Reduction and Equivariant Branching Lemma
without finite-dimensional reduction

Giuseppe Gaeta
Dipartimento di Fisica,
Università di Roma, 00185 Roma (Italy)

giuseppe.gaeta@roma1.infn.it

Summary. In the bifurcation study of nonlinear evolution PDEs with symmetry, one usually performs first a reduction to a finite dimensional critical space, thus obtaining the bifurcation equation (which inherits symmetries properties from the original problem), and then employs the symmetry – tipically through the reduction lemma and/or the equivariant branching lemma – to study this reduced problem. We argue that one could as well proceed in the opposite way, i.e. apply bifurcation analysis on a symmetry-reduced problem; this is done using some general results of Palais for variational analysis of G-invariant functionals. Such an approach presents some delicate points, which we discuss in detail.

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Introduction.

The equivariant branching lemma (EBL in the following) is, in its various versions [1-4] and together with the reduction lemma (RL in the following) [3], at the very basis of equivariant bifurcation theory [5]; we stress here that, although the EBL is usually formulated in the framework of linear group action, it also holds in the more general context of smooth (Lie-point) nonlinear group action [6].

When studying bifurcation in the presence of symmetry, one follows a standard strategy, common also to the generic (no symmetry) case: first of all one performs a reduction from the full space (e.g. an infinite dimensional function space) in which the original problem is set, to the (usually, finite dimensional) critical space, i.e. the space associated to the critical eigenvalues. This reduction can be performed using different techniques (e.g. the Lyapounov-Schmidt method, or using a Center Manifold reduction), depending on the problem at hand; in general, it is based on an implicit function theorem, and hence requires the existence of a "spectral gap": that is, there must be a gap separating the projection on the real axis of the critical eigenvalues and of the rest of the spectrum (of the appropriate linear operator entering in the bifurcation analysis).

Such a reduction belongs to "general" – i.e. not necessarily equivariant – bifurcation theory [7-15]; the symmetry present in the original problem (if any) plays a "passive" role in it: that is, while on the one side the symmetry determines the multeplicity of critical eigenvalues (up to accidental, and therefore nongeneric, degeneracies) and it is – in appropriate form – inherited by the reduced problem (notice that other symmetries, not present in the original problem, can be present in the reduced one [16]), i.e. by the *bifurcation equation*, on the other side in the reduction procedure one has to take into account the symmetry properties but does not have any help from them (except for handling the degeneration of critical eigenvalues enforced by the symmetry itself).

Once the above mentioned reduction – typically, to a finite dimensional problem – has been obtained, one can use the symmetry of the reduced problem, i.e. of the bifurcation equation; this use can go typically through further reduction into invariant subspaces determined on the basis of symmetry considerations (RL), and/or using on this (maybe reduced) equation tools which are essentially of topological origin (EBL). This procedure will be assumed to be known by the reader, and we will not discuss it any further, referring to the vast literature available (see e.g. [5-25] and references therein).

However – and this is the main observation we want to present in this note – one could also proceed in opposite order, and operate a symmetry reduction *directly* on the original, and thus possibly infinite dimensional, problem.

Once this is done, one can study bifurcation problems in the symmetry reduced setting, either directly by topological considerations, either by implementing in it the standard bifurcation analysis procedure (if possible), as discussed below.

It should be stressed that this direct symmetry reduction is *not* always possible; this is not only in the obvious sense that not all the problems present a symmetry, or a symmetry which gives a nontrivial reduction, but also in the sense that if a symmetry is present, this and the equation under study have to satisfy some further conditions, as discussed below. However, when the direct reduction is possible, it can easily deal with infinite dimensional problem, and even tackle situations in which the standard procedure can not even be started.

When we have to deal with G-equivariant flows in a function space or in a space of sections of a fiber bundle, with G a compact Lie group, the idea of direct symmetry reduction is a rather simple observation, amounting to an extension of the standard (Golubitsky-Stewart) reduction to suitable infinite-dimensional settings. However, the situation is more delicate when we wish to consider a non-compact Lie group G; and even more when we have to study variational problems defined by a G-invariant functional, as it will be explained below.

It is, of course, well known that this variational setting is actually the pertinent one in a number of physical situations. Also, although many groups arising in Physics are compact – e.g. the orthogonal, unitary or symplectic ones – other relevant groups, such as the Euclidean, Poincaré or Lorentz groups, are not compact; indeed, one is not able to study equivariant bifurcation theory for such groups, although relevant results can be obtained by assuming invariance under a subgroup of them (e.g. a lattice subgroup in the study of convection patterns). Such a reduction to a discrete function space also eliminates the continuous spectrum, and thus a major obstacle to bifurcation analysis.

In the present note we will at first consider equivariant evolution equations, for which our extension is rather easy; we will then, for the main part of this note, focus on the more difficult (and physically relevant) variational case.

Our direct symmetry reduction approach – particularly in the variational case – will be based on, and closely follow, the ideas providing the *"Symmetric Criticality Principle"* (SCP) [26,27] of Palais[1], and thus will have the same range of validity – and limitations – as this.

We would like to mention that an extension to the setting of *gauge theories* would also be possible, by use of the Michel's geometric theory[2] of symmetry breaking [34-37]. Such an extension will not be presented here as it would require the introduction of a rather heavy

[1] Actually, we will essentially just formalize the application of the SCP to bifurcation problems. See also [28,29] for a previous example of such an extension.

[2] Michel's theory aims at obtaining common solutions to all the problems in a given symmetry class (e.g. points which are critical for any G-invariant potential defined on a G-manifold M), and its developement was motivated by the study of model-independent $SU(3)$ symmetry breaking [28,29]; this has also been extended to study supersymmetry breaking [32]. See the references in [33] for a partial bibliography on Michel's theory and related matters.

formalism, which is maybe not of interest to most of the readers working in bifurcation theory; however, this extension would be obtained along the lines, and using the results, of [33] (see section 6 there) with very little additional effort. See also [38] for an application of Michel theory in infinite-dimensional non-variational problems.

In the following we will assume – as we have somewhat already done in this introduction – the reader is familiar with bifurcation and equivariant bifurcation theory [5], and freely refer to it; readers not being in this condition, however, should not have problems in following the main line of our discussion, which is reasonably self-contained.

The plan of the paper is as follows. As already mentioned, we will consider theories defined by a G-invariant (with G a Lie group) to be minimized. In section 1 below we will give an informal (and overly non-rigorous) discussion of the main idea behind our work. In the following section 2 we will recall the SCP of Palais; in section 3 we apply it to bifurcation problems, in which it can be seen as a "direct" RL, in order to obtain a "direct version" of the EBL; at this point we pause to mention how our discussion and result easily applies also to the non-variational setting. Finally, section 4 is devoted to the illustration of some examples, concerning both the variational and the non-variational setting.

1. An informal discussion.

As mentioned in the introduction, our results will require to tackle some relatively delicate matters concerning non-compact group actions and variational analysis. However, we believe that illustrating the rough idea to be implemented should help to grasp the point and to follow our discussion; thus, we will at first give an informal – and not rigorous at all – idea of the approach we want to follow.

Consider an infinite dimensional $\mathcal{V}$ in a function space, and an interval $\Lambda \subseteq R$; on this we have an action of a Lie group G. Let us now consider an equivariant flow on $\mathcal{V}$, say

$$\frac{\partial u}{\partial t} \ = \ \Phi[u;\lambda] \tag{1.1}$$

(the square brackets denote that Φ can depend on spatial derivatives of u as well); the equivariance means that

$$\Phi\,[gu;\lambda] \ = \ D_g\,\Phi\,[u;\lambda] \tag{1.2}$$

where D_g is the G-action on the tangent space to $\mathcal{V}$. Now, if $gu = g$ for all g in a subgroup $H \subseteq G$, we have that $D_g\Phi[u;\lambda] = \Phi[u;\lambda]$ for all $g \in H$; thus, considering the space $\mathcal{V}^H$ of points in $\mathcal{V}$ which are invariant under H, this is invariant under Φ, and our system can be reduced to $\mathcal{V}^H$. As the reader will have noticed, this is just the idea behind the reduction procedure of Golubitsky and Stewart.

Let us now consider another function space $\mathcal{F}$, on which G-action is defined; we give a smooth functional $L : \mathcal{F} \times \Lambda \to R$ and consider its functional gradient (also called variational derivative) δL. The idea is that if L is *invariant* under a Lie group G (acting in $\mathcal{F}$), then (δL) will be *equivariant* under G; this means that, with $L[g\varphi, \lambda] = L[\varphi, \lambda]$ for all $g \in G$, $\lambda \in \Lambda$ and $\varphi \in \mathcal{F}$, then

$$(\delta L)[g\varphi, \lambda] = ((\delta g)(\delta L))\,[\varphi, \lambda] \ ; \tag{1.3}$$

thus, if we compute $\delta\varphi$ at a $\varphi \in \mathcal{F}$ which is invariant under a subgroup $H \subseteq G$, we get

$$((\delta g)(\delta L))\,[\varphi, \lambda] \ = \ (\delta L)[g\varphi, \lambda] \ = \ (\delta L)[\varphi, \lambda] \tag{1.4}$$

for all $g \in H$. In other words, we should have that $(\delta L)[\varphi, \lambda]$ is directed along the space $\mathcal{F}^H$ of functions which admit H as (part of) their isotropy subgroup; thus the equation $(\delta L)[\varphi, \lambda] = 0$ can be reduced to $\mathcal{F}^H$, uniformly in $\lambda \in \Lambda$.

Clearly, this discussion – which has the advantage of depicting a direct generalization of the one applying in finite dimension [3,5,34] – is too cavalier to be correct in this form, i.e. in full generality; however, it is also clear that when it is indeed (under additional hypotheses and with a real proof) correct, one can reduce the variational problem, i.e. consider the restriction L_H of L to the space $\mathcal{F}^H$, claiming that critical points of $\mathcal{L}_H$ are automatically also critical points of L; this is a direct analogue of the RL [3,5].

Moreover, when the reduced space $\mathcal{F}^H$ has some favourable property, i.e. low dimension or suitable topology, one could use tools and results – essentially of topological nature – of the kind considered in the usual EBL in order to guarantee the existence of solutions, and smooth branches of solutions once the dependence on λ is taken into account, with given behaviour and located precisely in $\mathcal{F}^H$; this would be a direct analogue of the EBL [1-6,39].

It should also be remarked that our argument above does not really use the variational structure, but is just based on the equivariance of (δL); thus, it is to be expected that the same kind of results that can be obtained for a general (i.e. not necessarily stemming from a variational problem) G-equivariant application $f : \mathcal{F} \times \Lambda \to \mathrm{T}\mathcal{F}$ also hold in the G-invariant variational case, provided δL is indeed equivariant.

Actually, the variational structure should be apriori expected to make the reduction more difficult: in the reduced setting we are, by definition, not able to consider the variation of L in directions orthogonal to $\mathcal{F}^H \subset \mathcal{F}$; indeed, the above discussion used precisely the idea that the G-invariance of L guarantees that $(\delta L)[\varphi, \lambda]$ would be automatically zero in directions transversal to $\mathcal{F}^H$ [26,27].

In the following of the paper, we will make the above discussion rigorous; this will obviously require to impose suitable conditions on the groups, functions, spaces etc. to be considered. We will not attempt to give optimal conditions, but be satisfied with "reasonable" ones.

In order to show clearly that such conditions are indeed necessary, the following example (which is example 3.2 of [26]) should be quite convincing, and also showing how one should be careful even in the simpler finite dimensional setting.

Consider the space R^2 with coordinates (x, y), and in this the action of the group $G = R$ defined by $g_t : (x, y) \to (x + ty, y)$. Obviously, a potential $V(x, y)$ is G-invariant if and only if it is actually a function of y alone, $V = V(y)$, e.g. $V(x, y) = y^k$; we will consider in particular $V(x, y) = y$. The set $\mathcal{F}^G$ of points having the full group G as isotropy subgroup corresponds to the x axis, $\mathcal{F}^G = \{(x, 0)\}$. Notice that the restriction V_0 of a G-invariant V to the x axis is simply $V_0(x) = c = V(0, 0)$, and thus all points of $\mathcal{F}^G$ are critical for V_0; however, it is clear that for $V(x, y) = y$ there is no critical point, so that the Symmetric Criticality Principle – and therefore the reduction procedure sketched above – does not apply. Actually, the situation is even worse: indeed, the gradient $(\nabla V)(x, y) = (0, 1)$ of the G-invariant potential $V(x, y) = y$ is *not equivariant*. As shown later on in the examples section, the case in which we deal with an equivariant flow leads to a reduction. Notice that the group acting in this case is not compact.

2. Equivariant flow on G-manifolds and reduction.

Let us consider a Lie group G and a smooth manifold[3] M on which a G action is defined.

Recall that a chart at $\mu \in M$ is a diffeomorphism Φ of an open set $U \subset M$ (with $\mu \in U$) onto an open set $\Phi(U)$ in a Banach space V, such that $\Phi(\mu) = \{0\}$.

For any point $\mu \in M$ and any subgroup $H \subseteq G$ we can consider

$$G_x \ := \ \{g \in G \ : \ gx = x\} \tag{2.1}$$

$$M^H \ := \ \{x \in G \ : \ gx = x \ \forall g \in H \subseteq G\} \tag{2.2}$$

The subgroup G_x is called the isotropy subgroup of x, and the set M^H is the fixed points set under $H \subseteq G$. Notice that G_x is obviously a subgroup of G, but M^H is not guaranteed to be a submanifold of M.

Let us consider a point $\mu \in M$, such that $g\mu = \mu$ for all $g \in H \subseteq G$, i.e. $H \subseteq G_\mu$ and $\mu \in M^H$. Then to μ is associated an action of G in $T_\mu M$, denoted by (Dg_μ).

We say that G is *linearizable about* μ if there is such a Φ that for each $g \in G$ the map

$$\Psi \ = \ \Phi \cdot g \cdot \Phi^{-1} \ : \ \Phi(U) \to V \tag{2.3}$$

is the restriction to $\Phi(U)$ of a linear map $A_g : V \to V$. These A_g provide a linear representation of G in V, and $(D\Phi_\mu) : T_\mu M \to V$ sets up an equivalence of this with (Dg_μ), which is therefore called the linearization of G at μ.

[3] Throughout this paper, a manifold is possibly an infinite dimensional manifold in a function space; by *smooth* we mean $\mathcal{C}^\infty$.

As the A_g are linear maps, the set

$$W = \{v \in V \ : \ A_g v = v \ \forall g \in H\} \ \equiv \ V^H \tag{2.4}$$

is a closed linear subspace of V; as $\Phi : U \cap M^G \to \Phi(U) \cap W$, M^H is locally a smooth submanifold of M at μ. Therefore:

Lemma 1. *If M is a smooth G-manifold, and H a subgroup of G such that the G-action of H is linearizable about each point $x \in M^H$, then the set $M^H \subseteq M$ of symmetric points is a smooth submanifold of M.*

We stress that the converse of this lemma is not necessarily true, as shown by example 3.2 of [26]. Notice that if M is Riemannian and G acts isometrically, tyhe action is linearized about $\mu \in M^G$ by just passing to geodesic coordinates.

Thus, we are naturally led to ask when it is that the G action can be linearized about any point of M^G for M a G-manifold. It is known that when M is a Banach manifold and G is compact with $\mathcal{C}^1$ action on M, then the G action can be linearized about any point $\mu \in M^G$ [26,40]; however, the situation is much less clear for G non compact, even for G a connected and semisimple Lie group. It was conjectured by Palais and Smale in 1965 (see [26//PS]) that in this latter case the action is indeed linearizable about any $\mu \in M^H$, but such conjecture is, as far as I know, not proven; on the other side, it is known that the conjecture is true when G acts by a finite dimensional linear representation [41], and also when M is a real analytic manifold and the G action is real analytic [42,43].

This brief discussion shows how difficult it is to obtain general results for non-compact groups. On the other side, when we deal with a specific G-action we only have to check explicitely if the G-action can be linearized about points in M^H, which is a much easier task. Needless to say, when we have to deal with linear group actions – as it is often the case in physically relevant situations – we do not have to worry about this.

The condition of linearizability of the G-action is obviously needed to define an equivariance of $\Phi : M \to \mathrm{T}M$; however, it is *not* sufficient to infer that $\Phi : M^H \to \mathrm{T}M^H$.

In order to discuss this, we have to introduce the concept of admissible Banach space.

Consider a Banach linear G-space $\mathcal{V}$, i.e. a Banach space on which G acts linearly. Let $\mathcal{V}^*$ be the Banach space dual to $\mathcal{V}$; this is also a linear G-space, with G action given by $(g^* \varphi)(v) = \varphi(g^{-1} v)$, where $\varphi \in \mathcal{V}^*$ and $v \in \mathcal{V}$.

Denote by $\mathcal{W} \subseteq \mathcal{V}$ the linear subspace on which G acts as the identity, i.e.

$$\mathcal{W} \ = \ \{v \in \mathcal{V} \ : \ gv = v \ \forall g \in G\} \ . \tag{2.5}$$

Correspondingly, denote by $\mathcal{W}^* \subseteq \mathcal{V}^*$ the linear subspace of $\mathcal{V}^*$ on which G acts as the identity,

$$\mathcal{W}^* \ = \ \{\varphi \in \mathcal{V}^* \ : \ g\varphi = \varphi \ \forall g \in G\} \ ; \tag{2.6}$$

this is obviously the space of linear G-invariant functionals on $\mathcal{V}$.

To any linear subspace $V \subset \mathcal{V}$ is associated its annihilator $\mathrm{Ann}(V)$, also denoted as V_0, i.e. the linear subspace of $\mathcal{V}^*$ given by functionals which vanish identically on V,

$$\mathrm{Ann}(V) \equiv V_0 := \{\varphi \in \mathcal{V}^* \ : \ \varphi(v) = 0 \ \forall v \in V \subseteq \mathcal{V}\} = \{\varphi \ : \ V \subseteq \mathrm{Ker}(\varphi)\} \ . \tag{2.7}$$

Thus we can in particular consider $\mathcal{W}_0$, the annihilator of $\mathcal{W}$; the condition $\mathcal{W}^* \cap W_0 = \{0\}$ means that a linear G-invariant functional which vanishes identically on $\mathcal{W}$ does actually vanish identically on $\mathcal{V}$.

When this is the case, i.e. when

$$\mathcal{W}^* \cap W_0 = \{0\} \ \text{ for } \ \mathcal{W} = \{v \in V \ : \ gv = v \ \forall g \in G\} \tag{2.8}$$

we say that the Banach linear G-space $\mathcal{V}$ is *admissible*.

Notice that if G is compact, then any Banach linear G-space $\mathcal{V}$ is admissible. For general G, there is a simple criterion to check if a Banach linear G-space is admissible:

Lemma (Palais) *A Banach linear G-space $\mathcal{V}$ is admissible provided that for each nonzero G-invariant linear functional $\varphi :\in \mathcal{V}^*$ the invariant hyperplane φ^0 has an invariant complementary space.*

Corollary. *If $\mathcal{V}$ is completely reducible, so that every closed invariant subspace has a complementary closed invariant subspace, then $\mathcal{V}$ is admissible.*

This discussion is immediately applied to Banach G manifolds via the:

Definition. *A smooth G-manifold M is admissible if the G action is linearizable about each symmetric point $\mu \in M^G$ and the tangent space $V_\mu = \mathrm{T}_\mu M$ equipped the linearization Dg_μ is an admissible Banach linear G-space.*

This will be the necessary (and sufficient) condition to be able to consider a reduction of equivariant flows on M to M^G (and *a fortiori* to M^H for all $H \subseteq G$); see the discussion below in section 4.

3. The symmetric criticality principle

In this section we will briefly recall the "symmetric criticality principle" (SCP) of Palais; our discussion will closely follow [26]. We refer to [27] for further detail; see [40,44] for variational analysis in the presence of symmetries.

In short, the SCP says that if we have a functional L defined on a G-space $\mathcal{F}$, and we consider the restriction L_0 of this to the subspace $\mathcal{F}^G \subseteq \mathcal{F}$ of G-invariant functions, critical points of L_0 are also crirical points of L; one says also that "critical points on symmetric space are symmetric critical points" [26].

The SCP is **not** valid in general, but one knows conditions which guarantee it applies (as well as a number of counterexamples when these are not satisfied) [26,27]; thus, what we really have to discuss are conditions which can guarantee the validity of the SCP, and be satisfied in physically meaningful situations.

Let us first of all consider the case, standard in Physics (although not always explicitely formulated in this way) in which $\mathcal{F}$ is some space of sections of a fiber bundle [45-51]. Notice that this bundle could be trivial, e.g. $\mathcal{R}^n \times U$ if we consider a functional over functions $f : \mathcal{R}^n \to U$: in this case we can also see the functional as defined over the space $\mathcal{F}$ of sections of this bundle; imposing some condition on f, i.e; choosing a function space, is equivalent to selecting a subspace of sections of the bundle: e.g., considering L^2 functions corresponds to considering L^2 sections, or a finite energy condition will lead to considering a Sobolev space of sections.

We will thus consider a fiber bundle (E, π, B) over a smooth manifold B with total space E and fiber $\pi^{-1}(x) \simeq F$; we will then consider a space Σ of sections of E, and we will need this to be complete and equipped with a scalar product; thus, it is natural to require that Σ is a Sobolev space of sections (in physical applications, these appear naturally by finite energy conditions).

We will then consider a smooth functional $L : \mathcal{F} \times \Lambda \to R$, where $\lambda \in \Lambda \subseteq R^n$ will be parameters on which L (and $\mathcal{L}$ to be introduced below) can depend; we will be mainly interested in the case $\Lambda \subseteq R^1$.

We will, for definiteness, suppose L to be written in terms of a density $\mathcal{L} : J^p\mathcal{F} \times B \times \Lambda \to R$, with J^pF representing a jet over $\mathcal{F}$ of suitable (finite) order p, as

$$L[\varphi, \lambda] \ = \ \int_B \mathcal{L}[\varphi(x), \lambda; x] \, \mathrm{d}\mu(x) \quad ; \tag{3.1}$$

here $\mathrm{d}\mu(x)$ is a measure over the base space B, and the square brackets indicate that L and $\mathcal{L}$ depend on derivatives of φ up to some finite order. In (3.1) we have included a possible explicit dependence of $\mathcal{L}$ on the point $x \in B$.

As the parameters will not play any role in this section, we will drop them from the notation for ease of writing; the results obtained here will be uniform in $\lambda \in \Lambda$. In next section we will reintroduce λ in our discussion.

We will consider the case where there is a group Γ of operators $\gamma : \Sigma \to \Sigma$ acting regularly in Σ and such that

$$L[\gamma(\sigma), \lambda] \ = \ L[\sigma, \lambda] \quad \forall \gamma \in \Gamma, \ \forall \lambda \in \Lambda \ ; \tag{3.2}$$

in this case we say that Γ is a symmetry group of L.

Recall that the regularity of the group action means that the topologies of Γ and of Σ are compatible, i.e. that there are no sections which are near in the Σ topology but at a "large" distance along a Γ orbit. A simple example of non-regular action in finite dimension (here the role of sections is, of course, played by points), with $G = R$, is provided by the irrational flow on the torus T^2: now any neighbourhood, no matter how small, of a point x_0 contains points which are on the G-orbit of x_0, at large distance from x_0 along the orbit (but not, of course, in the metric on T^2).

`Remark 1.` That the regularity of the action plays an essential role can be easily seen by considering the finite dimensional example just mentioned, i.e. the action of $G = R$ on T^2 given by an irrational flow; this situation can be compared with the one obtained by considering a rational flow (in which case only $G/Z \simeq SO(2)$ acts effectively, so that we are actually dealing with a compact group action). $\odot$

`Remark 2.` If for all elements of $H \subseteq G$ the action of H is linearizable about each point of Σ^H, then Σ^H is a smooth submanifold of Σ [26]. Notice also that, as is seen by the example mentioned in the previous remark 1, this requires however the assumption of a regular action. $\odot$

If E is a G fiber bundle (i.e. a fiber bundle having G as structure group), it is natural to consider groups of the form $\Gamma = \Gamma_b \otimes_{\to} \Gamma_f$, where Γ_b is a group of transformations of the base space B, $\Gamma_f \subseteq G$ acts on the fiber, and $\otimes_{\to}$ denotes semidirect product; it is implicit in this that B is a Γ_b-manifold (i.e. a manifold on which it is defined an action of Γ_b). More generally, assuming that B is a Γ-manifold, we can assume that $\gamma \in \Gamma$ acts in Σ as

$$(\gamma\sigma)(x) \ = \ \gamma[\sigma(\gamma^{-1}(x))] \ . \tag{3.3}$$

`Remark 3.` As stressed in [26], the assumption that B is a Γ-manifold is to be understood to include the assumption that the map $(\gamma, x) \to \gamma(x)$ is a smooth map from $\Gamma \times B$ to B, and not just a smooth map $B \to B$ for each $\gamma \in \Gamma$. $\odot$

We can then consider subgroups $H \subseteq \Gamma$, and to any such subgroup we can associate the subset Σ^H of sections which are invariant under H, i.e.

$$\Sigma^H \ = \ \{\sigma \in \Sigma \ : \ \gamma(\sigma) = \sigma \ \forall \gamma \in H\} \ . \tag{3.4}$$

When Γ is a **compact** Lie group, the Σ^H are smooth submanifolds of Σ (union, possibly disjoint, of linear subspaces if Γ acts linearly), and we can consider the restriction L_H of L to $\Sigma^H \times \Lambda$.

`Remark 4.` Notice that assuming Γ compact does automatically guarantee the regularity of its action on Σ. $\odot$

Under the same condition of compactness of Γ, we have that: *The critical points of L_H are also critical points of L.* This statement constitutes indeed the "Symmetric Criticality Principle" of Palais [26,27], so that we are stating that it applies under the present conditions, and in particular for Γ compact. We summarize the discussion so far in the following

Theorem (Palais). [26] *Let G be a compact Lie group, B a smooth G-manifold, $(\mathcal{E}, \pi, B)$ a smooth G-fiber bundle over B with fiber $\pi^{-1}(x) \simeq F$, and M a Banach manifold of sections of $\mathcal{E}$. Let G act on M by $(g\mu)(x) = g[\mu(g^{-1}x)]$, and let $L : M \to R$ be a smooth G-invariant function on M. Then the set $\Sigma \subseteq M$ of G-equivariant sections is a smooth submanifold of M, and if α is a critical point for $L|_\Sigma$ it is in fact a critical point for L.*

It should be stressed again that, as mentioned above, the SCP is not valid in full generality, i.e. if we remove the assumptions considered above. The example of Palais mentioned at the end of section 1 is perhaps the best illustration of this.

Unfortunately, these assumptions can be too restrictive in a number of physically relevant cases; this applies in particular to the condition of having a compact group Γ. We have thus to consider if it is possible to have less restrictive conditions which still guarantee the validity of the SCP.

Indeed the assumptions we are considering here can be somewhat weakened; let us shortly mention how this generalization goes, again following closely [26]. For the sake of generality we will also consider Banach spaces (as already done in section 2 above) instead than restricting only to Hilbert ones. The discussion of section 2 is actually nearly completely transported in this frame; this is due to the fact that already there we had to focus on G-invariant functionals on a Banach space.

We consider now $\mathcal{F}$ a Banach G-manifold, and let $\mathcal{F}^G$ be the set of points $f \in \mathcal{F}$ such that $gf = f$ for all $g \in G$. As already mentioned in sect.2, if the G-action is linearizable about each point $f \in \mathcal{F}^G$, then $\mathcal{F}^G$ is a smooth submanifold of $\mathcal{F}$. When this condition is verified, we can enquire if $\mathcal{F}$ is an admissible Banach G-manifold, i.e. if $\mathrm{T}_f\mathcal{F}$ is an admissible Banach linear G-space for all $f \in \mathcal{F}^G$, see again the definitions and the discussion is section 2.

This is indeed a relevant feature for the SCP: in fact, we have the

Lemma (Palais) [26] *The symmetric criticality principle is valid for G-invariant functionals $L : \mathcal{F} \to \mathcal{R}$ defined on an admissible Banach G-manifold $\mathcal{F}$.*

By recalling the discussion in section 2 on admissible Banach G-manifolds, we also have immediately:

Corollary *If G is a compact Lie group, then any smooth G-manifold $\mathcal{F}$ is admissible and hence the symmetric criticality principle applies for any G-invariant functional $L : \mathcal{F} \to \mathcal{R}$.*

Corollary *If the Banach G-manifold $\mathcal{F}$ is real analytic and G is a connected semisimple*

Lie group with a real analytic action on $\mathcal{F}$, then the smooth G-manifold $\mathcal{F}$ is admissible and hence the symmetric criticality principle applies for any G-invariant functional $L : \mathcal{F} \to \mathcal{R}$.

`Remark 5.` Notice that in the above discussion, G is required to act linearly when we deal with a Banach space, but it can act nonlinearly – provided the action is separately linearizable around each point – when we deal with a smooth G-manifold. Needless to say, when the G action on a manifold is linear, the whole discussion is much easier. $\odot$

`Remark 6.` Notice that this discussion could be extended to the case where instead than having $L[\gamma\varphi, \lambda] = L[\varphi, \lambda]$, we have $L[\gamma\varphi, \lambda] = A[\varphi, \lambda] \cdot L[\varphi, \lambda]$, with A a scalar function: in this case the gradient (δL) at $\varphi \in \Sigma^H$ is also tangent to Σ^H and zero in transverse directions. This is similar to what happens when discussing the relation between symmetries of the lagrangian and conserved quantities (Noether theorem): when the lagrangian is not invariant under a group G, but gets multiplied by a scalar factor (as is the case e.g. for scaling applied to harmonic oscillators systems), we have invariance of the Euler-Lagrange equations, and useful results follow as well (see e.g. [52] for detail). $\odot$

`Remark 7.` It is maybe also of some interest to briefly mention the relation of the topics discussed here (and in the following) with the geometrical theory of symmetry breaking developed by physicists in the context of fundamental interactions.

The SCP can be seen, especially from a physicist's point of view, as an extension of results obtained (and methods employed) by Michel [34] in the study of the geometry of spontaneous symmetry breaking for fundamental interactions; see also the works of Michel and Radicati [30,31,35,36] for physical applications of Michel analysis, and [32,33,37] for later extensions of his approach.

As discussed in [53], it is possible *a posteriori* to see the "usual" RL and EBL as deriving from the approach and results of Michel; however, most of bifurcation theorists were unaware of physicists' work (a notable exception being provided by Sattinger [19]), and the physicists were unaware of the possible relevance of this work in the context of Nonlinear Dynamics and bifurcation theory (an exception being provided by Cicogna [1,39]). Notice that the present author is in the unconfortable situation of having being aware of Michel's results and working in bifurcation theory, for a long time without realizing the connection among the two; thus this remark should not be seen as critical toward anybody else.

Again from a physicist's point of view, it is remarkable that at the same time of Michel and Radicati work, Cabibbo and Maiani [54] developed a related approach; this approach constitutes in many ways an anticipation of the SCP.

As it should be clear by the previous discussion, the G-invariant variational case presents greater difficulties than the study of G-equivariant flows; it is thus maybe surprising to the more mathematically oriented reader that these early works attacked directly (and successfully !) this more difficult case. However, it should be recalled that on physical grounds, fields configurations which are related by a symmetry transformation should be seen as exactly the same: in this way, it is natural – and necessary – to discuss the theory in $\mathcal{F}/G$ rather than in $\mathcal{F}$. It is remarkable that obeying to the physical needs led (so to say, nearly intuitively) Michel, Radicati, Cabibbo and Maiani to a deep understanding of a mathematical theory which would not have been needed if they had chosen to work with gradients of the invariant functionals. $\odot$

4. The SCP in the bifurcation framework.

In the discussion of the previous section, the possible dependence on a parameter $\lambda \in \Lambda$ had no role. Indeed, such λ was confined to the role of a parameter, and a little thinking shows moreover that if the dependence of $\mathcal{L}$ on λ is smooth, and assuming that the symmetry holds for any value of λ, is completely independent of λ, and does not act on Λ, then also all the objects considered above – and in particular the fixed spaces Σ^H as embedded submanifolds of Σ, or the infinite dimensional generalization $\mathcal{V}_*^H \subset \mathcal{V}_*$ of these – also depend smoothly on λ *for regular values of* λ, i.e. at the exception of bifurcation points.

In this case, away from bifurcation points, everything depends smoothly on λ, and in particular the solutions to $(\delta\mathcal{L})(\varphi) = 0$, among which those determined via the SCP, will depend smoothly on λ.

Thus the fundamental fact discussed above, i.e. that we can consider the restriction L_H and be guaranteed that critical points of L_H are also critical points of L, holds true also in the case L depends on a parameter.

When we consider such restriction $L_H : \Sigma^H \times \Lambda \to R$, the critical points of this will depend on λ, i.e. we will have families of critical points φ_λ, depending smoothly on λ.

If we consider suitable but standard hypotheses, we can in this way recover the setting of bifurcation theory, and in particular guarantee that some given critical point bifurcates into a branch of critical points with given residual symmetry; although this will just follow the standard equivariant bifurcation theory [17-25], we will briefly recall what these “suitable assumptions” could be (referring to [17-25] for more detailed discussion and possible weaker sets of conditions).

First of all, as implicit in the fact we are using the SCP, we have to require that the space Σ on which the functional L of interest is defined is such that the SCP holds; this means we have to require the:

`Condition A.` Σ *is an admissible Banach manifold.*

(Obviously here the “A” stands for admissibility). This is the condition peculiar to the case of infinite dimensional spaces and possibly non-compact groups; the other conditions will be standard for bifurcation theory.

We denote by $\mathcal{B}_{c_0}$, or briefly $\mathcal{B}_0$, the ball of radius $c_0 > 0$ in Σ. The c_0 entering in this has been assumed, for the sake of simplicity, to be independent of λ, but some suitable dependence could also be allowed. We will then also require that:

`Condition B.` *There is some $\sigma_0 \in \mathcal{B}_0 \subset \Sigma$ which is a critical point of $L[\sigma, \lambda]$ for all $\lambda \in \Lambda$, and that there is a λ_0 in the interior of Λ such that this σ_0 is a linearly stable critical point for $\lambda < \lambda_0$, and becomes linearly unstable for $\lambda > \lambda_0$, with $d\mathrm{Re}(\sigma_0)/d\lambda > 0$ at $\lambda = \lambda_0$.*

Here the "B" stands for bifurcation; indeed, when Σ is a complete space and condition C is satisfied, condition B guarantees the existence of a branch of solutions bifurcating from σ_0 at λ_0.

Denoting by $\langle ., . \rangle$ the scalar product in Σ, in the following we will assume as a blanket hypothesis that:

`Condition C.` *$\mathcal{L}$ and therefore L are convex for $\langle \sigma, \sigma \rangle > c_0$, uniformly in $\lambda \in \Lambda$.*

The condition just introduced will be used to guarantee existence of solutions inside a ball of radius c_0, and rule out "explosive" behaviour at bifurcation, but will play no role in the other parts of our discussion. Obviously, the "C" stands for convexity.

Now, thanks to condition A, we can consider the directions in $\mathrm{T}_{\sigma_0}\Sigma$ along which the instability arises; we denote this linear subspace as $\mathrm{T}^+_{\sigma_0}\Sigma$ (this is tangent to the unstable manifold through σ_0 for the – in general, infinite dimensional – dynamical system defined by $\dot{\sigma} = (\delta L)(\sigma)$ [8,12]).

We denote by $\Gamma_0 \subseteq \Gamma$ the symmetry group of σ_0, i.e. $\Gamma_0 = \{\gamma \in \Gamma \, : \, \gamma(\sigma_0) = \sigma_0\}$. Let us now consider the subgroups of Γ_0; to each of these, say Γ_H, we associate the corresponding fixed space Σ^H defined as in (4). Obviously, for each $\Gamma_H \subset \Gamma_0$, $\sigma_0 \in \Sigma^H$.

It is a simple but relevant observation that if a direction $\gamma \in \mathrm{T}_{\sigma_0}\Sigma^H$ is unstable, i.e. $\gamma \in \mathrm{T}^+_{\sigma_0}\Sigma$, then all the directions in $\mathrm{T}_{\sigma_0}\Sigma^H$ are unstable: in other words, if there is a $\gamma \in \left[\mathrm{T}_{\sigma_0}\Sigma^H \cap \mathrm{T}^+_{\sigma_0}\Sigma\right]$, then $\mathrm{T}_{\sigma_0}\Sigma^H \subseteq \mathrm{T}^+_{\sigma_0}\Sigma$. Indeed, by the equivariance of $(\delta L)(\sigma)$ – defined in $\mathrm{T}_{\sigma_0}\Sigma$ – the action of Γ takes unstable directions into unstable directions (and similarly for stable ones) [55].

Thus, it makes sense to speak about subgroups Γ_H whose fixed space Σ^H is tangent in σ_0 to the unstable tangent space $T^+_{\sigma_0}\Sigma$; for the sake of brevity these will be called unstable subgroups at σ_0. An unstable subgroup at σ_0 will be called maximal if it is not contained in any other unstable subgroup at σ_0. We have thus proven the following result.

Proposition 1. *Let us consider the variational problem corresponding to the Γ-invariant functional $L[\sigma, \lambda]$, and let the above condition A, B and C be satisfied; then for each Γ_H which is a maximal unstable subgroup at σ_0, the set $\left[\left(\Sigma^H \backslash \sigma_0\right) \cap \mathcal{B}_0\right]$ is an invariant set under the dynamics of $\dot{\sigma} = -(\delta L)(\sigma)$.*

`Remark 9.` This means that we can restrict our variational problem to each of these sets, and the critical points of such restricted problems will automatically be also critical points of the original problem. This statement represents a **Reduction Lemma** [3,5] for the bifurcation problem in the considered (infinite dimensional) setting. $\odot$

Moreover, by construction – and due to the completeness of Σ – we are guaranteed of the existence of a stable critical point σ_λ (different from σ_0 for $\lambda > \lambda_0$) in Σ^H, and the smoothness of L in λ guarantees, as remarked above, that σ_λ depends smoothly on λ. Thus we obtain also the following proposition 2, and from this and proposition 1 the corollary below.

Proposition 2. *In the same case and under the same hypotheses as in Proposition 1 (in particular that conditions A,B and C are satisfied), let H be any maximal unstable subgroup of Γ_0 at σ_0. Then, the restricted variational problem given by $L_H[\sigma, \lambda]$ has a local branch of stable critical points σ_λ for λ in a right neighbourhood of λ_0, which represent a bifurcating branch of critical points all invariants under Γ_H.*

Corollary *Under the same hypotheses as in Proposition 1, and for any H a maximal unstable subgroup of Γ_0 at σ_0, the full variational problem given by $L[\sigma, \lambda]$ has a local branch of stable critical points $\sigma_\lambda^H \in \Sigma^H$ for λ in a right neighbourhood of λ_0.*

`Remark 10.` This statement represents an **Equivariant Branching Lemma** [1-6] for the variational problem in the considered (infinite dimensional) setting. $\odot$

`Remark 11.` It should be stressed that in the above discussion, not only the ambient space Σ was admitted to be infinite dimensional, but also the manifolds Σ^H to which the reduction operates could be infinite dimensional. $\odot$

Let us mention that these results could be rather easily generalized in several ways; some of the possible extensions are sketched in the following remarks.

`Remark 12.` When the topology of $\left[\left(\Sigma^H \backslash \sigma_0 \right) \cap \mathcal{B}_0 \right]$ is such to guarantee – e.g. by Morse theory [56,57;35,40,49] – the existence of several stable critical points or of critical points of different stability, one can similarly conclude the existence of local branches of critical points with corresponding invariance properties. $\odot$

`Remark 13.` Similar considerations could be applied to subsequent bifurcations, i.e. to study the case in which the bifurcating stable branches of solutions become themselves unstable for $\lambda > \lambda_1$; in this case the relevant sets obtained by intersection of the unstable manifolds with the convexity ball $\mathcal{B}_0$ and excluding the set which is already known to be unstable (the point σ_0 in the case considered above, the branch σ_λ in more complex cases) have a richer and richer topology: they are thus on the one side more difficult to study, and on the other side more prone to give information by application of topological tools. $\odot$

Finally, we notice that in the present section we dealt directly with the variational case. However, it is clear that we could proceed in the same way if, instead than considering a variational problem defined by a functional $L : \Sigma \to R$ invariant under $\Gamma : \Sigma \to \Sigma$, we were considering a problem defined by a dynamical system

$$\dot{\sigma} = \Phi(\sigma) \ , \tag{4.1}$$

where $\Phi : \Sigma \to \mathrm{T}\Sigma$ is a Γ-equivariant (not necessarily gradient) smooth vector field on the infinite dimensional manifold Σ, provided Σ is complete (otherways we cannot conclude the existence of fixed points), and provided we can define for any point $\sigma \in \Sigma$ a scalar product in $\mathrm{T}_\sigma\Sigma$, e.g. if Σ is locally modelled on a Hilbert space (or, in other words, is a Hilbert manifold) [40].

Both these conditions are automatically satisfied when Σ is a Sobolev space and, in particular, a Sobolev space of sections of a fiber bundle; we can restrict to this setting for the sake of concreteness (and for its relevance in applications).

If we are guaranteed that for such Φ – with the same notation as above for the space of H-invariant sections, with $H \subseteq \Gamma$ – we have

$$\Phi : \Sigma^H \to \mathrm{T}\Sigma^H \tag{4.2}$$

(i.e. in particular if Σ^H is a smooth submanifold of Σ), then we can at once repeat the discussion and conclusion of the variational case. The condition (2) is guaranteed by the condition of Γ acting regularly in Σ, and in particular, when Σ is a Sobolev space of sections of a G-fiber bundle P and Γ the corresponding Sobolev space (of matching order) of ad$P$, is verified automatically provided G is a compact group.

5. Examples.

`Example 1.` We start by considering again the finite dimensional situation dealt with at the end of section 1 in the variational case; we do now consider a nonvariational case, in order to point out how the symmetry reduction has a wider range of vailidity when we deal with equivariant flows (see also the next example 2 for the infinite dimensional case).

Thus, we consider the action of the group $G = R$ on $M = R^2$, in which we take coordinates (x, y), given by

$$g_a \ : \ (x,y) \ \to \ (x + ay, y) \ ; \tag{5.1}$$

thus, the action of G is given by the matrices

$$T_a \ = \ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \ . \tag{5.2}$$

It is readily seen that there is a submanifold M_0 invariant under the full group G, and this is precisely the x-axis: indeed, $g_a(x,0) = (x,0)$.

Obviously G is a maximal isotropy subgroup, and M_0 is a maximal invariant subspace; thus M_0 must be invariant under any equivariant flow in M. We write such a flow in coordinates as

$$\begin{aligned} \dot{x} &= X(x,y) \\ \dot{y} &= Y(x,y) \ , \end{aligned} \tag{5.3}$$

and the equivariance condition

$$\begin{aligned} X(x,y) + aY(x,y) &= X(x+ay,y) \\ Y(x,y) &= Y(x+ay,y) \end{aligned} \tag{5.4}$$

requires that

$$aY = ayX_x \ ; \quad 0 = ayY_x \ ; \tag{5.5}$$

this implies that $X_{xx} = 0$ and hence

$$\begin{aligned} X(x,y) &= A(y) + B(y)\,x \\ Y(x,y) &= B(y)\,y \ . \end{aligned} \tag{5.6}$$

On M_0, i.e. for $y=0$, we have $\dot{y}=0$, so that indeed the flow leaves the x axis invariant; the equation describing the restriction of the flow to this invariant manifold is simply

$$\dot{x} = \alpha + \beta x \tag{5.7}$$

with the real constants α, β given by $\alpha = A(0)$, $\beta = B(0)$.

Thus, the symmetry reduction to $\mathcal{F}^G = M_0$ is valid in this nonvariational case, although – as we have seen in section 1 – it would not generally apply to a variational problem in the same setting.

Notice that here we have not considered any dependence on a control parameter λ, for ease of notation, but it would not be difficult to include this.

`Example 2.` We will now consider an infinite dimensional version of the case considered in example 1 above.

Let us consider functions $u(x,y)$ defined on R^2, and let now M be a suitable[4] space of such functions, say $L^2(R^2,R)$ for definiteness. We consider $G=R$, acting in M by the unitary representation

$$g_a \ : \ u(x,y) \ \to \ u(x+ay,y) \ . \tag{5.8}$$

[4] Suitable means that we have the required structure of an Hilbert manifold, and that the G action to be considered is smooth and regular; thus the considerations to follow would also apply if we were considering, e.g., Sobolev spaces on M.

There is again a subspace M_0 invariant under the full group G: this is given simply by the functions which are independent of x,

$$M_0 \;=\; \{u \in M \;:\; u_x = 0,\; u(x,y) = u(y)\}\ . \tag{5.9}$$

Again $G_0 = G$ is a maximal isotropy subgroup and M_0 a maximal invariant subspace, and M_0 must be invariant under any G-equivariant flow in M, as we are going to check explicitely. In doing this, we can deal with the Lie generator γ for the group G, given simply by $\gamma = y\,\partial/\partial_x$.

We want now to consider PDEs of the form $u_t = F[u;\lambda]$ (square brackets denote, as earlier in this paper, dependence on u and its derivatives up to a finite order p), which will describe a flow in M.

In order to determine explicitely the PDEs invariant under G (that is, under γ), we have first to fix the order p of the equation and then apply standard computations [52,58,59]. Let us, for the sake of simplicity, consider $p = 1$, so that we write the equation $\mathcal{E}$ as

$$u_t \;=\; F(u, u_x, u_y; \lambda)\ . \tag{5.10}$$

We have then to consider the first prolongation [52,58,59] of γ, given by

$$\gamma^{(1)} \;=\; y\,\frac{\partial}{\partial x} \;-\; u_x\,\frac{\partial}{\partial u_y}\ \ ; \tag{5.11}$$

the symmetry condition is $\gamma^{(1)}[\mathcal{E}] = 0$, i.e.

$$\gamma^{(1)}\ \left[u_t - F(u, u_x, u_y; \lambda)\right] \;=\; 0\ , \tag{5.12}$$

which in view of (11) is simply $u_x \partial F/\partial u_y = 0$, so that the required invariant PDEs are those of the form

$$u_t \;=\; F(u, u_x; \lambda)\ . \tag{5.13}$$

It is clear that M_0 is invariant under these. Moreover, when we restrict to M_0, i.e. to $u(x,y;t) = w(y;t)$, we have that (13) reduces to an ordinary differential equation

$$\frac{dw}{dt} \;=\; \mathcal{F}(w,\lambda)\ , \tag{5.14}$$

with $\mathcal{F}(x,\lambda) = F(x,0;\lambda)$. Notice that the group G considered here is not compact, but is locally compact.

If we consider variational problems for a functional $L : M \to R$, say $L = \int \mathcal{L}[u]\,dx\,dy$ for definiteness, we could not apply the symmetry reduction. Indeed, it suffices to consider e.g. the case

$$L \;=\; \int\ \frac{1}{2}\ \left[u_y^2 - u^2\right]\ +\ u_x\ dx\,dy \tag{5.15}$$

which on M_0 gives the restricted functional

$$L_0 \;=\; \int \frac{1}{2}\,\left[u_y^2 - u^2\right]\; dy \tag{5.16}$$

leading to the Euler-Lagrange equations $u_{yy} = -u$; the u solution of this are not critical points for the full L. We stress again G is not compact.

`Example 3.` We will now consider the group $G = SO(2)$. First of all we notice that the analogue of the finite dimensional case considered in example 1, with $M = R^2$, and $M_0 = \{0\}$, would be trivial: the origin is invariant under any $SO(2)$-equivariant dynamics, as well known, so that the symmetry reduction for equivariant flow is a triviality; as for the variational case, invariant functions are of the form $V = V(r)$, and the gradient of these is of course equivariant. Let us thus consider directly the infinite dimensional setting.

We consider again $M = L^2(R^2, R)$, and coordinates (x, y) or (r, φ) in R^2, with the G-action given now by

$$g_\theta \;:\; u(x,y) \;\to\; u\left(\cos(\theta)x - \sin(\theta)y, \sin(\theta)x + \cos(\theta)y\right) \tag{5.17}$$

in cartesian coordinates, or

$$g_\theta \;:\; u(r,\varphi) \;\to\; u(r, \varphi+\theta) \tag{5.18}$$

in polar coordinates. The G-invariant functions are obviously those independent of the angular coordinate,

$$M_0 \;=\; \{u(r,\varphi) = w(r)\} \tag{5.19}$$

Repeating the check of the reduction procedure for first order equations as in example 2 would be immediate, leading again to an ODE for $w(r)$ (the same holds for higher order equations, with a more complicate expression for the prolongation $\gamma^{(p)}$ of the Lie generator of G).

Similarly, if we consider e.g. the $SO(2)$ symmetric equation

$$\Delta u \;-\; f(u,\lambda) \;=\; 0 \tag{5.20}$$

this is immediately restricted to M_0, yielding

$$\frac{1}{r}\frac{d}{dr}\left(r\,\frac{d}{dr}\right)\, w \;=\; f(w,\lambda)\ . \tag{5.21}$$

Notice that in this case M_0 is infinite dimensional, and we are not guaranteed of the existence of a solution to (21).

Let us now consider an example of variational problem in the same setting; we choose the simple functional on M given by

$$L = \int\int \left[\frac{1}{2} \left(u_x^2 + u_y^2 \right) - V(u;\lambda) \right] \mathrm{d}x \mathrm{d}y \ ; \tag{5.22}$$

in polar coordinates this reads

$$L = \int\int \left[\frac{1}{2} \left(u_r^2 + \frac{1}{r^2} u_\theta^2 \right) - V(u;\lambda) \right] r \mathrm{d}r \mathrm{d}\theta \ , \tag{5.23}$$

and thus the restriction of L to M_0 is simply

$$L = 2\pi \int \left[\frac{1}{2} u_r^2 - V(u;\lambda) \right] r \mathrm{d}r \ . \tag{5.24}$$

The Euler-Lagrange equations for (22),(23) are

$$\triangle u = - \frac{\partial V}{\partial u} \equiv f(u;\lambda) \tag{5.25}$$

or, recalling the expression of the Laplacian in polar coordinates,

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = f(u;\lambda) \tag{5.26}$$

while the Euler-Lagrange equations for (24) are

$$u_{rr} + \frac{1}{r} u_r = f(u;\lambda) \ ; \tag{5.27}$$

This shows that critical points $u_0 \in M_0$ of L_0 are also critical points of L.

Similar considerations would apply for $M = L^2(R^3, R)$ and $G = SO(3)$; a more specific application concerning $SO(3)$ will be presented in the next section.

`Example 4.` We will now consider again functions $u : R^2 \to R$, but $M = L^2(B, R)$ (B the square of side 2π in R^2), with now $G = Z_2 \times Z_2$ generated by two elements g_x, g_y which act in M by

$$\begin{aligned} g_x &: u(x,y) \to u(-x,y) \\ g_y &: u(x,y) \to u(x,-y) \ . \end{aligned} \tag{5.28}$$

We wish to consider the equation

$$u_t = \lambda u + a\, u_{xx} + b u_{yy} - u^3 \tag{5.29}$$

with boundary conditions on the square $B = [-\pi, \pi] \times [-\pi, \pi]$ corresponding to zero normal derivative; here a, b are real constants, $0 < a < b$.

Looking for stationary solutions to (29) is the same as looking for extremals for the functional

$$L = \int \left[\frac{\lambda}{2} u^2 - \frac{1}{4} u^4 \right] - \frac{1}{2} \left[a u_x^2 + b u_y^2 \right] dx\, dy \tag{5.30}$$

It is readily seen that G has four subgroups, listed here with their corresponding invariant subspaces (we give generators of H for the subgroup H):

$H \subseteq G$	Σ^H
$\{g_x, g_y\}$	$M_0 = \{u(x,y) = \alpha \}$
g_x	$M_y = \{u(x,y) = w(y)\}$
g_y	$M_x = \{u(x,y) = z(x)\}$
$\{e\}$	M

Here α are constants, and w, z functions of their argument alone.

The reductions of (29) to the corresponding fixed spaces Σ^H are readily obtained by cancelling the terms involving x and/or y derivatives.

The functions compatible with the required boundary conditions will be written as

$$u(x,y) = \sum_{k,\ell=0}^{\infty} \beta_{k\ell} \cos(kx) \cos(\ell y) \tag{5.31}$$

(we will also write $\alpha = \beta_{00}$); with this the linear part of (29) is simply

$$\dot{\beta}_{k\ell} = (\lambda - a k^2 - b \ell^2)\, \beta_{k\ell} \ . \tag{5.32}$$

Notice that $u = 0$ is a solution for all values of λ; for $\lambda < 0$ this is stable, and becomes unstable as λ crosses zero. For $\lambda \in [0, a)$ the space $M_0 = \Sigma^G$ (corresponding to $\beta_{k\ell} \equiv 0$) is not only invariant but also hyperbolically attractive, so that the stable solutions bifurcating from $u = 0$ are described by solutions to $\alpha_t = \lambda\alpha - \alpha^3$, i.e. are given by $u(x,y) = \pm\alpha$.

At $\lambda = a/4$, these two branches of constant solutions become first unstable in a direction tangent to M_x; the factor $1/4$ is due to the fact that we have to consider the linearization around $u = \alpha$, i.e. $u = \alpha + v$: the linearized equation for v is then $v_t = 4\lambda v + a v_{xx}$, using $\alpha^2 = \lambda$.

Our results guarantee that in analyzing this secondary bifurcation we can limit to consider functions in M_x, i.e. set all the $\beta_{k\ell}$ with $\ell \neq 0$ to zero. Notice that by a standard bifurcation argument we can then obtain the bifurcation equation involving α and β_{01} alone, but this further reduction does not follow from symmetry considerations.

The new branches of solutions lying in M_x will then become themself unstable in a direction tangent to M_y, at some higher value of λ.

`Example 5.` We consider now $M = L^2(R^2, R)$ and $G = Z_2 \times Z_2$ with generators g_x, g_y as in the previous example. We consider now the equation

$$u_t \;=\; -\lambda u_x x - (\lambda - c) u_{yy} - u^3 \qquad (5.33)$$

on which we do not impose boundary conditions. Passing to a spatial Fourier transform

$$u(x,y;t) \;=\; \frac{1}{2\pi} \int\int \Phi(k,\ell;t)\, e^{i(kx+\ell y)}\, dk\, d\ell \qquad (5.34)$$

the linear part of (33) reads

$$\dot\Phi(k,\ell;t) \;=\; \left[\lambda k^2 + (\lambda - c)\ell^2\right] \Phi(k,\ell;t) \qquad (5.35)$$

so that at $\lambda = 0$ the zero solution becomes unstable in the directions of the whole M_x space; the corresponding symmetry reduced equation is obviously

$$u_t \;=\; -\lambda u_{xx} - u^3 \;; \qquad (5.36)$$

notice that now we have a continuous critical spectrum, and we cannot apply bifurcation theory.

`Example 6.` Let us consider the complex Ginzburg-Landau equation,

$$\partial_t u \;=\; \partial_x^2 u + u - u\,|u|^2 \qquad (5.37)$$

which we consider as defining a flow in the space $M = L^2(R, C)$ of complex functions $u(x)$ on R, $u : R \to C$.

Passing to consider the Fourier transform of u,

$$u(x;t) \;=\; \frac{1}{\sqrt{2\pi}} \int \varphi(k,t)\, e^{ikx}\, dk \qquad (5.38)$$

and writing $||\varphi||^2 = (1/2\pi) \int |\varphi(k,t)|^2\, dk$, eq. (37) reads

$$\frac{d\varphi(k,t)}{dt} \;=\; (1 - k^2 - ||\varphi||^2)\varphi \qquad (5.39)$$

We consider here the group $G = R$ of translations in x, $g_a : u(x) \to u(x-a)$; this acts in M as giving an extra phase $\theta = ka$ to the coefficient of the base function e^{ikx}, that is

$$g_\tau : \varphi(k) \to e^{ik\tau}\varphi(k) \; ; \tag{5.40}$$

thus we have the maximal invariant subspace M_0, invariant under the full group G, which corresponds to constant functions; other invariant subspaces M_a, with isotropy subgroup Z corresponding to translations of multiples of $2\pi/a$, are given by functions for which (38) reduces to a Fourier series

$$u(x;t) \;=\; \sum_{k=-\infty}^{+\infty} f_k(t)\, e^{ikx} \; . \tag{5.41}$$

It is immediate to check that these subspaces are indeed invariants under (37). Notice that one is not able to perform a bifurcation analysis for (37), due to the presence of continuous spectrum, but the bifurcation analysis is completely standard (and trivial) for each of the symmetry reduction corresponding to these isotropy subgroups.

`Example 7.` In the same vein, we can consider the Swift-Hohemberg equation for $u : R \times R_+ \to R$,

$$\partial_t u \;=\; \lambda u - (1+\partial_x^2)^2 u - u^3 \; , \tag{5.42}$$

which we see again as a flow on $L^2(R,R)$.

Here the condition of real u requires $\varphi(k;t) = \varphi^*(-k;t)$, and g_τ acts as a rotation in the two dimensional spaces spanned by $\cos(kx)$ and $\sin(kx)$ (except of course for the one dimensional invariant subspace given by $k = 0$). We obtain in this case essentially the same picture as in example 6 above, i.e. for each subgroup Z of R corresponding to τ multiple of $2\pi/a$ we get an infinite dimensional invariant subspace given by the k multiples of a.

Once again, this equation is not amenable to standard bifurcation analysis due to continuous spectrum [60], but the bifurcation analysis is standard and pretty trivial in each of the invariant subspaces determined in this way.

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