The standard map and a probabilistic analogue

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Abstract

This paper is devoted to the study of statistical properties of trajectories of a class of dynamical systems which includes the standard map. A well known conjecture is formulated and its natural probabilistic analogue is defined in the framework of ergodic theory of random transformations. The main results is the proof of the probabilistic analogue of this conjecture.

Let $\varphi = \varphi(u)$ be a smooth function having the period 1 with respect to u, $\varphi(u) = \varphi(u+1)$. We consider the map $A_\varphi : (u,z) \to (u',z')$ of the cylinder $C = \{u, z : 0 \le u < 1, -\infty < z < \infty\}$ having standard type

$$A_\varphi : \begin{cases} u' = u + z' \bmod 1 \\ z' = z + \varphi(u) \end{cases} , \tag{1}$$

which in the special case $\varphi = \varphi_*(u) = h\sin(2\pi u)$ is the standard map

$$A_* : \begin{cases} u' = u + z' \bmod 1 \\ z' = z + h\sin(2\pi u) \end{cases} . \tag{2}$$

The maps A_φ, and especially A_* were studied in many papers (see [1] - [7]), and some conjectures connected with these maps were formulated and are very popular at the present time.

Conjecture. For sufficiently large value of the parameter h there exists a set $Q \subset C$ having positive Lebesgue measure such that if $z_n = z_n(u,v)$ is the coordinate of the point $(u_n, z_n) = A_*^n(u,z)$ (A_*^n is the n-th power of A_*) then the quality $I_n = \int_Q z_n^2(u,z)dudz$ behaves like cn as $n \to \infty$, where the constant $c \neq 0$.

The sense of this conjecture is that for initial data from Q the change of coordinate z_n behaves like a trajectory of random walk and as indicated in [7] at the present time we are very far from its proof. Originally this conjecture assumed that the Lebesgue measure of complement to sets Q on the cylinder C is equal to zero, and for this case it was disproved in the paper [5], where it was proved that on the half-line $h > 0$ there exists an open set H having infinite Lebesgue measure on the straight line, such that for any $h \in H$ there exists an open set $\Omega \subset C$ for which the quality I_n behaves like cn^2 as $n \to \infty$, where a constant $c \neq 0$.

The aim of present paper is to formulate and to prove a probabilistic analogue of the mentioned conjecture in the framework of ergodic theory of random transformations ([9], [10]). The essence of our approach is that instead of iterations of map A_* we consider consecutive actions of two maps A_* and A_φ with $\varphi \neq \varphi_*$ so that with probability $\frac{1}{2}$ we apply one of these maps. If the functions φ_* and φ are close in some metric sense then it is possible to say that such random dynamical system approximates original determinated system.

Let us introduce the following objects:
$T = \{u, z : 0 \leq u < 1, 0 \leq z < 1\}$ is the flat two-dimensional torus;
$S = \{0, 1\}$ is the set, consisting of two elements: 0 and 1;
ν is the measure on S such that measure of each element is equal to $\frac{1}{2}$;
Ω is the space, that is the direct product of countable number of sets S, and an arbitrary point of Ω is one-sided sequence $\omega = (\omega_1, \omega_2, \ldots)$ in which for any $n = 1, 2, \ldots$ the quantity ω_n takes one of two values: 0 or 1;
p is Bernulli measure on Ω, that is the countable product of measures ν;
σ is the map of Ω into itself, such that if $\omega = (\omega_1, \omega_2, \ldots) \in \Omega$, then $\sigma(\omega) = \omega' = (\omega_1', \omega_2', \ldots)$, where $\omega_n' = \omega_{n+1}$ $(n = 1, 2, \ldots)$;
$W = C \times \Omega$ and $\hat{W} = T \times \Omega$ are the direct products of the spaces C and Ω and the spaces T and Ω respectively;
μ is the measure on W and $\hat{W}$, being the product of the Lebesgue measure on C and the measure $p : d\mu = dudz \times dp$;
$\psi(u, \omega)$ is the function on W having the form

$$\psi(u, \omega) = \begin{cases} \varphi_*(u), & \text{if} \quad \omega_1 = 0 \\ \varphi(u), & if \quad \omega_1 = 1 \end{cases};$$

B is the map of the space W into itself such that if $(u, z, \omega) \in W$ then $B(u, z, \omega) = (u', z', \omega')$, where $(u', z') = A_{\psi(u,\omega)}(u, z)$, $\omega' = \sigma(\omega)$.

Definition 1. The map B of the space W is called the probabilistic analogue of the standard map A_* of the cylinder C.

Theorem 1. There exists the measurable and integrable with respect to measure μ

function $f(u,z,\omega)$ on $\hat{W}$ such that

$$\int_{\hat{W}} f(u,z,\omega)d\mu = \frac{1}{2}\int_0^1 \varphi(u)du$$

and for almost all points $(u,z,\omega) \in W$ (the measure μ of the complement is equal to zero) the coordinate $z^{(n)}(n=1,2,\ldots)$ of points $(u^{(n)}, z^{(n)}, \omega^{(n)}) = B^n(u,z,\omega)$ satisfy the equality

$$\lim_{n\to\infty} \frac{z^{(n)}}{n} = f(u,\{z\},\omega),$$

where $\{z\}$ is the fractional part of z.

Proof. We consider the function $\hat{\psi}(u,z,\omega) = \psi(u,\omega)$ on $\hat{W}$ and map $\hat{B}$ of the space $\hat{W}$ having the following form: if $(u,z,\omega) \in \hat{W}$ then $\hat{B}(u,z,\omega) = (\hat{u},\hat{z},\hat{\omega})$, where $\hat{u} = u + \hat{z} mod 1$, $\hat{z} = z + \hat{\psi}(u,z,\omega) mod 1$, $\hat{\omega} = \sigma(\omega)$. By virtue of the definition of the measure μ, the maps B, $\hat{B}$ and the conditions of theorem 1 $\mu(\hat{W}) = 1$, the map $\hat{B}$ preserves the measure μ on $\hat{W}$ (that is measure of a set is equal to the measure of it's inverse image) and for any $n = 1,2,\ldots$ the equality $z^{(n)} = \sum_{k=0}^{n-1} \hat{\psi}(\hat{B}(u,z,\omega))$ holds. Therefore the assertion of theorem 1 follows from the Birkhoff-Khinchin ergodic theorem applied to the function $\hat{\psi}(u,z,\omega)$ and map $\hat{B}$ of the space $\hat{W}$ with the measure μ.

Theorem 1 is proved.

Definition 2. We introduce the probabilistic analogue of the conjecture: for sufficiently large h in the definition of the map A_* there exists a set $\Pi \subset C$ having a positive and finite Lebesgue measure on C such that if we consider the coordinate $z^{(n)} = z^{(n)}(u,z,\omega)$ of point $(u^{(n)}, z^{(n)}, \omega^{(n)}) = B^n(u,z,\omega)$ then the quantity $\mathcal{J}_n = \int_{\Pi\times\Omega}(z^{(n)}(u,z,\omega))^2 d\mu$ behaves like cn as $n \to \infty$, where a constant $c \neq 0$.

Further we shall formulate and prove the theorem 2, from which it follows that on the half-line $h > 0$ there exists an open set H having infinite Lebesgue measure on the straight line such that for any $h \in H$ there exists a smooth function $\varphi(u)$ having the period 1 and differing from $\varphi_*(u)$ on the interval of arbitrarily small measure and conciding with φ_* for other points, such that there exists an open set $\Pi \subset C$ for which the probabilistic analog of conjecture holds.

In addition there exist infinitely-many such sets on C having the same measure which are mutually disjoint. If the function $\varphi(u)$ coinsides $\varphi_*(u)$ then the probabilistic analog of conjecture coinsides the conjecture. The difference of these function on the set of small measure is a natural approximation from probabilistic point of view. At the end of the paper a corollary is formulated in which states that the random map $\hat{B}$ introducing in the proof of theorem 1 is not ergodic. This corollary shows that the result of theorem 2 is not trivial because the general methods of probability theory based on using of martingale theory and central limit theorem are not applicable in this case.

Theorem 2. On the half-line $h > 0$ there exists an open set H having infinite Lebesgue measure on the straight line such that if $h \in H$ then there exists an integer $k = k(k) > 0$ such that the following assertion holds: for any ϵ satisfying the inequality $0 < \epsilon < 1$ there exist numbers α, β, infinitely differentiable function $\varphi(u)$ and an infinite number of mutually disjoint open sets $\Pi_m \subset C$ $(m \in \mathbf{Z})$ such that

1) $0 < \alpha < \beta < 1, \beta - \alpha \leq \epsilon$;

2) if $u \in [0, \alpha] u [\beta, 1)$ then $\varphi(u) = \varphi_*(u)$ and if $u \in (\alpha, \beta)$ then $|\varphi(u) - \varphi_*(u)| \leq 2k$;

3) for any $m \in \mathbf{Z}$ Lebesgue measure of set Π_m is equal to $\kappa = \int_{\Pi_0} dudz$, where $0 < \kappa < 1$, and the quantities $\mathcal{J}_n^{(m)} = \int_{\Pi_m \times \Omega} (z^{(n)}(u, z, \omega))^2 d\mu$ satisfy the equality

$$\lim_{n \to \infty} \frac{\mathcal{J}_n^{(m)}}{n} = k^2 \kappa \ , \tag{3}$$

(the quantities $z^{(n)}(u, z, \omega)$ are taken from definition 2).

Proof. Suppose that there exists u_0 satisfying the inequality $0 < u_0 < 1$, and integer number k such that

$$h \sin(2\pi u_0) = k > 0 \ , \tag{4}$$

$$-4 < 2\pi h \cos(2\pi u_0) = \rho < 0 \ , \tag{5}$$

$$\rho \neq -2 + 2\cos\frac{2\pi s}{q} \ , \quad (s = 0, \pm 1, \ldots, \pm q; q = 1, 2, 3, 4) \ , \tag{6}$$

and introduce the quantities

$$z_n^{(0)} = nk \ , 3n \in \mathbf{Z} \ . \tag{7}$$

By virtue of (2), (4) and (7) we have quality

$$A_*(u_0, z_n^{(0)}) = (u_0, z_{n+1}^{(0)}), n \in \mathbf{Z} \ . \tag{8}$$

Let

$$\delta = \frac{1}{2}\min(\epsilon, u_0, 1 - u_0) \ , \tag{9}$$

and define numbers

$$\alpha = u_0 - \delta \;, \beta = u_0 + \delta \;, \alpha' = u_0 - \frac{\delta}{2} \;, \beta' = u_0 + \frac{\delta}{2} \;, \qquad (10)$$

which by virtue of (9) satisfy the inequalities

$$0 < \alpha < \alpha' < u_0 < \beta' < \beta < 1 \;, \beta < \alpha \leq \epsilon \;. \tag{11}$$

We consider the infinite-differentiable function $\varphi = \varphi(u)$ having period 1 with respect u such that the following condition hold:

$$\varphi(u) = \varphi_*(u) \qquad \text{if} \qquad u \in [0, \alpha] \bigcup [\beta, 1) \;, \tag{12}$$

$$\varphi(u) = \varphi_*(u) - 2k \qquad \text{if} \qquad u \in [\alpha', \beta'] \;, \tag{13}$$

$$|\varphi(u) - \varphi_*(u)| \leq 2k \;, \qquad u \in [0, 1) \;. \tag{14}$$

According to (4) and (13) $\varphi(u_0) = -k$, and therefore by virtue of (1) and (7) we obtain:

$$A_\varphi(u_0, z_n^{(0)}) = (u_0, z_{n-1}^{(0)}) \;, n \in \mathbf{Z} \;. \tag{15}$$

For any $n \in \mathbf{Z}$ we introduce two sequences of maps

$$A^{(n)} : (u, z) \to (u^*, z^*) = A_*(u + u_0, z + z_{n-1}^{(0)}) - A_*(u_0, z_{n-1}^{(0)}), \tag{16}$$

$$\hat{A}^{(n)} : (u, z) \to (\hat{u}^*, \hat{z}^*) = A_\varphi(u + u_0, z + z_{n+1}^{(0)}) - A_\varphi(u_0, z_{n+1}^{(0)}), \tag{17}$$

and remark that from the equalities (1), (2), (13), (16) and (17) it follows that if $\alpha' - u_0 \leq u \leq \beta' - u_0$ then for any $n \in \mathbf{Z}$ maps $A^{(n)}$ and $\hat{A}^{(n)}$ are the same as the map $\hat{A} : (u, z) \to (\hat{u}, \hat{z})$ having the form

$$\hat{A} : \begin{cases} \hat{u} = u + \hat{z} \\ \hat{z} = z + h\sin(2\pi(u_0 + u)) - h\sin(2\pi u_0) \end{cases} \tag{18}$$

and $(0, 0)$ is the fixed point. The map $\hat{A}$ preserves the area and by virute of (5) the trace $Spd\hat{A}$ of matrix

$$d\hat{A}(0, 0) = \begin{pmatrix} 1 + 2\pi h\cos(2\pi u_0) & 1 \\ 2\pi\cos(2\pi u_0) & 1 \end{pmatrix} ,$$

defining the linear part of $\hat{A}$ at the point $(0, 0)$, satisfies the inequality $|Spd\hat{A}| < 2$. Therefore the point $(0, 0)$ is of elliptical type. Further we shall use the results of paper [5] (theorem 1 and its proof) from which it follows that on the half-line $h > 0$

there exists an open set H, having infinite Lebesgue measure on the straight line, such that for any $h \in H$ there exists u_0 and integer k such that the equations (4), (5), (6) hold and the fix point $(0,0)$ of the map $\hat{A}$ is a point of general elliptical type, i.e. in its normal Birkhoff form (see [11]) $\theta' = \theta + a_0 + a_1\tau + \ldots, \tau' = \tau$ (θ, τ are certain polar coordinates; θ is angle, τ is radius) the coefficient $a_1 \neq 0$. Therefore if $h \in H$ then Moser's theorem [12] can be applied to the map $\hat{A}$. According to this theorem in any neighborhood of the point $(0,0)$ there exists a continuous closed curve arounding the point $(0,0)$ and invariant with respect to $\hat{A}$. Let us take curve γ such that the points $(\tilde{u}, \tilde{z})$ lying inside the domain Γ bounded by this curve, satisfy the inequalities

$$\alpha' - u_0 < \tilde{u} < \beta' - u_0 \ , 2|\tilde{z}| < \epsilon \ , \tag{19}$$

and for any $m \in \mathbf{Z}$ we define the domain $\Pi_m \subset C$ of the following form:

$$\Pi_m = \{u, z : u = \tilde{u} + u_0 \ , z = \tilde{z} + z_m^{(0)} \ , (\tilde{u}, \tilde{z}) \in \Gamma\} \ , \tag{20}$$

where $z_m^{(0)}$ are the quantities introduced in (7) for $m = n$. ¿From the invariance of the curve γ with respect to $\hat{A}$ by virtue of (20), (19), (13), (7), (17) and definitions of maps $\hat{A}, A^{(n)}, \hat{A}^{(n)}$ in (18), (16) and (17) for any $m \in \mathbf{Z}$ we obtain the relations

$$A_*(\Pi_m) = \Pi_{m+1} \ , A_\varphi(\Pi_m) = \Pi_{m-1} \ , \tag{21}$$

and according to (19), (20) and (7) the coordinate z_m of any point $(u_m, z_m) \in \Pi_m$ satisfies the inequality

$$|z_m - mk| < \frac{\epsilon}{2} \ . \tag{22}$$

We prove now that the domains $\Pi_m (m \in \mathbf{Z})$ satisfy the assertions of theorem 2. By virtue of (22) for small ϵ the domains Π_m don't intersect each other for distinct n. Let $(u, z) \in \Pi_m$, $\omega \in \Omega$ and for all natural n we consider points $(u^{(n)}, z^{(n)}, \omega^{(n)}) = B^n(u, z, \omega)$. According to (21) and definition of B there exists integer m_n such that the point

$$(u^n, z^{(n)}) \in \Pi_{m_n} \ . \tag{23}$$

Therefore introducing the value $z^{(n)}$ by the formula $z^{(n)} = km_n + \xi_n$ by virtue of (23), (22), definition 2 and definition of the value $\mathcal{J}_n^{(m)}$ in the formulation of theorem 2 we have

$$|\xi_n| < \frac{\epsilon}{2} \ , \tag{24}$$

$$\mathcal{J}_n^{(m)} = \mathcal{J}_{n,1}^{(m)} + \mathcal{J}_{n,2}^{(m)} + \mathcal{J}_{n,3}^{(m)} + \mathcal{J}_{n,4}^{(m)} \ , \tag{25}$$

where

$$\mathcal{J}_{n,1}^{(m)} = k^2 \int_{\Pi_m \times \Omega} (m_n - m)^2 d\mu, \mathcal{J}_{n,2}^{(m)} = 2k \int_{\Pi_m \times \Omega} (m_n - m)\xi_n d\mu \ ,$$

$$\mathcal{J}_{n,3}^{(m)} = 2k^2 m \int_{\Pi_m \times \Omega} (m_n - m) d\mu, \mathcal{J}_{n,4}^{(m)} = \int_{\Pi_m \times \Omega} (km + \xi_n)^2 d\mu \ .$$

Let us estimate each of the integrals $\mathcal{J}_{n,s}^{(m)}$ $(s = 1, 2, 3, 4)$. To calculate the integrals $\mathcal{J}_{n,1}^{(m)}$ and $\mathcal{J}_{n,3}^{(m)}$ we note, that the quantity $\eta_n = m_n - m$ is the realization of the sum $\sum_{i=1}^{n} x_i$ of n independent random values $x_1, \dots, x_n$ taking on any of two values $x_i = \pm 1 \quad (i = 1, \dots, n)$ with probability $\frac{1}{2}$, and therefore according to well known statements of probability theory ([8]) we have the equalities $\int_\Omega (m_n - m)^2 dp = n$, $\int_\Omega (m_n - m) dp = 0$. Substituting these equalities into integrals $\mathcal{J}_{n,1}^{(m)}$ and $\mathcal{J}_{n,3}^{(m)}$ we obtain the following relations:

$$\mathcal{J}_{n,1}^{(m)} = k^2 n \int_{\Pi_m} dudz \ , \mathcal{J}_{n,3}^{(m)} = 0 \ . \tag{26}$$

By virtue of the independence of $m_n - m$ and ξ_n

$$\mathcal{J}_{n,2}^{(m)} = 0 \ , \tag{27}$$

and the inequality

$$|\mathcal{J}_{n,4}^{(m)}| < (km + \epsilon)^2 \int_{\Pi_m} dudz \tag{28}$$

are valid. According to (2), (21) and the formulation of theorem 2 the area $\int_{\Pi_m} dudz$ of the domain Π_m on the cylinder C does not depend on m and is equal to κ. Then substituting (26), (27) and (28) into (25) we obtain (3). Theorem 2 is proved.

Corollary. If H is the set constructed in theorem 2 and $h \in H$, then the map $\hat{B}$ of the space $\hat{W}$ constructed in the proof of theorem 1, is not ergodic.

The corollary follwos from the proof of theorem 2 since the set $\Pi_0 \times \Omega$, constructed there is invariant with respect to $\hat{B}$.

The result of theorem 2 in which instead of asymptotic equality (3) there are asymptotic inequalities was obtained in [6].

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