The Adiabatic Theorem of Quantum Mechanics

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Abstract

We prove the adiabatic theorem for quantum evolution without the traditional gap condition. We show that the theorem holds essentially in all cases where it can be formulated. In particular, our result implies that the adiabatic theorem holds also for eigenvalues embedded in the continuous spectrum. If there is information on the Hölder continuity of the spectral measure, then one can also estimate the rate at which the adiabatic limit is approached.

The adiabatic theorem of Quantum Mechanics describes the long time behavior of the solutions of an initial value problem where the Hamiltonian generating the evolution depends slowly in time. Traditionally, the theorem is stated for Hamiltonians that have an eigenvalue which is separated by a gap from the rest of the spectrum. Folk wisdom is that a gap condition is a *sine qua non* for the adiabatic theorem to hold. In particular, according to this folk wisdom, one does not expect a general adiabatic theorem for Hamiltonians that have an eigenvalue embedded in, say, the continuous spectrum. Our purpose is to show that this folk wisdom is wrong, and there is a general adiabatic theorem without a gap condition. All one really needs for the adiabatic theorem is a spectral projection for the Hamiltonian that depends smoothly on time.

To formulate the problem more precisely it is convenient, and traditional, to replace the physical time t by the scaled time $s = t/\tau$ where $\tau \to \infty$ is the adiabatic time scale. In this notation, the adiabatic theorem is concerned with the solution of the initial value problem

$$i\,\dot{\psi}_\tau(s) = \tau H(s)\,\psi_\tau(s), \quad s \in [0,1] \tag{1}$$

in the limit of large τ. $H(s)$ is a self-adjoint Hamiltonian which depends smoothly[1] on $s \in [0,1]$ and ψ is a vector (in Hilbert space) valued function. Hence $H(t/\tau)$ evolves slowly in time for a long interval of time with finite overall change in the Hamiltonian $H(s)$. The quantum adiabatic theorem says that the solution to the initial value problem is characterized, in the adiabatic limit $\tau \to \infty$, by spectral information.

The first satisfactory formulation and rigorous proof of an adiabatic theorem in the then new quantum mechanics was given in 1928 by Born and Fock [BF]. They were motivated by a point of view advocated by Ehrenfest, which identified classical adiabatic invariants with quantum numbers. The theorem they proved was geared to show that quantum numbers are invariant

[1] For the applications to quantum mechanics, $H(s)$ is a Schrödinger operator, which is unbounded, so the notion of smoothness requires some discussion, see e.g. [ASY]. To avoid getting into technical issues that may obscure the basic mechanism we want to expose, we assume that $H(s)$ is a smooth family of bounded operators. From a physical point of view the adiabatic theorem is an infrared (i.e. low energy) problem. As such, the unboundedness of the Hamiltonian, which is an ultraviolet property, should not play a role.

under adiabatic deformations. The class of Hamiltonian operators they considered was so that $H(s)$ has simple discrete spectrum. Their proof was a variant of the method of variation of constants. A formal version of this result, usually without the careful analysis of Born and Fock, is what one finds in most textbooks on quantum mechanics.

In 1958 Kato [K] initiated a new strategy for proving adiabatic theorems. He introduced a notion of adiabatic evolution which is purely geometric. It is associated with a natural connection in the bundle of spectral subspaces. Kato's method was to compare the geometric evolution with the evolution generated by $H(s)$ and to show that in the adiabatic limit the two coincide. Using this idea, Kato extended the results of Born and Fock to the case where $H(s)$ had non-simple spectrum and, more significantly, to operators that had more general types of spectra. This is of importance for applications to quantum mechanics of atoms and molecules where absolutely continuous spectrum is always present, see e.g. [CFKS]. Kato proved an adiabatic theorem when the initial data lie in a subspace corresponding to an isolated eigenvalue of $H(0)$, provided that the corresponding subspace of $H(s)$ has constant multiplicity and was separated by a gap from the rest of the spectrum for all s. No assumption on the spectral type of $H(s)$ in the rest of the spectrum, that is, beyond the gap, need be made.

Kato's results were further extended in [ASY] who showed that an adiabatic theorem can be formulated and proven without any assumption on the nature of the spectral type also for the initial data, thereby dropping the condition that the initial data lie in a subspace corresponding to an eigenvalue with fixed multiplicity. For example, the initial data could lie in a subspace corresponding to an energy band with, say, absolutely continuous spectrum. The one crucial condition that appears to make the theorem in [ASY] go is the gap condition. For the example above, the gap condition is that this energy band be separated by a gap from the rest of the spectrum. This result was applied to the study of quantization of transport in the Hall effect [T].

There are examples of adiabatic theorems without a gap condition. But, these examples all appear to be special, in that some special property of the Hamiltonian intervenes and appears to play the role of a gap. One such example is the adiabatic theorem for crossing eigenvalues which was studied by Born and Fock. We return to this example below. Another example is an adiabatic theorem for rank one perturbations of dense point spectra studied in [AHS].

The current status of the general adiabatic theorem is that, modulo technicalities, provided the initial data lie in a spectral subspace of $H(0)$ and the corresponding spectral subspace of $H(s)$ is separated by a gap from the rest of the spectrum for all s, the time evolution respects the spectral splitting in the adiabatic limit.

If one examines the existing adiabatic theorems one sees that while the gap condition plays a central and crucial role in the proofs, it appears to play no such role in the formulation. To formulate an adiabatic conjecture all one needs is a smooth family of spectral projections. If that is the case, the initial data have a distinguished spectral subspace to cling to. We shall show that this condition is essentially all one needs to prove the adiabatic theorem. More precisely, we prove an adiabatic theorem provided the Hamiltonian has a smooth and finite dimensional spectral projection. The gap condition is dispensed with.

The adiabatic theorem is sometimes understood to be the statement that the adiabatic limit is approached exponentially fast for all times that lie outside the support of $\dot{H}(s)$. A general result of this kind, using the gap condition, is described in [KS, N]. We do not prove such a strong result here. Rather, we stick to the traditional usage of Born, Fock and Kato, where by the adiabatic theorem we refer to the remarkable fact, quite unlike perturbation theory, that there is a precise control on the evolution for Hamiltonians that undergo a *finite* variation.

The spectral gap in the adiabatic theorem controls the rate at which the adiabatic limit is approached. A finite gap guarantees that the rate is at least $O(1/\tau)$. Giving up the gap condition does not go without price. To see what this price should be let us recall a result of Born and Fock who studied also crossing eigenvalues where the spectral projections have a smooth continuations through the crossing point [BF]. Born and Fock established an adiabatic theorem in this case, where the physical evolution clings to the spectral projection, and picks the smooth continuation at the crossing. The rate at which the adiabatic limit is approached is only $O(1/\sqrt{\tau})$ for linear crossing. This suggests that the price for giving up the gap condition is the sacrificing of knowledge about the rate at which the adiabatic limit is approached.

Our approach to an adiabatic theorem without a gap condition has some of the flavor of an operator analog of the Riemann-Lebesgue lemma. If a function and also its derivative are in $L^1(\mathbb{R})$ then it is an elementary exercise that its Fourier transform decays at infinity at least as fast as an inverse power of the argument. Riemann-Lebesgue lemma says that, in fact, the Fourier transform of any $L^1(\mathbb{R})$ function vanishes at infinity. The loss of *a-priori* information about the derivative translates to loss of information about the rate at which the function vanishes at infinity. In this analogy differentiability is the analog of the gap condition, and the $L^1(\mathbb{R})$ condition is the analog of the smoothness condition on the spectral projection.

To simplify the presentation, we shall stay away from making optimal assertions.

Let $H(s)$ be a family of bounded[2] self-adjoint Hamiltonians that generates unitary evolution as the solution of the initial value problem:

$$i\,\dot{U}_\tau(s) = \tau H(s) U_\tau(s), \quad U_\tau(0) = 1, \quad s \in [0,1]. \tag{2}$$

We assume that $H(s)$ has eigenvalue $\lambda(s)$ and this eigenvalue has finite multiplicity. For this eigenvalue we formulate and prove our main result:

Theorem 0.1 *Suppose that $P(s)$ is finite rank spectral projection, which is at least twice differentiable (as a bounded operator), for the Hamiltonian $H(s)$, which is bounded and differentiable for all $s \in [0,1]$. Then, the evolution of the initial state $\psi(0) \in Range P(0)$, according to Eq. (1), is such that in the adiabatic limit $\psi_\tau(s) \in Range P(s)$ for all s.*

Remark: In the case that there is eigenvalue crossing $P(s)$ is not smooth and $Tr\, P(s)$ is discontinuous. We shall not treat this case. The method we describe can, in fact, handle crossings provided these occur at finite number of points in time. But, keeping with our policy of avoiding making optimal results in order to keep the basic ideas in the forefront, we shall not treat crossings here.

We recall the notion of adiabatic evolution [K, ASY]. Let $U_A(s)$ be the solution of the initial value problem:

$$i\,\dot{U}_A(s) = \tau \left(H(s) + \frac{i}{\tau}\,[\dot{P}(s), P(s)] \right) U_A(s), \quad U_A(0) = 1, \quad s \in [0,1]. \tag{3}$$

It is known that this unitary evolution has the intertwining property [ASY]:

$$U_A(s)\, P(0) = P(s)\, U_A(s). \tag{4}$$

[2] See footnote 1.

That is $U_A(s)$ maps *Range* $P(0)$ onto *Range* $P(s)$. In particular, the solution of the initial value problem

$$i\,\dot\psi(s)=\tau\,\left(H(s)+\frac{i}{\tau}\,[\dot P(s),P(s)]\right)\,\psi(s),\;\;\psi(0)\in Range P(0), \tag{5}$$

has the property that $\psi(s)\in Range P(s)$. We shall show that the Hamiltonian evolution, $U_\tau(s)$, is close to the adiabatic evolution $U_A(s)$.

We first formulate the basic lemma:

Lemma 0.2 *Let $P(s),\;\; s\in[0,1]$, be a differentiable family of spectral projections for the self-adjoint Hamiltonian $H(s)$ with (operator) norm $\|\dot P(s)\|<\infty$. Suppose that the commutator equation*

$$[\dot P(s),P(s)]=[H(s),X(s)]+Y(s), \tag{6}$$

has operator valued solutions, $X(s)$ and $Y(s)$ with $X(s)$, $\dot X(s)$ and $Y(s)$ bounded. Then

$$\|(U_\tau(s)-U_A(s))\,P(0)\|\le\max_{s\,\in\,[0,1]}\left(\frac{2\,\|X(s)\,P(s)\|+\|\dot{\big(X(s)\,P(s)\big)}P(s)\|}{\tau}+\|\,Y(s)\,P(s)\|\right). \tag{7}$$

Remark: The commutator equation, Eq. (6), can be viewed as a definition of $Y(s)$. The issue is not to find a solution to this equation, but rather to find solutions that make Y as small as possible. In the case that there is a gap Δ separating the eigenvalue from the rest of the spectrum, a solution of the commutator equation is

$$X(s)=\frac{1}{2\pi i}\int_\Gamma R(z,s)\,\dot P(s)R(z,s)\,dz,\quad Y(s)=0. \tag{8}$$

Here Γ is a circle in the complex plane, centered at the eigenvalue, and of radius $\Delta/2$. $R(z,s)$ is the resolvent at scaled time s, see [ASY]. In this case the rate at which the adiabatic limit is obtained, is seen from Eq. (7) to be $1/\tau$.

The strategy for proving the adiabatic theorem without a gap condition is to show that one can pick Y so that its norm is arbitrarily small, possibly at the expense of large norm for X and $\dot X$. So long as the norm of X and $\dot X$ is finite, it can be compensated by taking τ large. This means that one can make the right hand side of Eq. (7) arbitrarily small. The price paid is that there is, generally speaking, no information about the rate at which the adiabatic limit is obtained.[3]

Proof: Let $W(s)=U_\tau^\dagger(s)U_A(s)$ be the wave operator comparing the adiabatic and Hamiltonian evolution. Since

$$\|U_\tau(s)-U_A(s)\|=\Big\|U_\tau(s)\Big(1-W(s)\Big)\Big\|=\|1-W(s)\|, \tag{9}$$

we need to bound $W(s)-1$. From the definition of the adiabatic evolution, the commutator equation, and the equation of motion

$$\dot W(s)\;\;=\;\;U_\tau^\dagger(s)\,\Big([\dot P(s),P(s)]\Big)\,U_A(s)$$

[3]With additional information about structure of the spectra, one can sometimes get estimates on the rate at which the adiabatic limit is approached. See corollary (0.8).

$$
\begin{aligned}
&= U_\tau^\dagger(s)\left([\dot P(s),P(s)]\right)\,U_\tau(s)\,W(s)\\
&= U_\tau^\dagger(s)\left([H(s),X(s)]+Y(s)\right)U_\tau(s)\,W(s) \qquad (10)\\
&= -\frac{i}{\tau}\left(\dot U_\tau^\dagger(s)X(s)U_\tau(s)+U_\tau^\dagger(s)X(s)\dot U_\tau(s)\right)W(s)+U_\tau^\dagger(s)Y(s)U_A(s)\\
&= -\frac{i}{\tau}\left((U_\tau^\dagger(s)X(s)U_\tau(s))^{\dot{}}-U_\tau^\dagger(s)\dot X(s)U_\tau(s)\right)W(s)+U_\tau^\dagger(s)Y(s)U_A(s)\\
&= -\frac{i}{\tau}\Big\{\left(U_\tau^\dagger(s)X(s)U_\tau(s)\,W(s)\right)^{\dot{}}-U_\tau^\dagger(s)X(s)U_\tau(s)\,\dot W(s)\\
&\qquad -U_\tau^\dagger(s)\dot X(s)U_\tau(s)\,W(s)\Big\}+U_\tau^\dagger Y(s)U_A(s)\\
&= -\frac{i}{\tau}\Big\{\left(U_\tau^\dagger(s)X(s)U_A(s)\right)^{\dot{}}-U_\tau^\dagger(s)X(s)\,[\dot P(s),P(s)]\,U_A(s)\\
&\qquad -U_\tau^\dagger(s)\dot X(s)U_A(s)\Big\}+U_\tau^\dagger(s)Y(s)U_A(s).
\end{aligned}
$$

The lemma then follows by integration since $W(s)$ is unitary with $W(0)=1$. □

Let us describe a solution of the commutator equation which is motivated by the solution Eq. (8) in the case of a gap. In order to have explicit error estimates and also in order to make the presentation simple and as elementary as possible, we choose a Gaussian regularizer.

Definition 0.3 *Let g and e denote the Gaussian and Error functions*[4]*, and Φ be the special function defined below:*

$$g(\omega)=e^{-\pi\omega^2},\quad e(t)=\int_{-\infty}^{t}ds\,g(s),\quad \Phi(t)=\theta(t)-e(t), \tag{11}$$

θ is the usual step function.

An elementary lemma is:

Lemma 0.4 *Φ has finite L^1 norm and finite moments. In particular:*

$$\|\Phi(t)\|_1=\frac{1}{\pi},\quad \|t\Phi(t)\|_1=\frac{1}{4\pi}. \tag{12}$$

Under scaling, $\Delta>0$:

$$\|\Phi\,(t\Delta)\|_1=\frac{1}{\pi\Delta},\quad \|t\Phi\,(t\Delta)\|_1=\frac{1}{4\pi\Delta^2}. \tag{13}$$

We assume, without loss, that the spectral projection $P(s)$ is associated with the eigenvalue zero.

Lemma 0.5 *Let $P(s)$ be a smooth spectral projection for $H(s)$ associated with the eigenvalue zero. Let Γ be an infinitesimal contour around the origin in the complex plane.*[5] *Let*

$$F_\Delta(s)=g\left(\frac{H(s)}{\Delta}\right)-P(s). \tag{14}$$

[4] The error function we use differs by a scaling of the argument, an overall factor and shift from the canonical error function.

[5] At this point the choice of Gaussian is not optimal. It would be more convenient to choose a regularizer which is a better approximant to a characteristic function and the reader may want to think of a Gaussian which is flattened at the top.

Then the commutator equation has the solution

$$
\begin{aligned}
X_\Delta(s) &= A + A^\dagger, \quad A = P(s)\,\dot P(s) R(0,s)\,\left(1 - g\left(\frac{H(s)}{\Delta}\right)\right);\\
Y_\Delta(s) &= -F_\Delta(s)\dot P(s) P(s) + P(s)\dot P(s) F_\Delta(s),
\end{aligned}
\tag{15}
$$

with

$$
\begin{aligned}
\|X_\Delta(s)P(s)\| &\le \frac{2\|\dot P(s)P(s)\|}{\Delta},\\
\|\dot{\Big(X_\Delta(s)P(s)\Big)}\| &\le \frac{2\|\ddot P(s)\|}{\Delta} + \frac{\pi\,\|\dot P(s)\|\,\|\dot H\|}{\Delta^2}.
\end{aligned}
\tag{16}
$$

Remark: In the case that the family of Hamiltonians is related by unitaries

$$
H(s) = V(s)\,H_0\,V^\dagger(s), \quad V(s) = \exp(i\,s\,\sigma)
\tag{17}
$$

such that σ is bounded operator, one can improve[6] the estimate to $\|\dot{\Big(X_\Delta(s)P(s)\Big)}\| \le \frac{\|[X,\sigma]\|}{\Delta}$.

Proof: We start with a formal calculation. Let

$$
X_\Delta(s) = \frac{1}{2\pi i}\int_\Gamma dz\,(1 - F_\Delta(s))\,R(z,s)\,\dot P(s) R(z,s)\,(1 - F_\Delta(s)).
\tag{18}
$$

Since $\dot P(s) = P(s)\dot P(s) + \dot P(s)P(s)$, $X_\Delta(s)$ can be written as a sum of two adjoint terms, one of them is

$$
\begin{aligned}
\frac{1}{2\pi i}\int_\Gamma dz\,(1 - F_\Delta(s))\,R(z,s)\,P(s)\dot P(s)R(z,s)\,(1 - F_\Delta(s)) &=\\
\frac{1}{2\pi i}(1 - F_\Delta(s))\,P\dot P(s)\left(\int_\Gamma dz\,\frac{R(z,s)}{z}\right)(1 - F_\Delta(s)) &=\\
P(s)\dot P(s)R(0,s)(1 - P(s))\,(1 - F_\Delta(s)) = P(s)\dot P(s)R(0,s)(1 - P(s) - F_\Delta(s)) &=\\
P(s)\dot P(s)R(0,s)\left(1 - g\left(\frac{H(s)}{\Delta}\right)\right) &= A.
\end{aligned}
\tag{19}
$$

We have used

$$
P(s)\,F_\Delta(s) = F_\Delta(s)\,P(s) = \Big(g(0) - 1\Big)\,P(s) = 0.
\tag{20}
$$

Using this integral representation of $X_\Delta(s)$ we now find $Y_\Delta(s)$. By our choice of $F_\Delta(s)$ we have $[F_\Delta(s), H(s)] = 0$. Hence,

$$
\begin{aligned}
[X_\Delta(s), H(s)] &= \frac{1}{2\pi i}\int_\Gamma dz\,\Big[(1 - F_\Delta(s))\,R(z,s)\,\dot P(s)R(z,s)\,(1 - F_\Delta(s)), H(s) - z\Big]\\
&= \frac{1}{2\pi i}\int_\Gamma dz\,(1 - F_\Delta(s))\Big[\,R(z,s)\,,\dot P(s)\Big](1 - F_\Delta(s))\\
&= (1 - F_\Delta(s))\,\Big[P(s), \dot P(s)\Big]\,(1 - F_\Delta(s))\\
&= [P(s), \dot P(s)] - \Big\{F_\Delta(s), [P(s), \dot P(s)]\Big\} + F_\Delta(s)[P(s), \dot P(s)]F_\Delta(s)\\
&= [P(s), \dot P(s)] + F_\Delta(s)\dot P(s)P(s) - P(s)\dot P(s)F_\Delta(s).
\end{aligned}
\tag{21}
$$

[6] see proposition (0.6)

So a solution of the commutator equation is

$$Y_\Delta(s) = -F_\Delta(s)\dot{P}(s)P(s) + P(s)\dot{P}(s)F_\Delta(s). \tag{22}$$

It remains to estimate the norms of X and $\dot{X}$. Using the fact the a Gaussian is its own Fourier transform,

$$g\left(\frac{H(s)}{\Delta}\right) = \Delta \int_{\mathbb{R}} g(\Delta\, t)\, \exp[2\pi i t H(s)] dt, \tag{23}$$

one checks that with our choice of Φ

$$R(0,s)\left(1 - g\left(\frac{H(s)}{\Delta}\right)\right) = 2\pi i \int_{\mathbb{R}} \Phi\,(t\Delta)\, \exp[2\pi i t H(s)]\, dt. \tag{24}$$

Hence

$$\left\| R(0,s)\left(1 - g\left(\frac{H(s)}{\Delta}\right)\right)\right\| \le 2\pi\, \|\Phi\,(t\Delta)\,\|_1 = \frac{2}{\Delta}. \tag{25}$$

Using the equation for $X(s)$ this estimate proves the bound on $X(s)$. To get a bound on $\dot{\left(X_\Delta(s)P(s)\right)}$, use Duhammel formula

$$\dot{\left(\exp(2\pi i t H(s))\right)} = 2\pi i\, t \int_0^1 dz\, e^{2\pi i z t H(s)}\, \dot{H}(s)\, e^{2\pi i(1-z)tH(s)}. \tag{26}$$

Collecting the various terms give the claimed estimate. □

Proposition 0.6 *Let $H(s)$ be the family*

$$H(s) = V(s)\, H\, V^\dagger(s), \quad V(s) = \exp(i\, s\, \sigma), \tag{27}$$

with P a finite-dimensional projection onto the $Ker\, H$. It is enough to solve for the commutator equation

$$[H, X] + Y = i\,\{\sigma, P\} - 2\, i\, P\sigma P, \tag{28}$$

for fixed X and Y. $X(s)$ and $Y(s)$ are then determined by the obvious unitary conjugation, and $\|\dot{X}(s)\| = \|[X,\sigma]\|$.

Proof: Since $P(s) = V(s)\, P\, V^\dagger(s)$, we have

$$\dot{P}(s) = i\, V(s)\, [\sigma, P]\, V^\dagger(s), \tag{29}$$

and

$$\begin{aligned} [\dot{P}(s), P(s)] &= i\, V(s)\, \Big[[\sigma, P], P\Big]\, V^\dagger(s) \\ &= i\, V(s)\, \Big(\{\sigma, P\} - 2P\,\sigma\, P\Big)\, V^\dagger(s). \end{aligned} \tag{30}$$

□

As lemma (0.5) shows, as Δ shrinks, the norms of $X(s)$ and $\dot{X}(s)$ may, and in general, will, grow. This, however is of no concern, as long as the norms remain finite, for one can always compensate for this growth by choosing τ large enough. The good thing about shrinking Δ is that this can be used to make the norm of Y_Δ arbitrarily small. Hence, we can always make the right hand side of Eq. (7) arbitrarily small.

Lemma 0.7 *Suppose that $H(s)$ is smooth with a zero eigenvalue with spectral projection $P(s)$ smooth and of finite rank. Let $F_\Delta(s)$ be as above. Then $\|Y_\Delta(s)\,P(s)\| = \|F_\Delta(s)\dot P(s)P(s)\| \to 0$ uniformly as Δ shrinks to zero.*

Proof: For the sake of simplicity suppose that $P(s)$ has rank one with $P(s)\psi(s) = \psi(s)$, $\langle\psi|\psi\rangle = 1$. Let $\varphi = \dot P(s)\psi$. Then, using property $P(s)\varphi(s) = P(s)\dot P(s)\psi(s) = P(s)\dot P(s)P(s)\psi(s) = 0$, we obtain

$$\begin{aligned}
\|F_\Delta \dot P(s)P(s)\|^2 = \|F_\Delta \dot P(s)\psi(s)\|^2 &= \|F_\Delta\varphi(s)\|^2 = \\
\left\|\left(g\left(\frac{H(s)}{\Delta}\right) - P(s)\right)\varphi(s)\right\|^2 = \left\|g\left(\frac{H(s)}{\Delta}\right)\varphi(s)\right\|^2 &= \left\langle\,\varphi\,|g^2\left(\frac{H(s)}{\Delta}\right)|\,\varphi\,\right\rangle = \\
\int_{\sigma(H(s))} g^2(x/\Delta)\,d\mu_\varphi(x), &
\end{aligned} \tag{31}$$

where μ_φ denotes the spectral measure. Now, $g\left(\frac{x}{\Delta}\right)$ is bounded by one, and goes monotonically to zero for all $x \neq 0$, and $g(0) = 1$. Hence

$$\lim_{\Delta\to 0}\int_{\sigma(H(s))} g^2(x/\Delta)\,d\mu_\varphi(x) = \mu_\varphi(0) = 0. \tag{32}$$

It follows that there is a sequence of Δ that make $Y_\Delta(s)$ arbitrarily small. □
By taking τ large enough one can therefore make the right hand side of Eq. (7) as small as one pleases. This completes the proof of the theorem. □

The physical interpretation of the general adiabatic theorem is that although the adiabatic theorem "always" holds, it does so for different physical mechanisms. In the case that there is a gap in the spectrum the adiabatic theorem holds because the eigenstate is protected by a gap from tunneling out of the spectral subspace. In the case that there is no gap and the spectrum near the relevant eigenvalue is essential, the adiabatic theorem holds for a different reason: The essential spectrum is associated with states that are supported near spatial infinity, and there is little tunneling to these states because of small overlap with the wave function corresponding to an eigenvalue which is supported away from infinity.

If one has additional information about the nature of the spectrum embedding the eigenvalue, one can sometimes get estimates on the rate at which the adiabatic limit is approached. An illustration of this is given below.

Recall [L] that a (Borel) measure μ is called (uniformly) α-Hölder continuous, $\alpha \in [0,1]$, if there is a constant C such that for every interval Δ with $|\Delta| < 1$ [7]

$$\mu(\Delta) < C|\Delta|^\alpha. \tag{33}$$

Corollary 0.8 *If the spectral measure $\mu_\varphi(\Delta)$, is α-Hölder continuous then the adiabatic limit is approached at least at rate $\left(\frac{1}{\tau}\right)^{\frac{\alpha}{2+\alpha}}$. In the case of a family of Hamiltonians related by unitaries*

$$H(s) = V(s)\,H_0\,V^\dagger(s), \quad V(s) = \exp(i\,s\,\sigma), \tag{34}$$

such that σ is bounded operator, the rate is at least $\left(\frac{1}{\tau}\right)^{\frac{\alpha}{1+\alpha}}$.

[7] $|\cdot|$ denote Lebesgue measure.

Proof: Let us note, first of all, that if the spectral measure $\mu_\varphi(\Delta)$, is α-Hölder continuous then rhs of (32) is bounded by $\tilde{C}|\Delta|^\alpha$. Indeed,

$$\begin{aligned}\int_{\sigma(H(s))} g^2(x/\Delta)d\mu_\varphi(x) &< \sum_{n=0}^{\infty} g^2(n)\Big(\int_{n\Delta}^{(n+1)\Delta} d\mu_\varphi(x) + \int_{-(n+1)\Delta}^{-n\Delta} d\mu_\varphi(x)\Big) \\ &< C|\Delta|^\alpha \sum_{n=-\infty}^{\infty} g^2(n) = \tilde{C}|\Delta|^\alpha. \end{aligned} \tag{35}$$

Collecting the various error estimates one gets for the right hand side of Eq. (7) the upper bound

$$\frac{A}{\Delta\tau} + \frac{B}{\Delta^2\tau} + \tilde{C}|\Delta|^\alpha. \tag{36}$$

A, B and C are constants. For the case of the family of Hamiltonians related by unitaries $B = 0$.[8] Optimizing the choice of Δ gives the result. □

The interest in such measures comes from the fact [L] that α-continuous measures are the limits of α-Hölder continuous measures. Knowing something about the Hausdorff dimension of the spectrum [L] translates to information about the rate of approach of the adiabatic limit. The above result about the rate of approach to the adiabatic limit in quantum evolution has some of the flavor of a result about the classical adiabatic limit for classical chaotic systems: It was shown by Ott [O] that for such systems, the approach to the adiabatic limit is with rate $O(1/\sqrt{\tau})$.

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[8] see proposition (0.6).

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