Homoclinic splitting, II. A possible counterexample to a claim by Rudnev and Wiggins on Physica D.

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Abstract: Results in the mentioned paper do not seem correct.

§**1.** *Introduction.*

In the paper [RW] it is claimed that the "quasi flat" estimates in the paper [G3] can be improved (see p. 9,17 of [RW]). We do not understand and below we explain why.

Using the notations of [RW], a point in phase space is $\vec\Gamma=(x,\vec\varphi,y,\vec I)$. In their definition the homoclinic splitting is the difference in the action coordinates $\vec I$ of the points $\tilde\Gamma^u$ and $\tilde\Gamma^s$ which are on the stable and unstable manifolds of an invariant torus and which at time $t=0$ have $x,\vec\varphi$ coordinates $x(0,\vec\alpha)=\pi+\xi(\alpha,\mu),\vec\varphi(0,\vec\alpha)=\vec\alpha+\vec\zeta(\vec\alpha,\mu)$ where μ is the perturbation parameter. The homoclinic splitting is $\vec\Delta(\vec\alpha,\mu)=\vec I^s(\vec\alpha)-I^u(\vec\alpha)$. The $\xi(\vec\alpha,\mu),\vec\zeta(\vec\alpha,\mu)$, "initial data", were not chosen 0 in [RW], so that $\vec\alpha$ is just a parameter for the initial data.

We found several points in [RW] that we could not understand and, thinking that they either were simply too difficult for us or that they were related to trivial typos, we thought that we could only check the conclusions in a simple case. We take, first, the model in eq. (8) of [RW] with a *one dimensional* rotator and with $F(x,\varphi)=(1-\cos x)^2\cos\varphi$:

$$\omega I+\frac{y^2}{2}+\varepsilon\,(\cos x-1)+\mu(1-\cos x)^2\cos\varphi \tag{1.1}$$

which is a special *Thirring model.* For purposes of comparison with [G3] model (1.1) translates, after rescaling actions by $\sqrt\varepsilon$ and time by ε: $\omega I\varepsilon^{-1/2}+\frac{y^2}{2}+(\cos x-1)+\mu(1-\cos x)^2\cos\varphi$ where μ is now the previous μ times ε^{-1}. What one has to prove, as a particular case of theorem 2.1 of [RW], is that the Fourier transform $\widehat\Delta_k$ of the splitting $\vec\Delta(\vec\alpha)$ defined in (20) of [RW] verifies the unlabeled inequality in theorem 2.1 of [RW]:

$$|\widehat\Delta_k|\le const\, e^{-|k|\omega\frac{\pi}{2\sqrt\varepsilon}} \tag{1.2}$$

where the constant depends upon k,μ,ε and can be bounded by a power of ε and of μ. Hence for $k=2$ it must be, to leading order as $\varepsilon\to0$:

$$|\widehat\Delta_2|\le const\, e^{-2\omega\frac{\pi}{2\sqrt\varepsilon}} \tag{1.3}$$

It is easy to check that this is true at first order. The second order is only sketched in [RW], and they seem to claim that it does not matter; *i.e.* that it obviously verifies (1.3) togheter with the higher orders. What one expects from [G3], *unless* the bound (8.1),(8.2) of [G3] is not optimal, would rather be:

$$|\widehat\Delta_2^{(2)}|\le const\, e^{-\omega\frac{\pi}{2\sqrt\varepsilon}} \tag{1.4}$$

to leading order as $\varepsilon\to0$, *i.e. much larger.* Since *of course* the same result arises when the number of rotators is >1 (it suffices to to think that (1.1) has also a third degree of freedom whose angle φ' is cyclic) then either there is a cancellation or there is an inconsistency between [RW] and [G3].

§**2.** *A second order analysis.*

A cancellation cannot be excluded without a calculation (or an *a priori* argument). The formalism of [RW] (borrowed from [G3]) allows easily to perform the calculation. To second order the splitting vector $\Delta^{(2)}$ is deduced from eq. (63),(64),(66) of [RW], and one finds:

$$\Delta^{(2)}(\alpha) = \int_{-\infty}^{\infty} dt\ \partial_{\varphi x} f(t)\, \mathcal{O}(\partial_x f)(t) \tag{2.1}$$

with $\xi_i(t)$ defined in eq. (56) of [RW] and: $\mathcal{O}(F)(t) = \int_{\rho\infty}^{t} \big(\xi_2(t)\xi_1(\tau) - \xi_2(\tau)\xi_1(t)\big) F(\tau) d\tau$ if $\rho = \rho(t)$ is the *sign* of t. Eq. (2.1), explictly spelled out *up to terms indicated by* $+\ldots$ *and contributing to the Fourier transform* $\widehat{\Delta}^{(2)}_k$*, for* $k = 2$*, quantities of order* $O(e^{-\pi\frac{\omega}{\sqrt{\varepsilon}}})$ *which can be neglected for our purposes,* becomes (see (1.3)):

$$\begin{aligned} \Delta^{(2)}(\alpha) = -2 \int_{-\infty}^{\infty} dt\, \rho(t) \cdot (\cos x(\tau) - 1) \cdot \sin x(\tau)\ \cdot \sin(\alpha + \omega t) \cdot \\ \cdot \int_{-\infty}^{\infty} d\tau \cdot \big(\xi_1(t)\xi_2(\tau) - \xi_2(t)\xi_1(\tau)\big) \cdot (\cos x(\tau) - 1) \cdot \sin x(\tau) \cdot \cos(\alpha + \omega\tau\big) + \ldots\ , \end{aligned} \tag{2.2}$$

with $x(\tau) = 4 \,\mathrm{arctg}\, e^{-\sqrt{\varepsilon}\tau}$. We want to show that $\widehat{\Delta}^{(2)}_2$ is of order $O(e^{-\frac{1}{2\sqrt{\varepsilon}}\pi\omega})$ and not vanishing. The integrals in (2.2) factorize: they are elementary and can be successively computed (or found on tables of integrals). We only give the final result:

$$\widehat{\Delta}_2 = \varepsilon^{-\frac{5}{2}} \frac{3\pi^2\omega^2}{4i} e^{-\frac{1}{2}\frac{\pi\omega}{\sqrt{\varepsilon}}}\ , \tag{2.4}$$

to leading order as $\varepsilon \to 0$. One can get also the subleading orders exactly, but there is no point to that since equation (2.4) contradicts (1.3) hence [RW], while agreeing with (1.4) *i.e.* with [G3]. We cannot explain this contradiction and, unless it arises from a misundertsanding by us of the ideas in [RW], it indicates that theorem 2.1 is invalid and theorem 2.3 cannot be deduced from it. We hope that this note will generate the curiosity of some colleague who will explain where we err in the above remark, if we do.

Acknowledgements: We are indebted to G. Benfatto for many discussions and suggestions.

References

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