An Approximate KAM-Renormalization-Group Scheme for Hamiltonian Systems

C. Chandre[1], H. R. Jauslin[1], and G. Benfatto[2]

[1]*Laboratoire de Physique, CNRS, Université de Bourgogne, BP 400, F-21011 Dijon, France*

[2]*Dipartimento di Matematica, Università di Roma "Tor Vergata", Via della Ricerca Scientifica, I-00133 Roma, Italy*

We construct an approximate renormalization scheme for Hamiltonian systems with two degrees of freedom. This scheme is a combination of Kolmogorov-Arnold-Moser (KAM) theory and renormalization-group techniques. It makes the connection between the approximate renormalization procedure derived by Escande and Doveil, and a systematic expansion of the transformation. In particular, we show that the two main approximations, consisting in keeping only the quadratic terms in the actions and the two main resonances, keep the essential information on the threshold of the breakup of invariant tori.

PACS numbers: 05.45.+b, 64.60.Ak

I. INTRODUCTION

In 1981, Escande and Doveil [1] set up an approximate renormalization scheme for Hamiltonian systems with two degrees of freedom, in order to study the breakup of invariant tori, and especially to compute the threshold of stochasticity. Their scheme was motivated by Chirikov's resonance overlap criterion [2], and by Greene's results [3] about the link between the existence of a torus with the stability of neighboring periodic orbits. They established the relevance of a sequence of these periodic orbits for the breakup of invariant tori, by setting an approximate transformation which focuses successively on smaller scales, i.e. acting like a microscope in phase space.

Due to the complexity of the phase space of a non-integrable Hamiltonian, their method requires strong approximations to obtain explicit expressions. Basically, two approximations were involved:

(1) a quadratic approximation in the actions: their transformation produces terms that are higher than quadratic in the actions; in order to remain in the same class of Hamiltonians, they neglect these higher order terms,

(2) a two-resonance approximation: they only keep the two main resonances at each iteration of the transformation.

The idea was to keep only the most relevant features of the mechanism of the breakup of a given torus.

In this article, we construct an approximate scheme using the same two approximations. We establish the connection between Escande's scheme [1,4] and the KAM-RG transformation derived in Refs. [5–7]. The aim is to show that an exact renormalization transformation can be approximated by a simple transformation: It can be useful to derive approximate explicit expressions of universal parameters, and to see what are the most relevant terms responsible for the breakup of invariant tori. The results we obtain support the general idea that the irrelevant terms of the renormalization transformation can be eliminated with little loss of accuracy in the parameters associated to the breakup of invariant tori.

The transformation $\mathcal{R}$ we define has two main parts: a KAM transformation which is a canonical change of coordinates that reduces the size of the perturbation from ε to ε^2, and a renormalization transformation which is a combination of a shift of the resonances and a rescaling of momentum and energy.

It acts on the following class of Hamiltonians with two degrees of freedom, quadratic in the action variables $\boldsymbol{A}=(A_1,A_2)$, and described by three even scalar functions of the angles $\boldsymbol{\varphi}=(\varphi_1,\varphi_2)$:

$$H(\boldsymbol{A},\boldsymbol{\varphi})=\frac{1}{2}\left(1+m(\boldsymbol{\varphi})\right)(\boldsymbol{\Omega}\cdot\boldsymbol{A})^2 + [\boldsymbol{\omega}_0+g(\boldsymbol{\varphi})\boldsymbol{\Omega}]\cdot\boldsymbol{A}+f(\boldsymbol{\varphi}), \tag{1.1}$$

where m, g, and f are of zero average. The vector $\boldsymbol{\omega}_0$ is the frequency vector of the considered torus and $\boldsymbol{\Omega}=(1,\alpha)$ is some other constant vector not parallel to $\boldsymbol{\omega}_0$. The perturbation (m,g,f) is of order $O(\varepsilon)$.

The renormalization-group approach is based on the following general picture: The idea is to construct the transformation $\mathcal{R}$ as a generalized canonical change of coordinates acting on some space of Hamiltonians such that the iteration of $\mathcal{R}$ converges to a fixed point. If the perturbation is smaller than critical, $\mathcal{R}$ should converge to a Hamiltonian of type (1.1) with $(m,g,f)=0$, which is integrable, and the equations of motion show that the torus with frequency vector $\boldsymbol{\omega}_0$ is located at $\boldsymbol{A}=0$. All Hamiltonians attracted by this trivial fixed point have an invariant torus of that frequency (this can be considered as an alternative version of the KAM theorem [8]). If the perturbation is larger than critical, the system does not have a KAM torus of the considered frequency and the iteration of $\mathcal{R}$ diverges. The domain of convergence to the trivial fixed point and the domain of divergence are separated by a *critical surface* invariant under the action of $\mathcal{R}$. The main hypothesis of the renormalization-group approach is that there should be another nontrivial fixed point (or more generally, a fixed set) on this critical surface, that is attractive for Hamiltonians on that surface. From its existence, one can expect to deduce universal properties in the mechanism of the breakup of invariant tori.

Many aspects of this general picture are still at the stage of conjecture, supported by some results in the perturbative regime [8], by numerical works [6,7,9,10], and by analogies with the related problem for area-preserving maps [11,12]. In particular, the relation between the properties of the nontrivial renormalization fixed point and the geometric properties of the invariant torus at the instability threshold are not well established. The coincidence of the critical coupling of one-parameter families at which a torus breaks up, with the boundary of attraction of the trivial fixed point is supported by numerical studies [6,7,9].

In Sec. II, we describe the KAM part of the transformation, and we make explicit the two approximations involved in this scheme: the quadratic approximation and the two-resonance approximation. In Sec. III, we present the renormalization transformation which is a combination of the KAM part, a shift of the resonances, and rescalings of actions and energy. In Sec. IV, we give our numerical results, and in particular, we show that the approximate scheme contains the essential features of the exact one.

II. KAM TRANSFORMATION

We perform a canonical transformation $\mathcal{U}_F : (\boldsymbol{\varphi}, \boldsymbol{A}) \mapsto (\boldsymbol{\varphi}', \boldsymbol{A}')$ defined by a generating function $F(\boldsymbol{A}', \boldsymbol{\varphi})$ [13,14] characterized by a scalar function X of the action and angle variables, of the form

$$F(\boldsymbol{A}', \boldsymbol{\varphi}) = \boldsymbol{A}' \cdot \boldsymbol{\varphi} + X(\boldsymbol{A}', \boldsymbol{\varphi}), \tag{2.1}$$

leading to

$$\boldsymbol{A} = \frac{\partial F}{\partial \boldsymbol{\varphi}} = \boldsymbol{A}' + \frac{\partial X}{\partial \boldsymbol{\varphi}}, \tag{2.2}$$

$$\boldsymbol{\varphi}' = \frac{\partial F}{\partial \boldsymbol{A}'} = \boldsymbol{\varphi} + \frac{\partial X}{\partial \boldsymbol{A}'}. \tag{2.3}$$

The function X is constructed such that in $\mathcal{H} \circ \mathcal{U}_F$ the perturbation terms of first order in ε are equal to zero. Inserting Eq. (2.2) into Hamiltonian (1.1), one obtains the expression of the Hamiltonian in the mixed representation of new action variables and old angle variables:

$$\begin{aligned}\tilde{H}(\boldsymbol{A}', \boldsymbol{\varphi}) = (\boldsymbol{\Omega} \cdot \boldsymbol{A}')^2/2 + \boldsymbol{\omega}_0 \cdot \boldsymbol{A}' \\ + \boldsymbol{\omega}(\boldsymbol{A}') \cdot \frac{\partial X}{\partial \boldsymbol{\varphi}} + h(\boldsymbol{A}', \boldsymbol{\varphi}) + O(\varepsilon^2),\end{aligned} \tag{2.4}$$

where

$$\boldsymbol{\omega}(\boldsymbol{A}') = \boldsymbol{\omega}_0 + (\boldsymbol{\Omega} \cdot \boldsymbol{A}')\boldsymbol{\Omega}, \tag{2.5}$$

$$h(\boldsymbol{A}', \boldsymbol{\varphi}) = \frac{1}{2} m(\boldsymbol{\varphi})(\boldsymbol{\Omega} \cdot \boldsymbol{A}')^2 + g(\boldsymbol{\varphi})\boldsymbol{\Omega} \cdot \boldsymbol{A}' + f(\boldsymbol{\varphi}). \tag{2.6}$$

The equation that determines X is thus:

$$\boldsymbol{\omega}(\boldsymbol{A}') \cdot \frac{\partial X}{\partial \boldsymbol{\varphi}} + h(\boldsymbol{A}', \boldsymbol{\varphi}) = 0. \tag{2.7}$$

We recall that the functions m, g, and f are of order $O(\varepsilon)$; as a consequence, X is also of order $O(\varepsilon)$. Equation (2.7) has the solution

$$X(\boldsymbol{A}', \boldsymbol{\varphi}) = \sum_{\nu \in Z^2} X_\nu(\boldsymbol{A}') \sin(\boldsymbol{\nu} \cdot \boldsymbol{\varphi}), \tag{2.8}$$

where, if we write $h(\boldsymbol{A}, \boldsymbol{\varphi}) = \sum_\nu h_\nu(\boldsymbol{A}) \cos(\boldsymbol{\nu} \cdot \boldsymbol{\varphi})$,

$$X_\nu(\boldsymbol{A}') = -\frac{h_\nu(\boldsymbol{A}')}{\boldsymbol{\omega}(\boldsymbol{A}') \cdot \boldsymbol{\nu}}. \tag{2.9}$$

The denominator of X_ν depends on the actions: thus, by power expansion, it generates terms that are higher than quadratic in the actions. In order to remain in the same space of Hamiltonians (1.1), we expand the Hamiltonian to the second order in the actions and neglect the order $O(A^3)$. The justification for such an approximation is that we are interested in the torus with frequency vector $\boldsymbol{\omega}_0$ which is located at $\boldsymbol{A} = 0$ for the trivial fixed point.
We consider Hamiltonians (1.1) with only two Fourier modes which are the two main resonances defined as follows: For a frequency vector $\boldsymbol{\omega}_0$, the resonances are given by the vectors $\boldsymbol{\nu}_n = (p_n, q_n)$ which are the sequence of the best rational approximations. They are characterized precisely by the following property: $|\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_n| < |\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}|$, for any $\boldsymbol{\nu} \equiv (p, q) \neq \boldsymbol{\nu}_n$ such that $|q| < q_{n+1}$. For the frequency vector $\boldsymbol{\omega}_0 = (1/\gamma, -1)$ with $\gamma = (1 + \sqrt{5})/2$, we define the two *main resonances* as $\boldsymbol{\nu}_1 = (1, 0)$ and $\boldsymbol{\nu}_2 = (1, 1)$. Well known properties of continued fractions (see, for example, [15]) imply that the vectors $\boldsymbol{\nu}_n$ satisfy the recursion relation $\boldsymbol{\nu}_{n+1} = N \boldsymbol{\nu}_n$, where $N = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$.
The scalar functions m, g and f will have expressions of the form

$$f(\boldsymbol{\varphi}) = f_{\nu_1} \cos(\boldsymbol{\nu}_1 \cdot \boldsymbol{\varphi}) + f_{\nu_2} \cos(\boldsymbol{\nu}_2 \cdot \boldsymbol{\varphi}). \tag{2.10}$$

The KAM part of the transformation $\mathcal{R}$ will generate a large set of Fourier modes from these two main resonances. The purpose of the renormalization is to relate these two main resonances representing the main scale, to the next pair of resonances (called the *daughter resonances*) representing the next smaller scale. For the frequency vector $\boldsymbol{\omega}_0 = (1/\gamma, -1)$, the daughter resonances are $\boldsymbol{\nu}_3 = (2, 1)$ and $\boldsymbol{\nu}_4 = (3, 2)$. As the canonical transformation (2.2)-(2.3) is linear in $\cos(\boldsymbol{\nu} \cdot \boldsymbol{\varphi})$ and $\sin(\boldsymbol{\nu} \cdot \boldsymbol{\varphi})$, $\boldsymbol{\nu}_n = \boldsymbol{\nu}_{n-1} + \boldsymbol{\nu}_{n-2}$ and the three scalar functions (m, g, f) are of order $O(\varepsilon)$, the lowest order to which the resonances $\boldsymbol{\nu}_3$ and $\boldsymbol{\nu}_4$ are produced, is respectively $O(\varepsilon^2)$ and $O(\varepsilon^3)$. Thus we neglect the order $O(\varepsilon^4)$ of the KAM transformation.
The next step is to express the Hamiltonian in the new angle variables using Eq. (2.3). We notice that this equation has to be inverted. As we need the Hamiltonian expressed in the new coordinates to order $O(\varepsilon^3)$, we have to

invert Eq. (2.3) up to order $O(\varepsilon)$, since the functions to be expressed in the new angles are already of order $O(\varepsilon^2)$. Moreover, these new angle variables depend on the actions, which we develop to order $O(A^2)$. The development can thus be summarized by expressing $\cos[\boldsymbol{\nu}.\boldsymbol{\varphi}(\boldsymbol{\varphi}')]$ as a function of $\cos(\boldsymbol{\nu}.\boldsymbol{\varphi}')$ and $\sin(\boldsymbol{\nu}.\boldsymbol{\varphi}')$, neglecting the orders $O(\varepsilon^2, A^3)$.
The final step is a translation in the $\boldsymbol{A}'$ variables such that the linear term is again of the form $\boldsymbol{\omega}_0 \cdot \boldsymbol{A}'$.
The Hamiltonian expressed in the new variables becomes:

$$H'(\boldsymbol{A}', \boldsymbol{\varphi}') = \frac{1}{2}\left(\Lambda + m'(\boldsymbol{\varphi}')\right)(\boldsymbol{\Omega}\cdot\boldsymbol{A}')^2 + (\boldsymbol{\omega}_0 + g'(\boldsymbol{\varphi}')\boldsymbol{\Omega})\cdot\boldsymbol{A}' + f'(\boldsymbol{\varphi}'), \qquad (2.11)$$

where m', g' and f' are given as functions of m, g and f. The constant Λ results of the fact that the mean value of m' is required to be equal to zero. We notice that the KAM transformation does not change $\boldsymbol{\Omega} = (1, \alpha)$.
The two-resonance approximation consists in retaining only the two daughter resonances and neglect all the other Fourier modes. Therefore the approximate KAM transformation $\tilde{\mathcal{U}}_F$ is a map acting on a low-dimensional space of Fourier coefficients:

$$\tilde{\mathcal{U}}_F\left(1; m_{\nu_1}, g_{\nu_1}, f_{\nu_1}; m_{\nu_2}, g_{\nu_2}, f_{\nu_2}\right) = \left(\Lambda; m'_{\nu_3}, g'_{\nu_3}, f'_{\nu_3}; m'_{\nu_4}, g'_{\nu_4}, f'_{\nu_4}\right). \qquad (2.12)$$

The explicit expression of this map is given in the Appendix.

III. RENORMALIZATION TRANSFORMATION

We construct the approximate transformation by combining two parts: a KAM transformation $(m, g, f, \alpha) \mapsto (m', g', f', \alpha)$ as defined above, and a renormalization (RG) consisting of a shift of the resonances and a rescaling of the actions and of time $(m', g', f', \alpha) \mapsto (m'', g'', f'', \alpha')$. The renormalization scheme described in this section is for a torus of frequency vector $\boldsymbol{\omega}_0 = (1/\gamma, -1)$ where $\gamma = (1+\sqrt{5})/2$. It is straightforward to adapt it to quadratic irrationals.
The approximate KAM-RG transformation is composed of four steps:
1) a KAM transformation described in the previous section, which is a change of coordinates that eliminates terms of order $O(\varepsilon)$, where ε is the size of the perturbation; this transformation produces terms of order $O(\varepsilon^2)$, terms that are higher than quadratic in the actions, and a large set of Fourier modes. We neglect terms of order $O(A^3, \varepsilon^4)$, and also, all the Fourier modes except the two daughter resonances $\boldsymbol{\nu}_3$ and $\boldsymbol{\nu}_4$. We notice that this transformation does not change $\boldsymbol{\Omega}$.
2) a shift of the resonances: a canonical change of coordinates that maps the pair of daughter resonances $(\boldsymbol{\nu}_3, \boldsymbol{\nu}_4)$ into the two main resonances $(\boldsymbol{\nu}_1, \boldsymbol{\nu}_2)$.
3) a rescaling of energy (or equivalently of time).
4) a rescaling of the action variables (which is a generalized canonical transformation).
The aim of this transformation is to treat one scale at the time. The steps 2), 3) and 4) are implemented as follows: The two main resonances $(1,0)$ and $(1,1)$ are replaced by the next pair of daughter resonances $(2,1)$ and $(3,2)$, i.e. we require that $\cos[(2,1)\cdot\boldsymbol{\varphi}'] = \cos[(1,0)\cdot\boldsymbol{\varphi}'']$ and $\cos[(3,2)\cdot\boldsymbol{\varphi}'] = \cos[(1,1)\cdot\boldsymbol{\varphi}'']$. This change is done via a canonical transformation $(\boldsymbol{A}', \boldsymbol{\varphi}') \mapsto (N^{-2}\boldsymbol{A}', N^2\boldsymbol{\varphi}')$ with

$$N^2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

This linear transformation multiplies $\boldsymbol{\omega}_0$ by γ^{-2} (since $\boldsymbol{\omega}_0$ is an eigenvector of N); therefore we rescale the energy by a factor γ^2 in order to keep the frequency fixed at $\boldsymbol{\omega}_0$. A consequence of the shift of the resonances is that $\boldsymbol{\Omega}$ is changed into $\boldsymbol{\Omega}' = (1, \alpha')$, where $\alpha' = (\alpha+1)/(\alpha+2)$.
Then we perform a rescaling of the action variables: we change the Hamiltonian H' into

$$\hat{H}'(\boldsymbol{A}', \boldsymbol{\varphi}') = \lambda H'\left(\frac{\boldsymbol{A}'}{\lambda}, \boldsymbol{\varphi}'\right)$$

with λ such that Hamiltonian (2.11) becomes of the form (1.1). Since the rescaling of energy and the shift N^2 transform the quadratic term of the Hamiltonian into $\gamma^2(2+\alpha)^2[\Lambda + m'(\boldsymbol{\varphi}')](\boldsymbol{\Omega}'\cdot\boldsymbol{A}')^2/2$, this condition leads to $\lambda = \gamma^2(2+\alpha)^2\Lambda$. This condition has the following geometric interpretation in terms of self-similarity of the resonances close to the invariant torus: the rescaling magnifies the size of the daughter resonances, and places them approximately at the location of the original main resonances [7].
In summary, the renormalization rescales m, g, f and $\boldsymbol{\Omega} = (1, \alpha)$ into

$$m''(\boldsymbol{\varphi}) = \frac{m'\left(N^{-2}\boldsymbol{\varphi}\right)}{\Lambda}, \qquad (3.1)$$

$$g''(\boldsymbol{\varphi}) = \gamma^2(2+\alpha)g'\left(N^{-2}\boldsymbol{\varphi}\right), \qquad (3.2)$$

$$f''(\boldsymbol{\varphi}) = \gamma^4(2+\alpha)^2\Lambda f'\left(N^{-2}\boldsymbol{\varphi}\right), \qquad (3.3)$$

$$\alpha' = \frac{1+\alpha}{2+\alpha}. \qquad (3.4)$$

The iteration of the transformation (3.4) converges to $\alpha_* = \gamma^{-1}$. It means that $\boldsymbol{\Omega}$ converges under successive iterations to $\boldsymbol{\Omega}_* = (1, 1/\gamma)$, which is orthogonal to $\boldsymbol{\omega}_0$ and is the unstable eigenvector of N^2 with the largest eigenvalue γ^2.

IV. DETERMINATION OF THE CRITICAL COUPLING; AN APPROXIMATE NONTRIVIAL FIXED POINT

We start with the same initial Hamiltonian as in Refs. [4,6,7]

$$H(\boldsymbol{A},\boldsymbol{\varphi}) = \frac{1}{2}(\boldsymbol{\Omega}\cdot\boldsymbol{A})^2 + \boldsymbol{\omega}_0\cdot\boldsymbol{A} + \varepsilon f(\boldsymbol{\varphi}) \ , \tag{4.1}$$

where $\boldsymbol{\Omega} = (1,0)$, $\boldsymbol{\omega}_0 = (1/\gamma,-1)$, $\gamma = (1+\sqrt{5})/2$, and a perturbation

$$f(\boldsymbol{\varphi}) = \cos(\boldsymbol{\nu}_1\cdot\boldsymbol{\varphi}) + \cos(\boldsymbol{\nu}_2\cdot\boldsymbol{\varphi}), \tag{4.2}$$

where $\boldsymbol{\nu}_1 = (1,0)$ and $\boldsymbol{\nu}_2 = (1,1)$.

We take successively larger coupling ε, and determine whether the approximate KAM-RG iteration converges to a Hamiltonian with $(m,g,f)=0$, or whether it diverges $(m,g,f)\to\infty$. By a bisection procedure, we determine the critical coupling $\varepsilon_c = 0.02885$. As a comparison, Escande's scheme [4] gives $\varepsilon_c = 0.02908$. The KAM-RG transformation [6,7,9] yields $\varepsilon_c = 0.02759$ (Greene's criterion gives also $\varepsilon_c = 0.02759$). The two-resonance scheme gives the result within 5%. Thus, the approximate scheme gives a fairly good description of the critical surface of the breakup of the torus.

The renormalization operator has two fixed points: a trivial fixed point which corresponds to the Hamiltonian $H(\boldsymbol{A},\boldsymbol{\varphi}) = (\boldsymbol{\Omega}_*\cdot\boldsymbol{A})^2/2 + \boldsymbol{\omega}_0\cdot\boldsymbol{A}$, and a nontrivial fixed point which lies on the boundary of the basin of attraction of the trivial fixed point. The critical surface which is the stable manifold of the nontrivial fixed point is of codimension 1. The relevant critical exponent is $\delta = 2.7135$. This value is close to the one obtained by Escande *et al.* [16] $\delta = 2.7480$, or by KAM-RG schemes [9,10] $\delta = 2.6502$, and by MacKay for area-preserving maps [11] $\delta = 2.6502$.

The fact that δ is close to γ^2 can be understood by the following heuristic arguments: To the first main resonance $\boldsymbol{\nu}_1$ corresponds a Fourier component $M\exp[i(1,0)\cdot\boldsymbol{\varphi}]$, and to the second one $P\exp[i(1,1)\cdot\boldsymbol{\varphi}]$. The first daughter resonance $\boldsymbol{\nu}_3$ is represented by $M'\exp[i(2,1)\cdot\boldsymbol{\varphi}]$. The transformation is a polynomial change of coordinates that generates a set of Fourier modes from the two main resonances. The way to generate the first daughter resonance at the lowest order is to combine one resonance $\boldsymbol{\nu}_1$ and one resonance $\boldsymbol{\nu}_2$. For the second daughter resonance, the change of coordinates must combine one resonance $\boldsymbol{\nu}_1$ and two resonances $\boldsymbol{\nu}_2$. We rescale the phase space in such a way that the daughter resonances become of the same size as the main resonances. These arguments give the following renormalization relations:

$$M' = k_1 MP, \tag{4.3}$$
$$P' = k_2 MP^2, \tag{4.4}$$

where k_1 and k_2 are two constants that depend on how the transformation is performed.

An analysis of this scheme shows that it has two fixed points: a trivial one $M=0$, $P=0$ and a nontrivial one $M_* = k_1 k_2^{-1}$, $P_* = k_1^{-1}$. The nontrivial fixed point has a stable manifold of codimension 1 characterized by a relevant critical exponent; the only eigenvalue greater than one of the linearized map at the nontrivial fixed point is $\delta = \gamma^2$. This relevant critical exponent does not depend on k_1 and k_2. This is in agreement with the general ideas of the renormalization group. Also, it shows that the relevant critical exponent of the approximate and exact KAM-RG transformation should be expected to be close to γ^2.

For the scaling factor at the nontrivial fixed point, we obtain numerically $\lambda_* = 19.1248$. This value can be compared with $\lambda_* = 18.8282$ obtained in Refs. [17,18,11,9,10]. Table 1 lists some of the universal parameters associated with the breakup of golden tori.

The approximate renormalization transformation has another fixed set which is a cycle of period three as it had also been encountered in area-preserving maps [12,19] and in the KAM-RG transformation [7]. This cycle is simply related to the nontrivial fixed point by symmetries. In particular, it belongs to the same universality class as the fixed point.

ACKNOWLEDGMENTS

We thank A. Celletti for providing the program to determine the critical coupling by Greene's criterion. We acknowledge useful discussions with G. Gallavotti, R. S. MacKay, and H. Koch. Support from EC contract No. ERBCHRXCT94-0460 for the project "Stability and universality in classical mechanics" and from the Conseil Régional de Bourgogne is acknowledged.

APPENDIX: TWO-RESONANCE SCHEME FORMULAE

We denote $c_\nu = \cos(\boldsymbol{\nu}\cdot\boldsymbol{\varphi})$, $c'_\nu = \cos(\boldsymbol{\nu}\cdot\boldsymbol{\varphi}')$, and $s'_\nu = \sin(\boldsymbol{\nu}\cdot\boldsymbol{\varphi}')$. The expression of the Hamiltonian in the mixed representation of new actions and old angles is

$$\begin{aligned}\tilde{H}(\boldsymbol{A}',\boldsymbol{\varphi}) = {} & \frac{1}{2}(\boldsymbol{\Omega}\cdot\boldsymbol{A}')^2 + \boldsymbol{\omega}_0\cdot\boldsymbol{A}' \\ & + \sum_{\nu_1,\nu_2} P_{\nu_1\nu_2}(\boldsymbol{A}')c_{\nu_1}c_{\nu_2} \\ & + \sum_{\nu_1,\nu_2,\nu_3} Q_{\nu_1\nu_2\nu_3}(\boldsymbol{A}')c_{\nu_1}c_{\nu_2}c_{\nu_3}, \end{aligned} \tag{4.5}$$

where

$$\begin{aligned}P_{\nu_1\nu_2}(\boldsymbol{A}) = \boldsymbol{\Omega}\cdot\boldsymbol{\nu}_1 X_{\nu_1}(\boldsymbol{A})[\boldsymbol{\Omega}\cdot\boldsymbol{\nu}_2 X_{\nu_2}(\boldsymbol{A})/2 \\ + g_{\nu_2} + m_{\nu_2}\boldsymbol{\Omega}\cdot\boldsymbol{A}], \end{aligned}\tag{4.6}$$

$$Q_{\nu_1\nu_2\nu_3}(\boldsymbol{A}) = \boldsymbol{\Omega}\cdot\boldsymbol{\nu}_1\boldsymbol{\Omega}\cdot\boldsymbol{\nu}_2 X_{\nu_1}(\boldsymbol{A})X_{\nu_2}(\boldsymbol{A})m_{\nu_3}/2. \tag{4.7}$$

We recall that X is of order $O(\varepsilon)$, and is given by Eq. (2.9); as a consequence, P is of order $O(\varepsilon^2)$, and Q is of order $O(\varepsilon^3)$.
Next, the expression of this Hamiltonian in the new angles requires the inversion of Eq. (2.3) to the order $O(\varepsilon)$. The expression of c_ν as a function of s'_ν and c'_ν is

$$c_{\nu_1} = c'_{\nu_1} + \sum_{\nu_2} R_{\nu_1\nu_2}(\boldsymbol{A}')s'_{\nu_1}s'_{\nu_2} + O(\varepsilon^2), \tag{4.8}$$

where $R_{\nu_1\nu_2}(\boldsymbol{A}) = \boldsymbol{\nu}_1\cdot\dfrac{\partial X_{\nu_2}}{\partial \boldsymbol{A}}$.
The Hamiltonian (1.1) expressed in the new variables becomes

$$\begin{aligned}H'(\boldsymbol{A}',\boldsymbol{\varphi}') = {} & \frac{1}{2}(\boldsymbol{\Omega}\cdot\boldsymbol{A}')^2 + \boldsymbol{\omega}_0\cdot\boldsymbol{A}' \\ & + \sum_{\nu_1,\nu_2} P_{\nu_1\nu_2}c'_{\nu_1}c'_{\nu_2} \\ & + \sum_{\nu_1,\nu_2,\nu_3} Q_{\nu_1\nu_2\nu_3}c'_{\nu_1}c'_{\nu_2}c'_{\nu_3} \\ & + \sum_{\nu_1,\nu_2,\nu_3} (P_{\nu_1\nu_2}+P_{\nu_2\nu_1})R_{\nu_2\nu_3}c'_{\nu_1}s'_{\nu_2}s'_{\nu_3} \\ & + O(\varepsilon^4). \end{aligned}\tag{4.9}$$

The next approximation is the Taylor expansion of $H'(\boldsymbol{A}',\boldsymbol{\varphi}')$ to the second order in the actions e.g.

$$P_{\nu_1\nu_2}(\boldsymbol{A}) = P^{(0)}_{\nu_1\nu_2} + P^{(1)}_{\nu_1\nu_2}\boldsymbol{\Omega}\cdot\boldsymbol{A} + P^{(2)}_{\nu_1\nu_2}(\boldsymbol{\Omega}\cdot\boldsymbol{A})^2 + O(A^3). \tag{4.10}$$

In the next step, we neglect all the resonances different from the daughter resonances $(2,1)$ and $(3,2)$ using the following relations

$$c'_{\nu_1}c'_{\nu_2} = \frac{1}{2}\left(c'_{\nu_1+\nu_2} + c'_{\nu_1-\nu_2}\right), \tag{4.11}$$

$$\begin{aligned}c'_{\nu_1}s'_{\nu_2}s'_{\nu_3} = \frac{1}{4}\big(&c'_{\nu_1+\nu_2-\nu_3} + c'_{\nu_1-\nu_2+\nu_3} \\ &- c'_{\nu_1+\nu_2+\nu_3} - c'_{\nu_1-\nu_2-\nu_3}\big), \end{aligned}\tag{4.12}$$

$$\begin{aligned}c'_{\nu_1}c'_{\nu_2}c'_{\nu_3} = \frac{1}{4}\big(&c'_{\nu_1+\nu_2-\nu_3} + c'_{\nu_1-\nu_2+\nu_3} \\ &+ c'_{\nu_1+\nu_2+\nu_3} + c'_{\nu_1-\nu_2-\nu_3}\big). \end{aligned}\tag{4.13}$$

In the following formulas, the subscripts 1, 2, 3 and 4 denote respectively the resonances $(1,0)$, $(1,1)$, $(2,1)$, and $(3,2)$. The Hamiltonian (4.9) becomes

$$\begin{aligned}H'(\boldsymbol{A}',\boldsymbol{\varphi}') = {} & \frac{1}{2}(\boldsymbol{\Omega}\cdot\boldsymbol{A}')^2 + \boldsymbol{\omega}_0\cdot\boldsymbol{A}' + \frac{1}{2}(P_{11}+P_{22}) \\ & + \frac{1}{2}(P_{12}+P_{21})c'_{\nu_3} \\ & + \frac{1}{4}(Q_{221}+Q_{212}+Q_{122}-S_{122})c'_{\nu_4}, \end{aligned}\tag{4.14}$$

where $S_{122} = 2P_{22}R_{21} + (P_{21}+P_{12})(R_{12}+R_{22})$. Equation (4.14) gives the expression of the Fourier coefficients of the daughter resonances

$$\Lambda = 1 + P^{(2)}_{11} + P^{(2)}_{22}, \tag{4.15}$$

$$m'_{\nu_3} = P^{(2)}_{12} + P^{(2)}_{21}, \tag{4.16}$$

$$m'_{\nu_4} = \frac{1}{2}\left(Q^{(2)}_{221} + Q^{(2)}_{212} + Q^{(2)}_{122} - S^{(2)}_{122}\right), \tag{4.17}$$

$$g'_{\nu_3} = \frac{1}{2}\left(P^{(1)}_{12} + P^{(1)}_{21}\right), \tag{4.18}$$

$$g'_{\nu_4} = \frac{1}{4}\left(Q^{(1)}_{221} + Q^{(1)}_{212} + Q^{(1)}_{122} - S^{(1)}_{122}\right), \tag{4.19}$$

$$f'_{\nu_3} = \frac{1}{2}\left(P^{(0)}_{12} + P^{(0)}_{21}\right), \tag{4.20}$$

$$f'_{\nu_4} = \frac{1}{4}\left(Q^{(0)}_{221} + Q^{(0)}_{212} + Q^{(0)}_{122} - S^{(0)}_{122}\right). \tag{4.21}$$

TABLE I. Universal parameters associated with the breakup of the golden mean torus.

Name	Value [a]	Approximate scheme	Simple scheme [b]
Unstable eigenvalue	2.6502	2.7135	$\gamma^2 \approx 2.6180$
Largest stable eigenvalue	0.3731	0.4087	$\gamma^{-2} \approx 0.3820$
Area multiplier λ_*	18.8282	19.1248	$\gamma^6 \approx 17.9443$
Inverse mass multiplier Λ_*	1.0493	1.0658	1
Time multiplier	γ^{-2}	γ^{-2}	γ^{-2}

[a]given in Refs. [12,4]
[b]derived in Refs. [4,16]

[1] D. F. Escande and F. Doveil, J. Stat. Phys. **26**, 257 (1981).
[2] B. V. Chirikov, Phys. Rep. **52**, 263 (1979).
[3] J. M. Greene, J. Math. Phys. **20**, 1183 (1979).
[4] D. F. Escande, Phys. Rep. **121**, 165 (1985).
[5] G. Gallavotti and G. Benfatto (unpublished, 1987).
[6] M. Govin, C. Chandre, and H. R. Jauslin, Phys. Rev. Lett **79**, 3881 (1997).
[7] C. Chandre, M. Govin, and H. R. Jauslin, Phys. Rev. E **57**, 1536 (1998).
[8] H. Koch, archived in mp_arc@math.utexas.edu, #96-383. To appear in Erg. Theo. Dyn. Syst. (1998).
[9] C. Chandre, M. Govin, H. R. Jauslin, and H. Koch, Phys. Rev. E **57**, XXX (1998).
[10] J. J. Abad, H. Koch, and P. Wittwer, archived in mp_arc@math.utexas.edu, #98-261.
[11] R. S. MacKay, Physica D **7**, 283 (1983).
[12] R. S. MacKay, *Renormalisation in Area-preserving Maps* (World Scientific, 1993).
[13] H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, Mass., 1980).
[14] G. Gallavotti, *The Elements of Mechanics* (Springer-Verlag, New York, 1983).
[15] H. Davenport, *The Higher Arithmetic* (Dover, New York, 1983).
[16] D. F. Escande, M. S. Mohamed-Benkadda, and F. Doveil, Phys. Lett. A **101**, 309 (1984).
[17] L. P. Kadanoff, Phys. Rev. Lett. **47**, 1641 (1981).
[18] S. J. Shenker and L. P. Kadanoff, J. Stat. Phys. **27**, 631 (1982).
[19] J. M. Greene and J. Mao, Nonlinearity **3**, 69 (1990).