An equivariant branching lemma for relative equilibria

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Summary The "equivariant branching lemma" of Cicogna and Vanderbauwhede is extended to consider the branching of relative equilibria via bifurcation in equivariant dynamical systems. The results rests on projection to group-orbit space.

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1. The Equivariant Branching Lemma (EBL) is a fundamental tool in equivariant bifurcation theory [1-5]; this was first formulated independently by Cicogna [6,7] and Vanderbauwhede [8] for the case of bifurcation of stationary solutions, and then extended in a number of ways, in particular to include Hopf bifurcation [9] and problems whose symmetry corresponds to a nonlinear group action [10].

The basic idea behind the EBL in its various versions is to use the symmetry properties of the system (or class of systems) under study to guarantee the dynamical invariance of a lower-dimensional space or manifold, and to apply on the restriction of the initial system to this invariant space standard (usually topological) tools to guarantee the existence of a bifurcating branch of solutions; thus any version of the EBL is strictly connected to a suitable version of the Reduction Lemma (RL) [4].

In the present short note we want to remark that a simple use of a rather old result by Michel [11] allows to obtain, under certain condition, am EBL for the bifurcation of branches of *relative equilibria* out of stationary solutions.

Relative equilibria – i.e. group orbits which are invariant under the dynamics – received in recent years extended attention, also in the context of bifurcation theory [12-14], leading to a good understanding of them; I do not claim the result given here is more powerful than those which are present in the literature, but its derivation – and the check that a given system satisfies the assumptions required for its validity – is so simple that I think it can however be of interest to report it.

2. In his work [11] motivated by the theory of strong interactions [15-17], Michel proved a remarkable general theorem, valid for the action of compact groups on finite dimensional manifolds. His investigation was naturally set in the variational frame, but he actually proved results on general equivariant vector fields, and I would like to quote his results in this language in view of the application to relative equilibria.

We denote by G the finite dimensional, compact Lie group acting on the manifold M; the quotient M/G is also called the orbit space and denoted by Ω (see e.g. [4,5,18,19] for its relevance in nonlinear dynamics and equivariant bifurcation theory). In general, Ω is not a manifold.

In more abstract terms, Ω is the set of classes of equivalence on M under the group action. The orbit $G(p)$ of a point in M will also be denoted as $\omega(p)$ or ω_p.

For each point $p \in M$ we can consider the set of elements in G which leave p unchanged,

$$G_p \ := \ \{ g \in G \mid gp \, = \, p \} \ ; \tag{1}$$

this is called the isotropy subgroup of p. We introduce another equivalence relation in M as follows: we consider points with isotropy subgroups conjugated in G as having the same *isotropy type*, and denote the set of points having the same isotropy type as a *stratum* in M:

$$\sigma(p) \ := \ \{ q \in M \mid \exists g \in G \ : \ G_q = g G_p g^{-1} \} \ . \tag{2}$$

Strata are submanifolds in M.

Notice that $q = gp$ implies $G_q = g G_p g^{-1}$, so that necessarily

$$\omega(p) \ \subseteq \sigma(p) \tag{3}$$

for any point $p \in M$; this means also that we can define strata in Ω; the stratum of an orbit $\omega \in \Omega$ will be denoted as $\Sigma(\omega)$.

When the group action in M is regular (this is automatically true if G is compact), the topology induced by the group action is compatible with the topology on M, and thus the topology in M induces a natural topology in Ω. In other words, it is then possible to define properly a distance $d(\omega_1, \omega_2)$ between group orbits (e.g. as the minimal distance in M between points $p \in \omega_1$ and $q \in \omega_2$).

We say that an orbit ω is *isolated in its stratum* if there is an $\varepsilon > 0$ such that $d(\omega, \mu) > \varepsilon$ for all the $\mu \neq \omega$ in $\Sigma(\omega) \subset \Omega$.

We refer to [17,20-22] for further detail on this geometric construction and its applications in Physics.

3. Let us now consider an equivariant vector field X on M; that is, a field such that for all $Y \in \mathcal{G}$ (the Lie algebra of G), satisfies $[X, Y] = 0$.

We will think, for ease of language, M as embedded in R^N, and use coordinates $\{x_1, ..., x_N\}$ in M; thus X can be written in coordinates as corresponding to the Dynamical System

$$\dot{x} \ = \ f(x) \tag{4}$$

where $f : M \to \mathrm{T}M$.

The equivariance of f means that for any $x \in M$ and all $g \in G$,

$$f(gx) \ = \ \varphi_g[(D_g f)(x)] \ . \tag{5}$$

Here $\varphi : \mathrm{T}M \to \mathrm{T}M$ is the isomorphism induced by g on tangent spaces, i.e. φ_g maps $\mathrm{T}_x M$ into $\mathrm{T}_{gx} M$.

It was proved by Michel [11] that

$$f(x) \ \in \ \mathrm{T}_x \sigma(x) \ ; \tag{6}$$

this also means that, as it was already proved by Bierstone and Schwarz [20,23-25] the vector field $X : M \to \mathrm{T}M$ can be projected to a vector field $X_\Omega : \Omega \to \mathrm{T}\Omega$ defined on the orbit space (recall each stratum is a smooth manifold).

It should be noticed that (6) also means that the closure of a stratum is an invariant set under any equivariant dynamics, and we can consider the restriction of (4) to it. This is a general "reduction lemma" in the spirit of Golubitsky and Stewart [4].

Now, from (6) it also follows that if an orbit ω is isolated in its stratum, then for all $x \in \omega$

$$f(x) \ \in \ \mathrm{T}_x \omega \ . \tag{7}$$

As noticed by Michel, in the case of invariant variational problem one can conclude that all the $x \in \omega$ are critical for any invariant potential [11,17] (necessarily the gradient is orthogonal to the group orbit, and thus must vanish).

In the case of general equivariant vector fields (7) can be interpreted as follows:

Lemma 1. *If a G-orbit ω on the G-manifold M is isolated in its stratum, then it is a relative equilibrium for any equivariant vector field X defined on M.*

4. Let us now consider the case where the equivariant vector field X depends on a control parameter $\lambda \in \Lambda$ (Λ is an interval on the real line), so that $X^{(\lambda)}$ is G-equivariant uniformly in λ; in other words, (4) is now replaced by

$$\dot{x} \ = \ f(x;\lambda) \tag{4'}$$

and (5) holds for all $\lambda \in \Lambda$.

We assume moreover that (a) $x \equiv x_0$ (call ω_0 the corresponding G-orbit) is a solution for all $\lambda \in \Lambda$, and that it is stable for $\lambda < \lambda_0$, and unstable for $\lambda > \lambda_0$ (this λ_0 is assumed to be in the interior of Λ). To avoid explosive behaviour, we assume also that (b) there is an open set $\mathcal{B}$, containing x_0 in its interior and of positive radius (in the metric defined on M), such that $X^{(\lambda)}$ points inward on $\partial\mathcal{B}$, for all values of $\lambda \in \Lambda$. As standard in bifurcation theory, we also assume that (c) the real part $\sigma_i^r(\lambda)$ of the critical eigenvalues of $L(\lambda) := (Df)(x_0;\lambda)$ satisfies

$$\left(\frac{\mathrm{d}\sigma_i^r}{\mathrm{d}\lambda} \right)_{\lambda_0} \ > \ 0 \ . \tag{8}$$

These assumption guarantee in a standard waythat x_0 bifurcates at $\lambda = \lambda_0$.

Assume now that there is a stratum $\Sigma_0 \subset \Omega$ which is a one-dimensional manifold: as $X_\Omega^{(\lambda)}$ is tangent to strata, it is directed along Σ_0 at all $\omega \in \Sigma_0$; moreover the above assumptions guarantee that for $\lambda > \lambda_0$ there is a branch of zeroes of $X_\Omega^{(\lambda)}$ lying precisely in the intersection of Σ_0 with $\mathcal{B}$ and distinct from ω_0. A zero of $X_\Omega^{(\lambda)}$ corresponds, as already remarked, to a relative equilibrium for the dynamics on M, described by $X^{(\lambda)}$; there is a branch of equilibria for $X_\Omega^{(\lambda)}$ bifurcating off ω_0, we can conclude there is a branch of relative equilibria for $X^{(\lambda)}$ bifurcating off x_0.

We can express the result of our discussion as follows; here we assume that x_0 is invariant under the full G:

Lemma 2. *Let us consider a one-parameter family of vector fields $X^{(\lambda)}$ on M, and let $x_0 \in M$ with $G_{x_0} = G$ be an isolated zero of $X^{(\lambda)}$ for all $\lambda \in \Lambda$; let assumptions (a),(b),(c) above be satisfied. Then, there is a local branch of relative equilibria bifurcating from x_0 at $\lambda = \lambda_0$.*

Notice that if $G_{x_0} \neq G$, then ω_0 does not reduce to x_0 alone, and x_0 is a degenerate zero of X; however, the discussion in terms of X_Ω does still hold, and we can again deduce the bifurcation of a branch of relative equilibria. Also, even if x_0 is not a zero of X, but ω_0 is a zero of X_Ω and we focus on the dynamics transversal to $\omega_0 \subset M$, we can again conduct the same discussion in terms of X_Ω, and deduce the existence of a branch of relative equilibria bifurcating off relative equilibria (notice relative equilibria can correspond to different subgroups of G, so that we can have a symmetry breaking leading from a type of relative equilibrium to another).

5. We can actually refine (6) – and the above remark – a little further: following Golubitsky and Stewart, let us define for each $H \subseteq G$ the submanifold

$$M^{(H)} \ := \ \{ x \in M \mid H \subseteq G_x \} \ . \tag{9}$$

Now, if in (5) we consider $g \in G_x$, we have immediately that

$$f(x) \; = \; (D_g f)(x) \quad \forall g \in G_x \tag{10}$$

and thus that

$$f(x) \; \in \; \mathrm{T}_x M^{(G_x)} \; . \tag{11}$$

Combining (7) and (11) we conclude immediately that for an orbit ω isolated in its stratum we have, for all $x \in \omega$,

$$f(x) \; \in \; \mathrm{T}_x \omega \cap \mathrm{T}_x M^{(G_x)} \; . \tag{12}$$

(notice in this case we could as well consider the manifold of points whose isotropy subgroup is exactly equal to G_x, as on ω we cannot have a G_y which strictly contains G_x).

However, $T_x \omega$ is by definition equivalent to the linear space spanned by the Lie algebra $\mathcal{G}$ of G, or more precisely to $\mathcal{G}/\mathcal{G}_x$ (where $\mathcal{G}_x$ is the Lie algebra of G_x). Thus the flow $\Phi_f(x;t)$ under an equivariant vector field will take x into $y \in \omega(x)$ such that $G_y = G_x$; such y can be written, as we are on $\omega(x)$, as $y = hx$ for some $h \in G$, and we conclude that necessarily $h \in N_G[G_x]$, the normalizer of G_x in G. Notice that, since G_x acts as the identity, we can actually consider $h \in D^0_x$, where $D \subseteq G$ is defined as

$$D_x \; := \; N_G[G_x]/G_x \tag{13}$$

and D^0 is the connected component of the identity in D. Thus, necessarily

$$f(x) \; \in \; \mathrm{T}_x[D^0(x)] \; . \tag{14}$$

The above construction and discussion are due to Golubitsky and Stewart; when G_x is a maximal isotropy subgroup of G, they obtained – as well known – powerful general results for linear actions: based on a homological classification of possible D^0, they obtained a classification of elementary bifurcations [4].

In the present context, the above discussion shows that to investigate the possible relative equilibria arising by the bifurcation mechanism we are considering, we have to focus attention on the subgroups D^0. In particular, if $D^0 = \{e\}$, then the relative equilibria whose existence is granted from Lemma 2 are actually equilibria *tout court*. Similarly, if $D^0 = SO(2)$ we are guaranteed that these relative equilibria are actually periodic orbits (possibly degenerate, i.e. stationary).

It should also be stressed that all the relative equilibria lying on the same G-orbit must be isomorphic: this poses some apriori restrictions on the possible kind of relative equilibria. E.g., we can have periodic orbits only if ω admits a fibration in circles S^1, and n-periodic orbits only if ω admits a fibration in tori $\mathbf{T}^n$.

Finally, we would like to notice that in the same way as the "standard" equivariant branching lemma (EBL) can be extended from one-dimensional strata to higher dimensional cases by the use of topological arguments guaranteeing the existence of zeroes of a vector field on an invariant set Σ based on the topology of Σ [7], the same kind of extensions can be obtained for our "relative equilibria EBL" (lemma 2): extensions of the standard EBL apply to bifurcations of stationary points for $X^{(\lambda)}_\Omega$, and these in turn correspond to relative equilibria of $X^{(\lambda)}$.

References

1. D. Ruelle, "Bifurcations in the presence of a symmetry group", *Arch. Rat. Mech. Anal.* **51** (1973), 136
2. D.H. Sattinger, *Group theoretic methods in bifurcation theory*, Lecture Notes in Mathematics **762**, Springer (Berlin) 1979
3. D.H. Sattinger, *Branching in the presence of symmetry*, S.I.A.M. (Philadelphia) 1983
4. M. Golubitsky, D. Schaeffer and I. Stewart, *Singularities and groups in bifurcation theory – vol. II*, Springer (Berlin) 1988
5. G. Gaeta, "Bifurcation and symmetry breaking", *Phys. Rep.* **189** (1990), 1
6. G. Cicogna, "Symmetry breakdown from bifurcation"; *Lett. N. Cim.* **31** (1981), 600
7. G. Cicogna, "Bifurcation from symmetry and topological arguments", *Boll. U.M.I.* **3A** (1984), 131
8. A. Vanderbauwhede, *Local bifurcation and symmetry*, Pitman (Boston) 1982
9. M. Golubitsky and I. Stewart, "Hopf bifurcation in the presence of symmetry"; *Arch. Rat. Mech. Anal.* **87** (1985), 107
10. G. Cicogna, "A nonlinear version of the equivariant bifurcation lemma"; *J. Phys. A* **23** (1990), L1339
11. L. Michel, "Points critiques de fonctions invariantes sur une G-variété", *Comptes Rendus Acad. Sci. Paris* **272-A** (1971), 433
12. M. Krupa, *SIAM J. Math. Anal.* **21** (1990), 1453
13. M. Koenig and P. Chossat, *C. R. Acad. Sci. Paris I* **318** (1994), 31
14. P. Chossat, M. Koenig and J. Montaldi, *C. R. Acad. Sci. Paris I* **320** (1995), 25
15. L. Michel and L. Radicati, "Properties of the breaking of hadronic internal symmetry", *Ann. Phys. (N.Y.)* **66** (1971), 758
16. L. Michel and L. Radicati, "The geometry of the octet", *Ann. I.H.P.* **18** (1973), 185
17. L. Michel, "Symmetry defects and broken symmetry. Configurations. Hidden symmetry", *Rev. Mod. Phys.* **52** (1980), 617
18. P. Chossat and R. Lauterbach, *C. R. Acad. Sci. Paris I* **325** (1997), 595
19. M. Koenig, *Math. Proc. Cambridge Phil. Soc.* **121** (1997), 401
20. M. Abud and G. Sartori, "The geometry of spontaneous symmetry breaking", *Ann. Phys. (N.Y.)* **150** (1983), 307
21. G. Sartorymmetry breaking and gauge theories", *Ann. Phys. (N.Y.)* **260** (1997), 149
23. E. Bierstone, *Topology* **14** (1975), 245
24. G.W. Schwarz, *Topology* **14** (1975), 64
25. G.W. Schwarz, *Publ. Math. I.H.E.S.* **51** (1980), 37