The Distribution of the first return time for rational maps

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Abstract: We obtain exponential error estimates for the approximation of the zeroth return time to the Poisson distribution for rational maps which might have critical points within the Julia set.

1 Introduction

Recently there has been some great interest in studying the rates of mixing in dynamical systems and how it translates in the distribution and convergence of return times. A rather general result of Galves and Schmitt [4] establishes the Poisson distribution of the zeroth return time for a general class of dynamical systems, namely those that are ϕ-mixing. They moreover provide error terms and used this to show in a follow up paper [1] that repetition times for subshifts of finite types are normal distributed. For subshifts of finite type Pitskel [9] proved that return times of all orders are in the limit Poisson distributed, but he does not give any error terms. Using approximations of transfer operators Hirata [7, 8] shows similar results for Axiom A maps. With respect to weaker mixing maps, Poisson distributed return times have been announced by Hirata, Saussol and Vaienti for a one parametric family of interval maps with an indifferent fixed point.

Here we look at rational maps on the Riemann sphere and their equilibrium states on the Julia set. Because of critical points, the mixing properties are weaker than in the cases mentioned above. However, using distortion theorems, it was shown in [3] that the Central Limit Theorem applies. We

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also know that correlations decay exponentially fast [5] and in [6] we proved that return times in the limit are Poisson distributed (for all orders). In this note we restrict ourselves to the zeroth return time and shall provide error terms for its deviation from the exponential distribution (theorem 1).

Let us consider rational functions and assume that μ is an equilibrium state for a Hölder continuous potential f which has a 'supremum gap' $P(f) - \sup f > 0$, where $P(f)$ is the topological pressure of f. Without loss of generality one can assume that $P(f) = 0$.

Let $T : \mathbf{C} \to \mathbf{C}$ be a rational map of degree $d \geq 2$, and denote by J its Julia set. Let $f : J \to \mathbf{R}$ is a Hölder continuous function which satisfies the condition $P(f) - f > 0$ ('supremum gap'), where $P(f)$ is the pressure of f. Then there exists an invariant measure μ on J (μ is conformal with respect to $P(f) - f$). The equilibrium state μ has extensively been studied (see e.g. [2, 3]).

With appropriate branch cuts on the riemann sphere one can define univalent inverse branches S_n of T^n on quasidisks Ω_n (which have piecewise smooth boundary) for all $n \geq 1$. We put $\mathcal{A}^n = \{\varphi(J) : \varphi \in S_n\}$ for the n-cylinders (for simplicity's sake we write $\varphi(J)$ for $\varphi(J \cap \Omega_n)$). Note that by [2] the 'boundary set' $\partial\mathcal{A}^n = \{\varphi(J \cap \partial\Omega_n) : \varphi \in S_n\}$ has zero μ-measure, that is $\mathcal{A}^n$ is a measure theoretic partition of J and the 'interiors' of its atoms are pairwise disjoint (the interior of $\varphi(J)$ is understood to be $\varphi(J \cap \mathrm{int}(\Omega_n))$).

Denote by $A_n(x)$ an atom in $\mathcal{A}^n$ for which $x \in A_n(x)$, and put χ_n for the characteristic function of $A_n(x)$. ($A_n(x)$ is almost always unique.)

In [6] (corollary 20) we showed that the return times are in the limit Poisson distributed for all orders, that is

$$\mu\left(\{y \in J : \xi_t(y) = r\}\right) \to \frac{t^r}{r!}e^{-t}, \tag{1}$$

for μ-almost every x, as n tends to infinity, where

$$\xi_t = \sum_{j=0}^{[t/\mu(\chi_n)]} \chi_n \circ T^j$$

is a 'random variable' whose value measures the number of times a given point returns to $A_n(x)$ within the normalised time $t/\mu(A_n(x))$.

In this note we address the question how fast the convergence is in the case of the zeroth (r=0) return time. If we put

$$\mathcal{N}_t = \{y \in J : \tau_n(y) > t/\mu(A_n(x))\}$$

(zero level set of ξ_t), where $\tau_n(y) = \inf\{k \geq 0 : T^k y \in A_n(x)\}$ is the return time for the set $A_n(x)$, then by equation (1) $\mu(\mathcal{N}_t) \to e^{-t}$ as $n \to \infty$ almost everywhere. This result is based on an application of a theorem of Sevast'yanov. Here however we will use more elementary arguments to get the following error estimate.

Theorem 1 *There exists a $\varsigma < 1$ so that for almost every $x \in J$ there exists a constant C_1 so that*

$$\left|\mu(\mathcal{N}_t) - e^{-t}\right| \leq C_1 \varsigma^n$$

for every t.

2 Mixing rates for rational maps

The following lemma is a pared down version of a result of Denker and Urbanski [2] on the inverse branches of rational maps. (One might have to replace T by a suitable iterate T^m.)

Lemma 2 *Let $\lambda < 1$. Then there exists a constant C_2 so that for every n the inverse branches S_n of T^n on simply connected regions Ω_n (with piecewise smooth boundaries) decompose into two classes, namely the contracting branches S_n' and the non-contracting branches S_n'' satisfying*
(i) $|S_n''| \leq C_2 \lambda^{-n}$
(ii) $\mathrm{diam}(T^k \varphi(\Omega_n)) \leq C_2 \lambda^{(n-k)/2}$ *for S_n' and $k \leq n$.*

We shall need some mixing properties for μ which is the equilibrium state for the potential f. Since f has the 'supremum gap', the number $\rho = e^{\sup f - P(f)}$ is less than 1. If we put $g_n = e^{f+fT+fT^2+\cdots+fT^{n-1}}$ then we obviously get $|g_n\varphi|_\infty \leq C_2 \rho^n$ for some constant C_2.

Lemma 3 *Let $\kappa > 1$. Then there exists a constant C_3 and $\sigma < 1$ so that*

$$\left|\mu(A) \cap T^{-k-n}Q) - \mu(A)\mu(Q)\right| \leq C_3 \sigma^k \kappa^n \mu(Q) |g_n\varphi|_\infty,$$

for all, $k, n > 0$, measurable Q and atoms $A = \varphi(J)$ of $\mathcal{A}^n$, where φ is a suitable inverse branch of T^n.

From now on let κ be so that $\kappa\sqrt{\rho} \leq 1$ and $\kappa\sqrt{\sigma} \leq 1$.

Let us note that if instead of the supremum norm on the right hand side one wants to estimate in terms of the measure of A, then one generally can't control the expanding term κ^n so well and make it grow at an arbitray slow exponential rate. If for instance one allows D be a union of atoms of $\mathcal{A}^n$ (not just contracting ones), then the corresponding mixing property is

$$\left|\mu(D \cap T^{-k-n}Q) - \mu(D)\mu(Q)\right| \leq C_3 \sigma^k \nu^n \mu(D)\mu(Q)$$

where $\nu > 1$ is determined by f, although if one only considers contracting branches, then ν can be replaced by κ. In either case one cannot achieve the ϕ-mixing property (which would require the coefficients on the right hand side to decay to zero independently of the 'cylinder length' n).

Let $0 < p < 1$ be so that $d^p\sqrt{\rho} \leq 1$. In the next lemma we show that those points that return 'too soon' to $A_n(x)$ constitute a zero measure set.

Lemma 4

$$\lim_{n\to\infty} \mu\left(\{y \in J : \tau_n(y) \leq pn\}\right) = 0$$

Proof. Let

$$U_m = \{y \in J : \tau_n(y) = m\},$$

that is $U_m \cap A_\varphi = A_\varphi \cap T^{-m}A_\varphi$ for all $\varphi \in S_n$. With $V = T^m U_{m,v} \cap A_\varphi$ we have $V = A_\varphi \cap T^m A_\varphi$. Let us write $\varphi = \psi^1\varphi^1$, where $\psi^1 \in S_m$ and $\varphi^1 = T^m\varphi \in S_{n-m}$. We proceed inductively and obtain

$$\varphi = \psi^k\psi^{k-1}\cdots\psi^1\varphi^k,$$

where $n = mk + \ell$, $0 \leq \ell < m$, $\psi^j \in S_m$ and $\varphi^k = T^{mk}\varphi \in S_\ell$. Let us note that $T^{mj}V = A_{\varphi^j} \cap A_{\varphi^{j+1}}$ for $j = 1, \ldots, k$, where $\varphi^j = T^{jm}\varphi = \psi^{j+1}\cdots\psi^1\varphi^k$. Since $\mu(A_{\psi^k\cdots\psi^1\varphi^k}) \leq \rho^{n+m}$ we can now estimate

$$\begin{aligned}\sum_{\varphi\in S_n} \mu(U_m \cap A_\varphi) &\leq \sum_{\psi^1,\cdots,\psi^k\in S_m} \mu(A_{\psi^k\cdots\psi^1\varphi^k}) \\ &\leq |S_m|\rho^{n+m},\end{aligned}$$

where there are at most $|S_m|$ choices for ψ^1 and then for every $j = 1, \ldots, k-1$ the $\psi^{j+1} \in S_m$ must satisfy $T^{jm}V \subset A_{\psi^{j+1}} \cap A_{\psi^j}$. For every ψ^j we get a unique ψ^{j+1} since the sets $\psi(J \cap \text{int}(\Omega_m), \psi \in S_m$ are disjoint. Hence the last inequality, where we also used the fact that $\mu(A_{\tilde{\varphi}}) \leq |\tilde{\varphi}|_\infty \leq \rho^{n+m}$ for $\tilde{\varphi} \in S_{n+m}$.

Since by assumption $d^p\sqrt{\rho} \leq 1$ we get

$$\sum_{\varphi \in S_n} \mu(U_m \cap A_\varphi) \leq d^m \rho^{n+m} \leq (d^p \rho^{1/2})^n \rho^{n/2} \rho^m \leq \rho^{n/2},$$

and therefore

$$\sum_{m=0}^{[pn]} \sum_{\varphi \in S_n} \mu(U_m \cap A_\varphi) \leq n\rho^{n/2},$$

which goes to zero as n goes to infinity. □

Lemma 5 *Let us put $B_n = \bigcup_{\varphi \in S_n''} \varphi(J)$. Then for almost every $x \in J$ there exists a number n_0 so that $x \notin B_n$ for all $n > n_0$.*

Proof. The probability of $x \in B_n$ is given by

$$\mu(B_n) = \sum_{\varphi \in S_n''} \mu(\varphi(J)) \leq |S_n''| \rho^n \leq c_1 \rho^{n/2},$$

if we choose λ in lemma 2 so that $\lambda\sqrt{\rho} \leq 1$. Thus

$$\sum_{n=0}^{\infty} \mu(B_n) \leq \frac{c_1}{1-\rho} < \infty$$

which by the Borel-Cantelli lemma implies the result. □

3 Proof of the Main Theorem

Put $h(t) = \mu(\mathcal{N}_t)$, where $\mathcal{N}_t$ is the zero level set of ξ_t. For simplicity put $A = A_n(x)$ and $\mathcal{M}_r = J \setminus \mathcal{N}_r = \{x \in J : \xi_r(x) > 0\}$. We immediately obtain the upper bound $\mu(\mathcal{M}_r) \leq t + \mu(A)$ and a lower bound in the following lemma.

Lemma 6 *There exists an $\eta < 1$ and a constant C_4 so that*

$$\mu(\mathcal{M}_r) \geq r(1 - C_4\eta^n)$$

for μ-almost all $x \in J$.

Proof. Let $A = A_n(x)$, put $B_0 = A$ and define for $j = 1, \ldots, [r/\mu(A)]$

$$\begin{aligned} B_j &= T^{-j}A \setminus \bigcup_{\ell=0}^{j-1} (T^{-j}A \cap T^{-\ell}A) \\ &= T^{-j}\left(A \setminus \bigcup_{\ell=0}^{j-1} (A \cap T^{-\ell+j}A)\right). \end{aligned}$$

Since $\mathcal{M}_r$ is the disjoint union of B_j, we get by invariance of the measure

$$\mu(B_j) \geq \mu(A) - \sum_{\ell=1}^{j} \mu(A \cap T^{-\ell}A).$$

To estimate $\mu(A \cap T^{-\ell}A)$ from above let us note that by lemma 4 for almost every $x \in J$ one has $A \cap T^{-\ell}A = \emptyset$ for $\ell \leq pn$, for some $p \leq 1/2$. Hence, if $\ell \in (pn, n]$, we obtain

$$\mu(A \cap T^{-\ell}A) \leq \rho^{n-\ell}\mu(A),$$

and if $\ell > n$, then we get by lemma 3

$$\mu(A \cap T^{-\ell}A) \leq \mu(A)\left(\mu(A) + C_3\sigma^{\ell-n}\kappa^n|g_n\varphi|_\infty\right),$$

where $\kappa > 1$ can be chosen arbitrarily and $C_3 = C_3(\kappa)$ is independent of n and ℓ. Since $|g_n\varphi|_\infty \leq \rho^n$ we can pick $\kappa = 1/\sqrt{\rho}$ to achieve

$$\begin{aligned} \mu(A \cap T^{-\ell}A) &\leq \mu(A)\left(\mu(A) + C_3\sigma^{\ell-n}\rho^{n/2}\right) \\ &\leq c_1\mu(A)\sigma^{\ell-n}\rho^{n/2}. \end{aligned}$$

Thus, for $j \geq 1$:

$$\begin{aligned} \mu(B_j) &\geq \mu(A) - \sum_{\ell=[pn]}^{n} \mu(A)\rho^{n-\ell} - \sum_{\ell=n+1}^{\infty} c_1\mu(A)\sigma^{\ell-n}\rho^{n/2} \\ &\geq \mu(A)\left(1 - \frac{\rho^{pn}}{1-\rho} - \frac{c_1\rho^{n/2}}{1-\sigma}\right) \\ &\geq \mu(A)\left(1 - c_2\rho^{pn}\right), \end{aligned}$$

and since $\mu(B_0) = \mu(A)$ we get

$$\mu(\mathcal{M}_r) = \sum_{j=0}^{[r/\mu(A)]} \mu(B_j) \geq \left(\left[\frac{r}{\mu(A)} \right] + 1 \right) \mu(A)(1 - c_2 \rho^{pn}) \geq r(1 - c_2 \eta^n),$$

where $\eta = \rho^p$ and $C_4 = c_2$. $\square$

We obtain the following mixing type theorem for the function h.

Lemma 7 *There exists a constant constant C_5 so that for all $t, r > 0$ and all n large enough*

$$|h(t+r) - h(t)h(r)| \leq C_5 \rho^{n/2}.$$

Proof. Let us first note that

$$\mathcal{N}_{t+r} = \mathcal{N}_t \cap T^{-[t/\mu(A)]} \mathcal{N}_{r-n'} \cap T^{-[t/\mu(A)]} \mathcal{N}_{n'},$$

where $n' = n\mu(A)$ and where we assumed that n is large enough so that $n\mu(A) - r$ is positive. Thus, by T-invariance of μ,

$$\left| \mu(\mathcal{N}_{t+r}) - \mu(\mathcal{N}_t \cap T^{-[t/\mu(A)+n]} \mathcal{N}_{r-n'}) \right| \leq \mu(\mathcal{M}_{n'}), \tag{2}$$

where a rough estimate yields

$$\mu(\mathcal{M}_{n'}) \leq n\mu(A)$$

and similarly

$$|\mu(\mathcal{N}_r) - \mu(\mathcal{N}_{r-n'})| \leq \mu(\mathcal{M}_{n'}) \leq n\mu(A). \tag{3}$$

Next we use the mixing property of μ. Note that

$$\mathcal{N}_{r-n'} = J \setminus \bigcup_{j=0}^{R-k-n} T^{-j} A,$$

where $R = [r/\mu(A)]$, and therefore

$$\begin{aligned} \mu(\mathcal{N}_{r-n'} \cap \mathcal{N}_t) &= \mu \left(\left(J \setminus \bigcup_{j=0}^{R-n} T^{-j} A \right) \cap \mathcal{N}_t \right) \\ &= \mu(\mathcal{N}_t) - \mu \left(\bigcup_{j=0}^{R-n} T^{-j} A \cap \mathcal{N}_t \right), \end{aligned}$$

while on the other hand one has

$$\mu(\mathcal{N}_t)\mu(\mathcal{N}_{r-n'}) = \mu(\mathcal{N}_t)\left(1-\mu\left(\bigcup_{j=0}^{R-n} T^{-j}A\right)\right).$$

Hence (the inverse branch φ of T^n is so that $A=\varphi(J)$) an application of lemma 3 yields

$$\begin{aligned}
&\left|\mu\left(\mathcal{N}_{r-n'}\cap T^{-R}\mathcal{N}_t\right)-\mu(\mathcal{N}_t)\mu(\mathcal{N}_{r-n'})\right|\\
&\quad= \left|\mu\left(\mathcal{N}_{r-n'}\cap T^{-R}\mathcal{N}_t\right)-\mu(\mathcal{N}_t)\mu\left(\bigcup_{j=0}^{R-k-n} T^{-j}A\right)\right|\\
&\quad\le \sum_{j=0}^{R-n}\left|\mu(T^{-j}A\cap T^{-R}\mathcal{N}_t)-\mu(\mathcal{N}_t)\mu(T^{-j}A)\right|\\
&\quad= \sum_{j=0}^{R-n}\left|\mu(A\cap T^{-(n+j)}\mathcal{N}_t)-\mu(\mathcal{N}_t)\mu(A)\right|\\
&\quad\le C_4\sum_{j=0}^{\infty}\kappa^n\sigma^j\mu(\mathcal{N}_t)|g_n\varphi|_\infty\\
&\quad\le c_1\rho^{n/2},
\end{aligned}$$

where we used that $\mu(\mathcal{N}_t)\le 1, \mu(A)\le\rho^n$ and $\kappa\sqrt{\rho}\le 1$. This estimate combined with equations (2) and (3) yields by the triangle inequality

$$|h(t+r)-h(t)h(r)|\le c_1\rho^{n/2}+2n\mu(A)\le C_5\rho^{n/2}.$$

□

An induction argument now shows (cf. [4] lemma 6):

$$|h(kr)-h(r)^k|\le\frac{C_5\rho^{n/2}}{1-h(r)}. \tag{4}$$

Proof of theorem 1 Put $A=A_n(x)$ and let us now estimate $h(r)^k-e^{-t}$, where we put $t=kr$. By lemma 6 $h(r)=1-\mu(\mathcal{M}_r)\le 1-r+rC_4\eta^n$, and thus

$$h(r)^k-e^{-t} \;\le\; (1-r+rC_4\eta^n)^k-e^{-t}$$

$$
\begin{aligned}
&\leq e^{k(-r+rC_4\eta^n)} - e^{-t} \\
&= e^{-t}\left(e^{krC_4\eta^n} - 1\right) \\
&\leq 2e^{-t}tC_4\eta^n
\end{aligned}
$$

if $krC_4\eta^n$ is small enough (say $\leq 1/2$). The lower bound is done similarly:

$$
\begin{aligned}
e^{-t} - h(r)^k &\leq e^{-t} - (1 - r - \mu(A))^k \\
&\leq e^{-t} - e^{-k(r+\mu(A)) - k(r+\mu(A))^2} \\
&\leq e^{-t}\left(k\mu(A) + k(r + \mu(A))^2\right)
\end{aligned}
$$

for $r + \mu(A)$ small enough. Thus

$$
|h(r)^k - e^{-t}| \leq c_1 t\eta^n e^{-t}.
$$

By lemma 5 we know that for almost every $x \in J$ the atom $A_n(x)$ is the image of J under a contracting branch, there exists a constant $c_2 > 1$ so that $c_2^{-1}\rho^n \leq \mu(A) \leq c_2\rho^n$ for all large enough n (see e.g. [2]). Now let us pick $r \in (\rho^{n/4}, 2\rho^{n/4})$ so that $k = t/r$ is an integer. We obtain using equation (4) and lemma 6:

$$
\begin{aligned}
|h(t) - e^{-t}| &\leq |h(t) - h(r)^k| + |h(r)^k - e^{-t}| \\
&\leq c_2 \frac{\rho^{n/2}}{1 - h(r)} + c_1 t\eta^n e^{-t} \\
&\leq c_3\rho^{n/4} + c_1 t\eta^n e^{-t} \\
&\leq C_1\varsigma^n,
\end{aligned}
$$

for $\varsigma < \min(\rho^{n/4}, \eta)$. $\square$

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