LIMITING ABSORPTION PRINCIPLE FOR SINGULARLY PERTURBED OPERATORS

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ABSTRACT. Given an operator H_1 for which a limiting absorption principle holds, we study operators H_2 which are produced by perturbing H_1 in the sense that the difference between some powers of the resolvents is compact. We show that (except for possibly a discrete set of eigenvalues) a limiting absorption principle holds for H_2.

We apply this theory to study potential and domain perturbations of Feller operators. While our theory mostly reproduces known results in the case of potential perturbations, for domain perturbations we get results which appear to be new. limité, de perturbation nouveaux.

1. INTRODUCTION

Suppose H is a self-adjoint operator in a Hilbert space $\mathcal{H}$, $R(z) = (H - z)^{-1}$ its resolvent, and λ some value in its spectrum $\sigma(H)$. In order to study spectral and scattering properties of H it is useful to have some control over the behavior of $R(z)$ as $z = \lambda \pm i\varepsilon$ tends to λ. Since the limit $\lim_{\varepsilon\downarrow 0} R(\lambda \pm i\varepsilon)$ cannot exist in the operator norm on $\mathcal{H}$ one tries to establish its existence in a weaker topology. Let $X \subset \mathcal{H}$ be a dense subspace of $\mathcal{H}$, X^* its dual. We say that a limiting absorption principle holds for H if the limit $\lim_{\varepsilon\downarrow 0} R(\lambda \pm i\varepsilon)$ exists in the norm topology of $\mathcal{B}(X, X^*)$, the space of bounded operators from X to X^*.

Our aim in this paper is to study the stability of such a limiting absorption principle under perturbations, especially under singular ones. That is, we suppose a limiting absorption principle holds for some operator H_1 and try to establish that it holds for a second operator H_2 which is constructed by perturbing H_1 in some way.

There is a huge number of publications on the limiting absorption principle for individual operators. Among those more closely related to our problem we would like to mention Agmon's pioneering work [1] for perturbations of the Laplacian, the abstract theory developed by Ben-Artzi and Devinatz [5] mainly for potential perturbations. Laplacians in exterior domains were treated, e.g., by Leis and Roach [15]. More recently Mourre estimates led to many new results, cf., e.g., Amrein, Boutet de Monvel, and Georgescu [2] for an overview and references. In particular Pearson [18] treats potential perturbations of the Laplacian with this approach, Iwashita [13] differential operators in exterior domains.

We want to consider the case where a differential operator is perturbed by introducing additional Dirichlet boundary conditions on some set, i.e., domain perturbations. This includes, but is not limited to, differential operators in exterior domains. We present an abstract theory that includes such perturbations as well

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as other types of regular and singular perturbations. Our approach is similar to the one presented in [5], but more general: The abstract theory in [5] is geared towards potential perturbations; while it does cover other situations as well it is not applicable in our case.

In what sense do we want to perturb H_1? [5] treats the case where $H_2 = H_1 + V$ with V some suitable operator, the main focus being on multiplication operators. The key assumption here is that $VR_1(z)$ is a compact operator from X^* to X. In the case of domain perturbations, however, the difference of the operators H_2 and H_1 becomes meaningless. For instance, if H_1 is the Laplacian on $MR2$ and H_2 the Dirichlet Laplacian on $MR2 \setminus B_1(0)$ with $B_1(0)$ the unit ball around the origin, then $H_2 - H_1$ is identically zero on the intersection of the domains.

To get around this difficulty one can in a first step consider the difference of the resolvents $R_2(z) - R_1(z)$ and suppose that this resolvent difference is compact from X^* to X. In general H_1 and H_2 will be defined on different Hilbert spaces, so we have to formulate a two-space theory; we refer to Section 2 for precise definitions.

There is one further complication: If one considers, e. g., domain perturbations of the Laplacian on MRd one easily finds sufficient conditions for the compactness of the semigroup differences $e^{-tH_2} - e^{-tH_1}$. From this one can directly conclude compactness of $R_2(z)^m - R_1(z)^m$ for $m \in \mathbb{N}$ large enough. For space dimension $d \geq 4$ proving compactness of $R_2(z) - R_1(z)$ itself, however, is more problematic. To avoid difficulties of this type we formulate our theory in terms of differences of powers of the resolvents rather than in terms of the differences of the resolvents themselves. This requires some new technical considerations which will be presented in Section 2 and at the beginning of Section 3.

In Section 3 we develope the abstract theory, leading to our main result, Theorem 3.5. Our approach is classical in concept, relying strongly on compactness arguments and in particular on the Fredholm alternative. In fact, aside from the considerations mentioned above and some technical simplifications, the line of argument is quite close to [5].

In Section 4 we specialize to the case where X is a weighted L^2-space. We formulate a proposition which essentially is the 'standard bootstrap argument'. This argument is used in many proofs of the limiting absorption principle; we prove it on an abstract level.

In Section 5, we apply our theory to study potential and domain perturbations of Feller operators. A Feller operator is an operator defined as the generator of a strong Markov process with the Feller property, cf. Demuth and van Casteren [8], [9], [10]. We assume that a limiting absorption principle holds for the unperturbed operator H_1 on $L^2(M)$ (M some appropriate measure space) and consider a perturbed operator $H_2 = (H_1 + V)_\Sigma$ that is produced by restricting $H_1 + V$ to a set $\Sigma \subset M$ via Dirichlet boundary conditions. We give sufficient conditions in terms of V and the equilibrium potential of the set $\Gamma = M \setminus \Sigma$ to ensure that a limiting absorption principle holds for H_2. If, for instance, H_1 is the Laplacian our theory allows to treat the usual short range potentials, but it also allows domain perturbations by sets Γ which may be unbounded – the condition we impose on Γ is slightly stronger than requiring that its capacity is finite.

2. Notation and Assumptions

Let $(\mathcal{H}_1, \langle\cdot,\cdot\rangle_1)$ and $(\mathcal{H}_2, \langle\cdot,\cdot\rangle_2)$ be Hilbert spaces, J_2 a bounded map from $\mathcal{H}_1$ to $\mathcal{H}_2$ with $J_2 J_2^*$ equal to the identity on $\mathcal{H}_2$, so $J_2^* J_2$ is an orthogonal projection on $\mathcal{H}_1$. To allow a more concise notation we also introduce $J_1 = J_1^* = 1$ the identity in $\mathcal{H}_1$.

The initial operator H_1 acts in $\mathcal{H}_1$, the perturbed operator H_2 in $\mathcal{H}_2$. As usual $\sigma(H_k)$, $\rho(H_k)$, and $R_k(z)$ denote spectrum, resolvent set, and resolvent, respectively, of the operator H_k, $k = 1, 2$.

Let the Hilbert space $(X, \langle\cdot,\cdot\rangle_X)$ be densely and continuously embedded in $\mathcal{H}_1$, $(X^*, \langle\cdot,\cdot\rangle_{X^*})$ its dual. Instead of identifying X and X^* we prefer to identify $u \in \mathcal{H}_1$ with the functional $\langle u, \cdot\rangle_1$ it generates on X, i.e., we consider X^* as the completion of $\mathcal{H}_1$ under the norm $\|\cdot\|_{X^*}$: $\|v\|_{X^*} = \sup_{u\in X} \frac{\langle v,u\rangle_1}{\|u\|_X}$. Thus $X \subset \mathcal{H}_1 \subset X^*$ with dense and continuous embeddings. Here in obvious notation $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_X$, and $\|\cdot\|_{X^*}$ denote the norms in $\mathcal{H}_1$, $\mathcal{H}_2$, X, and X^*, respectively. Generally, unless defined otherwise, whenever $\mathcal{K}$ is a normed space we will denote its norm by $\|\cdot\|_{\mathcal{K}}$.

As usual $\mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$ is the space of bounded operators from a Hilbert space $\mathcal{K}_1$ into another Hilbert space $\mathcal{K}_2$, $\mathcal{B}_\infty(\mathcal{K}_1, \mathcal{K}_2)$ the space of compact operators, $\mathcal{B}_2(\mathcal{K}_1, \mathcal{K}_2)$ the space of Hilbert Schmidt operators. The norm in $\mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$ is usually written as $\|\cdot\|_{\mathcal{B}(\mathcal{K}_1,\mathcal{K}_2)}$, the norm in $\mathcal{B}_2(\mathcal{K}_1, \mathcal{K}_2)$ as $\|\cdot\|_{HS}$.

We will abuse notation slightly and use the same symbol for bounded operators and their restrictions and extensions. For instance, $R_1(z)$ will not only be used for the resolvent of H_1 (in $\mathcal{B}(\mathcal{H}, \mathcal{H})$) but also for its restriction to $\mathcal{B}(X, X)$ as well as its extension to $\mathcal{B}(X^*, X^*)$. (That $R_1(z)$ indeed maps X to X is part of Hypothesis 2.1 below.) Similarly we use the same symbol for operators and for their integral kernels. For instance if e^{-tH_1} is a semigroup of operators which has an integral kernel we will denote this integral kernel by $e^{-tH_1}(\cdot,\cdot)$.

Our basic assumptions on the operators H_1 and H_2 can then be formulated as follows.

Hypothesis 2.1. *Suppose that for $k = 1, 2$ the operators H_k act as self-adjoint semibounded operators in $\mathcal{H}_k$ with domains $\mathcal{D}(H_k)$;*

$$\langle H_k u, u\rangle_k \geq c_0 \qquad \text{for all } u \in \mathcal{D}(H_k) \tag{2.1}$$

with some constant $c_0 \in MR : @ : .Moreover, suppose H_k$ are consistent with the space X in the sense that their resolvents $R_k(z) = (H_k - z)^{-1}$ satisfy

$$J_k^* R_k(z) J_k \in \mathcal{B}(X, X) \qquad \text{for all } z \in \rho(H_k) \tag{2.1}$$

the resolvents being continuous functions of z in $\mathcal{B}(X, X)$.

As explained above (2.1) is a slight abuse of notation, because we have used the same symbol for the restriction to X as for the operators in $\mathcal{H}_1$.

We will assume that a limiting absorption principle holds for the operator H_1.

Hypothesis 2.2. *Let Δ be a given open subset of MR*
Hypothesis 2.2. $: @ :: Assume that for all \lambda \in \Delta$

$$R_1^{\pm}(\lambda) := \lim_{\varepsilon\downarrow 0} R_1(\lambda \pm i\varepsilon) \tag{2.2}$$

exists and is continuous with respect to λ in $\mathcal{B}(X^*, X)$.

Finally we need our compactness assumption.

Hypothesis 2.3. *Suppose there exists an $a < c_0$, $a < \inf \Delta$ and an $m \in \mathbb{N}$ such that*

$$E := J_2^* R_2(a)^m J_2 - R_1(a)^m \in \mathcal{B}_\infty(X^*, X). \tag{2.3}$$

We will later on impose one further condition that in a sense intertwines H_1 and H_2, but before we can formulate this we have to introduce additional notation. We will begin by defining operators $F_k(z)$ which will be the key object in our analysis. Simultaneously we give a representation which is essentially due to Demuth and McGillivray [7].

Lemma 2.4. *Fix a constant $a < c_0$. Then for $k = 1, 2$ and for $z \in \mathbb{C} \setminus \{a\}$,*

$$R_k(a)^m - (z-a)^{-m} = \prod_{j=0}^{m-1} \left[R_k(a) - w_j (z-a)^{-1}\right]. \tag{2.4}$$

If $0 < |\arg(z-a)| < \frac{2\pi}{m}$ we can define

$$\begin{aligned} F_k(z) &:= \left[R_k(a)^m - (z-a)^{-m}\right]^{-1} \\ &= (z-a)^m \prod_{j=0}^{m-1} \left\{(-w_j)\left[1 + w_j(z-a) R_k\big(a + w_j(z-a)\big)\right]\right\} \end{aligned} \tag{2.5}$$

in $\mathcal{B}(\mathcal{H}_k, \mathcal{H}_k)$. Here $w_0 = 1$, $w_1 = e^{2\pi i (1/m)}$, $\ldots$, $w_{m-1} = e^{2\pi i (m-1)/m}$ denote the m-th complex roots of 1.

Proof. Equation (2.4) follows immediately from the formula

$$x^m - 1 = \prod_{j=0}^{m-1} (x - w_j) \tag{2.6}$$

and the spectral theorem. Rewriting

$$R_k(a) - w_j(z-a)^{-1} = -w_j \left[H_k - a - \tfrac{1}{w_j}(z-a)\right] R_k(a) (z-a)^{-1} \tag{2.7}$$

we see that the right hand side of (2.4) is invertible in $\mathcal{B}(\mathcal{H}_k, \mathcal{H}_k)$ for $0 < |\arg(z - a)| < \frac{2\pi}{m}$, because for such z we have $a + \frac{1}{w_j}(z-a) \in \mathbb{C} \setminus [a, \infty) \subset \rho(H_k)$ for $j = 0, 1, \ldots, m-1$, $k = 1, 2$. Formula (2.5) then follows by inverting (2.7) and rearranging the factors in the product, using

$$\begin{aligned} \left[R_k(a) - w_j(z-a)^{-1}\right]^{-1} &= (z-a)(H_k - a) R_k\big(a + \tfrac{1}{w_j}(z-a)\big)\big(-\tfrac{1}{w_j}\big) \\ &= (z-a)(-w_{n_j})\left[1 + w_{n_j}(z-a) R_k\big(a + w_{n_j}(z-a)\big)\right], \end{aligned} \tag{2.8}$$

where $n_0 = 0$, $n_j = m - j$ for $j \geq 1$ so $w_{n_j} = \frac{1}{w_j}$. □

We will also need the limit values $F_k^\pm(\lambda)$. These are directly related to the values $R_k^\pm(\lambda)$.

Lemma 2.5. *For $\lambda \in MR$*

Lemma 2.5. *: @ :,k=1,2, existsandiscontinuousinB(X,X*) if and only if* $J_k^* R_k^\pm(\lambda) J_k$ *exists and is continuous. If at least one of these conditions is satisfied the representation*

$$F_k^\pm \lambda = -(\lambda - a)^m \big[1 + (\lambda - a) R_k^\pm(\lambda)\big] \times \prod_{j=1}^{m-1} \Big\{(-w_j)\big[1 + w_j(\lambda - a) R_k\big(a + w_j(\lambda - a)\big)\big]\Big\} \tag{2.9}$$

holds. All factors in this formula (including the factor $\big[1 + (\lambda - a)R_1^\pm(\lambda)\big]$ *before the product) commute.*

Proof. For $\varepsilon > 0$ small enough $z = \lambda \pm i\varepsilon$ satisfies $0 < |\arg(z - a)| < \frac{2\pi}{m}$ and the representation (2.5) holds. Hence

$$J_k^* F_k(z) J_k = (z - a)^m \prod_{j=0}^{m-1} \Big\{(-w_j)\big[J_k^* J_k + w_j(z - a) J_k^* R_k\big(a + w_j(z - a)\big) J_k\big]\Big\} \tag{2.10}$$

The factors for $j \geq 1$ in this product are bounded and continuous with respect to z (even in the limit $z = \lambda \pm i\varepsilon \to \lambda$) in $\mathcal{B}(X, X)$ by (2.1), because $a + w_j(z - a)$ stays away from $[a, \infty)$ (even in the limit).

If $J_k^* R_k^\pm(\lambda) J_k$ exists and is continuous in $\mathcal{B}(X, X^*)$ the remaining factor ($j = 0$) in (2.10),

$$J_k^* J_k + (z - a) J_k^* R_k(z) J_k \tag{2.11}$$

has a well defined continuous limit in $\mathcal{B}(X, X^*)$ for $z = \lambda \pm i\varepsilon \to \lambda$. If, on the other hand, $J_k^* F_k^\pm(\lambda) J_k$ exists and is continuous in $\mathcal{B}(X, X^*)$ we can invert the factors for $j = 1, \ldots, m - 1$ in (2.10) in $\mathcal{B}(X, X)$ to conclude the existence and continuity of $J_k^* R_k^\pm(\lambda) J_k$ in $\mathcal{B}(X, X^*)$.

The representation (2.9) is an obvious consequence of (2.5) (resp. (2.10)), the existence of the limits and (2.1). The factors in (2.5) commute; by duality we can consider $J_k^* R_k(z) J_k$ as an operator in $\mathcal{B}(X^*, X^*)$ as well as in $\mathcal{B}(X, X)$, hence the factors in (2.10) commute as well even when we restrict $J_k^* F_k(z) J_k$ to an operator in $\mathcal{B}(X, X^*)$. By passing to the limit $z = \lambda \pm i\varepsilon \to \lambda \in$ MR : @ : *thesameholdsfor*(2.9). □

With this we can finally formulate our last hypothesis.

Hypothesis 2.6. *Suppose that for each compact subset* K *of* Δ *there is a constant* $c_K > 0$ *such that for all* $\lambda \in K$ *and for all* $u \in X$ *with* $\big(1 + EF_1^\pm(\lambda)\big)u = 0$, $R_1^\pm(\lambda)u$ *lies in* $\mathcal{H}_1$ *with*

$$\|R_1^\pm(\lambda)u\|_1 \leq c_K \|u\|_X\,. \tag{2.11}$$

Remark 2.7. Hypothesis 2.6, as it is presented here, is rather awkward, because it intertwines conditions on H_1 and conditions on H_2 via the operator E. Thus it is hard to check directly. One way to get around this difficulty is to replace it by the stronger condition (H).

(H) For each compact subset K of Δ there is a constant $c_K > 0$ such that (2.11) holds for all $u \in X$ with $R_1^+(\lambda)u = R_1^-(\lambda)u$.

That condition (H) does in fact imply Hypothesis 2.6 will follow from Lemma 3.1. When we work with this stronger assumption, however, we do not gain optimal results.

Let us illustrate this with the standard example, the Laplacian $-\Delta$ on MRd. There we can set $X = L^2_\varphi$ as defined in (4.1). Hypothesis 2.2 is satisfied for $\varphi = (1+|x|)^s$, $s > \frac{1}{2}$, but condition (H) only holds for $s > 1$ (cf. [5]). But we would like to use a space X that is as large as possible in order to keep Hypothesis 2.3 as weak as possible.

In Section 4 we will give a sufficient condition that guarantees Hypothesis 2.6 in the case of weighted L^2-spaces and is significantly weaker than condition (H). Namely, if we take $H_2 = -\Delta + V$ in the above example than if we choose $s > 1$ so that condition (H) is satisfied Hypothesis 2.3 only admits potentials which decay like $|x|^{-2-\gamma}$, $\gamma > 0$ as $|x|$ tends to infinity. The hypotheses from Section 4, on the other hand, allow to use $s > \frac{1}{2}$ and hence a decay as $|x|^{-1-\gamma}$ is sufficient.

Of course in the case of potential perturbations of the Laplacian better results than this are known in the literature (cf. [18] for the optimal conditions); our main goal, however, is the treatment of singular perturbations, not potential perturbations.

We note that, aside from Hypothesis 2.3, our basic hypotheses are quite similar to those in Ben-Artzi and Devinatz [5]. In particular Hypothesis 2.2 corresponds to Definition 2.1 in [5] and Hypothesis 2.6 to Assumption 3.2.

Our conditions are slightly better insofar as we need only continuity where [5] uses Hölder continuity, because we formulate the limiting absorption principle in terms of the resolvents rather than in terms of spectral measures.

The main change, however, is that we use the difference of powers of resolvents in Hypothesis 2.3. Using resolvents allows us to treat singular perturbations which could not be handled otherwise. (This trick was also used by Ben-Artzi, Dermenjian, and Guillot [4].) By considering their powers we can prove compactness in a wider range of cases. (It is not clear whether or not $E \in \mathcal{B}_\infty(X^*, X)$ implies $J_2^* R_2(a) J_2 - R_1(a) \in \mathcal{B}_\infty(X^*, X)$; we can neither prove the implication nor give a counterexample.)

3. Two-Space Theory for Limiting Absorption Principle

Throughout this section we will assume that the hypotheses from Section 2, namely Hypotheses 2.1, 2.2, 2.3, and 2.6 are satisfied.

The basic idea of our approach is to derive properties of $F_2(z)$ from those of $F_1(z)$. They are related via the following formula. For $z \in \mathbb{C} \setminus \{a\}$ with $0 < |\arg(z-a)| < \frac{2\pi}{m}$,

$$\begin{aligned} J_2^* J_2 F_1(z) - J_2^* F_2(z) J_2 &\\ = J_2^* F_2(z)\big(R_2(a)^m - (z-a)^{-m}\big) J_2 F_1(z) &\\ - J_2^* F_2(z) J_2 \big(R_1(a)^m - (z-a)^{-m}\big) F_1(z) &\\ = J_2^* F_2(z) J_2 \big[J_2^* R_2(a)^m J_2 - R_1(a)^m\big] F_1(z) & \end{aligned} \tag{3.1}$$

follows from the definition (2.5) of $F_k(z)$. With $E = J_2^* R_2(a)^m J_2 - R_1(a)^m$ this can be rewritten as

$$J_2^* J_2 F_1(z) = J_2^* F_2(z) J_2 \big[1 + E F_1(z)\big]. \tag{3.2}$$

This operator identity holds in $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_1)$ and by extension in $\mathcal{B}(X, X^*)$. By Hypothesis 2.2 $E \in \mathcal{B}_\infty(X^*, X)$ and hence $E F_1(z) \in \mathcal{B}_\infty(X, X)$. Thus we may use the Fredholm alternative: For fixed z either $\big(1+EF_1(z)\big)$ is invertible in $\mathcal{B}(X,X)$ or there exists an $u\in X$, $u\neq 0$ with $\big(1+EF_1(z)\big)u=0$.

Since we want to use this formula to study the limit $z=\lambda\pm i\varepsilon\to\lambda$ it will be sufficient to study the limit operator $1+EF_1^{\pm}(\lambda)$ for $\lambda\in\Delta$ and its invertibility in $\mathcal{B}(X,X)$. We note that one can in fact easily verify along the line of proof of Lemma 3.4 that for $0<|\arg(z-a)|<\frac{2\pi}{m}$ the operator $1+EF_1(z)$ is invertible, because otherwise H_2 would have a non-real eigenvalue.

Now we need to analyze the sets where $1+EF_1^{+}(\lambda)$ and where $1+EF_1^{-}(\lambda)$ are not invertible in more detail. First we note that these sets in fact coincide.

Lemma 3.1. *If for some $\lambda\in\Delta$ either $1+EF_1^{+}(\lambda)$ or $1+EF_1^{-}(\lambda)$ is not invertible in $\mathcal{B}(X,X)$ then there exists an $u\in X$, $u\neq 0$ such that $u=-EF_1^{+}(\lambda)u=-EF_1^{-}(\lambda)u$ and hence neither of the operators $1+EF_1^{\pm}(\lambda)$ is invertible.*

Proof. Without loss suppose $1+EF_1^{+}(\lambda)$ is not invertible. By the Fredholm alternative there is an $u\in X$, $u\neq 0$ with $u=-EF_1^{+}(\lambda)u$. Since E is self-adjoint we find that

$$\langle F_1^{+}(\lambda)u,u\rangle_1=-\langle F_1^{+}(\lambda)u,EF_1^{+}(\lambda)u\rangle_1 \tag{3.3}$$

is real. By inspection of (2.9) we see that each factor in this formula except for the first one $1+(\lambda-a)R_1^{\pm}(\lambda)$ is either self-adjoint or occurs together with its adjoint. Since $R_1^{+}(\lambda)^*=R_1^{-}(\lambda)$ we see $F_1^{+}(\lambda)^*=F_1^{-}(\lambda)$. Hence (3.3) implies that

$$\big\langle\big(F_1^{+}(\lambda)-F_1^{-}(\lambda)\big)u,u\big\rangle_1=0. \tag{3.4}$$

We claim that

$$\big\langle i\big(F_1^{+}(\lambda)-F_1^{-}(\lambda)\big)v,v\big\rangle_1\geq 0 \qquad \text{for all } v\in X. \tag{3.5}$$

This can be seen as follows. By (2.9)

$$\begin{aligned} F_1^{+}(\lambda)-F_1^{-}(\lambda) &= (\lambda-a)^{m+1}\big[-R_1^{+}(\lambda)+R_1^{-}(\lambda)\big] \\ &\quad\times\prod_{j=1}^{m-1}\Big\{(-w_j)\big[1+w_j(\lambda-a)R_1(a+w_j(\lambda-a))\big]\Big\} \\ &=\lim_{\varepsilon\downarrow 0}(\lambda-a)^{m+1}\big[-R_1(\lambda+i\varepsilon)+R_1(\lambda-i\varepsilon)\big] \\ &\quad\times\prod_{j=1}^{m-1}\Big\{(-w_j)\big[1+w_j(\lambda-a)R_1(a+w_j(\lambda-a))\big]\Big\}. \end{aligned} \tag{3.6}$$

Since all factors on the right hand side of (3.6) commute we only have to establish positivity for individual factors, (3.5) will then hold by passing to the limit. The first factor on the right hand side of (3.6) is

$$\begin{aligned} &-R_1(\lambda+i\varepsilon)+R_1(\lambda-i\varepsilon) \\ &=-2i\varepsilon R_1(\lambda+i\varepsilon)R_1(\lambda-i\varepsilon)=-2i\varepsilon\int \end{aligned} \tag{3.7}$$

MR $:\overline{@:E_1(d\nu)(\nu-\lambda)^2+\varepsilon^2\ ,}$

$E_1(\cdot)$ denoting the spectral measure associated with H_1. The integral on the right hand side of (3.7) is a positive operator for all $\varepsilon>0$.

If m is even $w_{m/2} = -1$, the corresponding factor in (3.6) is

$$1 - (\lambda - a)R_1(2a - \lambda). \tag{3.7}$$

This operator is positive, because $\|(\lambda - a)R_1(2a - \lambda)\|_1 \leq \frac{\lambda - a}{\lambda - 2a + c_0} < 1$.

For each of the remaining factors containing some w_j there is a corresponding adjoint factor containing $\overline{w_j}$. The product of an operator with its adjoint is positive, thus we indeed find

$$\begin{aligned} &\Big\langle i\big[-R_1(\lambda + i\varepsilon) + R_1(\lambda - i\varepsilon)\big] \\ &\quad \times \prod_{j=1}^{m-1} \Big\{(-w_j)\big[1 + w_j(\lambda - a)R_1(a + w_j(\lambda - a))\big]\Big\} v, v\Big\rangle_1 \geq 0 \end{aligned} \tag{3.8}$$

for all $v \in \mathcal{H}_1$ and a fortiori for all $v \in X$. Hence passing to the limit $\varepsilon \downarrow 0$ (3.5) holds. Together with (3.4) this implies

$$F_1^+(\lambda)u = F_1^-(\lambda))u, \tag{3.9}$$

proving the lemma. □

Now, as (3.2) suggests, we find that indeed the limit $F_2^\pm(\lambda)$ is well defined when $1 + EF_1^\pm(\lambda)$ is invertible in $\mathcal{B}(X, X)$. More precisely, we obtain the following result.

Lemma 3.2. *If $1 + EF_1^\pm(\lambda_0)$ is invertible in $\mathcal{B}(X, X)$ for some $\lambda_0 \in \Delta$ then there exists a neighborhood $U \subset \Delta$ of λ_0 such that for all $\lambda \in U$*

$$J_2^* R_2^\pm(\lambda) J_2 = \lim_{\varepsilon \downarrow 0} J_2^* R_2(\lambda \pm i\varepsilon) J_2 \tag{3.10}$$

exists and is continuous with respect to λ in $\mathcal{B}(X, X^*)$.

Proof. Let $z = \lambda_0 + \delta \pm i\varepsilon$, $\delta \in$ MR : @ :,$\varepsilon > 0$, be close to λ_0. Then

$$1 + EF_1(z) = \big(1 + EF_1^\pm(\lambda_0)\big)\Big[1 + \big(1 + EF_1^\pm(\lambda_0)\big)^{-1} E\big(F_1(z) - F_1^\pm(\lambda_0)\big)\Big]. \tag{3.10}$$

The continuity of $F_1(z)$ respectively $F_1^\pm(\lambda)$ in $\mathcal{B}(X, X^*)$ allows us to pick ε, δ small enough so that $\|\big(1 + EF_1^\pm(\lambda_0)\big)^{-1} E\big(F_1(z) - F_1^\pm(\lambda_0)\big)\|_{\mathcal{B}(X,X)} < 1$ and hence the expression in (3.10) is invertible with continuous inverse. That is, for $\lambda \in U$ where U is some small neighborhood of λ_0,

$$\big[1 + EF_1^\pm(\lambda)\big]^{-1} = \lim_{\varepsilon \downarrow 0} \big[1 + EF_1(\lambda \pm i\varepsilon)\big]^{-1} \in \mathcal{B}(X, X) \tag{3.11}$$

exists and is continuous with respect to λ. Hence we can conclude from (3.2) that

(3.12)
$$J_2^* F_2^\pm(\lambda) J_2 := \lim_{\varepsilon \downarrow 0} J_2^* F_2(\lambda \pm i\varepsilon) J_2 = \lim_{\varepsilon \downarrow 0} J_2^* J_2 F_1(\lambda \pm i\varepsilon)\big[1 + EF_1(\lambda \pm i\varepsilon)\big]^{-1}$$

exists and is continuous with respect to λ in $\mathcal{B}(X, X^*)$ for $\lambda \in U$. The claim then follows from Lemma 2.5 □

If $J_2^* R_2^\pm(\lambda) J_2$ exists in $\mathcal{B}(X, X^*)$ in some neighborhood $U \subset \Delta$ of λ_0 we know (cf. [20], Thm. XIII.19 and XIII.20) that the spectrum $\sigma(H_2)$ of H_2 is purely absolutely continuous in U. Hence we need to analyze the set where $1 + EF_1^\pm(\lambda)$ is not invertible in more detail. We begin with a simple observation.

Lemma 3.3. *If $\lambda \in \Delta$ is an eigenvalue of H_2 with eigenvector ψ then $u := -EJ_2^*\psi \in X$ satisfies $\big(1 + EF_1^\pm(\lambda)\big)u = 0$, $u \neq 0$.*

Proof. $u \in X$ is clear from the definition, since $E \in \mathcal{B}_\infty(X^*, X)$. Next we want to establish $u \neq 0$. We calculate

$$u = -\big[J_2^* R_2(a)^m J_2 - R_1(a)^m\big] J_2^* \psi = -\big[(\lambda - a)^{-m} - R_1(a)^m\big] J_2^* \psi, \tag{3.13}$$

where we have used the fact that $J_2 J_2^* \psi = \psi$ is an eigenfunction of H_2. Apply the factorization (2.4) and use the fact that all its factors commute. The expression in (3.13) cannot vanish unless

$$\big[R_1(a) - w_j(\lambda - a)^{-1}\big] J_2^* \psi = (\lambda - a)^{-1} \big[\lambda - a - w_j(H_1 - a)\big] R_1(a) J_2^* \psi = 0 \tag{3.14}$$

for some $j \in \{0, \dots, m-1\}$. But (3.14) would imply that $\zeta = a + \frac{1}{w_j}(\lambda - a)$ is an eigenvalue of H_1. For $j \neq 0$ this is impossible because then $\zeta \in \mathbb{C} \setminus$ MR : @ :, forj=0$\zeta = \lambda \in \Delta$ cannot be an eigenvalue of H_1 by (2.2).

It remains to verify $\big(1 + EF_1^{\pm}(\lambda)\big)u = 0$. We calculate

$$\begin{aligned} EF_1(\lambda \pm i\varepsilon)u &= -EF_1(\lambda \pm i\varepsilon)\big[(\lambda - a)^{-m} - R_1(a)^m\big] J_2^* \psi \\ &= E\big[R_1(a)^m - (\lambda \pm i\varepsilon - a)^{-m}\big]^{-1}\big[R_1(a)^m - (\lambda - a)^{-m}\big] J_2^* \psi \\ &= EJ_2^* \psi + EF_1(\lambda \pm i\varepsilon)\big[(\lambda \pm i\varepsilon - a)^{-m} - (\lambda - a)^{-m}\big] J_2^* \psi \,. \end{aligned} \tag{3.14}$$

In order to prove our claim we thus have to verify $EF_1(\lambda \pm i\varepsilon)\big[(\lambda \pm i\varepsilon - a)^{-m} - (\lambda - a)^{-m}\big] J_2^* \psi \underset{\varepsilon \downarrow 0}{\longrightarrow} 0$. By Lemma 2.5

$$F_1(\lambda \pm i\varepsilon)\big[(\lambda \pm i\varepsilon - a)^{-m} - (\lambda - a)^{-m}\big] \underset{\varepsilon \downarrow 0}{\longrightarrow} 0 \text{ in } \mathcal{B}(X, X^*) \tag{3.15}$$

since $(\lambda \pm i\varepsilon - a)^{-m} - (\lambda - a)^{-m} \underset{\varepsilon \downarrow 0}{\longrightarrow} 0$. With the factorization (2.5) and a similar factorization for $\big[(\lambda \pm i\varepsilon - a)^{-m} - (\lambda - a)^{-m}\big]$ we calculate

$$\begin{aligned} &F_1(\lambda \pm i\varepsilon)\big[(\lambda \pm i\varepsilon - a)^{-m} - (\lambda - a)^{-m}\big] \\ &\quad = \prod_{j=0}^{m-1} \Big\{(-w_j)\big[1 + w_j(\lambda \pm i\varepsilon - a) R_1\big(a + w_j(\lambda \pm i\varepsilon - a)\big)\big] \\ &\qquad\qquad \times \big(\lambda - a - w_j(\lambda \pm i\varepsilon - a)\big)(\lambda - a)^{-1}\Big\}. \end{aligned} \tag{3.16}$$

This operator remains bounded in $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_1)$ as ε tends to 0, because the only factor that could possibly be unbounded in this limit, namely the one for $j = 0$, is bounded because of

$$\big\| R_1(\lambda \pm i\varepsilon)(\mp i\varepsilon) \big\|_{\mathcal{B}(\mathcal{H}_1, \mathcal{H}_1)} \leq 1. \tag{3.17}$$

Boundedness in $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_1)$ and convergence on the dense subset $X \subset \mathcal{H}_1$ imply

$$F_1(\lambda \pm i\varepsilon)\big[(\lambda \pm i\varepsilon - a)^{-m} - (\lambda - a)^{-m}\big] \underset{\varepsilon \downarrow 0}{\overset{w}{\longrightarrow}} 0 \text{ in } \mathcal{B}(\mathcal{H}_1, \mathcal{H}_1) \tag{3.18}$$

($\overset{w}{\longrightarrow}$ denoting weak convergence). Thus (3.14) yields

$$EF_1^{\pm}(\lambda)u = \lim_{\varepsilon \downarrow 0} EF_1(\lambda \pm i\varepsilon)u = EJ_2^* \psi = -u. \tag{3.19}$$

$\square$

Using Hypothesis 2.6 we can establish the converse of the previous lemma.

Lemma 3.4. *Suppose $\big(1+EF_1^{\pm}(\lambda)\big)$ is* not *invertible in $\mathcal{B}(X,X)$ for some $\lambda \in \Delta$; let $u \in X$, $u \neq 0$ satisfy $u = -EF_1^{\pm}(\lambda)u$. Then λ is an eigenvalue of H_2 with eigenvector $\psi = R_2(a)J_2F_1^{\pm}(\lambda)u \in \mathcal{D}(H_2)$, $\psi \neq 0$.*

Proof. By Hypothesis 2.6 $R_1^{\pm}(\lambda)u \in \mathcal{H}_1$ and hence by (2.9) $v = F_1^{\pm}(\lambda)u \in \mathcal{H}_1$. Hence in the space X^*

$$\begin{aligned}(3.20)\qquad \big(R_1(a)^m - (\lambda-a)^{-m}\big)v &= \lim_{\varepsilon\downarrow 0}\big(R_1(a)^m - (\lambda-a)^{-m}\big)F_1(\lambda\pm i\varepsilon)u\\ &= u + \lim_{\varepsilon\downarrow 0}\big((\lambda\pm i\varepsilon-a)^{-m} - (\lambda-a)^{-m}\big)F_1(\lambda\pm i\varepsilon)u = u.\end{aligned}$$

Equation (3.20) implies $v \neq 0$ and also allows us to rewrite $\big(1+EF_1^{\pm}(\lambda)\big)u = 0$ as

$$(3.21)\qquad \big[\big(R_1(a)^m - (\lambda-a)^{-m}\big) + E\big]v = \big[J_2^*R_2(a)^mJ_2 - (\lambda-a)^{-m}\big]v = 0.$$

Multiplying (3.21) by the projection $1 - J_2^*J_2$ we see $(1 - J_2^*J_2)v = 0$, hence $J_2^*J_2v \neq 0$ and with (2.4) equation (3.21) becomes

$$\begin{aligned}(3.22)\qquad & J_2^*\big[R_2(a)^m - (\lambda-a)^{-m}\big]J_2v\\ &= J_2^*\prod_{j=0}^{m-1}\Big\{(\lambda-a)^{-1}\big[\lambda-a-w_j(H_2-a)\big]R_2(a)\Big\}J_2v = 0.\end{aligned}$$

Since $a + \frac{1}{w_j}(\lambda-a) \in \mathbb{C}\setminus[a,\infty) \subset \rho(H_2)$ for $j = 1,\dots,m-1$ this implies that the factor for $j=0$ satisfies

$$(3.23)\qquad (\lambda - H_2)R_2(a)J_2v = 0.$$

□

Lemmas 3.3 and 3.4 yield our main abstract theorem.

Theorem 3.5. *Suppose H_1 and H_2 satisfy Hypotheses 2.1, 2.2, 2.3, and 2.6. Then $\sigma(H_2)\cap\Delta$ is, except for possibly a discrete set $\sigma_p(H_2)\cap\Delta$ of eigenvalues of finite multiplicity, purely absolutely continuous. For all $\lambda \in \Delta\setminus\sigma_p(H_2)$*

$$(3.24)\qquad J_2^*R_2^{\pm}(\lambda)J_2 := \lim J_2^*R_2(\lambda\pm i\varepsilon)J_2$$

exists and is continuous in $\mathcal{B}(X,X^*)$.

Proof. (i) We begin by establishing that the set of eigenvalues of H_2, $\sigma_p(H_2)$ is discrete in Δ. Suppose not. Then there exists a sequence $(\lambda_k)_{k\in\mathbb{N}} \subset \big(\Delta\cap\sigma_p(H_2)\big)$ with $\lambda_k \xrightarrow[k\to\infty]{} \lambda \in \Delta$; the limit value λ may or may not be an eigenvalue. By Lemma 3.3 for each λ_k there exists a vector $u_k \in X$ with

$$(3.25)\qquad \big(1+EF_1^{\pm}(\lambda_k)\big)u_k = 0, \qquad k\in\mathbb{N};$$

without loss suppose $\|u_k\|_X = 1$. Since the sequence $(u_k)_{k\in\mathbb{N}}$ is bounded there exists a subsequence, without loss the sequence itself, such that $u_k - u_l \xrightarrow[k,l\to\infty]{w} 0$ in X ($\xrightarrow{w}$ denoting weak convergence). From (3.25) we infer

$$\begin{aligned}(3.26)\qquad u_k - u_l = &-EF_1^{\pm}(\lambda)(u_k-u_l)\\ &- E\big(F_1^{\pm}(\lambda_k) - F_1^{\pm}(\lambda)\big)u_k + E\big(F_1^{\pm}(\lambda_l) - F_1^{\pm}(\lambda)\big)u_l\,.\end{aligned}$$

the first term on the right hand side of (3.26) converges strongly to 0 in X, because $EF_1^{\pm}(\lambda) \in \mathcal{B}_\infty(X,X)$. The remaining two terms converge because $F_1^{\pm}(\lambda)$ is continuous in λ and $(u_k)_{k\in\mathbb{N}}$ is bounded. Hence we have in fact strong convergence,

$u_k - u_l \xrightarrow[k,l\to\infty]{} 0$ in X. Set $u = \lim_{k\in\mathbb{N}} u_k$. From $\|u_k\|_X = 1$ we conclude $\|u\|_X = 1$; from (3.25) and the continuity of $F_1^{\pm}(\cdot)$ that

$$\big(1 + EF_1^{\pm}(\lambda)\big)u = 0. \tag{3.27}$$

By Lemma 3.4 for $k \in \mathbb{N}$ there exist vectors $\psi_k, \psi \in \mathcal{D}(H_2)$ with $(H_2 - \lambda_k)\psi_k = 0$, $(H_2 - \lambda)\psi = 0$. Together with Hypothesis 2.6 this lemma also yields

$$\|\psi_k\|_2 \le \|R_2(a)\|_{\mathcal{B}(\mathcal{H}_2,\mathcal{H}_2)} \|J_2\|_{\mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)} \|F_1^{\pm}(\lambda_k)u_k\|_1 \le c \tag{3.28}$$

with $c > 0$. Since each ψ_k is eigenvector to a different eigenvalue λ_k it is orthogonal to all other eigenvectors ψ_l, $l \ne k$. Bessel's inequality then implies that $\widetilde{\psi}_k = \dfrac{\psi_k}{\|\psi_k\|_2} \xrightarrow[k\to\infty]{w} 0$ in $\mathcal{H}_2$ and hence by (3.28) $\psi_k \xrightarrow[k\to\infty]{w} 0$ in $\mathcal{H}_2$ and a fortiori

$$J_2^* \psi_k \xrightarrow[k\to\infty]{w} 0 \tag{3.29}$$

in X^*. On the other hand $u_k \to u$ and the continuity of $F_1^{\pm}(\cdot)$ imply

$$J_2^*\psi_k = J_2^* R_2(a) J_2 F_1^{\pm}(\lambda_k) u_k \xrightarrow[k\to\infty]{} J_2^*\psi \quad \text{in } X^*. \tag{3.30}$$

Since $\|u\|_X = 1$ we know $\psi \ne 0$ by Lemma 3.4 in contradiction to (3.29).

(ii) A similar argument may be used to show that each eigenvalue can have only finite multiplicity. Suppose not. Then there exists a sequence $(\psi_k)_{k\in\mathbb{N}}$ of orthogonal eigenvectors. We can define u_k by Lemma 3.3, norm by $\|u_k\|_X = 1$ and proceed as in (i).

(iii) If $\lambda_0 \in \Delta$ is not an eigenvalue of H_2 then by Lemma 3.4 $\big(1 + EF_1^{\pm}(\lambda_0)\big)$ is invertible in $\mathcal{B}(X,X)$ and by Lemma 3.2 the limit $J_2^* R_2^{\pm}(\lambda) J_2$ exists for all λ in some neighborhood $U \subset \Delta$ of λ_0. By the general theory (cf., e.g., [20], Thm. XIII.19, XIII.20) this establishes the claim. □

Remark 3.6. We have established that the point spectrum $\sigma_p(H_2)$ of H_2 does not accumulate in Δ. Of course it might still accumulate at the boundary of Δ. We can, however, exclude this behavior, if Hypothesis 2.2 holds for some Δ with $\sigma(H_2) \subset \Delta$.

4. A sufficient condition for Hypothesis 2.6 for weighted L^2-spaces

In most applications of the limiting absorption principle some weighted L^2-space is used for X. In this setup we can replace Hypothesis 2.6 by a different Hypothesis 4.1 if we strengthen the assumption on E, i.e., Hypothesis 2.3 slightly. The main advantage of this approach is that, while Hypothesis 2.6 intertwines conditions on H_1 and conditions on H_2 in a very implicit way and hence is hard to check, this new hypothesis will impose separate conditions which can be verified much more easily.

We work in the following setting. Let M be a measure space with Borel field $\mathcal{M}$, $\Sigma \in \mathcal{M}$ a measurable set. We define $\mathcal{H}_1 = L^2 := L^2(M)$, $\mathcal{H}_2 = L^2(\Sigma)$ and

$$X = L^2_\varphi := \left\{u \in L^2(E) : \ \|u\|_\varphi := \|\varphi u\| < \infty\right\}. \tag{4.1}$$

Here $\varphi : \ M \to [1,\infty)$ is a measurable function. (In this and the following section norm and scalar product without a subscript denote the usual L^2-norm and scalar product.) $L^2_{\varphi^s}$, $s \ge 0$ is defined by a formula analogous to (4.1), $L^2_{\varphi^{-s}}$ as its dual $(L^2_{\varphi^s})^*$; in particular $L^2_{\varphi^{-1}} = (L^2_\varphi)^*$. In this setup we can replace Hypotheses 2.2, 2.3, and 2.6 by

Hypothesis 4.1. *Suppose there is an open set $\Delta \subset MR$*
Hypothesis 4.1. : @ : $such that for all \lambda \in \Delta$,

$$R_1^{\pm}(\lambda) := \lim_{\varepsilon \downarrow 0} R_1(\lambda \pm i\varepsilon) \tag{4.2}$$

exists and is continuous in $\mathcal{B}(L^2_{\varphi}, L^2_{\varphi^{-1}})$. Furthermore, assume that for each compact subset K of Δ there exists a constant $c_K > 0$ such that for all $\lambda \in K$ and all $u \in L^2_{\varphi^2}$ with $R_1^+(\lambda)u = R_1^-(\lambda)u$

$$\|R_1^{\pm}(\lambda)u\| \leq c_K \|u\|_{\varphi^2}\,. \tag{4.3}$$

and

Hypothesis 4.2. *Suppose there exists an $a < c_0$, $a < \inf \Delta$ and an $m \in \mathbb{N}$ such that*

$$E := J_2^* R_2(a)^m J_2 - R_1(a)^m \in \mathcal{B}_\infty(L^2, L^2_{\varphi^{2+\gamma}}) \tag{4.4}$$

for some $\gamma > 0$.

Hypothesis 4.2 is, as we will see in the proof of Proposition 4.4 a slight strengthening of Hypothesis 2.3 formulated in the setting of weighted L^2-spaces. Equation (4.2) is a reformulation of Hypothesis 2.2. But what is the difference between (4.3) and Hypothesis 2.6?

On one hand in Hypothesis 2.6 some condition is imposed on those u for which $\big(1 + EF_1^{\pm}(\lambda)\big)u = 0$, while in (4.3) we impose a condition on the larger class of all those u for which $R_1^+(\lambda)u = R_1^-(\lambda)u$. This larger class has the advantage of being far easier to describe.

On the other hand the condition that is imposed in Hypothesis 2.6 is stronger: There we require that $\|R_1^{\pm}(\lambda)u\|$ can be estimated by the L^2_{φ}-norm of u, in (4.3) we merely require that it can be estimated by the (larger) $L^2_{\varphi^2}$-norm. In a sense (4.3) involves the same power of φ as (4.2) while Hypothesis 2.6 involves a smaller power: (4.2) implies $\langle R_1^{\pm}(\lambda)u, v\rangle \leq c_K\|\varphi u\|\,\|\varphi v\|$; (4.3) is of the form $\langle R_1^{\pm}(\lambda)u, v\rangle \leq c_K\|\varphi^2 u\|\,\|v\|$. Hypothesis 2.6, on the other hand, is of the form $\langle R_1^{\pm}(\lambda)u, v\rangle \leq c_K\|\varphi u\|\,\|v\|$.

In Section 5 we will discuss Hypothesis 4.2 in more detail for the case of potential and domain perturbations. We will give a sufficient condition for this Hypothesis in terms of the potential and the set on which we introduce additional boundary conditions. Since we will always consider Hypothesis 4.1 as a given let us mention some operators for which it holds.

Remark 4.3. (i) If $H_1 = -\triangle$ is the Laplacian on MRd then Hypothesis 4.1 holds with $\Delta = (0, \infty)$ and $\varphi = (1 + |x|)^s$, $s > \frac{1}{2}$, see [5].

(ii) Case (i) can be extended to functions of the Laplacian. If $f \in C^2(MR+)$, $f(t) > Ct^{\gamma/2}$ with some $C, \gamma > 0$, then $H_1 = f(-\triangle)$ satisfies the hypothesis with $\Delta = (f(0), \infty)$, $\varphi = \big(1 + |x|\big)^s$, $s > \frac{1}{2}$, see [6]. In particular this includes the relativistic Hamiltonian $\sqrt{-\triangle + 1} - 1$.

(iii) If H_1 is an elliptic operators with constant coefficients the hypothesis holds as well, see [5].

Proposition 4.4. *Hypotheses 4.1 and 4.2 imply Hypotheses 2.2, 2.3, and 2.6.*

Proof. We can rewrite (4.4) as $\varphi^{2+\gamma}E \in \mathcal{B}_\infty(L^2, L^2)$. Then by Schauder's theorem (cf., e.g., Kato [14] Thm. III.4.10) $E\varphi^{2+\gamma} = (\varphi^{2+\gamma}E)^* \in \mathcal{B}_\infty(L^2, L^2)$. Since $\varphi^{2+\gamma}E$

is a compact operator in a Hilbert space we can approximate it by operators of finite rank (cf., e. g., Pietsch [19] Section 10.2), say $\varphi^{2+\gamma}E_k$. Now we use the interpolation theory as presented by Lions and Magenes [16]. By the definition of the space L^2_φ (cf. (4.1)) the interpolation space according to this approach (cf. [16] Chapter 1, Sections 3–5) is $[L^2_\varphi, L^2]_\theta = \{u \in L^2 : \|\varphi^{1-\theta}u\| < \infty\} = L^2_{\varphi^{1-\theta}}$. Then by the interpolation theorem (cf. [16] Chapter I, Thm. 5.1) $\|E - E_k\|_{\mathcal{B}(L^2_{\varphi^{-\theta}}, L^2_{\varphi^{2+\gamma-\theta}})} \xrightarrow[k\to\infty]{} 0$ for all $\theta \in [0, 2+\gamma]$. Hence

$$E \in \mathcal{B}_\infty(L^2_{\varphi^{-\theta}}, L^2_{\varphi^{2+\gamma-\theta}}) \quad \text{for all } \theta \in [0, 2+\gamma]. \tag{4.5}$$

A fortiori this also means $E \in \mathcal{B}_\infty(L^2_{\varphi^{-1}}, L^2_\varphi)$, i. e., Hypothesis 2.3 is satisfied. Hypothesis 2.2 is equation (4.2), hence it only remains to verify Hypothesis 2.6.

We define the spaces

$$\widetilde{L}_s(\lambda) := \{u \in L^2_{\varphi^s} : R_1^+(\lambda)u = R_1^-(\lambda)u\}, \quad s \geq 1. \tag{4.6}$$

Since $R_1^\pm(\lambda) \in \mathcal{B}(L^2_\varphi, L^2_{\varphi^{-1}})$ by (4.2) we know that for $s \geq 1$ the space $\widetilde{L}_s(\lambda)$ is a closed subspace of $L^2_{\varphi^s}$. Further we note $\widetilde{L}_s(\lambda) = \widetilde{L}_1(\lambda) \cap L^2_{\varphi^s}$, i. e., we can consider $\widetilde{L}_s(\lambda)$, $s \geq 1$ as a subspace of $\widetilde{L}_1(\lambda)$. Again consider the interpolation spaces $[\widetilde{L}_1(\lambda), \widetilde{L}_2(\lambda)]_\theta$ defined according to [16]. Here $\widetilde{L}_2(\lambda)$ can be considered as $\widetilde{L}_2(\lambda) = \{u \in \widetilde{L}_1(\lambda) : \|\varphi u\|_{\widetilde{L}_1(\lambda)} < \infty\}$. So, for $0 \leq \theta \leq 1$,

$$[\widetilde{L}_1(\lambda), \widetilde{L}_2(\lambda)]_\theta = \{u \in \widetilde{L}_1(\lambda) : \|\varphi^\theta u\|_{\widetilde{L}_1(\lambda)} < \infty\} = \widetilde{L}_{1+\theta}(\lambda). \tag{4.7}$$

We know from (4.2) and (4.3) that $R_1^\pm(\lambda) \in \mathcal{B}(\widetilde{L}_1(\lambda), L^2_{\varphi^{-1}}) \cap \mathcal{B}(\widetilde{L}_2(\lambda), L^2)$ for all $\lambda \in \Delta$ and that for each compact subset K of Δ we can find a constant $c_K > 0$ such that for all $\lambda \in K$

$$\|R_1^\pm(\lambda)\|_{\mathcal{B}(\widetilde{L}_1(\lambda), L^2_{\varphi^{-1}})} \leq c_K, \qquad \|R_1^\pm(\lambda)\|_{\mathcal{B}(\widetilde{L}_2(\lambda), L^2)} \leq c_K. \tag{4.8}$$

Hence by the interpolation theorem ([16] Thm. 5.1) $R_1^\pm(\lambda) \in \mathcal{B}(\widetilde{L}_{1+\theta}(\lambda), L^2_{\varphi^{-1+\theta}})$ for all $\theta \in [0,1]$. Upon closer inspection of the proof of this theorem we see that we know more. The space $[\widetilde{L}_1(\lambda), \widetilde{L}_2(\lambda)]_\theta$ is the space of all traces of functions $w : \to MR \to \widetilde{L}_2(\lambda)$ in the space

$$W_1 = \{w \in L^2(;MR;\widetilde{L}_2(\lambda)) : |\tau|^{1/2\theta}\widehat{w} \in L^2(;MR;\widetilde{L}_1(\lambda)\}. \tag{4.9}$$

Here $\widehat{w}$ denotes the Fourier transform of w. (This is a function of τ, $\tau \in$ MR $: @ : .)Thenormon$W$_1$ is defined as the sum of the norm of w in $L^2(;MR;\widetilde{L}_2(\lambda))$ and the norm of $|\tau|^{1/2\theta}\widehat{w}$ in $L^2(;MR;\widetilde{L}_1(\lambda))$; for details we refer to [16], Chapter 1, Section 4. Then ([16] Theorem 4.2) if $u \in [\widetilde{L}_1(\lambda), \widetilde{L}_2(\lambda)]_\theta = \widetilde{L}_{1+\theta}(\lambda)$ then $u = w(0)$ with some $w \in W_1$ and

$$\frac{1}{c}\|w\|_{W_1} \leq \|u\|_{\widetilde{L}_{1+\theta}(\lambda)} \leq c\|w\|_{W_1} \tag{4.9}$$

with some $c > 0$. Moreover, one verifies that this constant c depends only on θ and is independent of the spaces between which one interpolates. Similarly $[L^2_{\varphi^{-1}}, L^2]_\theta = L^2_{\varphi^{-1+\theta}}$ is the space of traces of functions in a space W_2 and an estimate analogous to (4.9) holds. Define the action of $R_1^\pm(\lambda)$ on $w \in W_1$ by $\big(R_1^\pm(\lambda)w\big)(t) = R_1^\pm(\lambda)(w(t))$.

Then the interpolation theorem together with (4.8) and (4.9) yields

$$\begin{aligned}
\|R_1^{\pm}(\lambda)u\|_{L^2_{\varphi^{-1+\theta}}} &\le c\|R_1^{\pm}(\lambda)w\|_{W_2} \\
&\le c\max\{\|R_1^{\pm}(\lambda)\|_{\mathcal{B}(\widetilde{L}_1(\lambda),L^2_{\varphi^{-1}})}, \|R_1^{\pm}(\lambda)\|_{\mathcal{B}(\widetilde{L}_2(\lambda),L^2)}\}\|w\|_{W_1} \\
&\le c^2 c_K \|u\|_{\widetilde{L}_{1+\theta}(\lambda)}.
\end{aligned} \tag{4.10}$$

That is,

$$\|R_1^{\pm}(\lambda)\|_{\mathcal{B}(\widetilde{L}_{1+\theta}(\lambda),L^2_{\varphi^{-1+\theta}})} \le c(K,\theta) \tag{4.11}$$

with $c(K,\theta)$ depending on K and θ only.

Now suppose $u \in L^2_\varphi$ with $\big(1+EF_1^{\pm}(\lambda)\big)u = 0$. By Lemma 3.1 $u \in \widetilde{L}_1(\lambda)$, hence $R_1^{\pm}(\lambda)u \in L^2_{\varphi^{-1}}$ and by Lemma 2.5 also $F_1^{\pm}(\lambda)u \in L^2_{\varphi^{-1}}$. Hence from (4.5) with $\theta = 1$ we get $u = -EF_1^{\pm}(\lambda)u \in L^2_{\varphi^{1+\gamma}}$. Moreover, one sees that $\|u\|_{\varphi^{1+\gamma}} \le c_K\|u\|_\varphi$ with c_K some constant independent of λ for $\lambda \in K$.

We iterate this procedure: By Lemma 3.1 we now know $u \in \widetilde{L}_{1+\gamma}(\lambda)$, hence $R_1^{\pm}(\lambda)u \in L^2_{\varphi^{-1+\gamma}}$. While Lemma 2.5 does not apply directly any more we still have the factorization (2.9). By Hypothesis 2.1 and duality $R_1(z) \in \mathcal{B}(L^2,L^2) \cap \mathcal{B}(L^2_{\varphi^{-1}}, L^2_{\varphi^{-1}})$ for $z \in \rho(H_1)$. Thus we can interpolate to conclude $R_1(z) \in \mathcal{B}(L^2_{\varphi^{-\theta}}, L^2_{\varphi^{-\theta}})$ for $\theta \in [0,1]$. Again the norm can be bounded by a constant for z in a compact subset of $\rho(H_1)$. Hence $F_1^{\pm}(\lambda)u \in L^2_{\varphi^{-1+\gamma}}$, by (4.5) $u = -EF_1^{\pm}(\lambda)u \in L^2_{\varphi^{1+2\gamma}}$. After n steps we find $u \in L^2_{\varphi^{1+n\gamma}}$ with $\|u\|_{\varphi^{1+n\gamma}} \le c_{n,K}\|u\|_\varphi$ with some $c_{n,K} > 0$. Pick n with $1 \le n\gamma < 1+\gamma$ and we have $\|u\|_{\varphi^2} \le \|u\|_{\varphi^{1+n\gamma}} \le c_{n,K}\|u\|_\varphi$. Since $u \in \widetilde{L}_2(\lambda)$ by Lemma 3.1 (2.11) follows from (4.3). ◻

5. Application to Feller operators

In this section we want to give an outline how the abstract theory we have developed can be applied to singular as well as regular perturbations of Feller operators, i. e. operators defined as generators of strong Markov processes with the Feller property. For a detailed treatment of such operators we refer to Demuth and van Casteren [8], [9], [10], [11], and van Casteren [22]. See also Fukushima, Oshima, and Takeda [12], Ma and Röckner [17].

Let M be a locally compact second countable Hausdorff space with Borel field $\mathcal{M}$, dx a non-negative Radon measure on M. We define the operators H_1 and H_2 through their semigroups which are in turn defined via strong Markov processes.

Let $p:\ MR + \times M \times M \to MR+$ be a continuous function which satisfies the following assumptions (BASSA, cf. [8], [9], [10]).

(i) For all $s,t > 0$, $x,y \in M$,

$$\int_M p(t,x,y)dy \le 1, \quad \int_M p(s,x,z)p(t,z,y)dz = p(s+t,x,y), \quad p(t,x,y) = p(t,y,x). \tag{5.1}$$

(ii) If $f \in C_\infty(M)$, the space of continuous functions which vanish at infinity, then

$$\lim_{t\downarrow 0}\int_M p(t,x,y)f(y)dy = f(x), \quad \int (p(t,\cdot,y)f(y)dy \in C_\infty(M). \tag{5.2}$$

$\big(p(t,\cdot,\cdot)\big)_{t>0}$ is a Markovian transition function, see [12] Section 1.4 or [17], Section II.4. By the continuity condition (5.2) the corresponding operator semigroup is strongly continuous (cf. [12] Lemma 1.4.3) and we can define H_1 as the generator of the semigroup

$$(e^{-tH_1}f)(x) = \int_M p(t,x,y)f(y)dy. \tag{5.3}$$

H_1 is self-adjoint and positive by Hille-Yosida.

We need one additional assumption on p, namely, that this semigroup has finite spectral dimension, i. e., that there exist constants $c_S, \alpha_S, d_S > 0$ such that

$$\sup_{x,y\in M} p(t,x,y) = \big\|e^{-tH_1}\big\|_{\mathcal{B}(L^1,L^\infty)} \le c_S e^{\alpha_s t} t^{-d_s/2}\,. \tag{5.4}$$

By Kolmogorov's theorem p is the transition function of a strong Markov process. Let $E_x[\cdot]$ be the expectation for this process (starting point x), X_s the position at time s. Then we can perturb H_1 by a Kato Feller potential V, i. e., a potential satisfying

$$\lim_{t\downarrow 0}\sup_{x\in M}\int_0^t E_x\big[V_-(X_s)+1_B V_+(X_s)\big]ds = 0 \tag{5.5}$$

for all $B\in\mathcal{M}$ compact, where 1_B is the characteristic function of B. We define an operator H_V via the Feynman-Kac formula as the generator of

$$\big(e^{-tH_V}f\big)(x) = E_x\big[e^{-\int_0^t V(X_s)ds}f(X_t)\big]. \tag{5.6}$$

(Cf. [8] Thm. 2.5, [22] Thm. 2.8, [9], [10].) If $\mathcal{D}(H_1)\cap\mathcal{D}(V)$ is a form core for H_1+V (defined as the operator sum on $\mathcal{D}(H_1)\cap\mathcal{D}(V)$) our operator H_V coincides with the Friedrich's extension of H_1+V; confer [8] Chapter II or [10] for further details. In the following when we write an operator sum of the type H_1+V we will understand this to be defined through an equation analogous to (5.6).

Then we perturb this operator H_V by Dirichlet boundary conditions. Let $\Sigma\in\mathcal{M}$ be an open subset of M, S the first penetration time of the set $\Gamma = M\setminus\Sigma$,

$$S = \inf\Big\{t>0:\ \int_0^t 1_\Gamma(X_s)ds > 0\Big\}. \tag{5.7}$$

We define H_2 by

$$\big(e^{-tH_2}f\big)(x) = E_x\big[e^{-\int_0^t V(X_s)ds}f(X_t):\ S>t\big]. \tag{5.8}$$

Intuitively H_2 is a domain perturbation of H_V, namely $H_2 = (H_1+V)_\Sigma$, where $(\cdot)_\Sigma$ indicates restriction of the wavefunctions to the set Σ via Dirichlet boundary conditions. In fact one verifies that e^{-tH_2} is the strong limit of $e^{-t(H_V+\beta 1_\Gamma)}$ as β tends to infinity (cf. [8] Lemma 7.1). Now suppose H_V is local in the sense that $H_V f\in L^2(\Sigma)$ (i.e., vanishes on $\Gamma = M\setminus\Sigma$) for all $f\in L^2(\Sigma)\cap\mathcal{D}(H_V)$. Then one can show (cf. Baumgärtel and Demuth [3]) that H_2 coincides with the Friedrich's extension of the operator $H_{min} = H_V|_{L^2(\Sigma)\cap\mathcal{D}(H_V)}$. ($H_{min}$ is the operator which acts like H_V but has domain $L^2(\Sigma)\cap\mathcal{D}(H_V)$.)

As suggested in Section 4, X will be a weighted L^2-space, $X = L^2_\varphi$ with φ : $M\to[1,\infty)$ a measurable function which also satisfies

Hypothesis 5.1. *Suppose there exist constants $\gamma_\varphi > 0$ and $c_\varphi, \alpha_\varphi > 0$ such that for all $x \in M$*

$$E_x\left[\varphi(X_t)^{4+\gamma_\varphi}\right] \leq c_\varphi e^{\alpha_\varphi t} \varphi(x)^{4+\gamma_\varphi}\,. \tag{5.9}$$

Remark 5.2. (i) By Hölder's inequality (5.9) implies similar estimates for all exponents smaller than $4+\gamma_\varphi$.

(ii) While Hypothesis 5.1 may seem a bit technical it is satisfied in a wide range of examples. In particular one can easily verify that it holds for all polynomially bounded functions φ if p can be estimated by a Gaussian kernel in MRd, $p(t,x,y) \leq const. e^{-|x-y|^2/(2t)}$ or in the case where $H_1 = \sqrt{-\Delta+1} - 1$ is the relativistic Hamiltonian. We refer to [8], Remark 1 after Proposition 8.21 for details.

(iii) If $V \equiv 0$ we can replace (5.9) by the weaker condition $E_x[\varphi(X_t)] \leq c_\varphi e^{\alpha_\varphi t} \varphi(x)$.

Lemma 5.3. *If H_1 and H_2 defined by* (5.3) *respectively* (5.8) *satisfy Hypothesis 5.1 they satisfy Hypothesis 2.1.*

Proof. As noted above the Markovian transition function p defines a strongly continuous operator semigroup, by Hille-Yosida its generator H_1 is a positive selfadjoint operator. By Khas'minskii's lemma ([8], Prop. 3.5, see also Simon [21]) $H_V = H_1 + V$ and hence also H_2 is bounded from below by a constant c_0, without loss $c_0 \leq \inf(\sigma(H_1)) = 0$.

It remains to verify (2.1). By Hypothesis 5.1 H_1 satisfies

$$\left(e^{-tH_1}\varphi\right)(x) = E_x[\varphi(X_t)] \leq c_\varphi e^{\alpha_\varphi t}\varphi(x). \tag{5.10}$$

Hence $\|e^{-tH_1}\|_{\mathcal{B}(L^2_\varphi, L^2_\varphi)} = \|\varphi^{-1} e^{-tH_1}\varphi\|_{\mathcal{B}(L^2,L^2)} \leq c_\varphi e^{\alpha_\varphi t}$. Now consider e^{-tH_1} as an operator in $\mathcal{B}(L^2_\varphi, L^2_\varphi)$. We can define its resolvent in this space via Laplace transform. Clearly this resolvent coincides with $R_1(z)$ on L^2_φ, proving that $R_1(z)$ may be restricted to a bounded operator in $\mathcal{B}(L^2_\varphi, L^2_\varphi)$ for all $z \in \rho(H_1)$.

The argument for H_2 is similar. Here we replace (5.10) by

$$\begin{aligned}\left(e^{-tH_2}\varphi\right)(x) &\leq \left(e^{-tH_V}\varphi\right)(x) = E_x\left[e^{-\int_0^t V(X_s)ds}\varphi(X_t)\right] \\ &\leq \Big(E_x\left[\varphi(X_t)^2\right]\Big)^{1/2}\Big(E_x\left[e^{-2\int_0^t V(X_s)ds}\right]\Big)^{1/2} \leq c e^{\alpha t}\varphi(x)\end{aligned} \tag{5.11}$$

with some $c, \alpha > 0$. The last estimate in this equation uses the fact that

$$\sup_{x\in M} E_x\left[e^{-2\int_0^t V(X_s)ds}\right] = \left\|e^{-t(H_1+2V)}\right\|_{\mathcal{B}(L^\infty,L^\infty)} \tag{5.12}$$

is exponentially bounded by Khas'minskii's lemma. Proceed as above. ☐

Now suppose the operator H_1 satisfies Hypothesis 4.1, see Remark 4.3 for a few examples. Then we want to give a sufficient condition to obtain a limiting absorption principle for H_2, i. e., a sufficient condition for Hypothesis 4.2, in terms of V and Σ.

Hypothesis 5.4. *Assume there exists a $\gamma_v > 0$ such that for all $\delta > 0$ we can pick a potential $V_\delta \in L^\infty$ with*

$$\|\varphi^{2+\gamma_v} V_\delta\|_\infty < \delta \tag{5.13}$$

($\|\cdot\|_\infty$ denoting L^∞-norm) and

$$\varphi^{4+2\gamma_v}(V - V_\delta) \in L^1(M). \tag{5.14}$$

Furthermore, suppose Σ *satisfies*

$$\varphi^{4+2\gamma_v} v_\Gamma \in L^1(M), \tag{5.15}$$

where $\Gamma = M \setminus \Sigma$ *and* v_Γ *is the modified equilibrium potential*

$$v_\Gamma(x) = E_x[e^{-S},\ S < \infty]. \tag{5.16}$$

As before $S = \inf\{t > 0 :\ \int_0^t 1_\gamma(X_s)ds > 0\}$ *is the penetration time of* Γ.

We call v_Γ the modified equilibrium potential, because the equilibrium potential $\widetilde{v}_\Gamma$ is normally defined as

$$\widetilde{v}_\Gamma(x) = E_x[e^{-T},\ T < \infty], \tag{5.17}$$

where T is the first hitting time of Γ,

$$T = \inf\{t > 0 :\ X_t \in \Gamma\}. \tag{5.18}$$

These two definitions coincide if $\Gamma^r = (\operatorname{int}\Gamma)^r$, where $\Gamma^r = \{x \in M :\ P_x[T = 0] = 1\}$, $\operatorname{int}\Gamma$ denotes the open interior of Γ, and $(\operatorname{int}\Gamma)^r$ is defined analogously to Γ^r.

To better understand (5.15) recall that the capacity of Γ is equal to the L^1-norm of the equilibrium potential $\widetilde{v}_\Gamma$ (cf. [12], [17]) so (5.15) is slightly stronger than assuming that the capacity of Γ is finite.

Usually φ will be locally bounded; then the conditions on the potential V in (5.13) and (5.14) mean that V must be locally L^1 and must decay faster than φ^{-2} for $|x| \to \infty$.

As desired Hypothesis 5.4 together with the assumptions at the beginning of this section implies Hypothesis 4.2.

Lemma 5.5. *Let* H_1 *and* H_2 *be Feller operators defined by* (5.3) *and* (5.8), *respectively, which satisfy Hypotheses 5.1 and 5.4. Then there exists an* $m \in \mathbb{N}$ *and a* $\gamma > 0$ *such that for all* $a \in MR$
Lemma 5.5. $: @ : smallenough(a\text{¡}\inf \sigma(H_k),\ k = 1, 2)$

$$E = J_2^* R_2(a)^m J_2 - R_1(a)^m \in \mathcal{B}_\infty(L^2, L^2_{\varphi^{2+\gamma}}). \tag{5.19}$$

Proof. For each $\delta > 0$ we can decompose V into $V = V_\delta + V_+ + V_-$ with V_δ as in Hypothesis 5.4 and $\pm V_\pm \geq 0$. Via the Feynman-Kac formula we can define the following operators:

$$\begin{aligned} D_V &:= J_2^* e^{-tH_2} J_2 - e^{-tH_V} \\ D_+ &:= e^{-tH_V} - e^{-t(H_1+V_\delta+V_-)} \\ D_- &:= e^{-t(H_1+V_\delta+V_-)} - e^{-t(H_1+V_\delta)} \\ R_\delta(z) &= (H_1 + V_\delta - z)^{-1}, \qquad z \in \mathbb{C} \setminus [c_0, \infty). \end{aligned} \tag{5.20}$$

As a first step we want to establish $D_V, D_+, D_- \in \mathcal{B}_2(L^2, L^2_{\varphi^{2+\gamma}})$, the space of Hilbert Schmidt operators from L^2 to $L^2_{\varphi^{2+\gamma}}$. (Here $\gamma > 0$ may be chosen as small as necessary.) Actually we prefer to establish the equivalent statement $\varphi^{2+\gamma} D_V$, $\varphi^{2+\gamma} D_+$, $\varphi^{2+\gamma} D_- \in \mathcal{B}_2(L^2, L^2)$. Let $\|\cdot\|_{HS}$ denote the Hilbert Schmidt norm, i. e., the norm in $\mathcal{B}_2(L^2, L^2)$. We use

(5.21)

$$
\begin{aligned}
&\|\varphi^{2+\gamma} D_V\|_{HS}^2 \\
&\quad = \int_M \int_M dxdy\, \varphi(x)^{4+2\gamma} \Big(E_x \big[e^{-\int_0^S V(X_u)du} e^{-(t-S)H_V}(X_s, y) :\ S < t \big] \Big)^2 .
\end{aligned}
$$

Here, as usual, $e^{-(t-S)H_V}(\cdot,\cdot)$ denotes the integral kernel of the operator $e^{-(t-S)H_V}$. Following [8] we denote the pinned measure for paths starting from x and reaching y at time t by $d\mu_{0,x}^{t,y}$. Then, using [8] Theorem 2.9 and (5.4) we can estimate

$$
\begin{aligned}
(5.22) \qquad & E_x \big[e^{-\int_0^S V(X_u)du} e^{-(t-S)H_V}(X_s, y) :\ S < t \big] \\
&= \int_{S<t} e^{-\int_0^t V(X_u)du} d\mu_{0,x}^{t,y} \\
&\le \Big(\int_{S<t} d\mu_{0,x}^{t,y} \Big)^{1/2} \Big(\int e^{-2\int_0^t V(X_u)du} d\mu_{0,x}^{t,y} \Big)^{1/2} \\
&\le \Big(c e^{\alpha t} t^{-d_s/2} \Big)^{1/2} \Big(\int_{S<t} d\mu_{0,x}^{t,y} \Big)^{1/2}
\end{aligned}
$$

with some constants $c, \alpha > 0$. Thus (5.21) becomes

$$
\begin{aligned}
(5.23) \qquad \|\varphi^{2+\gamma} D_V\|_{HS}^2 &\le c e^{\alpha t} t^{-d_s/2} \int_M \int_M dxdy\, \varphi(x)^{4+2\gamma} \int_{S<t} d\mu_{0,x}^{t,y} \\
&= c e^{(\alpha+1)t} t^{-d_s/2} \int_M dx\, \varphi(x)^{4+2\gamma} E_x[e^{-t} :\ S < t] \\
&\le c e^{(\alpha+1)t} t^{-d_s/2} \int_M dx\, \varphi(x)^{4+2\gamma} v_\Gamma(x).
\end{aligned}
$$

To estimate D_+ and D_- we use Duhamel's formula ([8], Appendix B.6):

$$
(5.24) \qquad D_+(x,y) = \int_0^t ds \int_M dz e^{-sH_V}(x,z) V_+(z) e^{-(t-s)(H_1+V_\delta+V_-)}(z,y).
$$

With this and the estimate $|D_+(x,y)| \le e^{-t(H_1+V_\delta+V_-)}(x,y)$ we get

(5.25)

$$
\begin{aligned}
&\|\varphi^{2+\gamma} D_+\|_{HS}^2 \\
&\quad \le \int_M \int_M dxdy\, \varphi(x)^{4+2\gamma} D_+(x,y) e^{-t(H_1+V_\delta+V_-)}(x,y) \\
&\quad = \int_M \int_M dxdy \int_0^t ds \int_M dz\, \varphi(x)^{4+2\gamma} e^{-sH_V}(x,z) V_+(z) \\
&\qquad\qquad \times e^{-(t-s)(H_1+V_\delta+V_-)}(z,y) e^{-t(H_1+V_\delta+V_-)}(x,y) \\
&\quad = \int_0^t ds \int_M \int_M dxdz\, \varphi(x)^{4+2\gamma} e^{-s(H_V)}(x,z) V_+(z) e^{-(2t-s)(H_1+V_\delta+V_-)}(z,x).
\end{aligned}
$$

We estimate

$$
\begin{aligned}
(5.26) \qquad \int_M dx\, \varphi(x)^{4+2\gamma} e^{-sH_V}(x,z) &= E_z \big[e^{-\int_0^s V(X_u)du} \varphi(X_s)^{4+2\gamma} \big] \\
&\le \Big(E_z \big[e^{-p\int_0^s V(X_u)du} \big] \Big)^{1/p} \Big(E_z \big[\varphi(X_s)^{q(4+2\gamma)} \big] \Big)^{1/q}
\end{aligned}
$$

for all $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Picking $q > 1$ small enough we can obtain $q(4+2\gamma) \le 4 + \gamma_\varphi$ so that with Hypothesis 5.1 and [8] Theorem 2.9 we find

$$
\begin{aligned}
(5.27) \qquad & \int_M dx\, \varphi(x)^{4+2\gamma} e^{-sH_V}(x,z) \\
& \le (c_\varphi e^{\alpha_\varphi s})^{1/q} \varphi(z)^{4+2\gamma} \big\| e^{-s(H_1+pV)} \big\|^{1/p}_{\mathcal{B}(L^\infty, L^\infty)} \\
& \le c e^{\alpha s} \varphi(z)^{4+2\gamma}
\end{aligned}
$$

with some $c, \alpha > 0$ provided $\gamma > 0$ is small enough.

Thus, using [8] Theorem 2.9 again, (5.25) becomes

$$
\begin{aligned}
(5.28) \qquad & \|\varphi^{2+\gamma} D_+\|^2_{HS} \\
& \le \int_0^t ds\, c e^{\alpha s} \int_M dz\, \varphi(z)^{4+2\gamma} V_+(z) \big\| e^{-(2t-s)(H_1+V_\delta+V_-)} \big\|_{\mathcal{B}(L^1, L^\infty)} \\
& \le c' e^{\alpha' t} t^{-d_S/2} t \int_M dz\, \varphi(z)^{4+2\gamma} V_+(z)
\end{aligned}
$$

with constants $c', \alpha' > 0$. The estimate for D_- is completely analogous, it yields

$$
(5.29) \qquad \|\varphi^{2+\gamma} D_-\|^2_{HS} \le c e^{\alpha t} t^{1-d_S/2} \int_M dz\, \varphi(z)^{4+2\gamma} |V_-(z)|
$$

with $c, \alpha > 0$.

Combining (5.23), (5.28), and (5.29) we find

$$
\begin{aligned}
(5.30) \qquad & \big\| \varphi^{2+\gamma} \big(J_2^* e^{-tH_2} J_2 - e^{-t(H_1+V_\delta)} \big) \big\|_{HS} \\
& \le c e^{\alpha t} t^{-d_S/4} \Big(\|\varphi^{4+2\gamma}(V - V_\delta)\|^{1/2}_{L^1} + \|\varphi^{4+2\gamma} v_\Gamma\|^{1/2}_{L^1} \Big)
\end{aligned}
$$

with some $c, \alpha > 0$. If we pick $\gamma > 0$ small enough the right hand side of (5.30) is finite by Hypothesis 5.4. Now pick $m \in \mathbb{N}$ such that $m - \frac{d_S}{4} > -1$ and $a \in$ MR $: @ : suchthat$a$+\alpha < 0$ where α is the constant from (5.30). Then

$$
\begin{aligned}
(5.30) \qquad & \big\| \varphi^{2+\gamma} \big(J_2^* R_2(a)^m J_2 - R_\delta(a)^m \big) \big\|_{HS} \\
& \le \frac{1}{\Gamma(m)} \int_0^\infty dt\, t^{m-1} e^{at} \big\| \varphi^{2+\gamma} \big(J_2^* e^{-tH_2} J_2 - e^{-t(H_1+V_\delta)} \big) \big\|_{HS} < \infty
\end{aligned}
$$

(here $\Gamma(m)$ denotes the Γ-function). Finally we have

$$
\begin{aligned}
(5.31) \qquad \varphi^{2+\gamma} E &= \varphi^{2+\gamma} \big(J_2^* R_2(a)^m J_2 - R_1(a)^m \big) \\
&= \varphi^{2+\gamma} \big(J_2^* R_2(a)^m J_2 - R_\delta(a)^m \big) + \varphi^{2+\gamma} \big(R_\delta(a)^m - R_1(a)^m \big).
\end{aligned}
$$

For every $\delta > 0$ the first operator in (5.31) is Hilbert Schmidt, hence compact. By (5.13) the second operator tends to 0 in Operator norm as δ tends to 0, thus $\varphi^{2+\gamma} E$ is compact (i. e., element of $\mathcal{B}_\infty(L^2, L^2)$) as the norm limit of a sequence of compact operators. Hence $E \in \mathcal{B}_\infty(L^2, L^2_{\varphi^{2+\gamma}})$. ☐

So we have proved our main application theorem.

Theorem 5.6. *Suppose H_1 and H_2 are Feller operators satisfying Hypotheses 5.1 and 5.4, and H_1 also satisfies Hypothesis 4.1. Then a limiting absorption principle holds for H_2: Its spectrum inside the set Δ, $\sigma(H_2) \cap \Delta$ is, except for possibly* *a discrete set $\sigma_p(H_2) \cap \Delta$ of eigenvalues of finite multiplicity, purely absolutely continuous. For all $\lambda \in \Delta \setminus \sigma_p(H_2)$*

$$J_2^* R_2^{\pm}(\lambda) J_2 := \lim J_2^* R_2(\lambda \pm i\varepsilon) J_2 \tag{5.32}$$

exists and is continuous in $\mathcal{B}(L^2_\varphi, L^2_{\varphi^{-1}})$.

Let us illustrate the strength of our approach with the simplest example. Suppose $H_1 = -\Delta$ is the Laplacian on MRd. This operator satisfies Hypothesis 4.1 with $\Delta = (0, \infty)$ and $\varphi = (1 + |x|)^s$, $s > \frac{1}{2}$ (cf. [5]). Then Hypothesis 5.4 allows all potentials V which are at least locally L^1 and decay as $|x|^{-1-\gamma}$ as $|x|$ tends to infinity, i.e., the usual short range potentials. The condition (5.15) on the set Σ respectively $\Gamma = M \setminus \Sigma$ is a bit more implicit. For comparison the capacity of Γ is the L^1-norm of the equilibrium potential. For the Laplacian (5.15) becomes the condition $(1 + |x|)^{2+\gamma} v_\gamma \in L^1$, a slightly stronger restriction than the finiteness of the capacity. That is, Γ may be an arbitrary compact set or an unbounded set provided it is small enough in the sense of (5.15).

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