# Reminiscences on science at I.H.E.S.
# A problem on homoclinic theory and a brief review

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**Abstract.** *On the occasion of the 40-th anniversary of IHES I present a few scientific reminiscences: most of my scientific life has been marked by my visits and I run through them concluding with the analysis of a problem that originated during my last visit. The problem is to develop a convergent perturbative algorithm for the construction of the "Eliasson's potential" for the stable and unstable manifolds of an invariant torus: and to study its properties. A brief review follows.*

## §1. Reminiscences.

In 1966 I came to IHES as a young "professor": I was in fact just a Ph.D. student, by all standards. On the recommendation of Sergio Doplicher I was taken in the group of Ph.D.'s of Daniel Kastler. Of course I was fully aware of "being out of place": no time, however, was spent on this fact and soon I was eagerly working with Salvador Miracle–Solé on problems posed by David Ruelle. He went away for a few weeks and we met all possible difficulties so that we decided to write down the details of a proof that the problem given to us was not soluble: to realize, when the proof was complete, that we had in fact solved it. This was extending to many body interactions the Kirkwood–Salsburg equations (also known as "cluster expansion") for lattice system: it was a tremendous encouragement. It was for me the opening of a new world; that of perturbation theory: suddenly we had in our hands a most powerful instrument. We began using it to prove, with Dereck Robinson, that the Ising model had no phase transitions at high temperature, uniformly in the density.[1] At about the same time Jean Lascoux arrived with several papers from the Russian school; we saw that our work made them transparent and we could explain them around: in particular the work of Minlos and Sinai on phase coexistence appeared as a great achievement and as proof of the flexibility of perturbation theory which could *even* yield the analysis of the deep two phase region. I kept thinking to the matter until, quite a few years later, I could understand the theory of the fluctuations of the interface in the 2–dimensional Ising model at low temperature.

Leaving IHES in 1968 (I mean after may) I had met so many people (for instance Joel Lebowitz) and absorbed so many ideas and techniques that I was mature enough to pose problems without the help of Ruelle or of my collaborator and close friend Miracle–Solé. Saying farewell to Ruelle I saw on his blackboard (that he kept as black and powderless as a blackboard can possibly be until, much later, he gave it up, inexplicably) the Navier–Stokes equations: that he commented by stating that he was abandoning statistical mechanics for fluid mechanics "that seemed more interesting".

I brought along my copy of the fluids of Landau–Lifschitz and a few days later, in the US, I started studying it as I was sure I would soon need it. Even though I was discouraged by the difficulties (very transparent in spite of the book style which presents every problem as simple and as completely solved) I kept following Ruelle's work getting updates on the occasions of my frequent visits to IHES. Until, in 1973, I suddenly realized and was fascinated by his proposal of the existence of a probability distribution associated with chaotic motions in a conceptual generality comparable to the one, familiar in equilibrium

[1] We realized later that Dobrushin had preceded us.

statistical mechanics, of the Boltzmann–Gibbs distribution: this was, in my view, a development much more significant than the (timely and necessary) critique of Landau's Ptolemaic theory of fluid mechanics[2] stemming out of the Ruelle–Takens paper.

Ruelle's viewpoint on strange attractors made them simple objects via the Markov partitions and Sinai's theory of Anosov systems. But I could not go further: nevertheless the problem ("how to obtain some directly observable consequences" from Ruelle's principle) remained hunting me and I thought that it was the right approach and kept lecturing on the subject every year since while attempting even some numerical experiments[3] but mostly following other people experiments.

In the meantime I tried to understand Quantum Field Theory (I had cultivatied the feeling of a deep connection between renormalization theory and the Kirkwood Salsburg equations since the work with Miracle–Solé): in one of my visits to IHES I had met Francesco Guerra and in a few words, explaining his own basic work on Nelson's approach, he managed to make suddenly clear what I considered until then impenetrable (the works of Nelson and of Glimm–Jaffe and renormalization theory). So I started thinking to the matter and, also after a memorable lecture of Eckmann on the use of the cluster expansion in field theory, I was able to see the connection between the renormalization group of Wilson, constructive field theory and the cluster expansion developing a new interpretation of the mathematical works done until then in 2 and 3 dimensional field theory. I exposed the results at IHES with several people around (there by chance, because Gelfand and Sinai were visiting also) and I will never forget Niko Kuiper calling me in the Director's office and telling me that my talk had been successful: the first time I had been in that office was when Leon Motchane, a few months after my arrival and a few days after completing the first paper with Miracle–Solé (before the KS–equations), told me that I was given a 15% raise in my salary.

After the work in field theory I kept visiting regularly IHES where I was attracted also by the new young member Jurg Frölich: the policy of IHES started by Motchane of hiring young people at the highest rank ("member"), in spite of the obvious risks, is in my view what made the place so exceptional and so interesting and productive. But I was never able to collaborate with Frölich. Instead I worked with Henri Epstein in spite of the age difference: together with Pierre Collet we studied smooth conjugacy between flows on a surface of constant negative curvature stimulated by opinions by other visitors that what we wanted to do was "proved" to be impossible. The effervescent atmosphere around Dennis Sullivan always frightened me: his synthetic and powerful approach was in a way opposite to my nature. But of course I was among those who were impressed and attracted by his view on the theory of interval maps and Feigenbaum's theory and tried (without success) to imitate it in the theory of the tori breakdown in conservative systems: a subject into which I was drawn by Joel Lebowitz' request, while we were both at IHES, to explain in a talk the KAM theory that I had boasted, in refereeing a paper for him, to have understood from Arnold's celestial mechanics paper (the seminar took place a few days later at the Ecole Politechnique).

A long gap in my visits followed: which more or less coincided with a period in which I did not really work on new problems but looked at consequences of previous ones. I kept nevertheless thinking to the old questions and in my next long visit, in 1993, I was ready to attack a problem that E.G.D. Cohen (also a visitor at the time) proposed me to jointly work to find the connection between the experiments and theoretical ideas that he and his coworkers had been developing, in parallel with the works of Hoover and coworkers,

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[2] Unfortunately Ptolemy has become the villain of science: I do not share at all this superficial conception and I consider his work as great as possible. Hence a Ptolemaic theory is very respectable in itself and one needs work to criticize it, if at all possible.

[3] Learning programming on an archeological computer located in a little room of IHES, which had a marvelous object, a plotter, linked to it: if Oscar Lanford, then a member of IHES, considered numerical investigation of chaotic motions worth of devoting time to it *then* I should know too.

on nonequilibrium statistical mechanics. Two exciting years followed: in retrospect all that could have been done in one afternoon; but for us it was very hard work and at the end of the two years we could make a precise proposal (the "chaotic hypothesis") for the application of Ruelle's principle to some simple but concrete dynamical problems. My last visit to IHES was in 1997: I went there with bellicose projects to continue working on nonequilibrium statistical mechanics. But as usual I was taken away by other intervening projects[4] which ended having to do with the theory of homoclinic orbits and the transversality of their intersections: for this reason, although this is not a problem of the same size of the previously mentioned ones, I will devote to it the technical part of this note. I am sure that, as all problems that I started at IHES, this too will be fruitful and linked tightly with the previous works on the cluster expansion.

I also must remember here M.me Annie Rolland-Motchane: she, the first general secretary to IHES, clearly shared the merit of conceiving the structure of IHES and of making it work.

It is a structure that during the last 33 years always made me feel that IHES was an ideal place as a source of inspiration and of systematic work and it has been for me a privilege to be able to work there. I took this literally, as privileges must be earned: I was so absorbed in my work, taking up most of the nights (the day being reserved to wandering around trying to get the most out of the people present, with a few trips to the small library just to remind myself that even though it was small it was still completely full of things that I ignored): to the point that I could always manage avoiding the frustration due to my feeling of being there out of place among colleagues far more knowledgeable than I could possibly hope to be.

## §2. Homoclinic intersections in Hamiltonian systems. A "field theoretic" approach.

Consider a Hamiltonian system:

$$H = \underline{\omega}\cdot\underline{A} + \frac{I^2}{2J_0} + J_0 g^2(\cos\varphi - 1) + \varepsilon f(\underline{\alpha},\varphi) \tag{2.1}$$

where $\underline{\omega} = (\omega_1,\omega_2) \in R^2$, $\underline{A} = (A_1, A_2) \in R^2$, $\underline{\alpha} = (\alpha_1,\alpha_2) \in T^2$, $I \in R$, $\varphi \in T^1$ and $f(\underline{\alpha},\varphi)$ is an even analytic function of the angles $\underline{\alpha},\varphi$ which is a trigonometric polynomial in $\varphi$; its analyticity domain is supposed to be $|{\rm Im}\,\alpha_j| < \xi$. Therefore (2.1) represents a quasi–periodically forced pendulum.

What follows can be easily extended to the system obtained from (2.1) by adding to it $\frac12(J^{-1}\underline{A},\underline{A})$ where $J$ is a positive diagonal matrix. The case (2.1) is the limiting case as $J\to+\infty$ of the latter extension. The more general model is called *anisochronous* or *Thirring's model* as its peculiar properties were pointed out by Thirring, [T].

Supposing that $\underline{\omega}$ verifies a Diophantine property $|\underline{\omega}\cdot\underline{\nu}| > C^{-1}|\underline{\nu}|^{-1}$ for all $\underline{\nu} \in Z^2$, integer components *nonzero* vectors, it follows that for $\varepsilon$ small enough and for each $\underline{A}_0$ there is an invariant torus $\mathcal{T}(\underline{A}_0)$ which has equations parameterized by $\underline{\psi} \in T^2$ of the form $\underline{A} = \underline{A}_0 + \underline{H}(\underline{\psi})$, $\underline{\alpha} = \underline{\psi} + \underline{h}(\underline{\psi})$, $I = R(\underline{\psi}), \varphi = S(\underline{\psi})$. The motion on $\mathcal{T}(\underline{A}_0)$ is $\underline{\psi} \to \underline{\psi} + \underline{\omega}t$ and the average of $\underline{H}$ over $\underline{\psi}$ (hence on time) vanishes.

Hence we have a natural parameterization of the invariant tori by the (time) average value $\underline{A}_0$ of their action variables. In the more general Thirring's model there is also a relation between the rotation vector $\underline{\omega}$ on an invariant torus and the (time) average action $\underline{A}_0$ of the motion on it: namely $\underline{\omega} = J^{-1}\underline{A}_0$ (this is a remarkable property which prompted the appellative of "twistless" for such invariant tori).

Let $E(\underline{A}_0)$ be the energy of the motions on the invariant torus $\mathcal{T}(\underline{A}_0)$. The torus $\mathcal{T}(\underline{A}_0)$ is unstable and its stable and unstable manifolds $W^u(\underline{A}_0), W^s(\underline{A}_0)$ will consist of points with the same energy $E(\underline{A}_0)$; the intersection of $W^a(\underline{A}_0)$, $a = u, s$, with the

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[4] One was not a project but nevertheless took a lot of time: understanding the basics of Linux.

plane $\varphi = \pi$ and with the energy surface $H = E(\underline{A}_0)$ consists of points that can denoted as: $X(\underline{\alpha}) = (\pi, \underline{\alpha}, I^a(\underline{\alpha}), \underline{A}^a(\underline{\alpha}))$, for $a = u, s$ and $\underline{\alpha} \in T^2$. In general we denote a point in phase space $X = (\varphi, \underline{\alpha}, I, \underline{A})$.

Therefore on the 4–dimensional "section" $\varphi = \pi, H = E(\underline{A}_0)$ the manifolds have dimension 2. The vector $\underline{Q}(\underline{\alpha}) = \underline{A}^s(\underline{\alpha}) - \underline{A}^u(\underline{\alpha})$ defines the *splitting*. It has been remarked by Eliasson that the vector $\underline{Q}(\underline{\alpha})$ is a *gradient*: this means that there is a function $\Phi(\underline{\alpha})$ such that $\underline{Q}(\underline{\alpha}) = \underline{\partial}_{\underline{\alpha}} \Phi(\underline{\alpha})$. I will call $\Phi(\underline{\alpha})$ the *Eliasson's potential*.

Hence, since $\underline{\alpha}$ varies on $T^2$, there will be at least one point $\underline{\alpha}_0$ where $\underline{Q}(\underline{\alpha}_0) = \underline{0}$. The corresponding point $X(\underline{\alpha}_0)$ will be, of course, *homoclinic*, *i.e.* common to both manifolds. In the case at hand the parity properties of $f$ imply that $\underline{\alpha}_0 = \underline{0}$ is one such point, see for instance [G3].

It follows from the general theory of the tori $\mathcal{T}(\underline{A}_0)$, due to Graff, which is very close to KAM theory (particularly easy in the case (2.1), see for instance [G2]; see §5 of [CG] for the general anisochronous case), that the function $\Phi(\underline{\alpha})$ is analytic in $\varepsilon$ for $\varepsilon$ small, and such are the functions $\underline{A}^a(\underline{\alpha}), I^a(\underline{\alpha})$, for $a = u, s$.

Therefore we must be able to find the coefficients of their power series expansion in $\varepsilon$: in fact the expansion of $\underline{A}^a(\underline{\alpha}), I^a(\underline{\alpha})$ has been thoroughly discussed in all details in [G3], in the case of a polynomial $f$. It would be easy to derive from it the expansion for $\Phi(\underline{\alpha})$.

However here, §3, I will give a *self contained* exposition which leads to the expansion for $\Phi(\underline{\alpha})$ borrowing from [G3] only a few algebraic identities that it would be pointless to prove again. And I will dedicate §4 to a brief review of the results that are known (to me) on the matter or that can be easily derived from existing papers (which usually do not deal with Eliasson's potential but rather with its gradient). A question will be formulated at the end of this work.

The expansion below is the best tool, to my knowledge, to achieve a unified proof of various theorems dealing with the *homoclinic splitting*: the latter will be defined here as the Hessian determinant $\Delta$ of $\Phi(\underline{\alpha})$ evaluated at the homoclinic point at $\underline{\alpha} = \underline{0}$ (in our case). Evaluation of the latter determinant was begun by Melnikov who gave a complete solution (in spite of claims in other directions) in the "simple" case in which $\underline{\omega}$ is regarded as fixed. The quasi–periodic case was treated in the spirit of the present work, *i.e.* with attention to the Arnold's diffusion problem, in [HM] (togheter with the important realization of the usefulness of improper integrations analysis).

Considerable interest has been dedicated to the problem by various authors. The results are not easily comparable as every author seems to give his own definition of splitting: the most remarkable results have been developed in the russian school approach (based on the key works of Melnikov, Neishtadt, Lazutkin, [Ge]). And sometimes it has even been difficult to realize that some papers were just plainly incorrect (like the result in §10 of [CG]).

The interest of the above definition is its direct relation with the problem of showing the existence of *heteroclinic strings*, *i.e.* sequences of tori $\mathcal{T}(\underline{A}_i)$ such that $W^u(\underline{A}_i) \cap W^s(\underline{A}_{i+1}) \neq \emptyset$. Given a curve $\ell \to \underline{A}(\ell)$, with $\mathcal{T}(\underline{A}(\ell))$ being equienergetic tori (*i.e.* such that $\underline{\omega} \cdot \underline{A}(\ell) = const$ in the simple case (2.1), as one realizes from symmetry considerations), and calling $\underline{A}^a(\underline{\alpha}, \ell)$, $a = u, s$, the equations of the manifolds of $\mathcal{T}(\underline{A}(\ell))$, one finds such a string if one can show that $\underline{A}^u(\underline{\alpha}; \ell) - \underline{A}^s(\underline{\alpha}; \ell') = \underline{0}$ admits a solution for $\ell'$ close enough to $\ell$. Therefore one has to apply the implicit functions theorem around $\underline{\alpha} = \underline{0}$ and $\ell' = \ell$: the determinant of the Jacobian matrix controlling this problem is, clearly, the splitting defined above. Hence proving that the splitting is nonzero implies the existence of heteroclinic strings.

In the anisochronous case not all the $\underline{A}(\ell)$ are necessarily the average actions of an invariant torus (because of the resonances always present when the rotation vectors do depend on the actions) so that it becomes important to measure the size of the splitting compared to the size of the "gaps" on the curve parameterized by $\ell$. Hence we need to know quite well the dependence on $\underline{\omega}$ of the splitting and of the gaps and their relative sizes, see [GGM2].

I know of no paper in which the above definition of splitting is used in the case of quasi periodic forcing models, other than [CG] and the later [G3], [GGM1], [GGM2], [GGM3] or the related [BCG]: I will not discuss the papers using other definitions.

## §3. Feynman's graphs for Eliasson's potential.

We set $J_0=1$ for simplicity. Let $a=u,s$ and $X^a(0)=(\pi,\,\underline{\alpha}\,,I^a(\,\underline{\alpha}\,),\,\underline{A}^{\,a}(\,\underline{\alpha}\,))$; and let $X^a(t)=(\varphi^a(\,\underline{\alpha}\,,t),\,\underline{\alpha}+\underline{\omega}\,t,I^a(\,\underline{\alpha}\,,t),\,\underline{A}^{\,a}(\,\underline{\alpha}\,,t))$ be the solution of the equations of motion for (2.1): $\dot\varphi\,=\,I,\ \dot{\underline{\alpha}}\,=\,\underline{\omega}\,,\ \dot I\,=\,-\partial_\varphi f_0(\varphi)\,-\,\varepsilon\partial_\varphi f_1(\,\underline{\alpha}\,,\varphi),\ \dot{\underline{A}}\,=\,-\varepsilon\partial_{\underline{\alpha}}\,f_1(\,\underline{\alpha}\,,\varphi)$ where $f_0=g^2\cos\varphi$ and $f_1=f(\,\underline{\alpha}\,,\varphi)$. In the case $\varepsilon=0$ the stable and unstable manifolds of the torus with (average) action $\underline{A}_{\,0}$ coincide and the parametric equations of their stable and unstable manifolds are at $\varphi=\pi$ simply $X=(\varphi=\pi,\,\underline{\alpha}\,,I=-2g,\,\underline{A}\,=\,\underline{A}_{\,0})$ and the motion are $X^{(0)}(t)\,=\,(\varphi^0(t)\,=\,4\,\mathrm{arctg}\,e^{-gt},\,\underline{\alpha}\,+\,\underline{\omega}\,t,I(t)\,=\,-g\sqrt{2(1-\cos\varphi^0(t))},\,\underline{A}\,=\,\underline{A}_{\,0})$.

Hence if $X^a(\,\underline{\alpha}\,,t)=X^{(0)}(\,\underline{\alpha}\,,t)+\varepsilon X^{a,(1)}(\,\underline{\alpha}\,,t)+\varepsilon^2X^{a,(2)}(\,\underline{\alpha}\,,t)+\ldots$ we can immediately write the equations for $X^{(k)}\,=\,(\varphi^{a,(k)},\,\underline{0}\,,I^{a,(k)}\,\underline{A}^{\,a,(k)}),\ k\,\ge\,1$, where the angle components vanish identically because the equations for $\underline{\alpha}$ are trivially solved by the order 0 solution. The latter is a property of the isochrony of (2.1) and it does not hold in the anisochronous case, which is however equally easy to treat, see [G3]. The equations are, dropping for simplicity the label $a$:

$$\begin{aligned}\begin{pmatrix}\dot\varphi^{(k)}\\ \dot I^{(k)}\end{pmatrix}&=\begin{pmatrix}0&1\\ g^2\cos\varphi^0&0\end{pmatrix}\cdot\begin{pmatrix}\varphi^{(k)}\\ I^{(k)}\end{pmatrix}+\begin{pmatrix}0\\ F_0^{(k)}\end{pmatrix}=L(t)\cdot\begin{pmatrix}\varphi^{(k)}\\ I^{(k)}\end{pmatrix}+\begin{pmatrix}0\\ F_0^{(k)}\end{pmatrix}\\ \dot{\underline{A}}^{\,(k)}&=\underline{F}^{\,(k)}\end{aligned}\tag{3.1}$$

and the functions $F_0,\,\underline{F}$ are deduced from the equations of motion; recalling that $f_0=g^2\cos\varphi$ and $f_1=f(\,\underline{\alpha}\,,\varphi)$:

$$\begin{aligned}F_0^{(k)}=&\sum_{h=2}^{k}-\frac{1}{h!}\partial_\varphi^{h+1}f_0(\varphi^0(t))\sum_{\substack{k_1+\ldots+k_h=k\\ k_j\ge1}}\prod_{j=1}^{h}\varphi^{(k_j)}(t)+\\ &+\sum_{h=1}^{k}-\frac{1}{h!}\partial_\varphi^{h+1}f_1(\,\underline{\alpha}\,+\,\underline{\omega}\,t,\varphi^0(t))\sum_{\substack{k_1+\ldots+k_h=k-1\\ k_j\ge1}}\prod_{j=1}^{h}\varphi^{(k_j)}(t)\\ \underline{F}^{\,(k)}=&\sum_{h=1}^{k}-\frac{1}{h!}\partial_{\underline{\alpha}}\,\partial_\varphi^{h}f_1(\,\underline{\alpha}\,+\,\underline{\omega}\,t,\varphi^0(t))\sum_{\substack{k_1+\ldots+k_h=k-1\\ k_j\ge1}}\prod_{j=1}^{h}\varphi^{(k_j)}(t)\end{aligned}\tag{3.2}$$

It is convenient to rewrite the above expressions in a more synthetic and symmetric form:

$$\begin{aligned}F_0^{(k)}&=\sum_{\delta=0,1}\sum_{h=2-\delta}^{k}-\frac{1}{h!}\partial_\varphi^{h+1}f_\delta(\,\underline{\alpha}\,+\,\underline{\omega}\,t,\varphi^0(t))\sum_{\substack{k_1+\ldots+k_h=k-\delta\\ k_j\ge1}}\prod_{j=1}^{h}\varphi^{(k_j)}(t)\\ \underline{F}^{\,(k)}&=\sum_{\delta=0,1}\sum_{h=2-\delta}^{k}-\frac{1}{h!}\partial_{\underline{\alpha}}\,\partial_\varphi^{h}f_\delta(\,\underline{\alpha}\,+\,\underline{\omega}\,t,\varphi^0(t))\sum_{\substack{k_1+\ldots+k_h=k-\delta\\ k_j\ge1}}\prod_{j=1}^{h}\varphi^{(k_j)}(t)\end{aligned}\tag{3.3}$$

where of course in the second relation only the $\delta=1$ term does not vanish.

Hence if $W(t) = \begin{pmatrix} w_{00}(t) & w_{01}(t) \\ w_{10}(t) & w_{11}(t) \end{pmatrix}$ is the solution of the $2 \times 2$–matrix equation $\dot W = L(t)W, W(0) = 1$ we find: $\begin{pmatrix} \varphi^{(k)} \\ I^{(k)} \end{pmatrix} = W(t)\Big(X^{(k)}(0) + \int_0^t W(\tau)^{-1} \begin{pmatrix} 0 \\ F_0^{(k)}(\tau) \end{pmatrix} d\tau\Big)$.

The matrix $W(t)$ can be easily explicitly computed: the matrix elements are holomorphic for $g|\mathrm{Im}\, t| < \frac{\pi}{2}$ and $w_{00}, w_{10}$ tend to zero as $t \to \pm\infty$ as $e^{-g|t|}$ while the other column elements tend to $\infty$ as $e^{g|t|}$. We shall only need the matrix elements $w_{00}, w_{01}$ which have a simple pole at $\pm i\frac{\pi}{2g}$. It is $w_{00}(t) = \frac{1}{\cosh gt}$ and $w_{01}(t) = \frac{1}{2g}(\frac{gt}{\cosh gt} + \sinh gt)$.

Since we need only $\varphi^{(h)}$, $h = 1, \ldots, k-1$, to evaluate $F^{(k)}$ we spell out only the expressions of $\varphi^{(k)}(t)$ and $\underline{A}^{(k)}(t)$:

$$\begin{aligned} \varphi^{(k)}(t) =& w_{01}(t)\Big(I^{(k)}(0) + \int_0^t w_{00}(\tau)F^{(k)}(\tau)d\tau\Big) - w_{00}(t)\int_0^t w_{01}(\tau)F^{(k)}(\tau)d\tau \\ \underline{A}^{(k)}(t) =& \underline{A}^{(k)}(0) + \int_0^t \underline{F}^{(k)}(\tau)d\tau \end{aligned} \tag{3.4}$$

We want to determine, for each choice of $\underline{\alpha}$, the initial data $I^{(k)}(0)$, $\underline{A}^{(k)}(0)$ so that the motion is asymptotically quasi periodic, because we want to impose that the motion *tends* to the invariant torus $\mathcal{T}(\underline{A}_0)$. This means that the initial data must be determined in a different way depending on whether we impose this condition as $t \to +\infty$ or as $t \to -\infty$: in the first instance we determine the stable manifold and in the second the unstable one.

This condition is imposed simply by requiring $\underline{A}^{(k)}(0) = -\int_0^{\sigma\infty} \underline{F}^{(k)}(\tau)d\tau$ where $\sigma = +$ if we impose the condition at $t = +\infty$ or $\sigma = -$ if we impose the condition at $t = -\infty$ and, likewise, $I^{(k)}(0) = -\int_0^{\sigma\infty} w_{00}(\tau)F_0^{(k)}(\tau)d\tau$ *provided the integrals converge.* Therefore (3.4) become:

$$\begin{aligned} \varphi^{(k)}(t) =& \int_{\sigma\infty}^t \Big(w_{01}(t)w_{00}(\tau) - w_{00}(t)w_{01}(\tau)\Big)F_0^{(k)}(\tau)\,d\tau - \\ & - w_{00}(t)\int_{\sigma\infty}^0 w_{01}(\tau)F_0^{(k)}(\tau)\,d\tau \\ \underline{A}^{(k)}(t) =& \int_{\sigma\infty}^t \underline{F}^{(k)}(\tau)\,d\tau \end{aligned} \tag{3.5}$$

and we have a simple recursion relation *provided the integrals converge.*

The above expressions however may involve non convergent integrals and eventually they really do, in general. This has the consequence that (for instance) it is by no means true that $\underline{A}^{s,(k)}(t)\xrightarrow[t\to+\infty]{}0$ as the (3.5) would imply if the integrals were proper. Of course there will be no ambiguity about the meaning of such *improper* integrals. The meaning is uniquely determined simply by the requirement that the asymptotic form of (3.4) be a quasi periodic function.

Since the functions in the integrands can always be written as series of functions of the form $\sigma^\chi \frac{(\sigma t g)^j}{j!}e^{-g\sigma\tau h}e^{i\underline{\omega}\cdot\underline{\nu}\,t}$ for some $\chi = 0, 1, j, h, \underline{\nu}$ with $\sigma = \mathrm{sign}\,\tau$, it is easy to see that the rule for the evaluation of the integrals of such function is simply that of introducing a cut off factor $e^{-R\sigma\tau}$ with $\mathrm{Re}\,R$ large enough so that all the integrals converge, then one performs the (now convergent) integrals and then one takes the residue at $R = 0$ of the result: see §3 of [G3] for a more detailed discussion. To derive this rule one shoould try a few simple cases (like evaluating the second order explitcitly).

Improper integrals are very familiar in perturbation theory of quantum fields where they are normally introduced to obtain compact and systematic representations of the coefficients of perturbation expansions. Typically a Feynman diagram value is given by an improper integral: the algorithm is so familiar that it has become usual not to even mention which are the rules for the evaluation of such integrals.

Since the rules for the evaluation of the above imoroper integrals are discussed in detail in [G3] I shall not dwell on them and, instead, I proceed immediately to use the improper integrals in the same way they are used in quantum field theory: *i.e.* to find a simple diagrammatic representation of the iterative scheme described above. It is remarkable that such a scheme was found by Eliasson in his breakthrough theory of the KAM series, [E], without any reference to field theory: he has independently developed a diagrammatic representation of the KAM series.

We represent $\varphi^{(k)}(t)$ as:

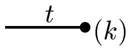


Fig.1

and, with the same "logic":

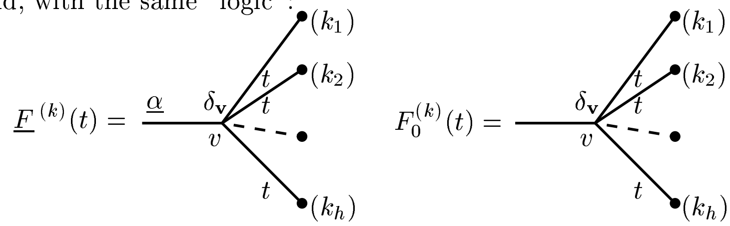


Fig.2

where it is $\sum_j k_j = k - \delta_v$, see (3.3); the label $\delta_v$ can be 0 or 1: the first drawing represents the term with $\delta = \delta_v$ in the expression for $\underline{F}^{(k)}$ in (3.3), and the second drawing represents the contribution to $F_0^{(k)}$ with $\delta = \delta_v$.

The node $v$ represents $-\partial_\varphi^{h+1} f_{\delta_v}$ times $\frac{1}{h!}$ in the second graph and $-\partial_{\underline{\alpha}}\partial_\varphi^h f_1$ times $\frac{1}{h!}$ in the first. Because of the $\partial_{\underline{\alpha}}$ derivative we can imagine that in the first graph thelabel $\delta_v$ on the node $v$ is *constrained* to be 1.

We can in the same way represent $\varphi^{(k)}(t)$ and $\underline{A}^{(k)}(t)$: we can in fact change the labels $t$ on the lines merging into the node $v$ into labels $\tau$ and interpret the node $v$ as representing an integration operation over the time $\tau$; one get in this way the following graphs:

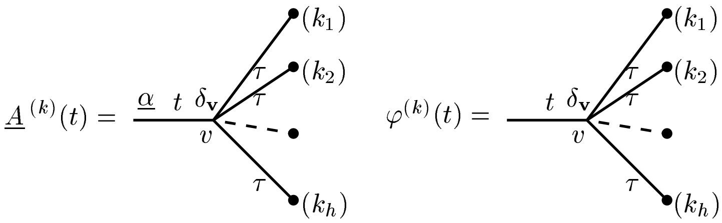


Fig.3

The node $v$ with the label $\delta_v$, which we noted that it must be 1 in the first drawing and that can be either 0 or 1 in the second, has to be thought as representing the operations acting on a generic function $F$:

$$\begin{aligned}
&\mathcal{I}_\sigma F(t) = \int_{\sigma\infty}^t F(\tau)\,d\tau && (3.6)\\
&\mathcal{O}_\sigma F(t) = \int_{\sigma\infty}^t \Big(w_{01}(t)w_{00}(\tau) - w_{00}(t)w_{01}(\tau)\Big)F(\tau)\,d\tau \;-\; w_{00}(t)\int_0^{\sigma\infty} w_{01}(\tau)F(\tau)\,d\tau
\end{aligned}$$

where $\sigma = +$ if we study the stable manifold and $\sigma = -$ if we study the unstable one.

In this way the graphs of Fig.3 represent respectively the *values*:

$$\frac{1}{h!}\mathcal{I}_\sigma\Big(-\partial_{\underline{\alpha}}\partial_\varphi^h f_1[\tau]\prod_{j=1}^h \varphi^{(k_j)}[\tau]\Big)(t), \qquad \frac{1}{h!}\mathcal{O}_\sigma\Big(-\partial_\varphi^{h+1} f_\delta[\tau]\prod_{j=1}^h \varphi^{(k_j)}[\tau]\Big)(t))x \qquad (3.7)$$

where an argument in square brackets means a dummy integration variable, inserted just as a reminder of the integration operation involved; furthermore $f_\delta[t]$ abbreviates $f_\delta(\underline{\alpha}+\underline{\omega}\,t,\varphi^0(t))$.

Clearly $I^{(k)}$, $\underline{A}^{(k)}$ can be expressed simply by summing over the labels $k_j$ and $\delta$ the values of the graphs in Fig.3: the summations should run over the same ranges appearing in (3.2), *i.e.* $h$ between $2-\delta$ and $k$, and $k_j\ge 1$ such that $\sum_j k_j=k-\delta$ and $\delta=0,1$). If we study the stable manifold we must take $\sigma=+$ and if we study the unstable one me must take $\sigma=-$ and $I^{(k)}$, $\underline{A}^{(k)}$ become respectively $I^{s,(k)}$, $\underline{A}^{s,(k)}$ or $I^{u,(k)}$, $\underline{A}^{u,(k)}$.

It is now immediate to iterate the above representation; one simply recalls that each symbol:

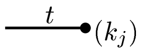


Fig.4

represents $\varphi^{(k_j)}(t)$ and that (3.7) is multi linear in the $\varphi^{(k_j)}(t)$. This leads to representing $\underline{A}^{(k)}(t)$ as sum of values of graphs $\vartheta$ of the form:

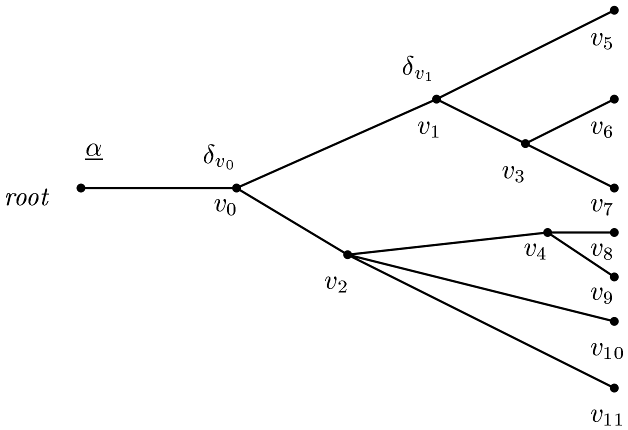


Fig.5

A graph $\vartheta$ with $p_{v_0}=2, p_{v_1}=2, p_{v_2}=3, p_{v_3}=2, p_{v_4}=2$ and $k=12$, and some labels. The lines length is drawn of arbitrary size. The nodes labels $\delta_v$ are indicated only for two nodes. The lines are imagined oriented towards the root and each line $\lambda$ carries also a (not marked) label $\tau_v$, if $v$ is the node to which the line leads; the root line carries the label $t$ but its "free" extreme, that we call the "root", is not regarded as a node.

The meaning of the graph is recursive: all nodes $v$, see Fig.5, represent $\mathcal{O}_\sigma$ operations except the "first" node $v_0$ which instead represents a $\mathcal{I}$ operation; the extreme of integration is $+\infty$ if we study the stable manifold and $-\infty$ if we stufy the unstable one. Furthermore each node represents a factor $-\frac{1}{p_v!}\partial_\varphi^{p_v+1}f_{\delta_v}[\tau_v]$ if $p_v$ is the number of lines merging into $v$, *except* the "first" node $v_0$ which represents $-\partial_{\underline{\alpha}}\,\partial_\varphi^{p_v}f_{\delta_v}[\tau_v]$ instead. The product $\prod_v\frac{1}{p_v!}$ is the "combinatorial factor" for the node $v$

The lines merging into a node are regarded as distinct: *i.e.* we imagine that they are labeled from 1 to $p_v$, but we identify two graphs that can be overlapped by permuting suitably and independently the lines merging into the nodes.

It is more convenient to think that all the lines are numbered from 1 to $m$, if the graph has $m$ lines, still identifying graphs that can be overlapped under the above permutation operation (including the line numbers). In this way a graph with $m$ lines will have a combinatorial factor simply equal to $\frac{1}{m!}$ provided we define 1 instead of $\prod_v\frac{1}{p_v!}$ the combinatorial factor of each node: *we shall take the latter numbering option.* Hence in Fig.5 one has to think that each line carries also a number label although the line numbers, distinguishing the lines, are not shown.

The endnodes $v_i$ should carry a $(k_i)$ label: but clearly unless $k_i=1$ they would represent a $\varphi^{(k_i)}$ which could be further expanded; hence the graphs in Fig.5 should have the labels $(k_i)$ with $k_i=1$: this however carries no information and the labels are not drawn. The interpretation of the endnodes is easily seen that has to be: $\mathcal{O}_\sigma(-\partial_\varphi f_{\delta_{v_i}})(\tau_{v'_i})$ if $v'_iv_i$ is the line linking the endnode $v_i$ to the rest of the graph. An exception is the trivial case of the graph with only one line and one node: this represents $\mathcal{I}(-\partial_{\underline{\alpha}} f_{\delta_{v_0}})(t)$ and it will be called the *Melnikov's graph.*

In this way we have a natural decomposition of $\underline{A}^{(a,(k)}(\underline{\alpha},t)$ as a sum of *values* of graphs. It is now easy to represent the power series expansion of the trajectories on the manifolds $W^a(\mathcal{T}(\underline{A}_0))$: one simply collects all graphs with labels $\delta_v$ with $\sum_v d_v = k \ge 1$ (they can have at most $2k$ lines, if one looks at the restrictions on the labels) and adds up their "values" obtaining the coefficient $\underline{A}^{a,(k)}(t)$. The $\mathcal{O}_\sigma$ and $\mathcal{I}_\sigma$ operations involve integrals with $\sigma\infty$ as an extreme and one has, obviously, to choose $\sigma = +$ if $a = s$ and $\sigma = -$ if $a = u$.

Since all the integration operations $\mathcal{O}$ or $\mathcal{I}$ are, in general, improper we see the convenience of the graphical representation and its analogy with the Feynman graphs of quantum field theory: in fact this is *more than an analogy* as the above graphs can be regarded as the Feynman graphs of a suitable field theory: see [GGM0] for the discussion of a similar case (*i.e.* the KAM theory representation as a quantum field theory).

An essential feature is missing: namely the graphs have no loops (they are in fact tree graphs). This major simplification is compensated by the major difficulty that the number of lines per node is *unbounded* (*i.e.* a field theory that generated the graphs would have to be "non polynomial").

Noting that the value of each graph is a function of $\underline{\alpha}$ we now have to check that each $\underline{A}^{a,(k)}(0)$ has the form $\underline{A}^{a,(k)}(0) = \partial_{\underline{\alpha}} \Phi^{a,(k)}$.

For this purpose we consider graphs like Fig.5 but with the root branch *deleted* keeping however a *mark* on the first node $v_0$ to remember that the line has been taken away. We call such a graph a *rootless* graph.

*It is convenient to define the value* $\mathrm{Val}_\sigma(\vartheta)$ *of such rootless trees*: its is defined as before but the marked node now represents the operation $\mathcal{I}_{\sigma,0}(F) \equiv \int_{\sigma\infty}^0 d\tau F(\tau)$ with $\sigma = +$ for the analysis of the stable manifold and $\sigma = -$ for the unstable, and the function $\partial^{p_{v_0}} f_{\delta_{v_0}}[\tau]$ (keeping in mind that the marked node *must* have $\delta_{v_0} = 1$, by construction).

The key remark is now the identity ("Chierchia's root identity", see [G3]):

$$\mathcal{I}_{\sigma,0}(F\,\mathcal{O}_\sigma(G)) = \mathcal{I}_{\sigma,0}(G\,\mathcal{O}_\sigma(F)) \tag{3.8}$$

which is an algebraic identity as our improper integrals only involve functions $F, G$ linear combinations of "monomials" of the form $\sigma^\chi \frac{(\sigma t g)^j}{j!} e^{-g\sigma\tau h} e^{i\,\underline{\omega}\cdot\underline{\nu}\,t}$ for some $\chi = 0, 1, j, h, \underline{\nu}$ with $\sigma = \mathrm{sign}\,\tau$, see above, for which both sides of (3.7) can be explicitly and easily evaluated.

This identity can be used to relate the values of different graphs: it means that the values of two rootless tress differing only because the mark is on different nodes and otherwise superposable are *identical*: this can be seen easily by successive applications of the identity (3.7), see [G3].

Therefore if we define:

$$\Phi^{\sigma,(k)}(\underline{\alpha}) = \frac{1}{k}\sum_\vartheta \mathrm{Val}_\sigma(\vartheta), \qquad \Phi^\sigma(\underline{\alpha}) = \sum_{k-1}^\infty \varepsilon^k \Phi^{\sigma,(k)}(\underline{\alpha}) \tag{3.9}$$

we see that the gradient with respect to $\underline{\alpha}$ of $\Phi^{\sigma,(k)}(\underline{\alpha})$ is precisely $\underline{A}^{a,(k)}(\underline{\alpha})$. And the splitting $\underline{Q}(\underline{\alpha})$ is the gradient of $\Phi(\underline{\alpha}) = \Phi^+(\underline{\alpha}) - \Phi^-(\underline{\alpha})$.

One can get directly a graphical representation of $\Phi^{(k)}$ as:

$$\Phi^{(k)}(\underline{\alpha}) = \sum_{\vartheta:\,k} \int_{+\infty}^{-\infty} dt\,\mathrm{Wal}_{\sigma(t)}(\vartheta) \tag{3.10}$$

where $\sigma(t) = \text{sign}\,(t)$ and $\text{Wal}_{\sigma(t)}(\vartheta)$ is just the *integrand* in the $\mathcal{I}_{\sigma,0}$ integral with respect to the first node variable $\tau = \tau_{v_0}$ appearing in the evaluation of $\text{Val}_\sigma(\vartheta)$. This concludes the construction of Eliasson's potential.

## §4. Properties of the potential.

Many properties of the gradient $\underline{Q}(\underline{\alpha}) = \partial_{\underline{\alpha}}\Phi(\underline{\alpha})$ have been studied in [G3], [GGM1], [GGM2], [GGM3]: they are immediately translated into properties of the potential $\Phi$, either by integration or by following the proofs of the corresponding statements for $\underline{Q}(\underline{\alpha})$. We just summarize them:

(1) If $\underline{\omega}$ is fixed then, generically, the first order dominates:

$$\Phi(\underline{\alpha}) = \varepsilon \int_{+\infty}^{-\infty} d\tau\, f(\underline{\alpha} + \underline{\omega}\, t, \varphi^0(t)) + O(\varepsilon^2) \tag{4.1}$$

this is the well known Melnikov's result. We shall say that there is "dominance of Melnikov's term" for some quantity every time that the lowest order perturbative term gives the dominant asymptotic behavior for it in a given limiting situation. Hence in (4.1) domination refers to $\varepsilon \to 0$.

In the "one dimensional" case not explicitly treated above, but much easier, of a periodic forcing in which there is only one angle $\underline{\alpha}$ and one action $\underline{A}$, Melnikov's domination remains true *even* if the parameter $g$ becomes small provided the $\varepsilon$ is chosen of the form $\mu g^q$ for some $q > 0$ (proportional to the degree $N_0$ of $f$ as a trigonometric polynomial in $\varphi$) and $|\mu|$ small enough.

This is somewhat nontrivial: in [G3] there is a proof based on the above formalism; other proofs are available as the problem is classical. The nontriviality is due to the necessity of showing the existence of suitable cancellations that eliminate values of graphs contributing to $\Phi(\underline{\alpha})$ higher order "corrections" (corresponding to special graphs) which are individually present and, in fact, larger than the first order contributions.

(2) The next case to study is the same case of $g$ small but with the Hamiltonian (2.1) (*i.e.* quasiperiodically, rather than periodically, and rapidly forced): let $g^2 = \eta$ and $\eta < 1$ be a parameter that we want to consider near 0. In this case, too, convergence requires that $\varepsilon = \mu\eta^q$ for some $q > 0$ (proportional to the degree $N_0$ of $f$ as a trigonometric polynomial in $\varphi$) and $|\mu|$ small enough.

The problem is discussed already in [G3] and, following it, we consider the graphs $\vartheta$ that contribute to $\Phi^{(k)}$ and at each node we decompose $f_\delta$ into Fourier harmonics $f_\delta(\underline{\alpha}, \varphi) = \sum_{\underline{\nu}} f_{\underline{\nu}}(\varphi) e^{i\underline{\nu}\cdot\underline{\alpha}}$. This leads to considering *new graphs* $\vartheta$ in which at each node $v$ a label $\underline{\nu}_v$ is added signifying that in the evaluation of the graph value the functions $f_{\delta_v}(\underline{\alpha}, \varphi)$ are replaced by $f_{\delta_v,\underline{\nu}}(\varphi) e^{i\underline{\alpha}\cdot\underline{\nu}}$. Of course at the end we shall have to sum over all the "momentum labels" $\underline{\nu}_v \in Z^2$. We call $F^{(k)}_\vartheta$ the contribution to $\Phi^{(k)}$ from one such more decorated graph. Then from §8 of [G3] one sees that:

$$\begin{aligned} |\Phi^{(k)}_\vartheta| &\le D\, B^{k-1}\, k!^{\,p} \Big(\sum_{\underline{\nu}'} e^{-|\underline{\nu}'\cdot\underline{\omega}|\frac{\pi}{2g}}\Big) \prod_v |f_{\underline{\nu}_v}| \\ |\Phi^{(k)}_{\underline{\nu}}| &\le (b\eta^{-q})^k \end{aligned} \tag{4.2}$$

where $g = \eta^{\frac{1}{2}}$ the sum over $\underline{\nu}'$ runs over the *nonzero* values of the sums of subsets of $\underline{\nu}_1, \ldots, \underline{\nu}_k$. The constants $B, D$ are bounded by an inverse power of $\eta$ and $p > 0$ is constant (depending on the degree of $f$ in $\varphi$); the constants $b, q$ can be bounded only in terms the maximum of $|f|$ in a strip $|\text{Im}\,\alpha_j|, |\text{Im}\,\varphi| < \xi$ on which the maximum is finite. The first property follows from the analysis in §8 of [G3]; the second is symply the statement that the stable and unstable manifolds are analytic function of $\varepsilon$ with radius of convergence proportional to $\eta^q$ for some $q$ (essentially a result of Graff, see §5 of [CG]).

(3) A consequence of (4.2) is that $\Phi(\underline{\alpha})$ can be represented as, see [GGM3] for details on the corresponding statement for the gradient of $\Phi$:

$$\begin{aligned}\Phi(\underline{\alpha}) =&\varepsilon \sum_{\underline{\nu}} e^{i\underline{\alpha}\cdot\underline{\nu}}(M_{\underline{\nu}} + \varepsilon D_{\underline{\nu}}(\underline{\alpha},\varepsilon))\\ M_{\underline{\nu}} =&\sum_n f_{\underline{\nu},n}\int_{+\infty}^{-\infty}(1-\frac{\dot\varphi^0(t)}{\cosh gt})\,\cos n\varphi^0(t)\,dt\\ |\partial^h_{\underline{\alpha}} D_{\underline{\nu}}(\underline{\alpha})| <&\eta^{-p_h}C_h|\underline{\nu}|^h, \qquad p_h,q_h>0\end{aligned}\tag{4.3}$$

which allows us to say, very easily, that the Eliasson's potential is "in some sense" dominated by Melnikov's value at least in the special cases in which $f_{\underline{\nu},n}$ are positive and "*as large as possible*", *i.e.* $f_{\underline{\nu},n} = c\,e^{-\kappa|\underline{\nu}|}\delta_{n,1}$ for $c>0$, and $\underline{\omega}$ has good Diophantine properties, *e.g.* if $\omega_1/\omega_2$ is the golden mean (here $|\underline{\nu}| \stackrel{def}{=} |\nu_1|+|\nu_2|$).
In the latter instance one verifies that, for all $\underline{\alpha}$, $|D_{\underline{\nu}}(\underline{\alpha},\varepsilon)| < \eta^{-q'}M_{\underline{\nu}}$ for all $|\underline{\nu}|<\eta^{-1}$ which togheter with the analyticity in $\varepsilon$ of $\Phi(\underline{\alpha})$ allows disregarding the contributions to $\Phi(\underline{\alpha})$ from the $\underline{\nu}$'s exceeding $\eta^{-1}$. The properties of the golden mean allow us immediately to see that in the sum only one pair $\pm\underline{\nu}$ dominates at $\underline{\alpha}=\underline{0}$: it is the pair $\underline{\nu}_0 = (f_k,-f_{k+1})$ if $f_j$ is the Fibonacci's sequence and $k$ such that $\kappa|\underline{\nu}_0| + \frac{\pi}{2\eta^{\frac12}}|\underline{\omega}\cdot\underline{\nu}_0|$ is minimum; *apart* from exceptional intervals of values of $\eta$ in correspondence of which there may be two pairs (or more) (see §2,6 of [DGJS]). The domination persists for all the $\underline{\alpha}$'s such that $|\sin\underline{\nu}_0\cdot\underline{\alpha}|>b$ where $b>0$ is any prefixed constant (the smaller $b$ the smaller has $\varepsilon$ to be to insure dominance).
Also the gradient of $\Phi(\underline{\alpha})$, and in fact any derivative of $\Phi$ is dominated by the Melnikov's term, by the same type of argument. But this is somewhat trivial: the real question is, in view of the remarks in §2 about the possible applications to heteroclinic strings and to Arnold's diffusion, whether the homoclinic splitting is dominated by Melnikov's integral. This seems to be, in the generality considered here, *still an open problem.* The reason is very simple; from (4.3) one easily deduces that:

$$\Delta = -\varepsilon^2\sum_{\underline{\nu},\underline{\nu}'} e^{-(|\underline{\omega}\cdot\underline{\nu}|+|\underline{\omega}\cdot\underline{\nu}'|)\frac{\pi}{2}\eta^{-\frac12}}\big((\underline{\nu}\wedge\underline{\nu}')^2 M_{\underline{\nu}}M_{\underline{\nu}'} + \varepsilon d_{\underline{\nu},\underline{\nu}'}\big)\tag{4.4}$$

*hence one realizes that the term that should be leading,* $\underline{\nu} = \pm\underline{\nu}' = \pm\underline{\nu}_0$, is missing in the lowest order part. Therefore the main contribution comes, *or may come*, from the remainder $d_{\underline{\nu},\underline{\nu}'}$ on which we have little information besides the above bounds (which would be plenty if the Melnikov's main term did not vanish). Curiously the above exceptional cases, *i.e.* when the value of $\eta$ is taken along a sequence $\eta_j\to0$ such that for each $j$ there are *two* minimizing vectors $\underline{\nu}_0$ and $\underline{\nu}_0'$, can be, instead, easily solved because $(\underline{\nu}_0\wedge\underline{\nu}_0')^2\ge1$ as no two Fibonacci's vectors can be parallel.
In the literature there are various claims about "proofs" of dominance of Melnikov's contribution to the splitting: they however seem to be always proofs of the "easy part" namely of the dominance of the Melnikov's term in *some components of* the splitting vector $\underline{Q}(\underline{\alpha})$ (implied by the above analysis).
The only known case of generic dominance of the Melnikov term *for the splitting* is the one discussed in [GGM1], see (5) below. And its analysis is already far more subtle than the above.
(4) In general the estimates (4.2), called "quasi flat" in [G3] are *optimal* (see [GGM4]): hence one *cannot hope* to have bounds on the Fourier transform of $\Phi$ of the form, for some $r>0$: $|\Phi_{\underline{\nu}}| < const\,\eta^{-r}e^{-\frac{\pi}{2g}|\underline{\omega}\cdot\underline{\nu}|}e^{-\kappa|\underline{\nu}|}$. Such estimates are called "exponentially small" and, occasionally, have been claimed to be possible.
(5) The above results are *very easy* compared to the ones that can be obtained by taking $g^2$ *fixed* and $\underline{\omega} = (\eta^{\frac12}\omega, \eta^{-\frac12}\omega')$, discussed in [GGM1] and called the *three time scales*

*problem*, because the system has three time scales of orders respectively $\eta^{-\frac{1}{2}}, 1, \eta^{\frac{1}{2}}$. In this case we consider the values of $\eta$ for which $\underline{\omega}$ verifies a Diophantine property of the form $|\underline{\omega}\cdot\underline{\nu}| > \eta^{\gamma}|\underline{\nu}|^{-t}$ with some $\gamma, \tau > 0$ and we take $\varepsilon$ equal to a suitably large power of $\eta$ so that the small divisors problems can be overcome and the invariant tori do exists.

The quasi flat estimates hold (for small $\eta$) but they *do not imply that the matrix* $\partial_{\underline{\alpha},\underline{\alpha}}\Phi(\underline{\alpha})|_{\underline{\alpha}=\underline{0}}$ *has three matrix elements of size exponentially small as* $\eta \to 0$. In fact all the four matrix elements are of the order of a power of $\eta$: this is so *in spite of the fact that the Melnikov term* $M(\underline{\alpha})$ *generates a contribution to the* $2 \times 2$ *splitting matrix with three exponentially small entries.*

In other words neither $\Phi$ nor the Hessian matrix $\partial_{\underline{\alpha},\underline{\alpha}}\Phi|_{\underline{\alpha}=\underline{0}}$ are dominated by the the Melnikov's "first order" contribution. *Nevertheless the Melnikov's contribution to the Hessian gives the leading term in the limit* $\eta \to 0$! (generically in the perturbation).

Of course if the above mentioned exponential estimates could be correct this would follow immediately from them: but they are not valid (as they would imply the wrong statement that the splitting matrix has 3 exponentially small entries) and the result holds *only because remarkable cancellations take place.* Hence, contrary to what is sometimes stated, the above case requires a delicate analysis, compared to the one in [G3] which solves easily the problems (1)÷(4) above at least as far as the domination of the first order in the derivatives of the Eliasson's function (hence the splitting vector) is concerned. In particular this means that $\Phi$ *is not a good measurement* of the splitting.

It is in the theory of this "three time scales problem" that the analogy with field theory and renormalization theory turns out to be particularly useful and the methods characteristic of such theories apply very well and turn into a rather simple matter the check of the infinitely many identities that are necessary in order that all terms in the Hessian that dominate the Melnikov contribution cancel each other leaving out just the Melnikov contribution as the leading one as $\eta \to 0$.

(6) The just described graphical technique seems not only very well suited for the questions analyzed or mentioned above but it seems quite promising also with respect to the solution of one of the main standing problems, namely: what is the asymptotic behavior of the splitting as $g \to 0$ and $\underline{\omega}$ fixed? The case mentioned in (3) above requires that all the Fourier components of the perturbation do not vanish: a finer analysis shows that this can be somewhat weakened but *not* to the extent of allowing polynomial perturbations. Hence such cases seem to have a rather limited interest: they appear in fact too special. But even so the only thing we know is the Melnikov's dominance in Eliasson's potential and in its derivatives.

On the other hand the theory discussed in [G3], [GGM3], and in §3 suggests the following question. First of all let us *define* an extension of Melnikov's function to *higher order.* We simply consider the function $\Phi^0(\underline{\alpha})$ which is obtained from the diagrammatic representation (3.9) *but replacing the operators* $\mathcal{O}$ *associated with the nodes of* $\tau$ *by the operator*:

$$\mathcal{O}_0(F)(t) = \frac{1}{2}\sum_{\chi=\pm}\int_{\chi\infty}^{t}\big(w_{01}(t)w_{00}(\tau) - w_{00}(t)w_{01}(\tau)\big)\,F(\tau)\,d\tau \tag{4.5}$$

then, supposing $\underline{\omega}_0$ with golden rotation number (or any number with very good Diophantine nature):

*Conjecture: In model (2.1) and assuming that* $g = \eta^{\frac{1}{2}}$, *will the Hessian of* $\Phi^0(\underline{\alpha})$ *give the leading asymptotics as* $\eta \to 0$ *of the splitting determinant at* $\underline{\alpha} = \underline{0}$ *"generically" in* $f$ *?*

Here generic means both genericity in the space of trigonometric polynomial pertubations of fixed degree (arbitrary) and in the space of the analytic perturbations, possibly with the constraint that the perturbation is of positive or negative type. However we require that the perturbation be polynomial in the $\varphi$ variable, see (9) below. As far as I know there is no proof even of the convergence of the series defining $\Phi^0$ (which is well defined only as a formal series and which may have to be regarded as an asymptotic series, see [G3], [GGM1]).

The conjecture can be extended to the case of three time scales considered in (5): in that case it is affirmatively answered in [GGM], where, however, one also sees that the Eliasson's potential and its derivatives is *not* dominated by the first order. It is only the splitting determinant that is dominated by the first order: *not surprisingly as this is the only quantity among the ones discussed which has a direct physical meaning.*

The conjecture could be strengthened by adding, for instance, that $\Phi^0$ can be replaced by the function $\tilde F^0$ obtained from $\Phi^0$ by developing in powers of $\varepsilon$ its Fourier coefficients $\Phi^0_{\underline\nu}$ and retaining only the lowest non vanishing order $\tilde\Phi^0_{\underline\nu}$ of each Fourier coefficents to form the Fourier transform of $\tilde\Phi^0$. In this stronger form it becomes, in the assumptions of (2) above (fast forcing and "maximal size of the Fourier coefficients of the perturbation), simply the statement that the splitting determinant can be computed by the first order Melnikov's integral: an open problem (as mentioned above). However in this form the conjecture is not really stronger than above because using $\Phi^0$ instead of $\Phi$ amounts to saying that the $\varepsilon d\,\underline\nu\,,\underline\nu\,'$ in (4.4) has the form $\varepsilon(\,\underline\nu\,\wedge\,\underline\nu\,')^2 d'_{\underline\nu\,,\underline\nu\,'}$.

Clearly in order that the answer to the question be affirmative one has to show the existence of suitable cancellations: I have checked that *they are indeed present at the order beyond the lowest* (since the lowest order for the homoclinic determinant is the second, this means that the answer is affirmative to third order). The check requires using the results in [GGM1], which may already imply a positive answer to all orders.

Denoting $\mathcal O$ the operator $\mathcal O(F)(t)\stackrel{def}{=}\mathcal O_{\sigma(t)}(F)(t)$ (with $\sigma(t)=\text{sign}\,(t)$) one remarks that in all the expressions involved in the graphs evaluations one always really uses $\mathcal O$; then it is useful to note the (algebraic) relation between the operator $\mathcal O$ and $\mathcal O_0$:

$$\begin{aligned}\mathcal O F(t)=&\mathcal O_0F(t)+|w_{01}(t)|\,G_0(F)+w_{00}(t)\,G(F)\\ \mathcal O_0F(t)=&\frac12\sum_{\chi=\pm}\int_{\chi\infty}^t\big(w_{01}(t)w_{00}(\tau)-w_{00}(t)w_{01}(\tau)\big)\,F(\tau)\,d\tau\\ G_0(F)=&\frac12\int_{+\infty}^{-\infty}d\tau\,w_{00}(\tau)\,F(\tau)\,d\tau\,,\qquad G(F)=\frac12\int_{+\infty}^{-\infty}d\tau\,|w_{01}(\tau)|\,F(\tau)\,d\tau\end{aligned}\tag{4.6}$$

and, as it is clear from [GGM1], the $G,G_0$ factors play the role of "counterterms" in the field theory interpretation of the diagrammatic expansion of $\Phi$. Hence the above question suggests that the leading behavior of the splitting determinant is due to graphs without counterterm contributions.[5] Here the "counterterms" contain non analytic functions and they are responsible for the impossibility of exponentially small estimates. A positive answer to the above question would state that they only give rise to subleading contributions to the splitting.

An explicit expression for the value contributing to $\Phi^0$ can be found in [GGM1]: see (6.2), for the isochronous case (2.1), and see the paragraph preceding (7.4) for the anisochronous case.

(7) The above theory can be immediately extended to anisochronous cases: one just has to consider a few new types of graphs, [G3], that contribute to the splitting vector $\underline Q\,(\,\underline\alpha\,)$ and to the splitting potential $\Phi(\,\underline\alpha\,)$.

(8) Most of the considerations above do not really use that the dimension of the quasi periodic motion is 2: if it is suppoo=sed larger it is however difficult to see what will

[5] Called in field theory "most divergent" graphs: rather improper an expression because in any reasonable field theory there should be no divergences at all; as it is the case in the theories that have been actually shown to exist on a mathematical basis

be the leading behavior of the splitting. One reason is that even the analysis of the Melnikov term is itself a quite difficult task: Diophantine approximation theory is in a very rudimentary stage if the dimension of the quasi periodic motion is $\ge 3$.

A glimpse of the difficulties that one should expect to meet is given by the three time scales problem (5) above. In this problem we can think that the slow frequency of order $\eta^{\frac12}$ is in fact obtained because the perturbation by a *three dimensional* quasi periodic motion with three *fast* frequences $\omega_1, \omega_2, \omega_3$ of order $\eta^{-\frac12}$ contains an almost resonant harmonic $\underline{\nu}$ such that $\nu_1\omega_1 + \nu_2\omega_2 = O(\eta^{\frac12})$. One would then naively think that in this case the homoclinic splitting can become "large" because we can have $\underline{\omega} \cdot \underline{\nu}$ small of order $\eta^{\frac12}$ with not too large $\nu_1, \nu_2$. But *this is illusory* precisely because from the results of the case (5) one sees that, although we could expect a large splitting vector and matrix, its Hessian at the homoclinic point will be exponentially small as $\eta \to 0$. Therefore in the three dimensional case we should expect that the resonances *do not enhance the splitting*: they can make large the spliting matrix but not its determinant! This remark also explains why the problem (5) above is so unexpectedly difficult to analyze (see [GGM1]).

(9) Finally there seems to be no reason whatsoever for having a small homoclinic splitting when the perturbation is not a polynomial (but "just" analytic) in the $\varphi$ variable, not even when the rotation vector $\underline{\omega}$ is very fast.

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