Breaking of linear symmetries and Michel theory: degeneracy, Grassmann manifolds, and invariant subspaces

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Summary. Michel's theory of symmetry breaking in its original formulation has some difficulty in dealing with problems with a linear symmetry, due to the degeneration in the symmetry type implied by the linearity of group action. One can usually circumvent this difficultyu by quotienting out linear relations, but here we propose a more intrinsic, and fully geometric, approach to the problem, making use of Grassmann manifolds. In this way Michel theory can also be applied to the determination of dynamically invariant manifolds for equivariant nonlinear flows.

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1. Introduction and motivations.

The L. Michel's geometric theory of symmetry breaking [1-5] allows to identify points which are critical for *any* G-invariant potential defined on a smooth G-manifold, where G is a compact Lie group. The fundamental result in this theory is stated by saying that a G-orbit is G-critical if and only if it is isolated in its stratum; this means essentially that all nearby orbits have different symmetry properties (see below for definitions, more detail, extensions and limitations).

However, when G acts in $\mathbf{R}^N$ via a linear representation, it is unavoidable that the only orbit which can possibly be isolated in its stratum is the trivial one, made of the origin alone, due precisely to the degeneration introduced by the linearity of the group action. In this case one can still extract useful information not only by the complex of Michel's theory – which indeed provides a quite detailed understanding of the orbit space in terms of variational properties – but also from the simple enunciation of the main theorem given above in non-rigorous way: this is usually done, explicitely or implicitely [6-9], by "quotienting out" the radial direction, i.e. considering the unit sphere S^{N-1} in $\mathbf{R}^N$ – as on this one can have orbits isolated in their stratum – and then reintroducing in the analysis the radial direction again.

This procedure has proved adequate to all the applications of Michel's theory in the case of linear G-action, but is quite unsatisfactory in that on the one side the easiest case (linear action in $\mathbf{R}^N$) appears more difficult – in principles if not in practice – to deal with than other ones, and also in that it requires to introduce analytical constraints (i.e. consider the subset $|x| = 1$) in the context of a theory which is purely geometrical.

The purpose of this note is to analyze Michel's theory in this case, namely linear G action in $\mathbf{R}^N$, resorting to the natural geometric framework for such a setting, i.e. in terms of *Grassmann manifolds*.

This will turn out to have some useful fallouts, in that we can extend the method and results of Michel to the search for invariant submanifolds under an equivariant dynamics (as these will be points in the appropriate Grassmannian).

It would also be possible to consider the case of tangent bundles to smooth G-manifolds M: in this frame, the fibers $\mathrm{T}_x M$ are linear spaces, on which we can consider the linearization $(DG)(x)$ (if well defined), so that the analysis developed here does actually extend beyond the linear frame.

It should also be mentioned that a different kind of motivation exists for setting Michel's theory in terms of Grassmannians: this has to do with the theory of integrable systems [10,11], and with recent work by L. Martinez et al. [12] in which they identify integrable hierarchies associated to singular strata of the infinite Grassmannian. The connection with their work will not be touched upon here, but will be discussed elsewhere.

2. The geometry of symmetry breaking.

In this section, I will briefly recall the basics of Michel's geometric theory of symmetry breaking; for further detail, the reader is referred to the original papers [1,2], or to some of the papers providing extensions and/or applications of it and including a detailed exposition [3-7,13-16], see in particular [5].

We consider a smooth[1] manifold M, on which is defined an action of a compact, connected, semisimple Lie group G, so that M is a G-manifold.

It should be mentioned that the compactness assumption could be somewhat relaxed, in particular when we consider variational problems defined on Sobolev spaces of sections of a fiber bundle [17-19]; however, we prefer to keep to as simple a (meaningful) setting as possible, in order to focus on the geometric ideas. Similarly, the extension to non-connected groups would be obvious, but we will not consider it for the same reason.

To any point $x \in M$ we associate an isotropy subgroup,

$$G_x \ := \ \{ g \in G \ : \ gx = x \} \tag{1}$$

and we will call the equivalence class of G_x under conjugation in G the *isotropy type* of x. Points on the same G-orbit have the same isotropy type, as for $y = gx$ we have immediately

$$G_y \ = \ g \, G_x \, g^{-1} \ . \tag{2}$$

The set of points having the same isotropy type is called a *stratum*; thus the stratum of x is defined as

$$\sigma(x) \ := \ \{ y \in M \ : \ G_y = h G_x h^{-1} \ , \ h \in G \} \ ; \tag{3}$$

it should be stressed that isomorphism of isotropy subgroup is *not* enough for belonging to the same stratum (for a simple example, consider $\mathbf{R}^2$ with $G = Z_2^x \times Z_2^y$ given by reflections across the x and y axes: nonzero points p on the axes all have $G_p = Z_2$, but Z_2^x is not conjugate to Z_2^y in G). Under the present hypotheses, strata are smooth submanifolds in M.

Let us now consider the *orbit space* $\Omega := M/G$ (this is also called the "configuration space" in physical literature, to stress the fact that G-related states should be seen as completely equivalent). To any orbit we can associate, thanks to (2), an isotropy type: thus, an isotropy stratification can also be defined in Ω.

The space M/G is not, except in highly exceptional cases, a manifold. However, at least for G compact, it is a *stratified manifold*, i.e. the disjoint union of manifolds.

[1] In this note, smooth will always mean $\mathcal{C}^\infty$.

As an example of a stratified manifold, the reader can think of the cube: this is the union of its interior (a three-dimensional manifold), the interior of the faces (six two-dimensional manifolds), the interior of the edges (twelve one-dimensional manifolds), and the vertices (six zero-dimensional manifolds). It happens that this example illustrates several general features of stratified manifolds: in particular, there is a unique generic stratum, open and dense, and strata of dimension k are in the frontier of strata of dimension $m > k$; both of these features are always true for stratified manifolds. An analysis, or even a very sketchy summary, of stratified geometry would be out of place here [20].

To avoid any confusion, we will call these strata *geometric strata*, or also Whitney strata; the strata previously defined in terms of isotropy types will be called *isotropy strata*, or also Michel strata. As we will mainly deal with the latter kind, whenever I write just "stratum" it is understood I am referring to an isotropy stratum.

Thus, Ω has a stratified structure both in terms of geometric strata and in terms of isotropy strata. The two stratifications happen to be compatible: any isotropy stratum of Ω is the union of geometric strata (a more detailed analysis, aimed at the study of symmetry breaking, in terms of algebraic invariants for G and of algebraic geometry is given e.g. in the works by Sartori and collaborators [5,13-16]; see also [21-23]). This means in particular that isotropy strata are smooth submanifolds of Ω.

In the study of equivariant dynamics, one also associates to a subgroup $H \subseteq G$ the space of fixed points (of M) under H,

$$M^H \ := \ \{x \in M \ : \ gx = x \ \forall g \in H\} \ = \ \{x \in M \ : \ H \subseteq G_x\} \tag{4}$$

(notice no G-conjugation enters in this definition); when we are specially interested in the case $H = G_x$ for some point $x \in M$, we write

$$M^{(x)} \ := \ \{y \in M \ : \ G_y \subseteq G_x\} \ . \tag{5}$$

Notice that $M^{(x)} \subseteq \sigma(x)$; the equality here corresponds to exceptional situations, and requires in particular that G_x is a normal subgroup of G, since $\omega(x) \subseteq \sigma(x)$, and $\omega(x) \subseteq M^{(x)}$ requires that $gG_xg^{-1} = G_x$.

Let us now consider a G-invariant potential on M, i.e. a differentiable function $V : M \to \mathbf{R}$ such that, for all $x \in M$ and all $g \in G$,

$$V(gx) \ = \ V(x) \ ; \tag{6}$$

this can also be seen as a potential $\Phi : \Omega \to \mathbf{R}$ defined on Ω, as it is constant along G-orbits. In particular, if x is a critical point of V, so is any point $y = gx$; thus, critical points of G-invariant potentials come in G-orbits, and we can speak of critical orbits.

`Example 1.` Let us consider a trivial example in order to fix ideas: let $M = \mathbf{R}$, and $G = \mathbf{Z}_2 = \{e, h\}$, where the nontrivial element acts as $h(x) = -x$. The orbits are made of two points, except for the exceptional orbit made of the origin $x = 0$ alone. The isotropy subgroup of any point is just $G_x = \{e\}$, except for the origin, for which $G_0 = G$. Thus, the orbit space is $\mathbf{R}_+ = \{x \geq 0\} = \{0\} \cup \{x > 0\}$, this decomposition providing the stratification. ⊙

In this example, one cannot avoid noticing that the special point $x = 0$, which constitutes a stratum by itself, also have the special property of being a critical point of *any* G-invariant (that is, even) potential; conversely, the only point which is critical for any G-invariant potential is precisely $x = 0$. One could wonder if this is a curious property of this trivial – and very special – example, or if it is possible to have the same situation in more interesting cases.

It is remarkable that the latter hypothesis is true, so that the geometrical theory of symmetry breaking can be seen as a way to generalize this situation; the reader tempted to think this is too trivial to give anything interesting is warned that the physical properties of particles in the $SU(3)$ theory of strong interactions can be predicted on the basis of such a generalization [6,7,24], and thus in a *model-independent* way.

We can now come back to discussing the general structure of Michel theory. For any x we consider the linearization of G_x, which acts naturally on the tangent space $\mathrm{T}_x M$ and leaves the origin of this invariant. (In more general case, one is not guaranteed that the G_x-action can be linearized, see [17-19,25]; as we want to deal with linear action, this point will not be of concern here.) The tangent space $\mathrm{T}_x M$ has then different natural splittings, according to different properties: first of all, denoting by ω the G-orbit of x, we have $\mathrm{T}_x M = \mathrm{T}_x \omega \oplus \mathrm{N}_x \omega$ (here $\mathrm{N}\omega$ denotes the normal bundle to ω, seen as a subbundle of $\mathrm{T}M$); moreover the normal space $\mathrm{N}_x \omega \subset \mathrm{T}_x M$ can be separated into a pointwise G_x-invariant part $\mathrm{N}_x^{(0)} \omega$ and a complementary, globally invariant, part $\mathrm{N}_x^{(1)} \omega = \mathrm{N}_x \omega \cap \mathrm{T}_x M^{(x)}$. Thus we write

$$\mathrm{T}_x M \;=\; \mathrm{T}_x \omega \,\oplus\, \mathrm{N}_x^{(0)} \omega \,\oplus\, \mathrm{N}_x^{(1)} \omega \tag{7'}$$

We can also split $\mathrm{T}_x M$ according to the stratification, denoting by σ the stratum of x, as

$$\mathrm{T}_x M \;=\; \mathrm{T}_x \sigma \,\oplus\, \mathrm{N}_x \sigma \;; \tag{7''}$$

indeed, as $\omega \subseteq \sigma$, it is clear that $\mathrm{T}_x \omega \subseteq \mathrm{T}_x \sigma$; notice that we cannot as of now affirm the same for $\mathrm{N}_x^{(0)} \omega$, since this could have a larger isotropy subgroup; however, this is not the case, as it follows by use of *slice theory* [26-28].

A discussion of this would go beyonds our needs here, but let us however recall what a slice is for the sake of completeness of this incomplete introduction. A slice at x is a local[2] submanifold $S_x \subseteq M$ such that $x \in S_x$ and:

(i) S_x is transversal to the orbit $\omega(x)$ and $\mathrm{T}_x S_x$ is complementary to $\mathrm{T}_x \omega(x)$ in $\mathrm{T}_x M$;

(ii) S_x is transversal to all the G orbits which meet S_x;

[2] We stress that in general it is not possible to construct a global slice.

(iii) S_x is (globally) invariant under G_x;
(iv) For $y \in S_x$ and $g \in G$, $gy \in S_x$ implies $g \in G_x$, i.e. G_x is the maximal subgroup which leaves S_x globally invariant; this also implies $G_y \subseteq G_x$.

The G-orbit of S_x gives a tubular neighbourhood $\mathcal{U}(x)$ of $\omega(x)$, which has a natural structure of a G_x fiber bundle over $\omega(x)$, the fiber over $y = gx$ corresponding to $S_y = g\,(S_x)$.

We can rewrite (i) as $\mathrm{T}_x M = \mathrm{T}_x S_x \oplus \mathrm{T}_x \omega(x)$; we also write $S_x^0 = S_x \cap M^{(x)}$. Then it follows from properties (iii) and (iv) of S_x, and the compactness of G_x, that $S_x^0 = \sigma(x) \cap S_x$, and that $\sigma(x) \cap \mathcal{U}(x) = \bigcup_{y \in \omega(x)} S_y^0$. These imply moreover, together with $\omega(x) \subseteq \sigma(x)$, that

$$\mathrm{T}_x \sigma(x) \ = \ \mathrm{T}_x \omega(x) \ \oplus \ \mathrm{T}_x S_x^0 \ . \tag{7'''}$$

Let us now consider the gradient of an invariant potential $V(x)$, which we will denote by $f(x) = \nabla V(x) \in \mathrm{T}_x M$; under the present hypotheses[3], we are guaranteed this is an equivariant vector field on M. In particular, if we consider $g \in G_x$, we have

$$f(x) \ = \ f(gx) \ = \ (Dg)_x \, f(x) \tag{8}$$

so that necessarily $f(x)$ is invariant under $(DG_x)_x$ and thus $f(x) \in \mathrm{T}_x M^{(x)}$.

Remark 1. Notice that if $f(x)$ is invariant under $(Dg)_x$, we have

$$f(x) \ = \ (Dg)_x \, f(x) \ = \ \phi^* g \, f(gx) \tag{9}$$

with $\phi_g : \mathrm{T}M \to \mathrm{T}M$ the lift of $g : M \to M$ to TM (so that ϕ_g^* maps $\mathrm{T}_{gx} M$ into $\mathrm{T}_x M$); thus we cannot conclude that $g \in G_x$, and in general we can only affirm that, in obvious notation, $G_x \subseteq G_{f(x)}$. $\odot$

Obviously, $f(x) \in \mathrm{T}_x M^{(x)}$ implies also that $f(x) \in \mathrm{T}_x \sigma(x)$. Actually, these statements – and the discussion above – hold for any G-equivariant vector field f; when f is the gradient of an invariant potential V, we know in addition that $f(x) \in \mathrm{N}_x \omega(x)$, as V is constant along ω. (Notice that $\mathrm{T}_x M^{(x)}$ could have nonzero intersection with $\mathrm{T}_x \omega(x)$, corresponding to the existence of a nontrivial centralizer $C_G[G_x]$.)

Thus, we have shown that

$$f(x) \ = \ (\nabla V)(x) \ \in \ \mathrm{N}_x^{(0)} \omega(x) \ = \ \mathrm{T}_x S_x^0 \ \subseteq \ \mathrm{T}_x \sigma(x) \tag{10}$$

This proof does essentially reproduce the argument of Michel [1].

[3] Somewhat surprisingly this is not true in full generality, as can be seen by considering $M = \mathbf{R}^2$, $G = \mathbf{R}$ acting by $g_t(x, y) = (x + ty, y)$, and $V(x, y) = y$: it suffices then to consider ∇V on the x axis to see that this is not equivariant [18,25].

One could also proceed the other way round and prove that if $\mathrm{T}_x\sigma(x) \neq \{0\}$, then there is a nonzero $f(x) = (\nabla V)(x) \in \mathrm{T}_x\sigma(x)$, i.e. $\omega(x)$ cannot be G-critical; the proof of this would require again use of slice theory [1].

We have thus in particular that:

Theorem 1. *If an orbit ω is isolated in its stratum, then necessarily $(\nabla V)(x) = 0$ for all points $x \in \omega$, and for* any *G-invariant potential V. Conversely, if $(\nabla V)(x) = 0$ for all points $x \in \omega$, for* all *G-invariant potential V, then ω is isolated in its stratum.*

This statement is also known as Michel's Theorem, although Michel proved quite more than this [1-4], as it is also clear from the above reproduction of his discussion.

Notice that if we have a general equivariant $f(x)$ (that is, not necessarily a gradient), having an orbit isolated in its stratum only guarantees that $f(x) \in \mathrm{T}_x\omega(x)$; recall that a group orbit which is invariant under the dynamical flow is said to be a *relative equilibrium.* Thus we have:

Theorem 2. *If an orbit ω is isolated in its stratum, then necessarily $f(x) \in \mathrm{T}_x\omega(x)$ for all points $x \in \omega$, and for* any *G-equivariant vector field $f : M \to \mathrm{T}M$; thus it is a relative equilibrium for any such f. Conversely, if ω is a relative equilibrium for* all *G-equivariant vector fields on M, then ω is isolated in its stratum.*

3. Linear group action.

The discussion of the previous section applies to any smooth action of a compact Lie group G on a smooth finite dimensional manifold M.

However, if we consider a linear G-action in $\mathbf{R}^n$, it is clear that the only orbit which could possibly be isolated in its stratum is the trivial one made of the origin alone, $\{O\}$. Indeed, the linearity of the G-action implies that x and λx (with $|x| \neq 0$ and $\lambda \neq 0$ a real number) have the same isotropy group: thus we have a degeneracy along radii, and any nontrivial G-orbit $\omega \in \Omega$ belongs to a set of orbits having the same symmetry properties and which is at least one-dimensional.

In this case, as Michel's theory also shows that ∇V will be directed along the stratum, and thus if the stratum is one-dimensional we can still use Michel's theory and reduce the problem to a one-dimensional one; this leads to results of the kind of the "Equivariant branching lemma" [8,24,25,29-34]. However, it is somewhat disturbing that the most simple situation would seem to lead to a less powerful use of the geometrical theory of symmetry breaking.

An alternative way of using the Michel's theorem in the form given above would go simply through selecting a sphere $S^{n-1} \subset \mathbf{R}^n$ (e.g. the unit sphere), analysing $(\nabla V)(x)$ for

$x \in S^{n-1}$, and dealing then separately with the radial direction. This approach is also readily implemented, but again it is unsatisfactory that we have to introduce anaytical considerations (the constraint $|x| = 1$) into an otherwise completely geometric approach.

We are now going to settle this theory in a format which is completely geometric and natural when dealing with linear symmetries, i.e. in terms of *Grassmann manifolds* [35-40]

We will consider $M = \mathbf{R}^n$; all of our considerations will extend immediately to $\mathbf{C}^n$. We recall that $\mathbf{Gr}_{k,n}(\mathbf{R})$ (in the following we just write $\mathbf{Gr}_{k,n}$) is the space of k-dimensional linear subspaces $\mathbf{R}^k$ of $\mathbf{R}^n$; this is given a standard manifold structure, see e.g. [35-40] or any textbook on differential geometry; see also the appendix. In particular, $\mathbf{Gr}_{n-1,n}$ is just the projective space $\mathbf{R}P^{n-1}$.

Thus, a point $p_k \in \mathbf{Gr}_{k,n}$ can be thought of as identified by k linearly independent vectors $\{\xi_1, ...\xi_k\}$, $\xi_i \in \mathbf{R}^n$, or equivalently by the set $\mathbf{P}_k$ of vectors $\eta = \sum_{i=1}^k a^i \xi_i$.

The *linear* G-action on $M = \mathbf{R}^n$, which we denote by L_g to emphsize its linearity, induces a well-defined G-action on $\mathbf{Gr}_{k,n}$: indeed an element $g \in G$ acts on η as

$$g(\eta) \ = \ \sum_{i=1}^k a^i L_g \xi_i \tag{11}$$

and thus a linear k-dimensional subspace $\mathbf{P}_k \subseteq \mathbf{R}^n$ (i.e. a point $p_k \in \mathbf{Gr}_{k,n}$) is transformed into a k-dimensional subspace $L_g(\mathbf{P}_k) \subseteq \mathbf{R}^n$ (i.e. into a point $\widetilde{g}(p_k) \in \mathbf{Gr}_{k,n}$), independent of the choice of basis vectors in p_k. [It is immediate to check that the linear independence of the ξ_i's, together with the group property, implies the linear independence of the $(L_g\xi_i)$'s.]

Notice that if L_g transforms vectors of $\mathbf{P}_k$ into different vectors of $\mathbf{P}_k$, we have $L_g(\mathbf{P}_k) = \mathbf{P}_k$ and $\widetilde{g}(p_k) = p_k$: thus, the representation $\{\widetilde{g}\}$ of G on $\mathbf{Gr}_{k,n}$ can well fail to be faithful or effective even if the representation L of G in M is such.

The isotropy subgroup G_p of $p \in \mathbf{Gr}_{k,n}$ is the set of elements $g \in G$ such that L_g transform all vectors of $\mathbf{P}$ into (possibly different) vectors in $\mathbf{P}$.

`Remark 2.` We stress that there could be points – or linear subspaces – in $\mathbf{P}$ with higher isotropy: thus, the origin will always be invariant under all of G, but belong to any linear subspace; and a subspace $\mathbf{P}_1$ having $G_1 \subset G$ as isotropy group could contain a smaller dimensional subspace $\mathbf{P}_0 \subset \mathbf{P}_1$ which admits G_0 as isotropy group, with $G_1 \subset G_0 \subseteq G$. As already stated in the definition above, G_p will thus be the larger subgroup of G under which *all* points in the subspace $\mathbf{P}_k$ represented by $p \in \mathbf{Gr}_{k,n}$ are mapped into points of the same subspace. $\odot$

`Remark 3.` Suppose now that all points of $\mathbf{P}_k$ are mapped into points of a different subspace $\widetilde{\mathbf{P}}_k$: this means that the two subspaces belong to the same G-orbit, and hence to the same isotropy stratum. $\odot$

Not surprisingly, such a lift from $\mathbf{R}^n$ to $\mathbf{Gr}_{k,n}$ is not possible when we consider a (equivariant) nonlinear flow. Indeed, let us consider $f : M \to \mathrm{T}M$ and the flow on M induced by e^{tf}; it is clear that *in general* a linear subspace $\mathbf{P}_k \subseteq \mathbf{R}^n$ is *not* mapped into a (possibly different) linear subspace by this, i.e.

$$e^{tf} \sum_{i=1}^{k} a^i \xi_i \ \neq \ \sum_{i=1}^{k} b^i \, e^{tf} \xi_i \ . \tag{12}$$

In other words, if we look at the "flow" ϕ induced by f on $\mathbf{Gr}_{k,n}$, this is ill-defined, as a point $p_k \in \mathbf{Gr}_{k,n}$ is in general taken into $p_n = \mathbf{Gr}_{n,n}$ (i.e. the whole space $\mathbf{R}^n$).

It is also clear, however, in particular in view of the discussion in sect.2, that the equivariance of f can entail the f-invariance of certain linear subspaces identified by symmetry properties.

`Example 2.` Thus, if $x_0 \in S^{n-1}$ is isolated in its stratum (upon restriction to S^{n-1}), we know that $f(x_0) \in \mathrm{T}_{x_0}\ell_0$, where ℓ_0 is the line through the origin and x_0. This is a linear one-dimensional space, i.e. a point in $\mathbf{Gr}_{1,n}$, and our discussion of Michel's theorem shows that indeed $e^{t\phi}\ell_0 \ \subseteq \ \ell_0$, for all $t > 0$. $\odot$

`Remark 4.` Notice that had we considered a *linear* vector field f, this would have induced a (local) flow on $\mathbf{Gr}_{k,n}$ (see remark 9 below). $\odot$

In summary, the discussion of this section shows how we can set Michel's theorem in terms of Grassmann manifolds: indeed, the G-action $\widetilde{g} : \mathbf{Gr} \to \mathbf{Gr}$ is well defined, and – as discussed in the next section – it can be used to determine not only f-invariant lines, as in the simple example above, but more general linear subspaces which are invariant under any G-equivariant flow $f : M \to \mathrm{T}M$.

4. Isolated points in Grassmannian stratification.

Let us now make the argument mentioned in the previous section more precise. We will deal with equivariant flows $f : M \to \mathrm{T}M$ and, to avoid unessential complications (i.e. not to have to discuss possible degeneracies), we assume that $f(0) = 0$; the same effect would be obtained by requiring that the G-action in $\mathbf{R}^n$ is effective, as in this case the only point such that $G_x = G$ is the origin, and Michel theorem ensures that $f(0) = 0$.

First of all, consider $\mathbf{Gr}_{1,n}$. A point $p \in \mathbf{Gr}_{1,n}$ isolated in its stratum corresponds to a line $\ell_0 \subset \mathbf{R}^n$ isolated in its stratum, and we know that in this case $f : \ell_0 \to \mathrm{T}\ell_0$, by Michel's theorem; thus, points $p \in \mathbf{Gr}_{1,n}$ isolated in their stratum identify one-dimensional linear subspaces $\mathbf{P}_1$ of $\mathbf{R}^n$ which are invariant under *any* G-equivariant flow.

`Remark 5.` Notice that we do not have to worry about $\{O\}$, which is invariant by itself and included in any linear subspace. In this respect, it is appropriate to mention that

$f : \sigma \to \mathrm{T}\sigma$ (here σ any stratum in M) does *not* guarantee that σ is invariant under e^{tf} for all $t \in [0, \infty]$: we can only conclude that $e^{tf} : \sigma \to \bar{\sigma}$ (this is the closure of σ). $\odot$

Let us now consider a higher dimensional subspace $\mathbf{P}_k$, $k \geq 2$. Suppose that all points in $\mathbf{P}_k$ belong to a single stratum σ_0 (so that the isotropy type of $p_k \in \mathbf{Gr}_{k,n}$ is obviously well defined), and that p_k is isolated in its stratum. Then, as we know that for all points $x \in \mathbf{P}_k$ it must be $f(x) \in \mathrm{T}_x\sigma_0$, we can conclude that $\mathbf{P}_k$ is invariant under *any* G-equivariant flow.

The apparent problem with this discussion is that an invariant subspace $\mathbf{P}_k$ could well contain points of higher symmetry (even apart from the origin). However, as mentioned above (remark 2), this is not really a problem as the isotropy group of $p_k \in \mathbf{Gr}_{k,n}$ is defined as the group which maps all points of the corresponding subspace $\mathbf{P}_k \subset \mathbf{R}^n$ into points of the same subspace. Thus, even in this case, the isotropy type of p_k (i.e. the conjugacy class of G_p in G) is well defined, and hence so is its stratum in $\mathbf{Gr}_{k,n}$. We can again conclude that if p_k is isolated in its stratum, it corresponds to a linear subspace $\mathbf{P}_k$ which is invariant under *any* G-equivariant flow.

For the converse (i.e. that these are the only linear subspaces invariant under all G-equivariant flows), rather than dealing directly with slices in $\mathbf{Gr}_{k,n}$ one can use the manifold structure of $\mathbf{Gr}_{k,n}$, or its relation with $O(n)/[O(k) \times O(n-k)]$ (see the appendix): with this, we have to deal with a finite dimensional (smooth, compact) G-manifold, on which Michel's theorem applies.

We summarize our discussion in the following form, in which the natural identification between a point $p \in \mathbf{Gr}_{k,n}$ and a linear subspace $\mathbf{P} \subseteq \mathbf{R}^n$ is understood.

Definition 1. *The isotropy group of $p \in \mathbf{Gr}_{k,n}$ is the larger subgroup $G_p \subseteq G$ such that for all points $x \in \mathbf{P}$ and for all $g \in G_p$, $y = L_g x \in \mathbf{P}$.*

Definition 2. *The (k,n) Grassmannian isotropy stratum of $p \in \mathbf{Gr}_{k,n}$, denoted by $\sigma_{k,n}(p)$ is the equivalence class of points $q \in \mathbf{Gr}_{k,n}$ for which there is an element $g \in G$ such that $G_q = g G_p g^{-1}$.*

Theorem 3. *If a point $p_k \in \mathbf{Gr}_{k,n}$ is isolated in its (Grassmannian isotropy) stratum $\sigma_{k,n}(p_k)$, then it corresponds to a linear subspace $\mathbf{P}_k \subseteq \mathbf{R}^n$ which is invariant under* any *G-equivariant flow f. Conversely, if a linear subspace $\mathbf{P}_k \subseteq \mathbf{R}^n$ is invariant under* all *G-invariant flows $f : \mathbf{R}^n \to \mathrm{T}\mathbf{R}^n$, then $p_k \in \mathbf{Gr}_{k,n}$ is isolated in its (Grassmannian isotropy) stratum.*

`Remark 6.` Notice that for the limit cases $k = 0$ (corresponding to $\mathbf{P}_0 \equiv \{O\}$) and $k = n$ (corresponding to $\mathbf{P}_n \equiv \mathbf{R}^n$) the theorem is trivially true. $\odot$

5. Isolated orbits in Grassmannian stratification.

The reader has possibly noticed that so far we have only considered *points*, rather than *orbits*, isolated in their stratum. This was done in order to discuss first the situation which is simplest to analyze and visualize; however, we are now going to consider G-orbits in $\mathbf{Gr}_{k,n}$ and strata in $\mathbf{Gr}_{k,n}/G$.

As stressed above (see sect.3), the G-action on $\mathbf{R}^n$, being linear, induces an action on $\mathbf{Gr}_{k,n}$. Thus we have $\widetilde{G}_k : \mathbf{Gr}_{k,n} \to \mathbf{Gr}_{k,n}$, and we can consider G-orbits (i.e. orbits under this G-action) in $\mathbf{Gr}_{k,n}$.

It is easy to see, e.g. using the canonical correspondence between points $p \in \mathbf{Gr}_{k,n}$ and k-dimensional linear subspaces $\mathbf{P} \subseteq \mathbf{R}^n$, that $q = \widetilde{g}p$ implies that $G_q = gG_pg^{-1}$. Thus the G-orbit of p is a submanifold of $\sigma_{k,n}(p)$. This means that – as it was the case for the manifold M and the orbit space M/G – the isotropy stratification in $\mathbf{Gr}_{k,n}$ can be lifted to an isotropy stratification of the orbit space $\Omega_{k,n} := \mathbf{Gr}_{k,n}/G$.

It should be clear that, as $\mathbf{Gr}_{k,n}$ is a smooth manifold, all the discussion of sect.2 immediately applies to it. Thus, in particular, the concept of an orbit $\omega \in \Omega$ isolated in its stratum is well defined.

`Remark 7.` Notice that an orbit in $\mathbf{Gr}_{k,n}$ – and hence also a stratum, which is the union of G-orbits – corresponds to the union of linear subspaces, and thus to a submanifold in $\mathbf{R}^n$. For all points $x \in \mathbf{R}^n$ which lie in subspaces $\mathbf{P}$ which are part of a stratum σ corresponding to an isotropy type represented by $G_0 \subset G$, the isotropy subgroup G_x is conjugate in G to a group G_1 such that

$$G_0 \ \subseteq \ G_1 \ . \tag{15}$$

We stress that, contrary to the case of stratification in M or in M/G, in (15) G_1 can be strictly greater than G_0. This is due to the fact that the isotropy subgroup of p (i.e. of a linear subspace $\mathbf{P}$) is the intersection of the isotropy subgroups of all points $x \in \mathbf{P}$: thus a point $x_0 \in \mathbf{P}$ (as already remarked in sects. 3 and 4) can have an isotropy group G_0 strictly containing G_p. $\odot$

It is remarkable that, thanks to the general theorems on geometric and isotropy stratifications, this situation is not any more difficult to deal with than the standard one.

Indeed, we know that points in M having an isotropy type strictly greater than the one of the stratum σ_0, will lie in strata which are on the border of σ_0. Thus, if we consider, as we have to do in our case, the union of σ_0 and of the strata with strictly higher isotropy type, this will just be the closure $\bar{\sigma}_0$ of σ_0. We denote this as $\sigma_0 \cup_{i=1}^b \sigma_0^i$, where it is understood that the strata σ_0^i belong to $\partial\sigma_0$.

Let us come back to the orbit $\omega = G(q) \subseteq \mathbf{Gr}_{k,n}$ and the linear spaces $\mathbf{P}$ corresponding to points $p \in \omega$. We know that all points x belonging to such $\mathbf{P}$ will lie in $\bar{\sigma}_0$, where

σ_0 is the stratum in $\mathbf{R}^n$ with isotropy type equal to that of $\sigma_{k,n}(q)$ (i.e. corresponds to "generic" points in these $\mathbf{P}$). The vector field $f(x)$ is always tangent to $\sigma(x)$, and thus we are guaranteed that for all points $x \in \mathbf{P}$ we have, in a somewhat folkloristic notation,

$$f(x) \ \in \ \mathrm{T}_x(\sigma_0) \cup_{i=1}^{b} \mathrm{T}_x(\sigma_0^i) \tag{16}$$

(obviously only one of these tangent spaces really exists, as x belongs to one and only one stratum in $\mathbf{R}^n$). We are therefore guaranteed that

$$e^{tf} \ : \ \sigma_{k,n}(p) \to \sigma_{k,n}(p) \ . \tag{17}$$

In other words, using the bordering relations among strata in $\mathbf{R}^n$ which corresponds to group/subgroup relations [1-5], we have proved that the isotropy Grassmannian strata $\sigma_{k,n} \subseteq \mathbf{Gr}_{k,n}$ are also f-invariant for any equivariant f.

The same holds for the isotropy Grassmannian strata in the orbit space $\Omega_{k,n} = \mathbf{Gr}_{k,n}/G$; thus, an orbit $\omega \in \Omega_{k,n}$ which is isolated in its stratum is invariant under any G-equivariant flow f.

Again, for the converse one can use the standard manifold structure of $\mathbf{Gr}_{k,n}$ discussed in the appendix, and make use of Michel theorem.

We summarize our discussion in a way similar to the one adopted at the end of the previous section 4.

Lemma 1. *The (k,n) Grassmannian isotropy stratification of $\mathbf{Gr}_{k,n}$ extends to an isotropy stratification of the orbit space $\Omega_{k,n} := \mathbf{Gr}_{k,n}/G$.*

Theorem 4. *If an orbit $\omega \in \Omega_{k,n} = \mathbf{Gr}_{k,n}/G$ is isolated in its [(k,n) Grassmannian isotropy] stratum $\Sigma_{k,n}(\omega)$, then it is invariant under* any *G-equivariant flow f. Conversely, if $\omega \in \Omega_{k,n}$ is invariant under* all *the G-equivariant flows, then it is isolated in its [(k,n) Grassmannian isotropy] stratum.*

6. Variational problems and Grassmann manifolds.

We have seen in sect.2 how Michel theorem allows to identify the points (the orbits) in M which are critical for any G-invariant potential. It is natural to ask if the "Michel theory on Grassmann manifolds" developed here could give some similar result.

First of all, it is clear that, unless V depends only on angular coordinates (and thus is either trivial or singular in the origin), we cannot define a potential on $G_{k,n}$, as $V(x)$ will take different values on different points of $\mathbf{R}^n$ lying in the subspace $\mathbf{P}$ which corresponds to $p \in \mathbf{Gr}_{k,n}$; notice that this also applies to the simplest case of $\mathbf{Gr}_{1,n}$, and of invariant lines ℓ_0 as seen above.

Thus, we can only conclude that, as $(\nabla V) = f$ is an equivariant vector field on M and a point p (an orbit ω) in $\mathbf{Gr}_{k,n}$ isolated in its stratum is invariant under an equivariant flow, at such point the gradient is necessarily directed along the space $\mathbf{P}$ corresponding to p (along the spaces $\mathbf{P}$ corresponding to points of the orbit).

Let us first consider the case of a point p isolated in its stratum: in this case, we can consider the restriction V_p of $V : \mathbf{R}^n \to \mathbf{R}$ to the subspace $\mathbf{P}$ corresponding to the point $p \in \mathbf{Gr}_{k,n}$ (i.e. $V_p : \mathbf{P} \to \mathbf{R}$). The above discussion and theorem 3 show that critical points of V_p are guaranteed to be also critical points of V. Obviously, minima (or maxima) of V_p could be saddle points for V.

Notice that we can use topological tool (e.g. Morse theory, or simpler ones) in order to guarantee the existence of a certain number and kind of critical points for V_p, even in the case V depends smoothly on external parameters. The simplest example of this use is provided, as already mentioned, by the *equivariant branching lemma* [8,24,25,29-34], dealing with the case of an invariant line ℓ_0 or – in the present language – with isolated points in the $(1,n)$ isotropy Grassmannian stratification of $\mathbf{Gr}_{k,n}$.

Notice that when $\mathbf{P}$ contains subspaces $\mathbf{P}_0$ with higher isotropy, one could apply similar considerations, and topological tools, to $\mathbf{P}/\mathbf{P}_0$; a recent – and physically relevant – example of application of this method is provided by [41].

The same kind of reduction approach would also apply to the case of orbits $\omega \subseteq \mathbf{Gr}_{k,n}$ isolated in their stratum $\Sigma_{k,n}$. In this case, it should be observed that the set of points x belonging to linear subspaces $p \in \omega$ can be quotiented in two ways, i.e. by the G action in $\mathbf{R}$, and by the action of $\mathbf{R}$ acting in $\mathbf{R}^n$ as multiplication (i.e. homogeneous dilation, taking a point $x_0 \neq 0$ into the line $\ell_0 = \{ax_0\}$). Now V is constant along G-orbits in $\mathbf{R}^n$, and thus one can reduce to considering the restriction V_p to the linear subspace $\mathbf{P}$ corresponding to *any one* point $p \in \omega$. In other words, we can in this way quotient out the G action, and we are then reduced to the previous case.

It should be mentioned, however, that the kind of "symmetry reduction" results which one can obtain in this way could also be obtained, more directly, using the *symmetric criticality principle* of Palais [18] or, as mentioned in sect.3, by restricting to S^{n-1} and dealing separately with the radial direction.

7. Examples.

In this section, we present some simple examples to illustrate applications of the theorem obtained above. As far as I know, the result obtained in example 6 is new.

`Example 3`. Let us consider $M = \mathbf{R}^2$, with the action of $G = \mathbf{Z}_2^{(x)} \times \mathbf{Z}_2^{(x)}$, corresponding to reflections, respectively, in x and in y. The orbit space correspond to one (say the first) quadrant,

$$\Omega \approx \{(x,y) \ : \ x \geq 0, \, y \geq 0\} \ ; \tag{18}$$

points $\xi = (x,y)$ with $xy \neq 0$ have isotropy group $G_\xi = \{e\}$, points on the x axis $\xi = (x,0)$ have isotropy group $G_\xi = \mathbf{Z}_2^{(y)}$, and points on the y axis $\xi = (0,y)$ have isotropy group $G_\xi = \mathbf{Z}_2^{(x)}$. Notice that $\mathbf{Z}_2^{(x)}$ and $\mathbf{Z}_2^{(y)}$ are not conjugated in G, so that they do not correspond to the same isotropy type, and thus the two axes are not in the same stratum.

In this case the only interesting grassmannian is $\mathbf{Gr}_1$, so we have to look at points $p \in \mathbf{Gr}_1$, i.e. at one-dimensional linear subspaces $\mathbf{P}_1 = (\lambda x_0, \lambda y_0)$, where $x_0^2 + y_0^2 \neq 0$.

If $x_0 y_0 \neq 0$, the isotropy subgroup G_p is just $\{e\}$, as it follows from definition 1 in section 4 (notice $\mathbf{P}_1$ always contains $\{0\}$ which has a larger isotropy, cfr. remark 7). For $y_0 = 0$, $\mathbf{P}_1$ corresponds to the x axis and thus $G_{p_x} = \mathbf{Z}_2^{(y)}$, and similarly for $x_0 = 0$ we have the y axis and $G_{p_y} = \mathbf{Z}_2^{(x)}$.

This means in particular that p_x and p_y are both isolated in their stratum (actually in both cases their stratum does not contain any other point), and according to theorem 3 they correspond to invariant subspaces under any G-equivariant flow on M.

It is immediate to check that indeed the G-equivariance for a vector field $X = f(x,y)\partial_x + g(x,y)\partial_y$ implies

$$\begin{array}{rclcrcl} f(x,-y) &=& f(x,y) & & g(x,-y) &=& -g(x,y) \\ f(-x,y) &=& -f(x,y) & & g(-x,y) &=& g(x,y) \ , \end{array} \tag{19}$$

and thus

$$f(x,y) \ = \ \alpha(x^2,y^2)\, x \ \ ; \ \ g(x,y) \ = \ \beta(x^2,y^2)\, y \ . \tag{20}$$

Thus, the axes are invariant lines, as stated. $\odot$

`Example 4`. Let us consider $M = \mathbf{R}^3$ with $G = SO(2)$ acting as rotations around the z axis. Now the orbit space is identified by the z coordinate, $\Omega = \mathbf{R}^1$. Points on the z axis $\xi = (0,0,z)$ have isotropy $G_\xi = G$, and points $\xi = (x,y,z)$ with $x^2 + y^2 \neq 0$ have isotropy group $G_\xi = \{e\}$.

The G-equivariant vector fields are given by $X = f(x,y,z)\partial_x + g(x,y,z)\partial_y + h(x^2+y^2,z)\partial_z$ such that

$$\begin{array}{rcl} y f_x \, - \, x f_y &=& g \\ y g_x \, - \, x g_y &=& - \, f \ ; \end{array} \tag{21}$$

this yields immediately

$$\begin{array}{rl} f(x,y,z) &= \alpha(x^2+y^2,z)\, x - \beta(x^2+y^2,z)\, y \\ g(x,y,z) &= \beta(x^2+y^2,z)\, x + \alpha(x^2+y^2,z)\, y \ . \end{array} \tag{22}$$

If we look at $\mathbf{Gr}_1$, all points $p \in \mathbf{Gr}_1$ corresponding to linear subspaces $\mathbf{P}_1 = (\lambda x_0, \lambda y_0, \lambda z_0)$ with $x_0^2 + y_0^2 \neq 0$ have isotropy $G_{p_0} = \{e\}$, while p_z corresponding to $\mathbf{P}_1 = (0, 0, \lambda)$ has isotropy group $G_{p_z} = G$. This point is isolated in its Grassmannian isotropy stratum and thus it corresponds to an invariant linear subspaces, as it is indeed immediate to check from (22).

Looking now at $\mathbf{Gr}_2$, all points $p \in \mathbf{Gr}_2$ have trivial isotropy group $G_p = \{e\}$; thus there is no point isolated in its stratum, and therefore no two-dimensional linear subspace invariant under all G-equivariant flows; again it is easy to check this is the case, i.e. that for any two-dimensional subspace $\mathbf{P}_2$ we can find a vector field of the form prescribed by (22) such that $\mathbf{P}_2$ is not invariant. $\odot$

`Example 5.` Let us now look at the case where M is the Lie algebra $\mathcal{G}$ of the group $G = SU(3)$, so that $M = \mathbf{R}^8 \approx su(3)$, on which G acts by the adjoint action: i.e., for $g \in G$, $A \in M$, $g : A \to gAg^{-1}$. We recall that M is the space of three dimensional traceless hermitian matrices.

We will use as a basis in M the Gell-Mann lambda matrices $\lambda_1, ...\lambda_8$ [42,2-7,24]; with these, the Cartan subalgebra $\mathcal{G}_0 = u(1) \oplus u(1)$ generating the center $Z = U(1) \times U(1)$ of G is spanned by λ_3 and λ_8; in physical terms, these correspond respectively to the third direction of isospin I_3 and to the hypercharge Y.

The orbit space $\Omega = M/G$ is two dimensional (corresponding to the two independent eigenvalues of the traceless matrices $A \in M$), and can be represented as the (λ_3, λ_8) plane.

As well known [4,7,24], in Ω there are three strata: a) the origin, with isotropy G; b) the three special lines $\ell_0 = \alpha\lambda_8$ and $\ell_\pm = \alpha(\lambda_3 \pm \lambda_8/\sqrt{3})$, where $\alpha \in \mathbf{R}\backslash\{0\}$, with isotropy $G_0 \simeq G_\pm \simeq SU(2) \times U(1)$; c) the open and dense (generic) stratum, including all other points, with isotropy $U(1)$. As for the orbits, generic ones are seven-dimensional, while those corresponding to $\omega \in \ell_i$ are four dimensional and obviously the origin of M corresponds to a zero-dimensional orbit [4,7,24]; explicit computations are reported in detail in [24].

We can now apply the theorems obtained above; considering $\mathbf{Gr}_{4,8}$ and using theorem 3, we obtain that the four-dimensional spaces corresponding to $G(\ell_i)$ are invariant for any flow in $M = \mathcal{G} = su(3)$ which is equivariant under the adjoint action of $G = SU(3)$.

Similarly, considering $\mathbf{Gr}_{4,8}/G = \Omega_{4,8}$ and using theorem 4, we would come to the same conclusion.

Notice that in the physically relevant [2-7] case where the equivariant flow f is the gradient flow of an invariant potential V (i.e. $f = -\nabla V$) and we look for critical points of V, and thus zeroes of f, this result in itself allows only to reduce to a four-dimensional problem. Actually, quotienting out the G-action we are reduced to one-dimensional problems along the lines ℓ_i (since ∇V is obviously orthogonal to G-orbits when V is G-invariant); if $V(x)$

is known to be convex for $|x| \to \infty$ we can immediately guarantee the existence of critical orbits on the lines ℓ_i, and if the origin corresponds to an unstable critical point we can also guarantee the existence of nontrivial critical orbits on the ℓ_i's [1-8,24]. $\odot$

`Example 6.` (The Nahm equations) As a final and, again, physically relevant example, we consider the Nahm equations [38,43-46]. From the point of view of use here, the Nahm equations describe a dynamical system in which the dynamical variables A_i $(i = 1, 2, 3)$ are the spatial components of a connection form $\alpha = A_\mu \mathrm{d}x^\mu$ on the adjoint bundle to a G principal bundle over $\mathbf{R}^4$ (we assume x^0 represents the time coordinate t); thus we have $M = \mathbf{R}^{3k} = \mathcal{G} \times \mathcal{G} \times \mathcal{G}$, where $\mathcal{G} \simeq \mathbf{R}^k$ is the Lie algebra of the group G.

The Nahm equations are written in the form

$$\frac{dA_i}{dt} \ + \ [A_0, A_i] \ = \ (1/2)\, \epsilon_{ijk}\, [A_j, A_k] \ ; \tag{23}$$

The gauge group Γ modelled on G [of smooth G-valued functions $g(t)$] acts on solutions to the Nahm equation as

$$\begin{aligned} g(A_0) \ &= g\, A_0\, g^{-1} \ - \ g_t\, g^{-1} \\ g(A_i) \ &= g\, A_i\, g^{-1} \ . \end{aligned} \tag{24}$$

Thus by a gauge fixing we can reduce the Nahm equations to

$$\frac{dA_i}{dt} \ = \ \frac{1}{2}\, \epsilon_{ijk}\, [A_j, A_k] \ , \tag{25}$$

(notice A_0 does not enter in the equations any more).

Notice that this flow is still G-equivariant, the G-action being now simply given by

$$g(A_i) \ = \ g A_i g^{-1} \tag{26}$$

i.e. the adjoint action of G on its Lie algebra. We also have a discrete symmetry corresponding to permutations of the three $\mathcal{G}$ factors in M, but we will not use this here.

In the case $G = SU(3)$, for which $M = \mathbf{R}^{24}$ we can use the results obtained above in example 5.

One is in particular interested in a reduction to the special subspaces ℓ_i considered in example 5; it is natural to ask if initial data with $A_i \in \ell_{p_i}$ (where $p_i = 0, +, -$) will originate a flow which remains in such subspaces. A direct proof of this would require involved computations, but we can answer (affirmatively) this question with very little effort by considering $\Omega = \mathbf{Gr}_{12,24}/G$ and in this the orbits corresponding to

$$\omega_{ijk} \ = \ \ell_{p_i} \ \times \ \ell_{p_j} \ \times \ \ell_{p_k} \ \subset \ \mathcal{G} \times \mathcal{G} \times \mathcal{G} \ \ . \tag{27}$$

These orbits are isolated in their stratum, as it follows from the results recalled above [1-7], and thus they are also invariant under any equivariant flow – as stated in theorem 4 – and thus in particular under the flow given by the Nahm equations. $\odot$

8. Discussion.

In this final section, we wish to present briefly some remarks, mainly in view of extensions and/or applications of the approach presented here.

`Remark 8.` First of all, we notice that although we have preferred to deal with $\mathbf{R}^n$ and with real Grassmann manifolds, all our discussion would be the same when dealing with $\mathbf{C}^n$ and with complex Grassmann manifolds. Actually, the real case could in principle present difficulties not present in the complex one, and that is why we preferred to treat it explicitely. $\odot$

`Remark 9.` In her thesis, M. Koenig has shown that one can study the stratification of a manifold (and its orbit space) by considering linear fields alone [47-49] (in this respect, see also the work of Sartori and collaborators [13-16], who follow a different but related approach). As mentioned in remark 4, if the vector fields $f : M \to \mathrm{T}M$ (where $M = \mathbf{R}^n$) we consider are linear, we can also extend them to vector fields $\phi : \mathbf{Gr}_{k,n} \to \mathrm{T}\mathbf{Gr}_{k,n}$. It seems thus reasonable to expect that applying the Koenig's approach in this case one could obtain better results concerning the stratification of Grassmann manifolds, invariant subspaces, and maybe also relevant applications to the variational case. $\odot$

`Remark 10.` Although in this note we only deal with linear G-action in $\mathbf{R}^n$, it appears that this approach can also be extended to a more general setting, i.e. to a general, possibly nonlinear, action of a group G (possibly non-compact) on a smooth manifold M (possibly infinite dimensional). In this case, provided the G-action is such to be linearizable at all points of M, or at least at all G-invariant points M^G, one can study the linearization $(DG)_x$ acting on $\mathrm{T}_x M \approx \mathbf{R}^n$ and repeat the same analysis conducted here. The invariant linear subspaces identified in this way would then be tangent subspaces to invariant (possibly, only local) submanifolds in M. $\odot$

`Remark 11.` Further extension is also possible to infinite dimensional situations, provided the (now, infinite dimensional) manifold M and the group (now, possibly noncompact) G and its action satisfy suitable hypotheses. In particular, the same results would continue to hold if we were considering a Sobolev space of sections of a fiber bundle with G compact, or even for G non-compact provided some other conditions are met. Suitable conditions are discussed e.g. by Palais in [18]; see also [25]. $\odot$

`Remark 12.` As shown in section 2, Michel theory has a natural setting in terms of equivariant – not necessarily gradient – vector fields, in which case isolated orbits correspond to relative equilibria for any equivariant vector field. For infinite dimensional cases, it is known [50] that one can have common solutions to all the differential equations with a common symmetry (in a given function space). It is not difficult to see, comparing the discussion given here and the one in [50] that such an extension would also go through to identify relative equilibria, i.e. invariant manifolds, in this wider case. However, we prefer to postpone such a discussion to a more complete treatment of the infinite dimensional case. $\odot$

`Remark 13.` As mentioned in the Introduction, recently Manas, Martinez Alonso and Medina noticed that a relation exists between singular strata in the infinite Grassmannian and integrable systems, which they call the "hidden KdV hierarchies"; these were known before [51,52], but their relation to singular strata was not noticed; on the other side, Adler and van Moerbeke already noticed connection between specific features of functions associated to the KP hierarchy and to the multiperiodic Toda flows [53,54]. This was indeed a main motivation for the present work, and we hope this can be used in the future as a first step towards an analysis of these questions related to the singular strata in the infinite Grassmannian. $\odot$

`Remark 14.` The main motivation for the present work was provided by the analysis of the degeneration implied by linearity of group actions in Michel's theory, and our whole discussion has been based on linear concepts, first of all the very one of Grassmann manifold. Thus asking if a "nonlinear version" of the theory developed here is possible, could at first appear to be contradictory, and – even worse – unuseful.

However our discussion, beside providing a proper geometrical setting for dealing with the "linear degeneration" in Michel's theory, also showed that the latter can be of use for the determination of flow-invariant (linear) subspaces for equivariant flows; such an application of Michel's theory (which, as far as I know, went unnoticed so far) calls naturally for an extension to the general case, i.e. the one where the G-action is not necessarily linear, the linear subspaces with same isotropy type are replaced by smooth submanifolds with this property, and even more the linear invariant subspaces are replaced by invariant smooth manifolds.

As mentioned above (see remark 10), one could proceed locally, i.e. considering the linear space $\mathrm{T}_x M$ and the linearization $(DG)_x$ of the G-action on it, at each point $x \in M$. However, a global formulation would obviously be more attractive; this would require to use *extended jet bundles* [55-57] rather than Grassmann manifolds, and would be out of place here, so that we defer such a discussion to a later contribution. $\odot$

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Appendix. Some basic properties of Grassmann manifolds.

Grassmann manifolds are a standard tool in Differential Geometry and many of its applications, including physical and/or physically relevant ones; on the other side, they have been rarely (or not at all) used in the study of Symmetry Breaking or in Nonlinear Dynamics: thus, it seems appropriate to collect here some basic facts about them, for the benefit of readers not familiar with the subject.

We will mainly follow [38]; details on Grassmann Manifolds are to be found in any book on Differential Geometry, e.g. [35-40]. We will discuss real Grassmann manifolds; the construction and results for complex ones are equivalent.

Consider $\mathbf{R}^n$; consider in here the set of k-dimensional linear subspaces $\mathbf{R}^k \subset \mathbf{R}^n$. This set can be given a structure of smooth manifold as described below. We denote this set by $\mathbf{Gr}_{k,n}$, in anticipation of the introduction of this structure, with which it will actually be the Grassmann manifold.

Denote by $\mathbf{P}^k$ a given k-dimensional linear subspace of $\mathbf{R}^n$, spanned by (say, orthogonal) vectors $\{\xi_1, ..., \xi_k\}$ (where $\xi_i \in \mathbf{R}^n$); we can obtain any other k-dimensional linear subspace by acting on $\mathbf{P}^k$ by an element of $O(n)$, the orthogonal group in n dimensions. Actually, it is clear that elements of the $O(k) \subset O(n)$ subgroup corresponding to transformation of $\mathbf{P}^k$ into itself will map $\mathbf{P}^k$ into itself and should be quotiented out; similarly, elements of the $O(n-k)$ subgroup acting in the space orthogonal to $\mathbf{P}^k$ will have no influence whatsoever. Thus,

$$\mathbf{Gr}_{k,n} \ \approx \ O(n) \, / \, [\, O(k) \ \times \ O(n-k) \,] \ . \qquad (A1)$$

The orbit space $O(n)/[O(k) \times O(n-k)]$ is a smooth manifold, as follows easily from general theorems in Lie group theory (see e.g. [5]) and thus (A1) shows immediately that $\mathbf{Gr}_{k,n}$ is a manifold once equipped with the manifold structure of $O(n)/[O(k) \times O(n-k)]$ (see e.g. [37-39] for details); its dimension is therefore

$$\dim \mathbf{Gr}_{k,n} \ = \ \dim O(n) \ - \ \dim O(k) \ - \ \dim O(n-k) \ = \ k(n-k) \ . \qquad (A2)$$

For a construction of $\mathbf{Gr}_{k,n}$ based on linear algebra rather than Lie groups, we can proceed as follows. Consider the set $\mathbf{M}_k$ of $(k \times n)$ matrices of maximal rank $k \le n$. To the k linearly independent vectors ξ_i of components $(\xi_i^1, ... \xi_i^n)$ identifying $\mathbf{P}^k$ we associate a matrix $M \in \mathbf{M}_k$ by $M_{ij} \ = \ \xi_i^j$.

Consider now $G_k = \mathrm{GL}(k, \mathbf{R})$: for any $g \in G_k$, the matrix $M' = gM$ represents the same $\mathbf{P}^k$, as it just provides a change of basis in $\mathbf{P}^k$. Thus

$$\mathbf{Gr}_{k,n} \ \approx \ \mathbf{M}_k \, / \, G_k \ . \qquad (A3)$$

Consider now $M \in \mathbf{M}_k$; we can write it as a block matrix $M \ = \ (A_1 \,|\, A_*)$, where A_1 is a $(k \times k)$ matrix – say for simplicity corresponding to the first k columns of M – of maximal rank (this is always possible as M is of rank k), and A_* is a $[k \times (n-k)]$ matrix. Then, acting by $A_1^{-1} \in G_k$ we can always obtain as representative of the G_k coset to which M belongs, the matrix

$$\widetilde{M} \ = \ \left(I \,|\, A_1^{-1}\, A_*\right) \ . \tag{A4}$$

Thus, $\mathbf{M}_k/G_k$ corresponds to the real matrices $A_1^{-1}\, A_*$; notice these are just the matrices A_* under a change of variables, and that no restriction whatsoever existed on the A_*; thus $\mathbf{M}_k/G_k$ corresponds to the space of $[k \times (n-k)]$ real matrices, which is isomorphic to $\mathbf{R}^{k(n-k)}$, and can be given its manifold structure.

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