Melnikov's approximation dominance. Some examples.

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ABSTRACT. *We continue a previous paper to show that Mel'nikov's first order formula for part of the separatrix splitting of a pendulum under fast quasi periodic forcing holds, in special examples, as an asymptotic formula in the forcing rapidity.*

§0. Introduction.

Recently there has been renewed interest in a problem treated in the paper [G3]. One of the several questions posed in [G3] was to find upper and lower bounds on the splitting between the stable and unstable manifolds of the invariant torus with rotation vector $\underline{\omega} = \eta^{-\frac{1}{2}}\underline{\omega}_0$ in the Hamiltonian system ("*Thirring model*"):

$$\eta^{-\frac{1}{2}}\underline{\omega}_0\cdot\underline{A} + \frac{1}{2J}\underline{A}\cdot\underline{A} + \frac{I^2}{2J_0} + g^2 J_0(\cos\varphi - 1) + \mu f(\underline{\alpha},\varphi)\ , \tag{0.1}$$

where $(I,\varphi)\in\mathbb{R}\times\mathbb{T}, (\underline{A},\underline{\alpha})\in\mathbb{R}^{l-1}\times\mathbb{T}^{l-1}$ are canonically conjugated variables, $\underline{\omega}_0\in\mathbb{R}^{l-1}$, and $J, J_0>0$ (respectively, "rotators' moments of inertia" and "pendulum's moment of inertia"), $g>0$ (g^2 is the "gravity"); $\underline{\omega}_0,\mu$ are parameters. Here J could be a scalar or a diagonal $(l-1)\times(l-1)$ matrix. And setting $\nu\equiv(n,\underline{\nu})\in\mathbb{Z}^l$, $|\nu| = |n| + |\underline{\nu}| = |n| + \sum_{i=1}^{l-1}|\nu_i|$:

$$f(\underline{\alpha},\varphi) = \sum_{\underline{\nu},n;\,|n|\le N_0} f_{\underline{\nu},n}\cos(\underline{\nu}\cdot\underline{\alpha} + n\varphi)\ , \tag{0.2}$$

where f_ν are fixed constants and $N_0>0$ is fixed.[1]

The paper [G3] did solve completely the upper bound question for f a trigonometric polynomial: in the sense that it derived generically optimal bounds for the Fourier transform of the 2–dimensional splitting function $\underline{\Delta}_\uparrow(\underline{\alpha})$ (a vector valued function defined in [G3], p. 366).

Subsequently it was rightly pointed out, [DGJS], that it would be interesting to just have examples in which some lower bound could be computed as given by the naive first (non trivial) order perturbation theory prediction (*i.e.* "*Mel'nikov integral*"). For instance one could study the difference in the free pendulum energy $h_0 = \frac{1}{2J_0}I^2 + J_0 g^2(\cos\varphi - 1)$ evaluated in two "corresponding points" of the stable and unstable manifolds of the invariant torus. Such difference is a very special case of the splitting functions considered in [G3].

Of course studying the splitting only via the variations of a single function ("observable") is very reductive. The stable and unstable manifolds are 2–dimensional surfaces, if observed on a (4–dimensional) section transverse to the motion and at fixed total energy, hence it is clear that studying the difference of the two manifolds at "corresponding points" on the chosen section requires the simultaneous analysis of 2 observables. Thus in

[1] It seems clear that the often sought "first order of perturbation theory" dominance as $\eta\to 0$ can only hold if $N_0<\infty$ and $f_{\underline{\nu},n}$ is nonzero and "quite large" for "many" $\underline{\nu}$'s. Hence $N_0<\infty$ is a reasonable assumption, together with the assumption that we introduce later on $f_{\underline{\nu}}$.

[G3] two independent observables are considered (more are not necessary: the dimension of the manifolds being 2).

One also needs to have good control over the Mel'nikov integral, *i.e.* on the *usually explicitly known* first order (in ε) expression of the splitting: not so easy in general. Control is possible in some rather special cases like the 3 degrees of freedom system (*i.e.* two rotators) with $f(\underline{\alpha},\varphi)=\sum_{\underline{\nu}}f_{\underline{\nu}}(\cos\underline{\nu}\cdot\underline{\alpha})(\cos\varphi)$, where $f_{\underline{\nu}}=e^{-\kappa|\underline{\nu}|}$ or more generally $F'e^{-\kappa|\underline{\nu}|}\le|f_{\underline{\nu}}|\le Fe^{-\kappa|\underline{\nu}|}$ for some $F',F,\kappa>0$, and $\underline{\omega}_0=(\omega_{01},\omega_{02})$ a rotation vector with a golden mean rotation number $\omega_{01}/\omega_{02}=\frac{1}{2}(\sqrt{5}-1)$.

The problem has been studied, [DGJS], without taking into account [G3] but without any saving of work because a theory of the upper bound equivalent to [G3] has to be derived one way or other: and in fact they had to restrict considerations to the *isochronous* cases ($J=+\infty$): taking $J<+\infty$ *does not* change the first order analysis but changes completely the higher orders treatment.

We show here that the analysis of [G3] yields an upper bound on the splitting *optimal and sufficient* to deduce the dominance of Mel'nikov's approximation for the part of the splitting defined by the variation of the function h_0 above defined, if the perturbation is:

$$f(\underline{\alpha},\varphi)=\sum_{\underline{\nu},\,|\underline{\nu}|\le N}f_{\underline{\nu}}\cos\underline{\nu}\cdot\underline{\alpha}\,\cos\varphi,\qquad N\stackrel{def}{=}\eta^{-1}\,, \tag{0.3}$$

with $f_{\underline{\nu}}\ge Fe^{-\kappa|\underline{\nu}|}$ for some $F,\kappa>0$ and all $\underline{\nu}$.[2] This differs from the function considered in [DGJS] by a quantity $O(e^{-\eta^{-1}\kappa})$ which is much smaller than any expected splitting. The latter is expected to be given by the Mel'nikov integral, hence to be of order $\ge O(e^{-O(\eta^{-\frac{1}{2}})})$.

Extending the sum over $\underline{\nu}$ to all $\underline{\nu}$'s would require extending the results in [G3] to the case in which the perturbation is analytic, while one of the "philosophical assumptions" of [G3] was that one should understand first only polynomial cases. However the purpose of [DGJS] was to provide an example of the Mel'nikov integral dominance and the (0.3) is as good for the purpose.

Generalizing [G3] to non polynomial analytic cases is easy: but to limit the length and the *simplicity* of the present comment on [G3] we do not provide the details. We can indicate that an extension to analytic cases has been performed in the very similar problem studied in [BCG] (providing again an example of the Mel'nikov integral dominance).

The fact that the bounds in [G3] are *uniform* in the matrix J (which is summarized in [G3] by calling *twistless* the invariant torus of (0.1) with rotation vector $\underline{\omega}$) will imply, below, that the perturbation f in (0.3) also generates a splitting measured by the variation of the function h_0 that is asymptotically exactly given by Mel'nikov's first order perturbation result for all $J\ge J_0$. The treatment given below being more general than what would be needed to just obtain the splitting of h_0 also solves (*in part*: see below) the problem posed in [RW]: a paper in which the basic strategy is adapted from [G3] but which seems to contain incorrect usage of [G3] leading to errors in intermediate steps and in the final results, see [GGM3].

The following section *assumes full* knowledge of [G3] and we consider it just as a (conceptually trivial) final comment to it, made interesting by the idea of [DGJS] of studying lower bounds for single observables: therefore we call it §9 (§10 below contains a few comments to relate the results to intervened papers), as the paper [G3] ends rather abruptly (because of exhaustion of the author, and of the problem) at §8. We use the notations of [G3] and we freely quote the formulae in the eight sections of [G3]. The following section has simply to be regarded as a new section of [G3]. The above is an introduction and formulae have been labeled by 0 to avoid confusion with the labels used in [G3].

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[2] This note is made necessary by recent claims that [G3] would not be sufficient to get an upper bound which would yield the full result of [DGJS], see [St].

The relation between the splitting of the observable h_0 and the 2–dimensional splitting $\underline{\Delta}_{\uparrow}(\underline{\alpha})$ introduced in [G3] is, on the Poincaré section considered in [G3] (*i.e.* at $\varphi=\pi$, and fixed energy) and if $J=+\infty$, simply $-\underline{\omega}\cdot\underline{\Delta}_{\uparrow}(\underline{\alpha})$. In the anisochronous case ($J<+\infty$) it is slightly more involved. Unless explicitly stated, we shall mean that the "splitting" is the vector $\underline{\Delta}_{\uparrow}(\underline{\alpha})$ of [G3], p.366.

§9. Homoclinic splitting in presence of several fast rotators.

An interesting consequence of the theory of §8 is an expression of the splitting in the $l=3$ case and when the vector $\underline{\omega}$ has all components *fast*, *i.e.* $\underline{\omega}=\eta^{-\frac{1}{2}}\underline{\omega}_0$ and $\eta\to 0$, while $\underline{\omega}_0$ is fixed and verifies the Diophantine condition (1.3).

Examining the tree expansion for $\underline{\Delta}_{\uparrow}(\alpha)$ one realizes that $\underline{\Delta}^{h}_{\uparrow\underline{\nu}}$ is given by a sum of tree values *each of which* is bounded by $\xi\equiv e^{-|\underline{\omega}\cdot\underline{\nu}'|g^{-1}(\frac{\pi}{2}-d)}$ (where d is prefixed and chosen equal to $\eta^{\frac{1}{2}}$, as in §8, to treat the case $l=2$) with a $\underline{\nu}'$ in general different from $\underline{\nu}$ but *nonzero*, see p. 379.

Each tree will in general contain several bubbles (see §6, p. 376). Here it will be convenient not to lump together the contributions from the trees with the same *free* structure: rather it will be convenient to leave them separate: thus a tree value will be described as in §6 but its fruits will contain their *seed* inside (see §6, p. 377). We should call them *seeded fruits*, for obvious reasons; below we shall simply call them just "*fruits*": they can be either ripe or dry, as defined in §6, p. 376. Therefore we redecompose the resummation trees into single trees by specifying the seed inside each fruit. This is a minor change and it is convenient as it will show that the bound in (8.1) can be trivially and greatly improved in the case of perturbations like (0.3).

The small factor ξ arises for some $\underline{\nu}'$ with $|\underline{\nu}'|\le Nh$ (N being the *cut–off* parameter $N=\eta^{-1}$) in two different ways.

(1) In trees with free momentum $\underline{\nu}'$ (see §6, p.375) and *only* ripe fruits (see §6, p.376) it arises by bounding carefully (*i.e.* by complex integration) the integrals relative to the free nodes (called v) variables τ_v and by using the trivial bounds to bound the fruit values (*i.e.* with no excursion into the complex plane to estimate them, which of course would not be allowed because of lack of analyticity of the integrals that express the values of ripe fruits).

(2) In trees with at least one dry fruit it arises from a fruit located inside it, perhaps "very deeply" (*i.e.* inside many other fruits), which only contains ripe fruits. It exists necessarily (note that no fruit at all is a special case of only ripe fruits) and in this case the free momentum cannot vanish (see §6): this was the reason why the induction described in §6 worked.

Therefore there is a natural resummation of the splitting series $\sum_{\underline{\nu},h}\varepsilon^h e^{i\underline{\nu}\cdot\underline{\alpha}}\,\underline{\Delta}^{h}_{\uparrow\underline{\nu}}$: namely we separate the first order contribution and *we collect together all contributions from trees whose value is bounded by the same small factor* ξ. In this way:

$$\underline{\Delta}_{\uparrow}(\underline{\alpha})=\sum_{\underline{\nu}\ne\underline{0}}e^{i\underline{\nu}\cdot\underline{\alpha}}e^{-\frac{\pi}{2g}|\underline{\omega}\cdot\underline{\nu}|}\Big(\mu\,\tilde{\underline{\Delta}}^{1}_{\uparrow\underline{\nu}}+\sum_{h=2}^{\infty}\mu^h\,\tilde{\underline{\Delta}}^{h}_{\uparrow\underline{\nu}}(\underline{\alpha})\Big)\,, \tag{9.1}$$

where $e^{i\underline{\nu}\cdot\underline{\alpha}}e^{-\frac{\pi}{2g}|\underline{\omega}\cdot\underline{\nu}|}\,\tilde{\underline{\Delta}}^{1}_{\uparrow\underline{\nu}}$ isolates the first order terms, while the quantity $e^{i\underline{\nu}\cdot\underline{\alpha}}$ $e^{-\frac{\pi}{2g}|\underline{\omega}\cdot\underline{\nu}|}\,\tilde{\underline{\Delta}}^{h}_{\uparrow\underline{\nu}}(\underline{\alpha})$ is defined as the sum of the values $e^{i\underline{\nu}\cdot\underline{\alpha}}e^{-\frac{\pi}{2g}|\underline{\omega}\cdot\underline{\nu}|}\,\tilde{\underline{\Delta}}^{h,\vartheta}_{\uparrow\underline{\nu}}(\underline{\alpha})$ of all trees ϑ of order h containing either only ripe fruits (or no fruits at all) or a dry fruit which has inside a seed that is a tree with only ripe fruits (or no fruits at all); see items (1) and (2) above.

Since a tree can have several fruits which can be "exponentially bounded" there is ambiguity in attributing the terms which contain several small factors $e^{-\frac{\pi}{2g}|\underline{\omega}\cdot\underline{\nu}_j|}$ with momenta $\underline{\nu}_1,\underline{\nu}_2,\ldots$: in the latter cases we just make an arbitrary choice to attribute such terms to $\tilde{\underline{\Delta}}^{h}_{\uparrow\underline{\nu}}(\alpha)$ with some $\underline{\nu}$ among $\underline{\nu}_1,\underline{\nu}_2,\ldots$ (*e.g.* $\underline{\nu}$ is the first among $\underline{\nu}_1,\underline{\nu}_2,\ldots$ in some lexicographic order).

The series involving $\mu^h\,\tilde{\underline{\Delta}}^{h}_{\uparrow\underline{\nu}}(\alpha)$ starts at least at order two because the trees must be at least of second order, having already separated out the first order contribution. Furthermore each tree contains as a factor the product of the couplings $f_{\underline{\nu}_v}$ that correspond to its nodes v with order label $\delta_v=1$ (see p. 367); and the set of the $\underline{\nu}_v$ must be such that *a suitable subset* W of the set of nodes verifies $\sum_{v\in W}\underline{\nu}_v=\underline{\nu}$. This means that we can bound the contribution of each tree ϑ with h nodes v of order $\delta_v=1$ (and hence total number of nodes $m\le 2h$) by the bounds in Appendix A1, in which the part in the constant c_1 due to the factors $f_{\underline{\nu}_v}$ is explicitly factored out:

$$|\,\tilde{\underline{\Delta}}^{h,\vartheta}_{\uparrow\underline{\nu}}(\,\underline{\alpha}\,)|\le J_0gD_0d^{-\beta}N^{4h}(B_0d^{-\beta})^{h-1}h!^p\,|f_{\underline{\nu}_1}|\cdots|f_{\underline{\nu}_h}|\,, \tag{9.2}$$

where β,p are the parameters in (8.1),(8.2) with D_0,B_0 bounded in appendix A1, see (A1.2)÷(A1.15), by $\overline{B}$, for some constant $\overline{B}$ (which is not the same as $c_3^2M_*^2N^{-4}$ as in (A1.15) only because we have factored out the product of the factors $f_{\underline{\nu}_v}$ associated with each of the h nodes or order 1).

Note that in general the label $\underline{\nu}$ in $\tilde{\underline{\Delta}}^{h,\vartheta}_{\uparrow\underline{\nu}}(\,\underline{\alpha}\,)$ is not equal to the total momentum of the tree ϑ (*i.e.* $\tilde{\underline{\Delta}}^{h}_{\uparrow\underline{\nu}}(\,\underline{\alpha}\,)$ *is not* a Fourier component, as it depends explicitly on $\underline{\alpha}$).

It is not difficult to see (by working out a special example, see [GGM3] for a trivial example; furthermore the proof of the theorem in §10 of [CG], although incorrect as a proof of the theorem, nevertheless provides a much less trivial example) that the above formula and bounds are optimal, so that a bound like:

$$|\,\underline{\Delta}^{h}_{\uparrow\underline{\nu}}|<B'^{h}\eta^{-\beta'}e^{-\frac{\pi}{2g}|\,\underline{\omega}\cdot\underline{\nu}\,|} \tag{9.3}$$

for the $\underline{\nu}$-th Fourier component of $\Delta_\uparrow(\,\underline{\alpha}\,)$ would be incorrect.[6a]

Looking at (9.1) we can therefore bound the sum over $\underline{\nu}$ and h of the *second* addend by first truncating the series to order $h\le N$: the remainder is estimated, as in §8, via the first of (8.1),[6b] and will be of the order $O((\eta^{-Q}\mu)^N)$ so that if $|\mu|<\eta^{Q+1}$ it has size not exceeding $O(\eta^{\eta^{-1}})$, much smaller than the generically expected contribution from the first order terms.

The first N orders can be bounded (as in (8.6) and using $d=\eta^{\frac12}$) by:

$$\begin{aligned}&J_0gD_0\eta^{-\beta/2}N^4\sum_{h=2}^{N}\mu^hF^h\sum_{\underline{\nu}_1+\ldots+\underline{\nu}_h=\underline{\nu}}e^{-\kappa\sum_j|\,\underline{\nu}_j|}(B_0N^4\eta^{-\beta/2})^{h-1}h!^p\le\\&\qquad\le 2J_0gD_0\eta^{-\beta/2}c^lN^4\big(\mu FB_0N^{l+p+4}l!\eta^{-\beta/2}\big)^2e^{-\kappa|\,\underline{\nu}\,|}\,,\end{aligned} \tag{9.4}$$

where we have used that $\sum_{\underline{\nu}_1+\ldots+\underline{\nu}_h=\underline{\nu}}e^{-\kappa\sum_j|\,\underline{\nu}_j|}\le c^le^{-\kappa|\,\underline{\nu}\,|}(lh)^{lh}$ for some c, because the sum has dimension $\le hl$ (in fact $=h(l-1)$) and we have assumed that μ is so small that $(\mu FB_0N^{l+p+4}l!\eta^{-\beta/2})^2<\frac12$. Hence under the condition:

$$|\mu|\mu_0^{-1}\overset{def}{=}|\mu|\,2D_0c^lN^4J_0g\eta^{-\beta/2}\left(FB_0N^{l+p+4}l!\eta^{-\beta/2}\right)^2<1 \tag{9.5}$$

[6a] The above bound, after summation over h, is claimed to hold in [RW], theorem 2.1, for a quantity that the authors appeared to think, in private correspondence, to be quite simply related to the $\underline{\Delta}_\uparrow(\alpha)$ of [G3] and of this note.

[6b] Which descends, for instance, from the KAM–type theorems in §5 of [CG], hence it holds for *analytic perturbations* and it is uniform in the perturbation as long as that is analytic in $|\mathrm{Im}\,\alpha_j|<\kappa$ and $|f_{\underline{\nu}}|<Fe^{-\kappa|\,\underline{\nu}\,|}$, if κ is a prefixed quantity, *no matter how small.*

the sum in (9.4) is bounded by: $\mu e^{-\kappa|\underline{\nu}|}$. Since $\tilde{\underline{\Delta}}^1_{\uparrow\underline{\nu}}$ is essentially[6c] $e^{-\kappa|\underline{\nu}|}$, we can write

$$\begin{aligned}\underline{\Delta}_{\uparrow}(\underline{\alpha}) = \mu \sum_{\underline{\nu}\neq\underline{0};|\underline{\nu}|\le N} e^{i\underline{\nu}\cdot\underline{\alpha}} e^{-\frac{\pi}{2g}|\underline{\omega}\cdot\underline{\nu}|}\Big(\tilde{\underline{\Delta}}^1_{\uparrow\underline{\nu}} + \mu e^{-\kappa|\underline{\nu}|}\,\underline{\Gamma}_{\underline{\nu}}(\underline{\alpha})\Big)\\ + O(\eta^{\eta^{-1}}) + O(\mu^2 e^{-\frac{\kappa}{2}\eta^{-1}})\,, \qquad |\underline{\Gamma}_{\underline{\nu}}(\underline{\alpha})| < 1\,,\end{aligned} \tag{9.6}$$

where $O(\eta^{\eta^{-1}})$ is a bound on the terms with $h > N$ and $O(\mu^2 e^{-\frac{\kappa}{2}\eta^{-1}})$ is a bound on the terms with $h \le N$ with a bound proportional to $e^{-\frac{\pi}{2g}|\underline{\omega}\cdot\underline{\nu}|}$ with $|\underline{\nu}| > N$.

By using the expression (9.6) for the splitting vector we can compute the determinant of the splitting matrix or, similarly to [DGJS], the variation of the pendulum energy evaluated on the stable and unstable manifolds, both at $\varphi = \pi$.

In both cases, by conforming to the traditional terminology, we call Mel'nikov integral the first (non trivial) order contribution.

If one computes the determinant of the splitting matrix, one has to differentiate once with repect to $\underline{\alpha}$ the splitting vector $\underline{\Delta}_{\uparrow}(\underline{\alpha})$. Since the derivative of the terms arising from the first order contributions is trivial we only need to say that the derivatives of $\underline{\Gamma}_{\underline{\nu}}(\underline{\alpha})$ can be computed graph by graph and in each graph they amount to a multiplication by a component of the total momentum. Therefore under a slightly stronger requirement on μ (since $|\underline{\nu}| < \eta^{-1}$ one has to require that μ is smaller than η times the μ_0 defined above: $|\mu| < \mu_0\eta$) one finds

$$\begin{aligned}\det\partial_{\underline{\alpha}}\,\underline{\Delta}_{\uparrow}(\underline{\alpha})\Big|_{\underline{\alpha}=\underline{0}} = -2\Big(\frac{\pi\,\mu}{g}\Big)^2 \sum_{|\underline{\nu}|,|\underline{\nu}'|\le N} \frac{\underline{\omega}\cdot\underline{\nu}}{\sinh\big[\frac{\pi}{2}\underline{\omega}\cdot\underline{\nu}\,g^{-1}\big]}\cdot\frac{\underline{\omega}\cdot\underline{\nu}'}{\sinh\big[\frac{\pi}{2}\underline{\omega}\cdot\underline{\nu}'\,g^{-1}\big]}\cdot\\ \cdot\Big(f_{\underline{\nu}}f_{\underline{\nu}'}(\underline{\nu}\wedge\underline{\nu}')^2 + \mu e^{-\kappa(|\underline{\nu}|+|\underline{\nu}'|)}\Gamma'_{\underline{\nu},\underline{\nu}'}\Big) + O(\mu^2 e^{-\frac{\kappa}{2}\eta^{-1}})\,,\end{aligned} \tag{9.7}$$

with $|\Gamma'_{\underline{\nu},\underline{\nu}'}| < 1$, under the stated condition on $|\mu|$ (*i.e.* $|\mu| < \eta\mu_0$).

As to the variation of the pendulum energy, by calling $M(\underline{\alpha})$ the Mel'nikov integral (depending on $\underline{\alpha}$, contrary to the *homoclinic determinant* (9.7), which is computed at the homoclinic point $\varphi = \pi$, $\underline{\alpha} = \underline{0}$), one has

$$M(\underline{\alpha}) = \mu \sum_{\underline{\nu}\neq\underline{0};|\underline{\nu}|\le N} e^{i\underline{\nu}\cdot\underline{\alpha}} e^{-\frac{\pi}{2g}|\underline{\omega}\cdot\underline{\nu}|}\,\underline{\omega}\cdot\underline{\Delta}^1_{\uparrow\underline{\nu}} \tag{9.8}$$

so that, for some constant C,

$$\big|\underline{\omega}\cdot\underline{\Delta}_{\uparrow}(\underline{\alpha}) - M(\underline{\alpha})\big| \le C|\mu|^2 \sum_{\underline{\nu}\neq\underline{0};|\underline{\nu}|\le N} e^{-\frac{\pi}{2g}|\underline{\omega}\cdot\underline{\nu}|}|\underline{\omega}|e^{-\kappa|\underline{\nu}|} \tag{9.9}$$

which can be compared with the results in [DGJS], and, unlike [DGJS], it holds also in the anisochronous case.

Note that so far no use has been made of the assumption that $l = 3$ nor that $\underline{\omega}$ has a rotation number equal to the golden mean nor of the contents of the paper [DGJS]. The above (9.8) and (9.7) also seem well suited for extensions to $l > 3$ (or to more general rotation numbers) of the Mel'nikov integral dominance.

Remark. Assuming $f_{\underline{\nu}} = e^{-\kappa|\underline{\nu}|}$ for $|\underline{\nu}| < N$, by an argument of [DGJS] (exposed in §6 of [DGJS], see also §2) we see that, if $l = 3$ and $\underline{\omega}_0$ has rotation number equal to the golden mean ($\omega_{01}/\omega_{02} = \frac{1}{2}(\sqrt{5}-1)$), in (9.8) the Mel'nikov integral dominates the

[6c] Precisely $\underline{\Delta}^1_{\uparrow\underline{\nu}} = -i\underline{\nu}\,f_{\underline{\nu}}\,\frac{2\pi}{g}\frac{\underline{\omega}\cdot\underline{\nu}}{\sinh[\frac{\pi}{2}g^{-1}\,\underline{\omega}\cdot\underline{\nu}]} = -i\underline{\nu}\,f_{\underline{\nu}}\,\Phi_{\underline{\nu}}\,e^{-\frac{\pi}{2}g^{-1}|\underline{\omega}\cdot\underline{\nu}|}$ with $1 \le |\Phi_{\underline{\nu}}| < (2|\underline{\omega}\cdot\underline{\nu}|g^{-1}+1)$.

asymptotics as $\eta \to 0$ if μ is small as indicated. In fact, apart for some "special" small intervals of values of η and of α, *only two addends matter* in (9.8) (see [DGJS]): they correspond to $\underline{\nu}$ equal to $\pm\underline{\nu}_k = \pm(f_{k+1}, -f_k)$ where f_j is Fibonacci's sequence and $k = k(\eta)$ is the value that minimizes

$$\kappa|\underline{\nu}_j| + \frac{\pi}{2}\eta^{-\frac{1}{2}}|\underline{\omega}_0 \cdot \underline{\nu}_j| \tag{9.10}$$

(of course $|\nu_k| \le N$): the corresponding miminum has the form $c(\eta)\eta^{-\frac{1}{4}}$, with $c(\eta) > 0$ an explicitly computable quantity bounded above and below by positive constants, uniformly in $\eta \to 0$. The special values of η are those for which the minimum is taken over *two* values of k, or of η's close enough to such values; the special values of $\underline{\alpha}$ are those for which $|\sin\underline{\alpha}\cdot\underline{\nu}_k| < b$, where $b > 0$ is an arbitarily prefixed quantity (if b is too small then the size of η for which dominance occurs becomes correspondingly smaller). Excluding the just mentioned values of $\underline{\alpha}$ and η the asymptotics of the Mel'nikov integral $M(\underline{\alpha})$ is proportional to $\mu\eta^{-\frac{1}{2}}e^{-c(\eta)\eta^{-\frac{1}{4}}}\sin\underline{\nu}_k\cdot\underline{\alpha}$. Moreover to the r.h.s. of (9.9) the same considerations trivially apply, *i.e.* only two (or four) addends matter so that it is bounded by a constant times $\mu^2\eta^{-\frac{1}{2}}e^{-c(\eta)\eta^{-\frac{1}{4}}}$ and the first order dominance for $\underline{\omega}\cdot\underline{\Delta}_{\uparrow}(\underline{\alpha})$ follows, for all $\underline{\alpha}$ except the special ones. So [G3] and the first order analysis (in [DGJS][6d]) prove the first order dominance not only in the isochronous case, as done in [DGJS], but also in the anisochronous case.

But of course the really interesting quantity is (9.7): *i.e.* the actual splitting at $\underline{\alpha} = \underline{0}$. For this quantity we *cannot* conclude dominance of the first order term, unless some (*likely*, see [GGM1], but yet undiscovered) cancellations take place: i*t is manifest from (9.7) that the would be leading term ($\underline{\nu} = \underline{\nu}' = \underline{\nu}_k$) vanishes.* Hence no conlusion can be drawn, in the class of models considered here, about the splitting at the homoclinic point (besides the one already derived in §8 and saying that it is smaller than any power of η). Unless one considers $\eta \to 0$ along a special sequence in correspondence of which there are *two pairs* of successive Fibonacci's vectors with the same, or almost the same, $\kappa|\underline{\nu}_k| + \frac{\pi}{2}|\underline{\omega}\cdot\underline{\nu}_k|$: in this case, obviously, we have dominance of the Mel'nikov term because no two Fibonacci's vectors are parallel, see (9.7). Note that the expressions "almost the same" and "close enough", used above for the variable η, simply mean that the two Fibonacci's vectors such that the quantity $\beta_k = \kappa|\underline{\nu}_k| + \frac{\pi}{2}|\underline{\omega}\cdot\underline{\nu}_k|$ is closest to its minimum over k for two values of k, say $k, k+1$, and $\beta_k - \beta_{k+1}$ is a quantity that is uniformly bounded as $\eta \to 0$.

§10. Comments.

(1) The above theory is different from that of [DGJS] in a radical way; in fact it works in the isochronous case ($J = +\infty$) as well as in the anisochronous one. *This is a new result* (see however the acknowledgements below). And the last remark shows that the problem of the homoclinic splitting asymptotics with 2 fast forcing frequences is still open if it is intended as defined by (9.7) (as one should in view of the possible applications to heteroclinic chains and Arnol'd diffusion). This is so in spite of apparent claims to the contrary, [St].

(2) In *three* time scales problems (*i.e.* when $\underline{\omega} = (\eta^{-\frac{1}{2}}\omega_{01}, \eta^{\frac{1}{2}}\omega_{02})$ and $g = O(1)$) the above arguments *do not lead to any conclusion* because $\underline{\omega}\cdot\underline{\nu}$ is not always proportional to $\eta^{-\frac{1}{2}}$ but it can *even* be proportional to $\eta^{\frac{1}{2}}$ also for small $|\underline{\nu}|$: so that the exponentially small factor is not always present. This means that *even the individual components of*

[6d] The rest of the paper [DGJS] provides an interesting partial alternative to the work in [G3]; the partiality being mainly due to the isochrony assumption, essential in the theory of [DGJS].

the splitting $\underline{\Delta}_{\uparrow}(\underline{\alpha})$ in general are not only not dominated by the Mel'nikov integral but have size of the order of a power of η!

Regarding the intersection matrix, *if* (9.3) *could be assumed valid three matrix elements would be exponentially small* and also the determinant would be (exponentially small and) dominated by the Mel'nikov integral. But, as we said, (9.3) is false. In fact the intersection matrix has, in this case, *all matrix elements that have polynomial size* as $\eta \to 0$, starting with second order: this part of the theorem in §10 of [CG] is, of course, valid.[6e] And it is perhaps natural to think that *also* the splitting (9.7) has size of order of a power of η. This is however *an error*, as the determinant is indeed *exponentially small* in η, *although the matrix elements have size of the order of a power of* η, due to the presence of cancellations in the determinant. This was proved in [GGM1], where such cancellations are exploited.

The above error was in fact suffered in [CG]: a *trivial computational error* in one of the about twenty terms contributing to the third order crept in and led the authors of [CG] to believe they had checked that *even the determinant of the splitting matrix* would be polynomially bounded away from zero as $\eta \to 0$ (*i.e.* bounded as the individual matrix elements are) under the convergence condition that μ is itself a (large) power of η. In any case these considerations show that there is apparently no hope, without exploiting cancellations in the determinant, to obtain a exponentially small bound (the matrix elements have polynomial size); the derivation of the exponentially small bounds for this case from (9.3), to which many authors seems apparently to believe, is false because (9.3) is false, in general and on the section considered here.

(3) But the paper [CG] *had, nevertheless, laid the foundations of the theory developed in* [G3]: a very simple and systematic theory, because of its field theoretic viewpoint, and very flexible thanks to the possibility of rapid (graphical) comparison of arbitrarily high order terms of perturbation expansions. Enough to cope with the error and to correct it, as shown in [GGM1] (an achievement scarsely appreciated in [St]). In fact the papers [G3], [GGM1] suggest which could be the general solution to the problem of the splitting (9.7) *even when the interaction is a trigonometric polynomial*: see the conjecture in [G4]; a "few" cancellations remain to be checked to prove (or to disprove) the conjecture.

In fact the reason the error was not spotted by the authors of [CG] (it was pointed out by Gelfreich) was due to the *apparent* matching of the result with the intuitive idea that the splitting in three time scale problems would be "large". The splitting, defined as the smallest eigenvalue of the intersection matrix, has the dimension of an action (being the derivative with respect to an angle of the difference in the actions for the stable and for the unstable manifolds) *hence it has to be measured with respect to an action.* In [CG] the relevant action with which to compare the splitting was the size of the gaps, in action space, between the average actions of the invariant tori surviving the perturbation.

The latter gap was estimated in [CG] by a power, as large as wished, of η times $\max\{J_0 g, Jg\}$: hence the splitting was larger than any power of η^{-1} *with respect to the gaps.*

The proof was wrong and it was corrected in [GGM1]. And in [GGM2] it has been shown (to the skeptics, see [St])[6f] that the gaps have *in this case* size of almost $O(e^{O(-\eta^{-1})})$ while the splitting is dominated by the Mel'nikov integral *for generic perturbations*, even if of polynomial type: the Mel'nikov integral being of order $O(e^{-\eta^{-\frac{1}{2}}\frac{\pi\omega}{2g}})$ if the frequency vector is $\underline{\omega} = (\eta^{\frac{1}{2}}\omega_1, \eta^{-\frac{1}{2}}\omega)$. So the splitting is still very large.[6g]

[6e] The theorem was "proved" by showing that three matrix elements had polynomial size while the fourth was "exponentially small": the last claim is incorrect because of a trivial computational error which, if corrected, yields that even the fourth matrix elements is of polynomial size.

[6f] Extending a property and a method that was noted long ago by Neishtadt, [Ne].

[6g] And *even larger than value claimed* in [CG]: the ratio between the splitting and the gap being now

(4) Recently we learnt of Eliasson's remark that the homoclinic splitting $\underline{Q}(\underline{\alpha})$ is a *gradient* $\underline{Q}(\underline{\alpha}) = \underline{\partial}_{\underline{\alpha}}\Phi(\underline{\alpha})$ of some *potential* Φ: this is due to the Lagrangian nature of the stable and unstable manifolds. Once this is known one recognizes that the tree expansion in sections 1÷8 of [G3] provides a power series expansion of the potential Φ, see [G4].

(5) One can apply the above theory to the case in which the dimension of the angles $\underline{\alpha}$ is 1: this is the periodic forcing case discussed in §8. In this case there is no distinction between the (9.7) and (9.8) and one finds an asymptotic expansion for such splittings (see §8). This case is closely related to a corresponding problem in the theory of the standard map: the basic approach is due to Lazutkin and recently there has been renewed interest in it. The paper [Ge], which is nice and accurate, also fills some (apparently) missing points in the original work. The approach in [Ge] is also based on formal power series expansions for the separatrix; but unlike [G3] the control of the remainders is not solely made via the analysis of formal series, thus providing an interesting alternative, although (of course) the amount of work is comparable.

The drawback, at the moment, is the need of an "integrable normal form" near the invariant tori (circles in this case). Therefore, although adapting the method of [Ge] to cover the cases of Hamiltonian systems appears straightforward in the case of isochronous systems (which trivially admit an integrable normal form), a possible extension to the anisochronous case seems to require substantial extra work, at least if one proceeds along the lines of [Ge]. The method of [G3] has not (yet) been applied to study the standard map case.

Likewise the extension to higher dimension seems to require further extra work (a beginning of the theory is in [DGJS]); and so does the extension to three time scales problems.

Once the above extensions will exist they will be be very interesting, and, solving in a substantially different way the problems solved in the latter papers, they will permit a (better) comparison with the methods in [G3] and [GGM1].

(6) Finally the reason why one may hope to get large *absolute*[6h] splitting in three degrees of freedom systems is *not* because some $\underline{\omega}\cdot\underline{\nu}$ (in the small factors $e^{-|\underline{\omega}\cdot\underline{\nu}|\frac{\pi}{2g}}$) becomes small when $\underline{\nu}$ becomes a good rational approximation of $\underline{\omega}^{\perp}$, as sometimes hinted in the literature, see [St]. This is shown by the three time scales problem in (2) above in which $\underline{\omega}\cdot\underline{\nu}$ can be very small even for small $\underline{\nu}$ (*i.e.* if the fast component of $\underline{\nu}$ vanishes) and *nevertheless* the splitting does not have a polynomial absolute size, due to remarkable cancellations (see [GGM1]). One has rather to think that the splitting can become large because the perturbation $f(\underline{\alpha},\varphi)$ has a small analyticity band in complex φ plane: in fact the higher the degree N_0 in φ the smaller it seems that one has to choose μ with respect to η if one wants to guarantee Mel'nikov's approximation to dominate.

Acknowledgements. *The content of this paper was (and is) considered by us a trivial remark on the work [G3]: as one can see in the story [St] this view is violently not shared by other specialists. Therefore we wrote it up to clarify our methods and techniques, which in spite of apparently preconceived claims to the contrary (see [St]), are proving more and more well suited for the problems under analysis. We are greatly indebted to G. Benfatto, G. Benettin and A. Carati for clarifying discussions and for support.*

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$O(e^{O(\eta^{-1})})$, while in [CG] it was claimed to be "only" $O(\eta^{-a})$ for arbitrary a.

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[St] Collection of the comments received by us from colleagues and from referees about the paper [GGM1], and our replies: in http://ipparco.roma1.infn.it.

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