\in Q\backslash\partial D. \tag{3.52}$$

Remark 3.18. *Since (2.28) implies that*

$$\widetilde{p}_{\Lambda}^{B}\left(\beta,\mu;c^{\#}=0\right)=p_{\Lambda}^{I}\left(\beta,\mu\right), \tag{3.53}$$

by (2.36), (2.46) and (3.2) we get

$$D_0=\left\{\left(\theta,\mu\right):\ \left|\widehat{c}\left(\beta,\mu;\nu\right)\right|>0\right\}=\left\{\left(\theta,\mu\right):\ \rho_0^{B}\left(\beta,\mu\right)>0\right\}=D. \tag{3.54}$$

Therefore, (see Remark 2.18) the condition (C) is sufficient and necessary for $D\neq\{\emptyset\}$.

4. Thermodynamics of the Weakly Imperfect Bose-Gas

Since the pressure $\widetilde{p}_{\Lambda}^{B}$ (2.28) and $\lim_{\Lambda}\widetilde{p}_{\Lambda}^{B}=\widetilde{p}^{B}$ are known explicitly :

$$\begin{aligned}\widetilde{p}^{B}\left(\beta,\mu;c^{\#},\nu^{\#}\right) &= \frac{1}{\beta\left(2\pi\right)^{3}}\int\limits_{\mathbb{R}^{3}}d^{3}k\ln\left(1-e^{-\beta E_{k}\left(|c|^{2}\right)}\right)^{-1}\\ &\quad-\frac{1}{\beta\left(2\pi\right)^{3}}\int\limits_{\mathbb{R}^{3}}d^{3}k\left[E_{k}\left(|c|^{2}\right)-f_{k}\left(|c|^{2}\right)\right]\\ &\quad+\mu\left|c\right|^{2}-\frac{1}{2}v\left(0\right)\left|c\right|^{4}+\left(\nu\overline{c}+\overline{\nu}c\right),\end{aligned} \tag{4.1}$$

Theorem 3.14 and Corollaries 3.16, 3.17 give exact solution of the model (1.4) on the level of thermodynamics. Therefore, (3.52) gives access to thermodynamic properties of the Weakly Imperfect Bose-Gas for all $(\theta,\mu)\in Q$ except the line of transitions ∂D, see Fig. 1, and Fig. 3. The aim of this section is to discuss thermodynamic properties of the model (1.4) and in particular the Bose-condensate which appears in domain D. The first statement concerns the gauge-symmetry breaking in domain D.

Theorem 4.1. *Let $D\neq\{\emptyset\}$. Then quasi-averages*

$$\lim_{\{\nu\to0:\arg\nu=\varphi\}}\lim_{\Lambda}\left\langle a_0^{\#}/\sqrt{V}\right\rangle_{H_{\Lambda}^{B}\left(\nu^{\#}\right)}=e^{\pm i\varphi}\left|\widehat{c}\left(\beta,\mu\right)\right|=\left\{\begin{array}{c}\neq0,\left(\theta,\mu\right)\in D\\ =0,\left(\theta,\mu\right)\notin D\end{array}\right\} \tag{4.2}$$

Proof :
As in the proof of Theorem 3.14 by the gauge transformation

$$\mathcal{U}_\varphi a_0 \mathcal{U}_\varphi^* = a_0 e^{-i\varphi} = \widetilde{a}_0,\ \varphi = \arg\nu$$

we get

$$\begin{aligned}\widetilde{H}_\Lambda^B(\mu,r) &= \mathcal{U}_\varphi H_\Lambda^B\left(\mu,\nu^\#\right)\mathcal{U}_\varphi^* = \widetilde{H}_\Lambda^B - \mu\widetilde{N}_\Lambda - \sqrt{V}r\left(\widetilde{a}_0+\widetilde{a}_0^*\right),\\ p_\Lambda\left[H_\Lambda^B\left(\nu^\#\right)\right] &= p_\Lambda\left[\mathcal{U}_\varphi H_\Lambda^B\left(\mu,\nu^\#\right)\mathcal{U}_\varphi^*\right] = p_\Lambda^B\left(\beta,\mu;r=|\nu|\right).\end{aligned} \tag{4.3}$$

In virtue of

$$0 = \left\langle\left[\widetilde{H}_\Lambda^B(\mu,r),\widetilde{N}_\Lambda\right]\right\rangle_{\widetilde{H}_\Lambda^B(r)} = r\sqrt{V}\left\langle\widetilde{a}_0-\widetilde{a}_0^*\right\rangle_{\widetilde{H}_\Lambda^B(r)},$$

and (cf. (3.10))

$$0 \le \left\langle\left[\widetilde{N}_\Lambda,\left[\widetilde{H}_\Lambda^B(\mu,r),\widetilde{N}_\Lambda\right]\right]\right\rangle_{\widetilde{H}_\Lambda^B(r)} = r\sqrt{V}\left\langle\widetilde{a}_0+\widetilde{a}_0^*\right\rangle_{\widetilde{H}_\Lambda^B(r)},$$

we get that

$$\left\langle\widetilde{a}_0\right\rangle_{\widetilde{H}_\Lambda^B(r)} = \left\langle\widetilde{a}_0^*\right\rangle_{\widetilde{H}_\Lambda^B(r)} \ge 0. \tag{4.4}$$

Since (cf. (3.9))

$$\begin{aligned}&\partial_r^2 p_\Lambda^B(\beta,\mu;r) = \beta\left(\left\{(\widetilde{a}_0+\widetilde{a}_0^*)-\langle\widetilde{a}_0+\widetilde{a}_0^*\rangle_{\widetilde{H}_\Lambda^B(r)}\right\},\left\{(\widetilde{a}_0+\widetilde{a}_0^*)-\langle\widetilde{a}_0+\widetilde{a}_0^*\rangle_{\widetilde{H}_\Lambda^B(r)}\right\}\right)_{\widetilde{H}_\Lambda^B(r)}\\ &\ge 0\end{aligned} \tag{4.5}$$

by the Theorem 3.14 and Corollary 3.16 the sequence of the convex (for $r \ge 0$) functions $\left\{p_\Lambda^B(\beta,\mu;r)\right\}_\Lambda$ converges to the (convex function)

$$\begin{aligned}\widehat{p}^B(\beta,\mu;r) &\equiv \sup_{c\in\mathbb{C}}\widetilde{p}^B\left(\beta,\mu;c^\#,\nu^\#\right) = \sup_{\substack{|c|\ge 0\\ \psi=\arg c}}\widetilde{p}^B\left(\beta,\mu;|c|\,e^{\pm i\psi},|\nu|\,e^{\pm i\varphi}\right)\\ &= \widetilde{p}^B\left(\beta,\mu;|\widehat{c}(\beta,\mu;r)|\,e^{\pm i\varphi},|\nu|\,e^{\pm i\varphi}\right),\end{aligned} \tag{4.6}$$

see (3.38) and (4.1), uniformly on $D\times[0,r_0]$. By explicit calculations one gets that derivatives

$$\begin{aligned}0 &\le \partial_r\widehat{p}^B(\beta,\mu;r) = 2\left|\widehat{c}(\beta,\mu;r)\right| \le C_1,\\ 0 &\le \partial_r^2\widehat{p}^B(\beta,\mu;r) = 2\partial_r\left|\widehat{c}(\beta,\mu;r)\right| \le C_2\end{aligned} \tag{4.7}$$

are continuous and bounded in $D\times[0,r_0]$. Therefore, by the Griffiths lemma (see e.g. [6] and Appendix B)

$$\lim_\Lambda \partial_r p_\Lambda\left[\widetilde{H}_\Lambda^B(r)\right] = \lim_\Lambda\left\langle\frac{\widetilde{a}_0+\widetilde{a}_0^*}{\sqrt{V}}\right\rangle_{\widetilde{H}_\Lambda^B(r)} = 2\left|\widehat{c}(\beta,\mu;r)\right|,$$

or by (4.4),

$$\begin{aligned}\lim_\Lambda\left\langle\widetilde{a}_0/\sqrt{V}\right\rangle_{\widetilde{H}_\Lambda^B(r)} &= \left|\widehat{c}(\beta,\mu;r)\right|,\\ \lim_\Lambda\left\langle\widetilde{a}_0^*/\sqrt{V}\right\rangle_{\widetilde{H}_\Lambda^B(r)} &= \left|\widehat{c}(\beta,\mu;r)\right|.\end{aligned} \tag{4.8}$$

Returning in (4.8) back to original creation/annihilation operators, one gets

$$\begin{aligned}\lim_{\Lambda}\left\langle a_0/\sqrt{V}\right\rangle_{H_\Lambda^B(\nu^\#)} &= e^{+i\varphi}\left|\widehat{c}\,(\beta,\mu;r)\right|,\\ \lim_{\Lambda}\left\langle a_0^*/\sqrt{V}\right\rangle_{H_\Lambda^B(\nu^\#)} &= e^{-i\varphi}\left|\widehat{c}\,(\beta,\mu;r)\right|.\end{aligned}\tag{4.9}$$

Then the first part of the statement (4.2) follow from (4.9) and the continuity of the solution $\widehat{c}\,(\beta,\mu;r)$ at $r=0$. Whereas the second part follows from (3.54). ■

Corollary 4.2. *Note that by the gauge invariance*

$$\left\langle \frac{a_0^\#}{\sqrt{V}}\right\rangle_{H_\Lambda^B(\nu^\#=0)} = 0.\tag{4.10}$$

Therefore, we have the gauge-symmetry breaking :

$$\lim_{\nu\to 0}\lim_{\Lambda}\left\langle \frac{a_0^\#}{\sqrt{V}}\right\rangle_{H_\Lambda^B(\nu^\#)} \neq \lim_{\Lambda}\lim_{\nu\to 0}\left\langle \frac{a_0^\#}{\sqrt{V}}\right\rangle_{H_\Lambda^B(\nu^\#)},\tag{4.11}$$

as soon as the Bose-condensation $\rho_0^B(\beta,\mu)\neq 0$.

Corollary 4.3. *Since by (4.5), (4.7)*

$$\partial_r^2\left(\inf_{|c|}\widetilde{\Delta}_\Lambda(r)\right) = \partial_r^2\left(p_\Lambda^B(\beta,\mu;r)-\widehat{p}^B(\beta,\mu;r)\right)\geq -C_2,$$

the Kolmogorov lemma (see [12] and Appendix B) implies that

$$\left|\left\langle\frac{\widetilde{a}_0}{\sqrt{V}}\right\rangle_{\widetilde{H}_\Lambda^B(r)} - \left|\widehat{c}_\Lambda(\beta,\mu;r)\right|\right| \leq 2\sqrt{\delta_\Lambda C_2}\tag{4.12}$$

for $r\in[l_\Lambda, r_0-l_\Lambda]$, $l_\Lambda = 2\sqrt{\delta_\Lambda/C_2}$, see (3.43) and (4.8).

Note that the Cauchy-Shwartz inequality gives

$$\left\langle\frac{a_0^*}{\sqrt{V}}\right\rangle_{H_\Lambda^B(\nu^\#)}\left\langle\frac{a_0}{\sqrt{V}}\right\rangle_{H_\Lambda^B(\nu^\#)} \leq \left\langle\frac{a_0^* a_0}{V}\right\rangle_{H_\Lambda^B(\nu^\#)}.$$

Hence, by (2.45) and (4.2) one gets

$$\left|\widehat{c}_\Lambda(\beta,\mu)\right|^2 \leq \lim_{\nu\to 0}\lim_{\Lambda}\left\langle\frac{a_0^* a_0}{V}\right\rangle_{H_\Lambda^B(\nu^\#)} = \rho_0^B(\beta,\mu),\tag{4.13}$$

which is in coherence with definitions of domains D_0 and D, cf. Theorem 2.16 and Corollary 2.17. To prove equality in (4.13) we proceed as follows.

Theorem 4.4. *Let*

$$\begin{array}{l} H^B_{\Lambda,\alpha} = H^B_\Lambda + \alpha a^*_0 a_0 \\ H^B_{\Lambda,\alpha}\left(\nu^{\#}\right) = H^B_{\Lambda,\alpha} - \sqrt{V}\left(\nu a^*_0 + \overline{\nu} a_0\right) \end{array} \tag{4.14}$$

for $\alpha \in \mathbb{R}^1$. Then

$$p^B_\alpha\left(\beta, \mu; \nu^{\#}\right) = \lim_\Lambda p_\Lambda\left[H^B_{\Lambda,\alpha}\left(\nu^{\#}\right)\right] = \lim_\Lambda \left[\sup_{c\in\mathbb{C}} \widetilde{p}^B_{\Lambda,\alpha}\left(\beta, \mu; c^{\#}, \nu^{\#}\right)\right], \tag{4.15}$$

for $|\nu| \leq r_0, r_0 > 0$ and $(\theta, \mu) \in Q \backslash \partial D_\alpha$ where domain

$$D_\alpha \equiv \left\{(\theta, \mu) : \; p^B_\alpha\left(\beta, \mu; \nu^{\#} = 0\right) > p^I(\beta, \mu)\right\}. \tag{4.16}$$

Remark 4.5. *Since $H^B_{\Lambda,\alpha=\frac{1}{2}\varphi(0)} = \widehat{H}^B_\Lambda$, see (2.47), by the Theorem 2.20 we get that $D_{\alpha=\frac{1}{2}\varphi(0)} = \{\emptyset\}$.*

Our reasoning below is a translation of some results of Sections 2 and 3 to the perturbed Hamiltonian $H^B_{\Lambda,\alpha}$ for small α.

Lemma 4.6. *If potential $v(k)$ satisfies (A), (B) and (C), then*

$$D_{0\alpha} \equiv \left\{(\theta, \mu) : \; \sup_{c\in\mathbb{C}} \widetilde{p}^B_\alpha\left(\beta, \mu; c^{\#}\right) > p^I(\beta, \mu)\right\} \neq \{\emptyset\}. \tag{4.17}$$

for $\alpha < -\mu_0$, where μ_0 is defined in Lemma 2.15.

Proof :
Since the $\eta_{\Lambda,\alpha}(\mu; x)$ for the Hamiltonian (4.14) (cf. (2.28)) has the form

$$\eta_{\Lambda,}(\mu; x) = -\frac{1}{2V} \sum_{k\in\Lambda^*, k\neq 0} (E_k - f_k) + (\mu - \alpha)x - \frac{1}{2}v(0)x^2, \tag{4.18}$$

one can follow the line reasoning of Lemma 2.15 and Theorem 2.16 to get (4.17) for $\mu \leq 0$ such that $(\mu - \alpha) > \mu_0$. Therefore, the value of $\mu_0 + \alpha$ must be negative. ■
By continuity of (4.18) which respect to α it is clear that $\lim_{\alpha\to 0} D_{0\alpha} = D_0$. Now we turn to the
Proof of Theorem 4.4 :
(1) Since the Bogoliubov approximation (2.24) gives the estimate of the pressure $p_\Lambda\left[H^B_{\Lambda,\alpha}\left(\nu^{\#}\right)\right]$ from below (see Proposition 2.12) :

$$\sup_{c\in\mathbb{C}} \widetilde{p}^B_{\Lambda,\alpha}\left(\beta, \mu; c^{\#}, \nu^{\#}\right) \leq p_\Lambda\left[H^B_{\Lambda,\alpha}\left(\nu^{\#}\right)\right],$$

by the Bogoliubov inequality we get (cf. (3.4))

$$\begin{aligned} 0 &\leq \Delta_{\Lambda,\alpha}\left(\beta, \mu; c^{\#}, \nu^{\#}\right) \equiv p_\Lambda\left[H^B_{\Lambda,\alpha}\left(\nu^{\#}\right)\right] - \widetilde{p}^B_{\Lambda,\alpha}\left(\beta, \mu; c^{\#}, \nu^{\#}\right) \\ &\leq \frac{1}{V}\left\langle H^B_{\Lambda,\alpha}\left(\widehat{c}^{\#}, \mu, \nu^{\#}\right) - H^B_{\Lambda,\alpha}\left(\mu, \nu^{\#}\right)\right\rangle_{H^B_{\Lambda,\alpha}(\nu^{\#})}. \end{aligned} \tag{4.19}$$

(2) For operators $A^{\#}\equiv a_0^{\#}-\sqrt{V}c^{\#}$ and for a Taylor expansion of $H_{\Lambda,\alpha}^{B}\left(\widehat{c}^{\#},\mu,\nu^{\#}\right)$ around $a_0^{\#}$ one gets the estimate

$$\begin{aligned}0 \;\le\; & \inf_{c\in\mathbb{C}}\Delta_{\Lambda,\alpha}\left(\beta,\mu;c^{\#},\nu^{\#}\right)=\Delta_{\Lambda,\alpha}\left(\beta,\mu;\widehat{c}_{\Lambda,\alpha}^{\#}\left(\beta,\mu;\nu^{\#}\right),\nu^{\#}\right)\le u_\alpha+\\ & +\frac{w_\alpha}{2}\left\langle\left\{\left(a_0^{*}-\sqrt{V}\overline{c}\right),\left(a_0-\sqrt{V}c\right)\right\}\right\rangle_{H_\Lambda^B\left(\nu^{\#}\right)}.\end{aligned}\tag{4.20}$$

by carrying through verbatim the arguments developed starting on Remark 3.2 and finishing by Remark 3.13. The only difference with the case $\alpha=0$ comes from

$$\left[A,\left[H_{\Lambda,\alpha}^{B}\left(\mu,\nu^{\#}\right),A^{*}\right]\right]=\left[A,\left[H_{\Lambda}^{B}\left(\mu,\nu^{\#}\right),A^{*}\right]\right]+\alpha,$$

cf. (3.35) and the note that $\lim_{\alpha\to0}u_\alpha=u$ and $\lim_{\alpha\to0}w_\alpha=w$.

(3) Put $c^{\#}=\left\langle a_0^{\#}/\sqrt{V}\right\rangle_{H_{\Lambda,\alpha}^{B}\left(\nu^{\#}\right)}$ in the left-hand side of (4.20). The same line of reasoning as in Theorem 3.14 gives asymptotic estimate

$$0\le\inf_{c\in\mathbb{C}}\Delta_{\Lambda,\alpha}\left(\beta,\mu;c^{\#},\nu^{\#}\right)\le\delta_{\Lambda,\alpha}=const.\frac{1}{\sqrt{V}}\tag{4.21}$$

valid for $(\theta,\mu)\in Q\backslash\partial D_\alpha$, $|\alpha|<-\mu_0$, and $|\nu|\le r_0$ which ensures the proof of (4.15) for $D_\alpha\neq\{\emptyset\}$. ∎

Corollary 4.7. *Since*

$$\begin{aligned}\partial_\alpha^2 p_\Lambda\left[H_{\Lambda,\alpha}^{B}\left(\nu^{\#}\right)\right] &= \frac{\beta}{V}\left(\left(a_0^{*}a_0-\langle a_0^{*}a_0\rangle_{H_{\Lambda,\alpha}^{B}\left(\nu^{\#}\right)}\right),\left(a_0^{*}a_0-\langle a_0^{*}a_0\rangle_{H_{\Lambda,\alpha}^{B}\left(\nu^{\#}\right)}\right)\right)_{H_{\Lambda,\alpha}^{B}\left(\nu^{\#}\right)}\\ &\ge 0,\end{aligned}$$

functions $\left\{p_\Lambda\left[H_{\Lambda,\alpha}^{B}\left(\nu^{\#}=0\right)\right]\right\}_\Lambda$ *are convex for* $\alpha\in\mathbb{R}^1$. *The same is obviously true for (cf. (4.1), (4.14) and (4.15))*

$$\begin{aligned}\lim_{\Lambda}p_\Lambda\left[H_{\Lambda,\alpha}^{B}\left(\nu^{\#}=0\right)\right] &= \sup_{c\in\mathbb{C}}\widetilde{p}_\alpha^{B}\left(\beta,\mu;c^{\#},\nu^{\#}=0\right)\\ &= \widetilde{p}_\alpha^{B}\left(\beta,\mu;\widehat{c}_\alpha^{\#}\left(\beta,\mu\right),0\right)\\ &= \widetilde{p}^{B}\left(\beta,\mu;\widehat{c}_\alpha^{\#}\left(\beta,\mu\right),0\right)-\alpha\left|\widehat{c}_\alpha\left(\beta,\mu\right)\right|^2.\end{aligned}\tag{4.22}$$

By explicit calculations one gets that

$$\partial_\alpha\widetilde{p}_\alpha^{B}\left(\beta,\mu;\widehat{c}_\alpha^{\#}\left(\beta,\mu\right),0\right)=-\left|\widehat{c}_\alpha\left(\beta,\mu\right)\right|^2<const\tag{4.23}$$

for $(\theta,\mu)\in Q$ *and* $|\alpha|\le-\mu_0$. *Therefore, by the Griffiths lemma ([6] and Appendix B) we obtain :*

$$\begin{aligned}\lim_{\Lambda}\partial_\alpha p_\Lambda\left[H_{\Lambda,\alpha}^{B}\left(\nu^{\#}=0\right)\right] &= \lim_{\Lambda}\left(-\left\langle\frac{a_0^{*}a_0}{V}\right\rangle_{H_{\Lambda,\alpha}^{B}\left(\nu^{\#}=0\right)}\right)\\ &= -\left|\widehat{c}_\alpha\left(\beta,\mu\right)\right|^2.\end{aligned}\tag{4.24}$$

Corollary 4.8. *By the continuity in $\alpha \to 0$, equations (4.2) and (4.24) imply that*

$$\rho_0^B(\beta,\mu) = \lim_{\Lambda} \left\langle \frac{a_0^* a_0}{V} \right\rangle_{H_\Lambda^B} = \lim_{\Lambda} \left\langle \frac{a_0^*}{\sqrt{V}} \right\rangle_{H_\Lambda^B} \lim_{\Lambda} \left\langle \frac{a_0}{\sqrt{V}} \right\rangle_{H_\Lambda^B} = \left|\widehat{c}(\beta,\mu)\right|^2 . \tag{4.25}$$

We conclude this section by analysis of the Bose-condensate $\rho_0^B(\beta,\mu)$ behaviour. In virtue of (4.25) it reduces to the analysis of the behaviour of $|\widehat{c}(\beta,\mu)|$ which corresponds to the $\sup_{c\in\mathbb{C}}$ of the trial pressure (4.1) :

$$\widetilde{p}^B\left(\beta,\mu;c^{\#},\nu^{\#}=0\right) = \xi\left(\beta,\mu;x\equiv|c|^2\right) + \eta\left(\mu;x\equiv|c|^2\right) \equiv \widetilde{p}^B\left(\beta,\mu;c^{\#}\right), \tag{4.26}$$

where (cf. (2.28),(2.29))

$$\begin{aligned}
\xi(\beta,\mu;x) &= \frac{1}{(2\pi)^3\beta}\int_{\mathbb{R}^3} d^3k \ln\left(1-e^{-\beta E_k}\right)^{-1} \\
\eta(\mu;x) &= \frac{1}{2(2\pi)^3}\int_{\mathbb{R}^3} d^3k\left(f_k - E_k\right) + \mu x - \frac{1}{2}v(0)x^2, \\
f_k &= \varepsilon_k - \mu + x\left[v(0)+v(k)\right], h_k = xv(k), E_k = \sqrt{f_k^2 - h_k^2}.
\end{aligned} \tag{4.27}$$

Below we collect some properties of the trial pressure (4.26) :
(1) For $\mu \le 0$ the function (4.26) is differentiable with respect to $x = |c|^2 \ge 0$ and

$$\lim_{|c|^2\to\infty} \widetilde{p}^B\left(\beta,\mu;c^{\#}\right) = -\infty, \tag{4.28}$$

Hence, $\sup_{x\ge0}(\xi+\eta)(\beta,\mu;x)$ is attained either at $x=0$, or at a positive solution of the equation

$$\begin{aligned}
0 &= \partial_x(\xi+\eta)(\beta,\mu;x) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} d^3k\left(1-e^{\beta E_k}\right)^{-1}\partial_x E_k \\
&\quad - \frac{1}{2(2\pi)^3}\int_{\mathbb{R}^3} d^3k\left(\partial_x E_k - \partial_x f_k\right) + \mu - xv(0),
\end{aligned} \tag{4.29}$$

see Fig. 2.
(2) By definitions (4.27) and the properties (A) and (B) of the potential $v(k)$ one gets that

$$\partial_x f_k = v(0) + v(k), \partial_x E_k = E_k^{-1}\left(f_k v(0) + (f_k - h_k)v(k)\right) \ge 0,$$

for $\mu \le 0$, $x \ge 0$ and any $k \in \mathbb{R}^3$. Therefore, by (4.29) we have

$$\partial_x \widetilde{p}^B\left(\beta,\mu;c^{\#}=0\right) \le \partial_x\eta(\mu;x=0) \equiv \partial_x\widetilde{p}^B\left(\beta=\infty,\mu;c^{\#}=0\right) = \mu\ . \tag{4.30}$$

(3) By explicit calculation one finds that $\partial_\mu\partial_x\eta(\mu;x) \ge 0$ for $\mu \le 0$ and $x \ge 0$. Hence

$$\partial_x\eta(\mu;x) \le \partial_x\eta(\mu=0;x), \tag{4.31}$$

and $\partial_x \eta(\mu = 0; x)$ is a concave function of $(0, \infty)$.
(4) Now let potential $v(k)$ satisfy the condition (C). Then

$$\partial_x^2 \eta_\Lambda(\mu = 0; x) = -v(0) + \frac{1}{2(2\pi)^3} \int_{\mathbb{R}^3} \frac{[v(k)]^2}{\varepsilon_k} d^3k \geq 0, \tag{4.32}$$

Since $\eta_\Lambda(\mu = 0; x = 0) = 0$, (4.32) means that the trivial pressure

$$\widetilde{p}^B\left(\beta = \infty, \mu; c^\#\right) = \eta_\Lambda(\mu = 0; x)$$

attains $\sup_{x \geq 0}$ for $\widehat{x}(\theta = 0, \mu = 0) > 0$, and, by continuity for $(\theta \geq 0, \mu \geq 0)$, the domain

$$D_0 = \{(\theta, \mu) : \ \widehat{x}(\theta, \mu) > 0\} \neq \{\emptyset\},$$

see Lemma 2.15, Theorem 2.16 and Fig. 2.
(5) Fix $\mu \in D_0$ and $\theta = 0$. Then, according to (4.30),

$$\partial_x \eta_\Lambda(\mu; x = 0) = \partial_x \widetilde{p}^B\left(\beta = \infty, \mu; c^\# = 0\right) = \eta_\Lambda(\mu = 0; x) < 0,$$

but $\partial_x^2 \widetilde{p}^B\left(\beta = \infty, \mu; c^\#, \nu^\# = 0\right) > 0$ ensures $|\widehat{c}(\beta = \infty, \mu)|^2 = \widehat{x}(\theta = 0, \mu) \equiv \overline{x}(\mu) > 0$ (see Fig. 2), i.e.

$$\widetilde{p}^B\left(\beta = \infty, \mu; c^\# = 0\right) < \widetilde{p}^B\left(\beta = \infty, \mu; |\widehat{c}(\beta = \infty, \mu)|^2\right). \tag{4.33}$$

(6) Since $\partial_x \xi(\beta, \mu; x) < 0$ (see (4.29)) and

$$\partial_\theta \partial_x \xi(\beta, \mu; x) = \frac{(-1)}{(2\pi)^3} \int_{\mathbb{R}^3} d^3k \frac{\beta^2 E_k e^{\beta E_k}}{(1 - e^{\beta E_k})^2} \partial_x E_k < 0, \tag{4.34}$$

there is $\theta_0(\mu)$ (cf. Theorem 2.16) such that for $\mu \in D_0$, one gets :

$$\begin{aligned} \sup_{x \geq 0} [\xi(\beta_0(\mu), \mu; x) + \eta(\mu; x)] &= \xi(\beta_0(\mu), \mu; 0) + \eta(\mu; 0) \\ &= \xi(\beta_0(\mu), \mu; \widehat{x}(\theta_0(\mu), \mu) > 0) \\ &\quad + \eta(\mu; \widehat{x}(\theta_0(\mu), \mu) > 0), \end{aligned} \tag{4.35}$$

whereas for $\theta < \theta_0(\mu)$ the supremum is attained at $x = \widehat{x}(\theta, \mu) > 0$ and for $\theta > \theta_0(\mu)$ it "jumps" to $\widehat{x}(\theta, \mu) = 0$, see Fig. 1 and Fig. 2. Therefore, we have proved the following theorem.

Theorem 4.9. *If interaction potential $v(k)$ satisfies conditions (A), (B), and (C), then domain $D \neq \{\emptyset\}$ and the Bose-condensate undergo a jump on the boundary ∂D :*

$$\rho_0^B\left(\theta^{-1}, \mu\right) = \left\{ \begin{array}{c} > 0, \ (\theta, \mu) \in D \\ = 0, \ (\theta, \mu) \in Q \backslash \overline{D} \end{array} \right\}. \tag{4.36}$$

Behaviour of the trail pressure (4.26) and the condensate (4.36) are illustrated by Figures 2 and 3.

5. Concluding Remarks

The paper presents exact solution of the Bogoliubov Weakly Imperfect Bose-Gas model (1.4) invented for explanation of the superfluity [1, 2].
We show that the thermodynamic properties of the model drastically depend on the interaction potential. We found that it is non-diagonal part of interaction that makes the model non-trivial (non-equivalent to the Ideal Bose-Gas) and we formulated necessary and sufficient conditions (A), (B), and (C) on potential to ensure that $D \neq \{\emptyset\}$ (Theorems 2.16 and 3.14).
Thus we answer to the problem formulated in [5] by showing that its solution depends on the potential. In particular we establish that conditions (C) (formulated in [5] as necessary) is sufficient for the Weakly Imperfect Bose-Gas be equivalent to the Ideal Bose-Gas in the hole domain of its stability $Q = \{\theta \geq 0\} \times \{\mu \leq 0\}$.
We show that Bogoliubov approximation for the Weakly Imperfect Bose-Gas is exact (Theorem 3.14). It gives an access to explicit calculation of the pressure $p^B(\beta, \mu)$. On the other hand this exact solution is rather different from the result of the Bogoliubov treatment [1, 2] of the Hamiltonian (1.4). The reason is that this treatment besides the Bogoliubov approximation involves additional hypothesis which are equivalent to modifications of the original Bogoliubov Hamiltonian (1.4), see discussion in [8, 9], [13, 14] and references there.
We found that for interactions satisfying (A), (B), and (C) there is a domain (Fig. 1)

$$D = \{(\theta, \mu) : \mu_0 < \mu \leq 0, 0 \leq \theta < \theta_0(\mu)\} \subset Q$$

where the pressure $p^B_\Lambda(\beta, \mu) \neq p^I(\beta, \mu)$. We show (Theorem 2.20) that in fact

$$D = \left\{(\theta, \mu) : \rho^B_0(\beta, \mu) \neq 0\right\},$$

where $\rho^B_0(\beta, \mu)$ is the density of the Bose-condensate in the Bogoliubov model. It is shown (Theorem 4.1) that the gauge symmetry is broken in D and that ρ^B_0 change its value on ∂D discontinuously (Theorem 4.9). Moreover, the Hamiltonian $H^B_\Lambda\left(\widehat{c}^{\#}_\Lambda, \mu\right)$, which is thermodynamic equivalent to $H^B_\Lambda(\mu)$ (Corollary 3.16), has a gap in the spectrum for $\lim_{k \to 0} E_k$ in domain D, i.e., in the presence of the Bose-condensate, see (4.27). This again indicates that the original Bogoliubov Hamiltonian (1.4) is far from its modifications [1, 2, 5, 8, 9, 13, 14] invented for description of the superfluidity.

Appendix A. : Proof of Theorem 4.1

1. Note that the pressure $p_\Lambda\left[H^B_\Lambda\left(\nu^{\#}\right)\right]$ is bounded from below and from above uniformly on any compact $K \subset Q \times \{\nu : |\nu| < r_0\}$ see (2.25) and (3.20). Since the family

$$\left\{p_\Lambda\left[H^B_\Lambda\left(\nu^{\#}\right)\right] = p^B_\Lambda\left(\beta, \mu; \nu^{\#}\right)\right\}_{\Lambda \subset \mathbb{R}^3}$$

consists of convex functions of the chemical potential $\mu < 0$, by compactness argument (see e.g. Ch.II, §10, [15]) there is a subsequence $\left\{p^B_{\Lambda_j}\left(\beta, \mu; \nu^{\#}\right)\right\}_{j=1}^{\infty}$ which converges uniformly in μ on any compact $C_\mu \subset \mathbb{R}^1_-$ and fixed β, ν to the convex function $p^B\left(\beta, \mu; \nu^{\#}\right)$.

2. The grand-canonical pressure has the form :

$$
\begin{aligned}
p_\Lambda^B\left(\beta,\mu;\nu^{\#}\right) &= \frac{1}{\beta V}\ln\left\{\sum_{N=0}^{\infty} e^{\beta V\left(\mu\frac{N}{V}-f_\Lambda^B\left(\beta,\frac{N}{V};\nu^{\#}\right)\right)}\right\} = \\
&= (\beta V)^{-1}\ln \Xi_\Lambda^B\left(\beta,\mu;\nu^{\#}\right),
\end{aligned}
\tag{A.1}
$$

where

$$
f_\Lambda^B\left(\beta,\rho=N/V;\nu^{\#}\right) = -\frac{1}{\beta V}Tr_{\mathcal{F}_N}e^{-\beta H_\Lambda^B\left(\nu^{\#}\right)},\ \rho\geq 0, \tag{A.2}
$$

is the free-energy density. By conditions of the Theorem 4.1, $(\theta,\mu)\in D$, which corresponds to the one-phase domain : $\rho_0^B>0$. Consequently, $\left\{\partial_\mu p_\Lambda^B = \langle \frac{N}{V}\rangle_{H_\Lambda^B(\nu^{\#})}>0\right\}_\Lambda$ and $\partial_\mu p^B =$ (the Griffiths lemma) $=\lim_\Lambda \partial_\mu p_\Lambda^B$ are continuous functions of $\mu\in[\mu_0(\beta)+\varepsilon,0]$, $\varepsilon>0$. Then by the Tauberian theorem proved in [16] the existence of the limit $p^B\left(\beta,\mu;\nu^{\#}\right)$ entails the existence of the limit

$$
f^B\left(\beta,\rho;\nu^{\#}\right) = \lim_\Lambda f_\Lambda^B\left(\beta,\rho;\nu^{\#}\right), \tag{A.3}
$$

which is uniform on the interval

$$
\rho\in I_{\varepsilon_{1,2}} = \left[\partial_\mu p^B\left(\beta,\mu_0(\beta)+\varepsilon_1;\nu^{\#}\right),\partial_\mu p^B\left(\beta,-\varepsilon_2;\nu^{\#}\right)\right],\ \varepsilon_{1,2}>0. \tag{A.4}
$$

In fact on this interval (A.3) coincides with its convex envelop (the Legendre transformation) :

$$
\widetilde{f}^B\left(\beta,\rho;\nu^{\#}\right) = C.E.\left\{f^B\left(\beta,\rho;\nu^{\#}\right)\right\} = \sup_{\mu\leq 0}\left\{\mu\rho - p^B\left(\beta,\mu;\nu^{\#}\right)\right\}. \tag{A.5}
$$

3. In virtue of (A.3) and (A.5), for $|\Lambda_j|$ large enough functions $\left\{f_{\Lambda_j}^B\left(\beta,\rho;\nu^{\#}\right)\right\}_{j=1}^\infty$ are strictly convex on $I_{\varepsilon_{1,2}}$ and for $\mu\in[\mu_0(\beta)+\varepsilon_1,0]$

$$
\sup_{\frac{N}{V}}\left(\mu\frac{N}{V} - f_\Lambda^B\left(\beta,\frac{N}{V};\nu^{\#}\right)\right) = \mu\overline{\rho}_\Lambda - f_\Lambda^B\left(\beta,\overline{\rho}_\Lambda;\nu^{\#}\right) \equiv -F_\Lambda\left(\beta,\mu;\overline{\rho}_\Lambda,\nu^{\#}\right), \tag{A.6}
$$

$\overline{\rho}_\Lambda(\mu)\in I_{\varepsilon_{1,2}}$. Then for $\left|\frac{N}{V}-\overline{\rho}_\Lambda\right|>\xi>0$ one gets

$$
F_\Lambda\left(\beta,\mu;\frac{N}{V},\nu^{\#}\right) > F_\Lambda\left(\beta,\mu;\overline{\rho}_\Lambda,\nu^{\#}\right)+\gamma \equiv \overline{F}_\Lambda+\gamma,\ \gamma>0, \tag{A.7}
$$

and for $\left|\frac{N}{V}-\overline{\rho}_\Lambda\right|<\xi'<\xi$ one has

$$
\overline{F}_\Lambda \leq F_\Lambda\left(\beta,\mu;\frac{N}{V},\nu^{\#}\right) \leq \overline{F}_\Lambda+\frac{\gamma}{2}. \tag{A.8}
$$

By (A.1) one gets that for $\frac{N}{V}-\overline{\rho}_\Lambda>\xi$ there are two constants $a_{1,2}>0$ such that

$$
F_\Lambda\left(\beta,\mu;\frac{N}{V},\nu^{\#}\right) > a_1 + a_2\left(\frac{N}{V}-\overline{\rho}_\Lambda-\xi\right) \tag{A.9}
$$

4. (*The Large Deviation Principle for the particle density*). By the standard reasoning (see e.g. [17]) one gets from the grand-canonical distribution of particles and (A.7)-(A.9) that

$$\begin{aligned}
p_{\Lambda,I} &= \mathbb{P}_\Lambda\left\{0 \le \frac{N}{V} < \overline{\rho}_\Lambda - \xi\right\} = \left(\Xi^B_\Lambda\right)^{-1} \sum_{0\le N<V(\overline{\rho}_\Lambda-\xi)} e^{-\beta V F_\Lambda\left(\beta,\mu;\frac{N}{V},\nu^{\#}\right)} \le \\
&\le V\left(\overline{\rho}_\Lambda - \xi\right) e^{-\beta V\left(\overline{F}_\Lambda+\gamma\right)}, \qquad \text{(A.10)}\\
p_{\Lambda,II} &= \mathbb{P}_\Lambda\left\{\overline{\rho}_\Lambda - \xi \le \frac{N}{V} < \overline{\rho}_\Lambda + \xi\right\} \ge \left(\Xi^B_\Lambda\right)^{-1} \sum_{V(\overline{\rho}_\Lambda-\xi')\le N<V(\overline{\rho}_\Lambda+\xi')} e^{-\beta V F_\Lambda\left(\beta,\mu;\frac{N}{V},\nu^{\#}\right)} \ge \\
&\ge 2\left(\Xi^B_\Lambda\right)^{-1} \xi' V e^{-\beta V\left(\overline{F}_\Lambda+\frac{\gamma}{2}\right)}, \qquad \text{(A.11)}\\
p_{\Lambda,III} &= \mathbb{P}_\Lambda\left\{\frac{N}{V} \ge \overline{\rho}_\Lambda + \xi\right\} \le \left(\Xi^B_\Lambda\right)^{-1} \sum_{N\ge V(\overline{\rho}_\Lambda+\xi)} e^{-\beta V\left[a_1+a_2\left(\frac{N}{V}-\overline{\rho}_\Lambda-\xi\right)\right]}. \qquad \text{(A.12)}
\end{aligned}$$

Since $p_{\Lambda,I} + p_{\Lambda,II} + p_{\Lambda,III} = 1$, these estimates imply that

$$\lim_\Lambda p_{\Lambda,II} = 1 \qquad \text{(A.13)}$$

for any $\xi > 0$.

5. Now we can apply the Large Deviation Principle for the particle density in domain D to obtain (3.19). By (3.35) for $A^{\#}$ one gets

$$\left\langle A^*\left[A,\left[H^B_\Lambda\left(\mu,\nu^{\#}\right),A^*\right]\right]A\right\rangle_{H^B_\Lambda\left(\nu^{\#}\right)} \le 2v(0)\left\langle A^*\frac{N_\Lambda}{V}A\right\rangle_{H^B_\Lambda\left(\nu^{\#}\right)} - \mu\left\langle A^*A\right\rangle_{H^B_\Lambda\left(\nu^{\#}\right)}. \qquad \text{(A.14)}$$

Since

$$\left\langle A^*\frac{N_\Lambda}{V}A\right\rangle_{H^B_\Lambda\left(\nu^{\#}\right)} = \left\langle\left(\frac{N_\Lambda}{V}-\overline{\rho}_\Lambda\right)A^*A\right\rangle_{H^B_\Lambda\left(\nu^{\#}\right)} + \overline{\rho}_\Lambda\left\langle A^*A\right\rangle_{H^B_\Lambda\left(\nu^{\#}\right)}, \qquad \text{(A.15)}$$

we have to estimate from above the first term in the left-hand side of (A.15). To this end we follow (A.10)-(A.13) :

(I)

$$\left(\Xi^B_\Lambda\right)^{-1} \sum_{0\le N<V(\overline{\rho}_\Lambda-\xi)} \left(\frac{N}{V}-\overline{\rho}_\Lambda\right) e^{-\beta V F_\Lambda}\left\langle A^*A\right\rangle_{H^B_\Lambda\left(\nu^{\#}\right)}\left(\beta,N;\nu^{\#}\right) \le 0. \qquad \text{(A.16)}$$

(II)

$$\begin{aligned}
&\left(\Xi^B_\Lambda\right)^{-1} \sum_{V(\overline{\rho}_\Lambda-\xi)\le N<V(\overline{\rho}_\Lambda+\xi)} e^{\beta\mu N}\left(\tfrac{N}{V}-\overline{\rho}_\Lambda\right) Tr_{\mathcal{F}_N}\left(e^{-\beta H^B_\Lambda\left(\nu^{\#}\right)}A^*A\right) \le \\
&\le 2\xi\left(\Xi^B_\Lambda\right)^{-1} \sum_{N=0}^{\infty} e^{\beta\mu N} Tr_{\mathcal{F}_N}\left(e^{-\beta H^B_\Lambda\left(\nu^{\#}\right)}A^*A\right) = \xi\left\langle A^*A\right\rangle_{H^B_\Lambda\left(\nu^{\#}\right)}\left(\beta,\mu;\nu^{\#}\right)
\end{aligned} \qquad \text{(A.17)}$$

(III)

$$
\begin{aligned}
&\left(\Xi_\Lambda^B\right)^{-1} \sum_{N\geq V(\overline{\rho}_\Lambda+\xi)} \left(\tfrac{N}{V}-\overline{\rho}_\Lambda\right) e^{-\beta V F_\Lambda} \left\langle A^*A\right\rangle_{H_\Lambda^B(\nu^\#)} \left(\beta, N; \nu^\#\right) \leq \\
&\leq \left(\Xi_\Lambda^B\right)^{-1} \sum_{N\geq V(\overline{\rho}_\Lambda+\xi)} \left(\tfrac{N}{V}-\overline{\rho}_\Lambda\right) e^{-\beta V\left[a_1+a_2\left(\frac{N}{V}-\overline{\rho}_\Lambda-\xi\right)\right]} 2\left(N+|c|^2 V\right) = \\
&= 2\left(\Xi_\Lambda^B\right)^{-1} \sum_{N\geq V(\overline{\rho}_\Lambda+\xi)} \left(\tfrac{N}{V}-\overline{\rho}_\Lambda\right)\left(N-V\overline{\rho}_\Lambda\right) e^{-\beta V a_1} e^{-\beta a_2(N-V(\overline{\rho}_\Lambda+\xi))} + \\
&+2\left(\Xi_\Lambda^B\right)^{-1} V\left(\overline{\rho}_\Lambda+|c|^2\right) e^{-\beta V a_1} \sum_{N\geq V(\overline{\rho}_\Lambda+\xi)} e^{-\beta a_2(N-V(\overline{\rho}_\Lambda+\xi))} = \\
&= p_{\Lambda,II}\left(\xi' V^2\right)^{-1} e^{-\beta V\frac{\gamma}{2}} \sum_{N\geq V(\overline{\rho}_\Lambda+\xi)} \left(N-V\overline{\rho}_\Lambda\right)^2 e^{-\beta a_2(N-V(\overline{\rho}_\Lambda+\xi))} + \\
&+p_{\Lambda,II} V^{-1}\left(\overline{\rho}_\Lambda+|c|^2\right) e^{-\beta V\frac{\gamma}{2}} \sum_{N\geq V(\overline{\rho}_\Lambda+\xi)} e^{-\beta a_2(N-V(\overline{\rho}_\Lambda+\xi))} \leq const.e^{-\beta V\frac{\gamma}{2}}.
\end{aligned} \tag{A.18}
$$

Combining (A.16)-(A.18) with (A.14), (A.15) we find that for any compact $C_\mu \subset (\mu_0(\beta), 0)$, $|\nu| \leq r_0$ and $C_\beta \subset \mathbb{R}^1_+$ one gets

$$
\left\langle A^*\left[A,\left[H_\Lambda^B\left(\mu,\nu^\#\right), A^*\right]\right] A\right\rangle_{H_\Lambda^B(\nu^\#)} \leq a\left\langle A^*A\right\rangle_{H_\Lambda^B(\nu^\#)} + b, \tag{A.19}
$$

for positive a, b which depend on C_μ, C_β, and r_0.

Appendix B. : The Griffiths and the Kolmogorov lemata

For the reader convenience we state below the two above Lemata with their proofs.

Lemma B.1 (Griffiths [6]). *Let $\{f_n(x)\}_{n\geq 1}$ be a sequence of convex functions on a compact $I \subset \mathbb{R}$. If exists a pointwise limit*

$$
\lim_{n\to\infty} f_n(x) = f(x), \; x \in I, \tag{B.1}
$$

then

$$
\lim_{n\to\infty} \inf \partial_x f_n(x-0) \;\; \geq \;\; \partial_x f(x-0), \tag{B.2}
$$

$$
\lim_{n\to\infty} \sup \partial_x f_n(x+0) \;\; \leq \;\; \partial_x f(x+0). \tag{B.3}
$$

Proof:
By convexity one has

$$
\partial_x f_n(x) \;\; \leq \;\; \frac{f_n(x+l)-f_n(x)}{l}, \tag{B.4}
$$

$$
\partial_x f_n(x) \;\; \geq \;\; \frac{f_n(x)-f_n(x-l)}{l}, \tag{B.5}
$$

for $l > 0$. Then taking the limit $n \to \infty$ in (B.5) we obtain :

$$
\lim_{n\to\infty} \sup \partial_x f_n(x+0) \;\; \leq \;\; \frac{f(x+l)-f(x)}{l}, \tag{B.6}
$$

$$
\lim_{n\to\infty} \inf \partial_x f_n(x-0) \;\; \geq \;\; \frac{f(x)-f(x-l)}{l}. \tag{B.7}
$$

Now taking the limit $l \to +0$, in (B.7), one gets (B.3). ■

Remark B.2. *In particular, if $x_0 \in I$ is such that $\partial_x f_n(x_0 - 0) = \partial_x f_n(x_0 + 0)$ and $\partial_x f(x_0 - 0) = \partial_x f(x_0 + 0)$, then*

$$\lim_{n\to\infty} \partial_x f_n(x_0) = \partial_x f(x_0). \tag{B.8}$$

Lemma B.3 (Kolmogorov [12]). *Let functions $\{\Delta_n(x)\}_{n\geq 1} \in C^2(I)$ defined on $[a,b] \equiv I$ satisfy :*

$$|\Delta_n(x)| \leq \varepsilon_n(I) \underset{n\to\infty}{\longrightarrow} 0, \ \forall x \in I, \tag{B.9}$$

and

$$(a)\, \partial_x^2 \Delta_n(x) \leq D_n(I) < +\infty,$$

or

$$(b)\, \partial_x^2 \Delta_n(x) \geq -D_n(I) > -\infty.$$

Then :
(i) if $D_n(I) \neq 0$, one has

$$|\partial_x \Delta_n(x)| \leq 2\sqrt{\varepsilon_n(I) D_n(I)}, \tag{B.10}$$

for $x \in [a + l_n, b - l_n]$, where l_n is defined by $l_n = 2\sqrt{\varepsilon_n(I)/D_n(I)}$ for $n > n_0 : (b-a)/2 \geq l_n$,
(ii) if $D_n(I) = 0$, we have

$$|\partial_x \Delta_n(x)| \leq 2 l^{-1} \varepsilon_n(I), \tag{B.11}$$

for any l such that $(b-a)/2 \geq l$ and for $x \in [a+l, b-l]$.

Proof:
Let us consider the function

$$\begin{aligned} G_{\xi,l}(x) \equiv\ & \Delta_n(\xi + l) - \Delta_n(x) - (\xi + l - x)\, \partial_x \Delta_n(x) \\ & - \frac{(\xi + l - x)^2}{l^2} \left[\Delta_n(\xi + l) - \Delta_n(\xi) - l \partial_x \Delta_n(\xi)\right] \end{aligned} \tag{B.12}$$

under the condition that $x \in [\xi, \xi + l] \subset I$ (with $l > 0$). By the fact that $\Delta_n(x) \in C^2(I)$ we get

$$G'_{\xi,l}(x) = -(\xi + l - x)\, \partial_x^2 \Delta_n(x) + 2\frac{(\xi + l - x)}{l^2} \left[\Delta_n(\xi + l) - \Delta_n(\xi) - l \partial_x \Delta_n(\xi)\right]. \tag{B.13}$$

Note that $G_{\xi,l}(\xi) = G_{\xi,l}(\xi + l) = 0$, so, there are two points ξ_1 and $\xi_2 \in [\xi, \xi + l]$ such that $G'_{\xi,l}(\xi_1) \leq 0$ and $G'_{\xi,l}(\xi_2) \geq 0$. Then using (B.13), one gets :

$$l^{-1}\left[\Delta_n(\xi + l) - \Delta_n(\xi)\right] - \frac{1}{2} l \Delta''_n(\xi_1) \leq \Delta'_n(\xi) \leq l^{-1}\left[\Delta_n(\xi + l) - \Delta_n(\xi)\right] - \frac{1}{2} l \Delta''_n(\xi_2) \tag{B.14}$$

for $l > 0$ and for any $\xi \in [a, b - l]$.
One can define also the function $G_{\xi,-l}(x)$ on $[\xi - l, \xi] \subset I$ (with $l > 0$). Hence, we get for $l > 0$ and for any $\xi \in [a, b - l]$

$$l^{-1}\left[\Delta_n(\xi) - \Delta_n(\xi - l)\right] - \frac{1}{2} l \Delta''_n(\eta_1) \leq \Delta'_n(\xi) \leq l^{-1}\left[\Delta_n(\xi) - \Delta_n(\xi - l)\right] - \frac{1}{2} l \Delta''_n(\eta_2), \tag{B.15}$$

where η_1, $\eta_2 \in [\xi - l, \xi]$ such that $G'_{\xi,-l}(\eta_1) \leq 0$ and $G'_{\xi,-l}(\eta_2) \geq 0$.
Suppose (a), then, by $|\Delta_n(x)| \leq \varepsilon_n(I)$, we obtain from the inequality (B.14) that

$$-2l^{-1}\varepsilon_n(I) - \frac{1}{2}lD_n(I) \leq \Delta'_n(\xi) \tag{B.16}$$

for $\xi \in [a, b - l]$ and from (B.15) that

$$\Delta'_n(\xi) \leq 2l^{-1}\varepsilon_n(I) + \frac{1}{2}lD_n(I) \tag{B.17}$$

for $\xi \in [a + l, b]$.
(i)Let $D_n(I) \neq 0$. Then by (B.16) and (B.17) we get

$$|\Delta'_n(\xi)| \leq \min_l \left[2l^{-1}\varepsilon_n(I) + \frac{1}{2}lD_n(I)\right] \tag{B.18}$$

and thus

$$|\Delta'_n(\xi)| \leq 2\sqrt{\varepsilon_n(I)\,D_n(I)} \tag{B.19}$$

for $\xi \in [a + l_n, b - l_n]$, where $l_n = 2\sqrt{\varepsilon_n(I)/D_n(I)}$ (under the condition that $(b-a)/2 \geq l_n$).
(ii)Let $D_n(I) = 0$. Then by (B.16) and (B.17), we obtain

$$|\partial_x \Delta_n(x)| \leq 2l^{-1}\varepsilon_n(I) \tag{B.20}$$

for any l such that $(b-a)/2 \geq l$ and for any $x \in [a + l, b - l]$.
If we suppose (b), we obtain the same result by considering functions $\{-\Delta_n(x)\}_{n\geq 1}$. ■

Appendix C. : Study of $\langle\{\delta a_0^*, \delta a_0\}\rangle_{H_\Lambda^B(\nu\#)}$

Here we derive the estimate (3.35), which was obtained in [11].
By the spectral decomposition of the Hamiltonian : $H_\Lambda^B(\mu, \nu^\#)\,\psi_n = E_n\psi_n$ one gets

$$\langle\{\delta a_0^*, \delta a_0\}\rangle_{H_\Lambda^B(\nu\#)} = \frac{1}{\Xi_\Lambda^B(\beta, \mu; \nu^\#)} \sum_{m,n} |A_{mn}|^2 \left(e^{-\beta E_n} + e^{-\beta E_m}\right) \tag{C.1}$$

where $A_{mn} = (\psi_m, \delta a_0 \psi_n)$. Note that

$$\begin{aligned}
\left(e^{-\beta E_n} + e^{-\beta E_m}\right) &= 2\frac{\left(e^{-\beta E_n} - e^{-\beta E_m}\right)}{\beta(E_m - E_n)} \frac{\beta(E_m - E_n)}{2} \frac{\left(e^{-\beta E_n} + e^{-\beta E_m}\right)}{\left(e^{-\beta E_n} - e^{-\beta E_m}\right)} \\
&= 2\frac{\left(e^{-\beta E_n} - e^{-\beta E_m}\right)}{\beta(E_m - E_n)} \frac{\beta(E_m - E_n)}{2} cth\left(\frac{\beta(E_m - E_n)}{2}\right).
\end{aligned}$$

Since for $x \in \mathbb{R}$,

$$chx - \frac{shx}{x} = \sum_{n=1}^{\infty} \frac{2n}{(2n+1)!}x^{2n} \leq \frac{xshx}{3},$$

we obtain that $xcthx \leq 1 + x^2/3$. Thus one gets for (C.1) the estimate

$$\begin{aligned}\langle\{\delta a_0^*, \delta a_0\}\rangle_{H_\Lambda^B(\nu^\#)} &\leq \frac{2}{\Xi_\Lambda^B(\beta,\mu;\nu^\#)}\sum_{m,n}|A_{mn}|^2\frac{\left(e^{-\beta E_n}-e^{-\beta E_m}\right)}{\beta(E_m-E_n)} \\ &\quad + \frac{1}{6\Xi_\Lambda^B(\beta,\mu,\nu^\#)}\sum_{m,n}|A_{mn}|^2\left(e^{-\beta E_n}-e^{-\beta E_m}\right)\beta(E_m-E_n) \\ &\leq 2\left(\delta a_0^*, \delta a_0\right)_{H_\Lambda^B(\nu^\#)} + \frac{\beta}{6}\left\langle\left[\delta a_0^*, \left[H_\Lambda^B\left(\mu,\nu^\#\right), \delta a_0\right]\right]\right\rangle_{H_\Lambda^B(\nu^\#)}. \quad (C.2)\end{aligned}$$

■

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FIGURE CAPTIONS

Figure 1.
(1) Phase diagram of the Bogoliubov Weakly Imperfect Bose-Gas.
(2) Estimation of the phase diagram given in [5] and in Section 2; schematic : deviation of the boundary $\partial\Gamma$ from the straight lines $\partial\Gamma_{\mu_*}$, $\partial\Gamma_{\theta_*}$ is exaggerated in the vinicity of (μ_*, θ_*).
Here $\theta =$ *temperature*, $\mu =$ *chemical potential.*
Remark : Numerical estimate gives : $\mu_*/\mu_0 \sim 10^3$, $\theta_*/\theta_0 \sim 10$.
Figure 2.
Illustration to the Bogoliubov approximation variational problem : behaviour of the difference between the trial pressure $\widetilde{p}^B(\beta, \mu; x)$ and the IBG pressure $p^I(\beta, \mu)$ as the function of the variational parameter $x = |c|^2$ for different values of (θ, μ). Non-trivial suprema are indicated by empty circles.
Figure 3.
Discontinuous behaviour of the Bose-condensate density $|\widehat{c}(\beta, \mu)|^2 = \rho_0^B(\theta^{-1}, \mu)$ for the Bogoliubov WIBG : $\rho_0^B > 0$ in domain $D = D_0$ and $\rho_0^B = 0$ in the rest of the stability domain $Q \backslash \overline{D}$.