Hamilton-Jacobi equation, heteroclinic chains and Arnol'd diffusion in three time scales systems

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ABSTRACT. *Interacting systems consisting of two rotators and a point mass near a hyperbolic fixed point are considered, in a case in which the uncoupled systems have three very different characteristic time scales. The abundance of quasi periodic motions in phase space is studied via the Hamilton–Jacobi equation. The main result, a high density theorem of invariant tori, is derived by the classical canonical transformation method extending previous results. As an application the existence of long heteroclinic chains (and of Arnol'd diffusion) is proved for systems interacting through a trigonometric polynomial in the angle variables.*

Keywords: Hamilton Jacobi, KAM, Arnold diffusion, homoclinic splitting

§1. The system.

Let $(\alpha_1,\alpha_2) = \underline{\alpha} \in \mathbf{T}^2$ be a pair of angles; let $\underline{A} = (A_1,A_2) \in \mathbf{R}^2$ be their conjugate momenta or "*actions*", and let $(p,q) \in \mathbf{R}^2$ be a further pair of canonically conjugate coordinates. We consider the Hamiltonian function, depending on two dimensionless parameters, $\varepsilon,\eta > 0$, defined by:

$$\mathcal{H} = h(\eta^{\frac12}A_1) + \eta^{-\frac12}\omega_2 A_2 + G(pq,\eta^{\frac12}A_1) + \varepsilon f(\eta^{\frac12}A_1,\,\underline{\alpha}\,,p,q) \tag{1.1}$$

We suppose that h,G,f are real analytic functions of their arguments in the domain $\eta^{\frac12}|A_1| \le R$, $|p|,|q| \le r$, $\underline{\alpha} \in \mathbf{T}^2$ for some $R,r > 0$. We *also suppose* that f is a trigonometric polynomial of degree N in the variables $\underline{\alpha}$.

For $\varepsilon = 0$ the Hamiltonian (1.1) represents, physically, a system consisting of two rotators and a point mass near an unstable fixed point ($p = q = 0$).

The instability time scale of the unstable equilibrium is $g \equiv g(pq,\eta^{\frac12}A_1) \stackrel{def}{=} \partial_x G(x,\,\eta^{\frac12}A_1)|_{x=pq} = O(1)$ if, as we shall suppose, $g > 0$.

Define $\eta^{\frac12}\omega_1 \equiv \eta^{\frac12}\omega_1(pq,\eta^{\frac12}A_1) = \partial_{A_1}[h(\eta^{\frac12}A_1) + G(pq,\eta^{\frac12}A_1)]$: the two rotators rotate with angular velocities $\eta^{\frac12}\omega_1$ and $\eta^{-\frac12}\omega_2$ with ratio $O(\eta)$ if, as we shall suppose, $0 \le \tilde\omega_a \le |\omega_1| \le \tilde\omega_b$ for suitable constants $\tilde\omega_a,\tilde\omega_b$.

Hence the system has three time scales of respective orders $\eta^{-\frac12},1,\eta^{\frac12}$ called respectively: *slow* (rotator #1), *normal* (unstable motion), *fast* (rotator #2). We suppose $\eta < 1$ so that the fast rotator really is the *isochronous* one and the slow rotator is the *anisochronous* one.

§2. Characteristic parameters.

A few *characteristic parameters* can be associated with the system because of the analyticity assumptions.

Let $2\rho_0, 2\xi_0, (2\kappa_0)^{\frac12}, (2\kappa_0)^{\frac12}, 2\kappa_0$ measure the size in the complex planes of a holomorphy domain in the A_1, α_i, p, q and $x = pq$ variables, respectively. We denote:

$$\begin{aligned} Q_\rho =& \{|\mathrm{Re}\, A_1| \le \eta^{-\frac{1}{2}}(R+\rho),\ |\mathrm{Im}\, A_1| \le \eta^{-\frac{1}{2}}\rho\} \\ U_\xi =& \{|\mathrm{Im}\, \alpha_j| \le \xi\}, \qquad V_\kappa = \{|p|,|q| \le \kappa^{\frac{1}{2}}\}, \qquad S_\kappa = \{|x| \le \kappa\} \end{aligned} \tag{2.1}$$

Our analyticity hypotheses imply the existence of $\rho_0, \xi_0, \kappa_0 > 0$ such that $h(\eta^{\frac{1}{2}}A_1)$ is holomorphic in $Q_{2\rho_0}$, $G(x, \eta^{\frac{1}{2}}A_1)$ is holomorphic in $S_{2\kappa_0} \times Q_{2\rho_0}$ and f in $Q_{2\rho_0} \times U_{2\xi_0} \times V_{2\kappa_0} \stackrel{def}{=} \mathcal{D}_{2\rho_0, 2\xi_0, 2\kappa_0}$.

Suppose:

(i) $g(pq, \eta^{\frac{1}{2}}A_1) \stackrel{def}{=} \partial_x G(x, \eta^{\frac{1}{2}}A_1)|_{x=pq} \ne 0$;

(ii) $\omega_1(pq, \eta^{\frac{1}{2}}A_1) \stackrel{def}{=} \partial h(\eta^{\frac{1}{2}}A_1) + \partial G(pq, \eta^{\frac{1}{2}}A_1) \ne 0$, where ∂ denotes differentiation with respect to the argument $a = \eta^{\frac{1}{2}}A_1$;

(iii) $m_1(pq, \eta^{\frac{1}{2}}A_1) \stackrel{def}{=} \partial^2 h(\eta^{\frac{1}{2}}A_1) + \partial^2 G(pq, \eta^{\frac{1}{2}}A_1) \ne 0$.

Hence we can define $\Gamma_0, E_0, M_0 > 0$ so that:

$$\begin{aligned} &0 < \lambda_0 = \min\{\rho_0, \kappa_0\}, \qquad |f|, |h|, |G| < E_0 \\ &0 < \Gamma_0 < |\omega_1(pq, \eta^{\frac{1}{2}}A_1)|,\ |\omega_2|, |g(pq, \eta^{\frac{1}{2}}A_1)| < E_0\lambda_0^{-1} \\ &0 < M_0^{-1} < |m_1(pq, \eta^{\frac{1}{2}}A_1)| \end{aligned} \tag{2.2}$$

in the holomorphy domain $\mathcal{D}_{2\rho_0, 2\xi_0, 2\kappa_0}$. We suppose $\eta < 1$ (see §1) and, for simplicity, $\xi_0, \varepsilon < 1$.

The parameters $E_0, \Gamma_0, M_0, \lambda_0, \xi_0$ will be called *characteristic parameters* of the system (note that $\Gamma_0\lambda_0, \tilde{\omega}_b\lambda_0 < E_0$). *Only* the dimensionless parameters ξ_0, $E_0\Gamma_0^{-1}\lambda_0^{-1}$ and $M_0E_0\lambda_0^{-2}$ will control the constants in the main bounds (see (4.3) and (5.5)).

§3. Hamilton–Jacobi equation.

The Hamilton–Jacobi equation for the Hamiltonian (1.1) will be an equation with two unknown functions $\Phi(\eta^{\frac{1}{2}}A_1', \underline{\alpha}, p', q)$ and $\tilde{G}(x, \eta^{\frac{1}{2}}A_1')$ given by:

$$\begin{aligned} &h(\eta^{\frac{1}{2}}(A_1' + \partial_{\alpha_1}\Phi)) + \eta^{-\frac{1}{2}}\omega_2(A_2' + \partial_{\alpha_2}\Phi) + G((p' + \partial_q\Phi)q, \eta^{\frac{1}{2}}(A_1' + \partial_{\alpha_1}\Phi)) \\ &\quad + \varepsilon f(\eta^{\frac{1}{2}}(A_1' + \partial_{\alpha_1}\Phi), \underline{\alpha}, p' + \partial_q\Phi, q) = \\ &= h(\eta^{\frac{1}{2}}A_1') + \eta^{-\frac{1}{2}}\omega_2A_2' + G(p'(q + \partial_{p'}\Phi), \eta^{\frac{1}{2}}A_1') + \tilde{G}(p'(q + \partial_{p'}\Phi), \eta^{\frac{1}{2}}A_1') \end{aligned} \tag{3.1}$$

If the functions $\Phi, \tilde{G}$ existed one could set:

$$\begin{aligned} \underline{A} &= \underline{A}' + \partial_{\underline{\alpha}}\Phi, \qquad & p &= p' + \partial_q\Phi \\ \underline{\alpha}' &= \underline{\alpha} + \partial_{\underline{A}'}\Phi, \qquad & q' &= q + \partial_{p'}\Phi \end{aligned} \tag{3.2}$$

and the change of variables $(\underline{A}, \underline{\alpha}, p, q) \longleftrightarrow (\underline{A}', \underline{\alpha}', p', q')$, *where defined*, would cast the Hamiltonian (1.1) into the form:

$$\mathcal{H}_0 = h(\eta^{\frac{1}{2}}A_1') + \eta^{-\frac{1}{2}}\omega_2A_2' + G(p'q', \eta^{\frac{1}{2}}A_1') + \tilde{G}(p'q', \eta^{\frac{1}{2}}A_1') \tag{3.3}$$

thereby integrating the system, because the motions in the new coordinates would become $\underline{A}' = \text{const.}$, $\underline{\alpha}' \to \underline{\alpha}' + \underline{\omega}'t$, $p' \to p'e^{-\overline{g}'t}$ and $q' \to q'e^{\overline{g}'t}$ with:

$$\begin{aligned} &\overline{g}' \equiv \overline{g}'(x', \eta^{\frac{1}{2}}A_1') = \partial_{x'}G(x', \eta^{\frac{1}{2}}A_1') + \partial_{x'}\tilde{G}(x', \eta^{\frac{1}{2}}A_1') \qquad (3.4) \\ &\underline{\omega}' \equiv \underline{\omega}'(x', \eta^{\frac{1}{2}}A_1') = (\eta^{\frac{1}{2}}\partial' h(\eta^{\frac{1}{2}}A_1') + \eta^{\frac{1}{2}}\partial' G(x', \eta^{\frac{1}{2}}A_1') + \eta^{\frac{1}{2}}\partial'\tilde{G}(x', \eta^{\frac{1}{2}}A_1'), \eta^{-\frac{1}{2}}\omega_2) \end{aligned}$$

where $x' = p'q'$ and ∂' denotes differentiation with respect to the argument $a' = \eta^{\frac{1}{2}} A'_1$.

This is in fact, as it is well known, *not possible* in general. However one proves the following result.

PROPOSITION. *There exists a canonical change of variables like (3.2) such that*
(a) it is of class C^∞ in all variables and in ε for ε small enough;
(b) it is close to the identity within $O(\varepsilon)$ in C^∞;
(c) there is a function $\tilde G(x', \eta^{\frac{1}{2}} A'_1)$ which is of class C^∞ in all variables, including ε, which is divisible by ε and such that in the new variables the Hamiltonian coincides together with its derivatives, with (3.3), on the set (without interior points) of the $(A'_1, p,' q', \underline{\alpha}')$'s such that the rotation vector $\underline{\omega}' \equiv \underline{\omega}'(p'q', \eta^{\frac{1}{2}} A'_1)$ defined in (3.4) verifies the Diophantine condition:

$$|\,\underline{\omega}' \cdot \underline{\nu}\,| > C(\eta) |\,\underline{\nu}\,|^{-\tau} \qquad \textit{for all } \underline{0} \neq \underline{\nu} \in \mathbf{Z}^2 \tag{3.5}$$

with $\tau \geq 1$ prefixed and $C(\eta) > 0$, provided ε is small enough, depending on $C(\eta)$ and the other parameters of the model.

The above proposition is essentially proven in [CG], §5, Lemma 1', (see [G2] and [P] for cases without gaps). The proof in [CG] deals really with the subset of the set the variables $\underline{A}'$ and $x' = p'q'$ where $\underline{\omega}'(p'q', \eta^{\frac{1}{2}} A'_1) = (1 + \gamma(p'q', \eta^{\frac{1}{2}} A'_1))\ \underline{\omega}'(0, \eta^{\frac{1}{2}} A'_1)$ for some $\gamma(x', a')$ of class C^∞ in its arguments: by examining the proof one sees that the result holds under the above more general condition.

Remarks. (1) Calling $W = W_{\varepsilon, C(\eta)}$ the set in the phase space such that the corresponding rotation vectors $\underline{\omega}'$ verify (3.5), one says that the Hamilton–Jacobi equation is soluble on W and, in W, casts the Hamiltonian (1.1) into the *normal form* given by the r.h.s. of (3.3).

(2) Since the function $\underline{\omega}'$ has non zero gradient with respect to $\underline{A}'$ (under the mentioned condition of smallness of ε; see (2.2)), the volume W of phase space where (3.5) holds has a complement with measure bounded proportionally to $C(\eta)$. And at fixed $x' = p'q'$ the measure of the subset of the interval $[-\eta^{-\frac{1}{2}} R, \eta^{-\frac{1}{2}} R]$ on the A'_1 axis has also measure bounded proportionally to $C(\eta)$. The latter subset consists of a sequence of small intervals (whose total length has size $< O(C(\eta))$) that are called "*gaps*" for natural reasons.

(3) The values of A'_1 such that (3.5) holds with $x' = p'q' = 0$ are points where the Hamiltonian coincides with one like (3.3) *together with its first derivatives.* Thus they clearly correspond to invariant tori (if $p' = q' = 0$, also called "persistent tori") and to their stable and unstable manifolds (if $q' = 0$ or $p' = 0$, respectively, sometimes called *whiskers*). On such manifolds the motion takes place following the evolution described after (3.3).

If we simply apply the results in [CG], the bounds on the gaps between the invariant tori (which persist notwithstanding the perturbation) are found to be "too large" with respect to the "*homoclinic splitting*" size (it has become usual to say that the splitting is "exponentially small in $\eta^{-\frac{1}{2}}$" as $\eta \to 0$). The latter, see below (remark 3 in §5), is the quantity of main interest in the theory of drift and diffusion. A simple way to use this result to discuss diffusion in phase space is to choose ε very small: but it turns out that it has to be chosen too small for the purpose of several interesting applications (and one says that it has to be chosen "exponentially small with $\eta^{-\frac{1}{2}}$"); we postpone to §5 a more technical discussion.

This paper is dedicated to overcome the limitations that have been briefly alluded to and to show that the restrictions to be put on the size of ε can be *greatly improved* with respect to the ones that follow directly from previous work, (§5 and Appendix A10 of [CG]). Therefore we shall proceed in a somewhat different way with respect to the one outlined above, and we shall perform a preliminary step (§4) before applying the analysis of [CG]. As a consequence we shall bound from above (§5) the spacing between invariant tori, *i.e.* the gaps sizes, and apply the results to show that the homoclinic angles are (generically) *very large* compared to the (dimensionless) spacing: this will imply (§6) the existence of long heteroclinic chains for a system consisting of a pendulum interacting with two rotators *thereby deducing (by using §8 of [CG]) the existence of drift and diffusion in the system* when the perturbation parameter ε is *smaller than a power of* η, *rather than exponentially small in* $\eta^{-\frac{1}{2}}$. Finally we show (§7) that the perturbation can have a large component which, provided it depends only on the fast rotator angle (or more generally provided it is "monochromatic", see §7), does not alter some of the results.

A problem with three time scales naturally arises in celestial mechanics: in the theory of the precession motion of a satellite axis due to its non sphericity. Under certain approximations, see [CG], it is described by a Hamiltonian of the form (1.1) plus a large "monochromatic" term. In [CG], §12, it was shown that Arnol'd diffusion occurs, but a preceding error on the lower bound of a homoclinic splitting infirms the conclusions. In [GGM1] the correct lower bound is given and combining [GGM1] with the new results of §7 one sees that the analysis in [CG] applies, provided that the monochromatic term is absent (this means that the normal form method described in §8 of [CG] applies to the heteroclinic chains that, below, we show to exist). Finally, in the last section, we outline a *possible* strategy on how to approach and complete the analysis of the full problem of §12 of [CG].

The non standard part of this work relies strongly on the explicit formula for the homoclinic splitting derived in [GGM1] and called the "*large angles theorem*".

§4. Averaging.

We shall first perform a canonical transformation, that we shall denote $\Phi_{\mathcal{N}}$, to "reduce" the perturbation to order $O(\varepsilon^{\mathcal{N}})$, with $\mathcal{N}$ an integer suitably large, but such that no "small divisors" appear in the definition of $\Phi_{\mathcal{N}}$. This is easily achieved by requiring $\mathcal{N} \le \zeta\eta^{-1}N^{-1}$, with $\zeta = |\omega_2|(2\tilde{\omega}_b)^{-1}$, (note that the existence and the relative sizes of three time scales is essential here). Subsequently we shall apply the results of [CG] to the Hamiltonian expressed in the new variables; in this way we shall be able to prove a more refined bound on the size of the gaps between invariant tori, (and show that it is smaller than the homoclinic splitting without the need to require ε to be exponentially small in $\eta^{-\frac{1}{2}}$).

Fixed an integer number $\mathcal{N}$, we look for a function $\Phi_{\mathcal{N}}(\eta^{\frac{1}{2}}A_1', \underline{\alpha}, p', q)$ generating a canonical transformation which casts the Hamiltonian (1.1) into the form (3.3) up to order $\mathcal{N}$. This means that, in the new variables, the Hamiltonian will have the form (3.3), but with a different function $\tilde{G}$ (that we shall denote $\tilde{G}_{\mathcal{N}}$) and *an extra additive term* (the new perturbation) which will not be of order $O(\varepsilon)$ but of order $O(\varepsilon^{\mathcal{N}})$.

The functions $\Phi_{\mathcal{N}}, \tilde{G}_{\mathcal{N}}$ will be polynomials in ε:

$$\begin{aligned}\Phi_{\mathcal{N}} &= \varepsilon\Phi^{(1)} + \varepsilon^2\Phi^{(2)} + \ldots + \varepsilon^{\mathcal{N}}\Phi^{(\mathcal{N})}\\ \tilde{G}_{\mathcal{N}} &= \varepsilon\tilde{G}^{(1)} + \varepsilon^2\tilde{G}^{(2)} + \ldots + \varepsilon^{\mathcal{N}}\tilde{G}^{(\mathcal{N})}\end{aligned} \tag{4.1}$$

We proceed to determine recursively the coefficients for $\mathcal{N} \le \frac{1}{2}\eta^{-1}N^{-1}$ in order to prove the following *averaging theorem*.

THEOREM 1. *Fixed $\mathcal{N} \le \zeta\eta^{-1}N^{-1}$, with $\zeta = |\omega_2|(2\tilde\omega_b)^{-1}$, the functions $\Phi^{(j)}, \tilde G^{(j)}$ exist for all $1 \le j \le \mathcal{N}$ and are holomorphic in the "decreasing" domains $\mathcal{D}_{\rho_j,\xi_j,\kappa_j}$ with $\xi_j = \xi_0 - j\delta$, $\rho_j = \rho_0 e^{-j\delta}$, $\kappa_j = \kappa_0 e^{-j\delta}$ for $\delta > 0$ such that $\mathcal{N}\delta < \frac12\xi_0$, and they verify the bounds:*

$$|\Phi^{(j)}| < DB^{j-1}(j-1)!, \qquad |\tilde G^{(j)}| < E_0 B^{j-1}(j-1)!, \qquad j \le \mathcal{N} \tag{4.2}$$

where $D = bE_0\Gamma_0^{-1}D_0$ and $B = b\,(E_0\Gamma_0^{-1}\lambda_0^{-1}\xi_0^{-1})^2 D_0^2\mathcal{N}^2$, if the constant b is large enough, and $D_0 = \eta^{-\frac12}\xi_0^{-1}(\mathcal{N}\xi_0^{-1})^2\log(\mathcal{N}\xi_0^{-1})$.
Furthermore:

- *Map (3.2) with $\Phi = \Phi_{\mathcal{N}}$ generates a canonical transformation from $\mathcal{D}_{\rho_{\mathcal{N}},\xi_{\mathcal{N}},\kappa_{\mathcal{N}}} \supseteq \mathcal{D}_{\rho_0 e^{-\xi_0/2},\xi_0/2,\kappa_0 e^{-\xi_0/2}}$ to the domain $\mathcal{D}_{\rho_0,\xi_0,\kappa_0}$ if:*

$$|\varepsilon| < \varepsilon_0 = b'\left(\frac{\Gamma_0\lambda_0}{E_0}\right)^2 \frac{1}{\mathcal{N}}\frac{\eta}{(\mathcal{N}\xi_0^{-1})^6\log^2(\mathcal{N}\xi_0^{-1})} \tag{4.3}$$

with b' small enough.

- *And in the new variables $(\underline{A}', \underline{\alpha}', p', q')$ the Hamiltonian (1.1) becomes:*

$$\begin{aligned} &h(\eta^{\frac12}A_1') + \eta^{-\frac12}\omega_2 A_2' + G(p'q', \eta^{\frac12}A_1') + \tilde G_{\mathcal{N}}(p'q', \eta^{\frac12}A_1') + \\ &+ \left(\frac{\varepsilon}{2\varepsilon_0}\right)^{\mathcal{N}} f_{\mathcal{N},\varepsilon}(\eta^{\frac12}A_1', \underline{\alpha}', p', q') \end{aligned} \tag{4.4}$$

with $|\varepsilon^{-1}\tilde G_{\mathcal{N}}|, |f_{\mathcal{N},\varepsilon}| < 2E_0$.

The proof follows immediately (for instance) from the arguments used to prove Nekhoroshev's theorem in [G1,BG] and is given in Appendix A1, for completeness, extending similar considerations in [CG].

Remarks. (1) An interesting consequence is obtained by choosing $\mathcal{N} = \gamma\eta^{-\frac12}$ with $\gamma > 0$. In this case under a condition $|\varepsilon| < \varepsilon_0 = O(\eta^{\frac92}/\log^2\eta^{-1})$, see (4.3), the Hamiltonian (1.1) can be put in the form (4.4).
(2) Another interesting consequence (of the proof of the above theorem given in Appendix A1) concerns the case with $\mathcal{N}$ fixed independently on η and a perturbation f whose Fourier transform with respect to $\underline{\alpha}$ does not vanish only for wave vectors $\underline{\nu}$ multiples of a given wave vector $\underline{\nu}_0$ ("*monochromatic*", or "*integrable*", perturbation). In the latter case the condition is $|\varepsilon| < O(1)$ for all $\mathcal{N}$, prefixed; see the corollary discussed in the last paragraph of Appendix A1, (although it would be easy to prove directly the simple statement made there).
(3) As a quantitative estimate of the parameters of the new Hamiltonian (4.4) it follows from the proof in Appendix A1 that the parameters $\lambda_0, \Gamma_0, E_0, M_0$ can all be taken the same as those of the initial Hamiltonian within a factor 2: also η is the same while, of course, ε becomes $(\varepsilon(2\varepsilon_0)^{-1})^{\mathcal{N}}$.

§5. Abundance of tori. Interpolation.

Given the Hamiltonian (4.4), define the *free spectrum* $\mathcal{S}_0$ as the collection of the rotation vectors

$$\underline{\omega} = \underline{\omega}(x', \eta^{\frac12}\underline{A}') \stackrel{def}{=} \left(\eta^{\frac12}\partial' h(\eta^{\frac12}A_1') + \eta^{\frac12}\partial' G(x', \eta^{\frac12}A_1'), \eta^{-\frac12}\omega_2\right) \tag{5.1}$$

as A_1' varies in $[-R\eta^{-\frac12}, R\eta^{-\frac12}]$ and $x' = 0$ and ∂' denotes differentiation with respect to the argument $a' = \eta^{\frac12}A_1'$. More generally we define $\mathcal{S}$ to be the collection of vectors (5.1) as $A_1' \in [-R\eta^{-\frac12}, R\eta^{-\frac12}]$ and $|x'| < \kappa_0 e^{-\xi_0/2}$.

Let us consider the values A'_1 such that the rotation vector $\underline{\omega} \in \mathcal{S}_0$ in (5.1) satisfies at $x' = 0$ the *diophantine condition*:

$$|\eta^{\frac{1}{2}}\omega_1\nu_1 + \eta^{-\frac{1}{2}}\omega_2\nu_2| > C(\eta)|\,\underline{\nu}\,|^{-\tau} \qquad \text{for all } \underline{0} \neq \underline{\nu} \in \mathbf{Z}^2 \tag{5.2}$$

where $\eta^{\frac{1}{2}}\omega_1 \equiv \eta^{\frac{1}{2}}\omega_1(x', \eta^{\frac{1}{2}}A'_1)$ is the first component of $\underline{\omega}$ in (5.1), and $\tau \geq 1$, $C(\eta) > 0$ are Diophantine constants: ω_1 has size of order 1 (see §1).

Such A'_1 occupy a set in $[-R\eta^{-\frac{1}{2}}, R\eta^{-\frac{1}{2}}]$ whose complement has measure bounded above by:

$$b'' M_0 E_0 \lambda_0^{-1} C(\eta) \tag{5.3}$$

with a suitably large b'', because the measure of the set of ω_1's in the interval of variability of ω_1 ($[\tilde{\omega}_a, \tilde{\omega}_b] \subseteq [0, E_0\lambda_0^{-1}]$, see §1) verifying (5.2) has complement bounded by $\tilde{b}'' E_0 \lambda_0^{-1} \eta^{\frac{1}{2}} C(\eta)$), for some constant $\tilde{b}''$, and the derivative of the map from action to frequency is $\eta\partial^2 h(pq, \eta^{\frac{1}{2}}A_1)$, bounded below by ηM_0^{-1} (see (2.2)). Hence any interval in $[-R\eta^{-\frac{1}{2}}, R\eta^{-\frac{1}{2}}]$ of size $\Delta(\eta) > b'' M_0 E_0 \lambda_0^{-1} C(\eta)$ will contain points verifying (5.2).

If $C(\eta) = \Omega e^{-s\eta^{-\frac{1}{2}}}$ with $s, \Omega > 0$ (the factor Ω is here just to fix the dimensions and it could be any constant with the right dimensions,*e.g.* ω_2 or E_0/λ_0 or other), then it follows from the combination of theorem 1 above and lemma 1' of [CG] that for each $\underline{\omega} \in \mathcal{S}_0$ verifying (5.2) the Hamiltonian system (3.3), hence (4.4), will have an invariant torus such that the rotation vector $\underline{\omega}\,'$ assumes the value $\underline{\omega}$, for η small enough and $|\varepsilon| < \varepsilon_0$.

In fact the "effective coupling", by applying theorem 1 with $\mathcal{N} = \gamma\eta^{-\frac{1}{2}}$, is of order $O((\varepsilon/2\varepsilon_0)^{\gamma\eta^{-\frac{1}{2}}})$ and, for γ large, this is so small that even if multiplied by $[C(\eta)/\Omega]^{-q}$ for $q > 0$ arbitrarily fixed gives a still very small result when η gets small enough.

The stability of the tori and their stable and unstable manifolds or of the motions in their vicinity depends on the size of the latter product (for a suitably large q, see [CG], eq. (5.76), where $q = 6$ is an estimate) which is small if $\gamma > qs(\log 2)^{-1}$.

Note that if we simply applied lemma 1' of [CG] to the Hamiltonian (1.1) we would have found, as a bound on the convergence radius for the parametric equations of the tori and of some of their nearby motions, a quantity of order $O(\varepsilon_0)$ times $[C(\eta)/\Omega]^q$, which would have been exponentially small in $\eta^{-\frac{1}{2}}$ if $C(\eta) = \Omega e^{-s\eta^{-\frac{1}{2}}}$ with $\Omega, s > 0$.

Another way of interpreting the analysis (see [CG]) is by saying that there is a change of coordinates $(\underline{A}, \underline{\alpha}, p, q) \longleftrightarrow (\underline{A}\,', \underline{\alpha}\,', p', q')$, defined in the vicinity of $|\mathrm{Re}\,A_1| < \eta^{-\frac{1}{2}}R$ and $|p|, |q|$ small, which is of class C^∞ (in $(\underline{A}\,', \underline{\alpha}\,', p', q')$ and ε) which casts Hamiltonian (1.1) or (4.4) in the form (3.3) on the set of $(\underline{A}\,', \underline{\alpha}\,', p', q')$'s for which the vector $\underline{\omega}\,' = \underline{\omega}\,'(p'q', \eta^{\frac{1}{2}}A'_1)$ defined in (3.4) verifies (3.5), *i.e.* it is equal to some $\underline{\omega} \in \mathcal{S}_0$ verifying (5.2).

Furthermore, fixed $\underline{A}\,'$ and p', q' in such a set, the coordinate transformation (of the remaining two coordinates $\underline{\alpha}$) is analytic in ε for $|\varepsilon| < \varepsilon_0 O([C(\eta)/\Omega]^{q\gamma^{-1}\eta^{\frac{1}{2}}})$.

And the identity between (1.1) and (3.3) with (3.4) holds, on the set on which the functions agree, *also between their first derivatives with respect to the* $(\underline{A}\,', \underline{\alpha}\,', p', q')$ *variables*. So that the sets with $\underline{A}\,'$ and $p'q' = x'$ fixed and $\underline{\omega}\,'$ verifying (3.5) are invariant and the motion on them is very simple, and described after (3.3).

We stress again that the latter statement is a slight generalization of the quoted results in [CG]: namely the latter really covers the statements in the preceding paragraph for vectors $\underline{\omega}\,'(x', \eta^{\frac{1}{2}}A'_1)$ which verify (3.5) *and* have the special form $\underline{\omega}\,'(x', \eta^{\frac{1}{2}}A'_1) = (1 + \gamma(x', \eta^{\frac{1}{2}}A'_1))\,\underline{\omega}\,'(0, \eta^{\frac{1}{2}}A'_1)$ for some small $\gamma(x', \eta^{\frac{1}{2}}A'_1)$. This restricts us, *de facto*, to considering cases less general than those that verify (3.5). However, as mentioned above, if one goes through the proof of lemmata 1,1',2 of

[CG] one realizes that the proof covers, unchanged, the more general case we quote here.

As a consequence the following *high density theorem of invariant tori* also holds (corollary of theorem 1 and of the above analysis).

THEOREM 2. *Fixed $\Omega, s>0$ let $C(\eta)=\Omega e^{-s\eta^{-\frac{1}{2}}}$; and let $\mathcal{N}=\gamma\eta^{-\frac{1}{2}}$, $\gamma>0$. For η small and $|\varepsilon|<\varepsilon_0$, where ε_0 is given by (4.3), the Hamiltonian (1.1) has, for all η small enough, a family of hyperbolic invariant tori, with stable and unstable manifolds of dimension 3, whose rotation vectors $\underline{\omega}'=(\eta^{\frac{1}{2}}\omega_1',\eta^{-\frac{1}{2}}\omega_2)\in\mathcal{S}_0$ verify (3.5) and therefore have first components which fill the interval $[\tilde\omega_a,\tilde\omega_b]$ within $O(\Omega e^{-s\eta^{-\frac{1}{2}}})$ for all η small enough, provided $\varepsilon<\varepsilon_0$.*

More precisely the parametric equations of such tori can be written, for ε small enough and $\underline{\omega}'\in\mathcal{S}_0$ verifying (3.5), as:

$$\begin{aligned}\underline{A}&=\underline{H}_{\underline{\omega}'}(\underline{\alpha}') & I&=\mathcal{I}_{\underline{\omega}'}(\underline{\alpha}')\\ \underline{\alpha}&=\underline{\alpha}'+\underline{h}_{\underline{\omega}'}(\underline{\alpha}') & \varphi&=\Psi_{\underline{\omega}'}(\underline{\alpha}')\end{aligned}\tag{5.4}$$

with $\underline{H}_{\underline{\omega}'}, \underline{h}_{\underline{\omega}'}, \mathcal{I}_{\underline{\omega}'}, \Psi_{\underline{\omega}'}$, at fixed $\underline{\omega}'$, analytic in ε and in $\underline{\alpha}'\in\mathbf{T}^2$ and divisible by ε, so that such functions can be found by perturbation expansions. The smallness condition on ε is $|\varepsilon|<O(\varepsilon_0[C(\eta)]^{q\gamma^{-1}\eta^{\frac{1}{2}}})$ (a very weak condition, as γ can be arbitrarily prefixed) and the motion of the data (5.4) is simply $\underline{\alpha}'\to\underline{\alpha}'+\underline{\omega}'t$.

It is interesting to write the complete condition that ε is small enough (including the constants) and a more quantitative expression for the filling of the action axis. Since the above theorem relies on [CG] one has to find bounds for several quantities associated with (4.4). If M_0^{-1} denotes a lower estimate for the minimum value of the second derivative of (1.1) evaluated at $\varepsilon=0$ with respect to $a=\eta^{\frac{1}{2}}A_1$ (*i.e.* if M_0 is defined as in (2.2)) one finds, applying the condition (5.90) of [CG], for some $B,q>0$:

$$|\varepsilon|<\frac{1}{B}\varepsilon_0\left(\frac{\lambda_0\Gamma_0}{E_0}\,\frac{\lambda_0^2\eta}{E_0M_0}\,\xi_0\,C(\eta)\right)^{q/\mathcal{N}}=\text{const.}\,\varepsilon_0\,[C(\eta)]^{q\gamma^{-1}\eta^{\frac{1}{2}}}=O(\eta^{\frac{9}{2}+})\tag{5.5}$$

having unified various constants to simplify the expression (and after translation to the present symbols and conventions) and denoting $\frac{9}{2}+$ a prefixed number bigger than $\frac{9}{2}$. And under the same conditions the filling of phase space is within $O(M_0E_0\lambda_0^{-1}C(\eta))$.

A more detailed description of the above analysis is the following strengthening of the above theorem.

THEOREM 3. *Under the same hypotheses of theorem 2 above there exists a C^∞ function $\overline{\mathcal{H}}_0(\underline{A}',x')$, $x'=p'q'$, defined in $\mathcal{D}_{\rho_0/2,\xi_0/2,\kappa_0/2}$ and a C^∞ canonical transformation, defined on the latter domain, and having the form:*

$$\begin{aligned}\underline{A}&=\underline{A}'+\underline{H}(\underline{A}',\underline{\alpha}',p',q'), & I&=\mathcal{I}(\underline{A}',\underline{\alpha}',p',q')\\ \underline{\alpha}&=\underline{\alpha}'+\underline{h}(\underline{A}',\underline{\alpha}',p',q'), & \varphi&=\Psi(\underline{A}',\underline{\alpha}',p',q')\end{aligned}\tag{5.6}$$

with $\underline{H},\underline{h},\mathcal{I},\Psi$ of class C^∞ divisible by ε and analytic in ε, $\underline{\alpha}'$ at fixed $\underline{A}', x'=p'q'$ such that if $\underline{\omega}'=\partial_{\underline{A}'}\overline{\mathcal{H}}_0(\underline{A}',x')$ verifies (3.5) then in the new coordinates the Hamiltonian coincides, together with its first order derivatives, with $\overline{\mathcal{H}}_0(\underline{A}',x')$. And $\overline{\mathcal{H}}_0$ is close within $O(\varepsilon)$ to $h(\eta^{\frac{1}{2}}A_1')+\eta^{-\frac{1}{2}}\omega_2A_2'+G(p'q',\eta^{\frac{1}{2}}A_1')$. Hence the motion of the data $(\underline{A}',\underline{\alpha}',p',q')$ for which $\underline{\omega}'$ verifies (3.5) are $\underline{A}'\to\underline{A}'$, $\underline{\alpha}'\to\underline{\alpha}'+\underline{\omega}'t$, $p'\to p'e^{-g't}$, $q'\to q'e^{+g't}$ with $g'=\partial_{x'}\overline{\mathcal{H}}_0(\underline{A}',x')$ if $x'=p'q'$.

Remarks. (1) The value $C(\eta)=\Omega e^{-s\eta^{-\frac12}}$ is particularly interesting in view of the results in [GGM1], where the Hamiltonian

$$\mathcal{H}=\eta^{\frac12}A_1+\eta\frac{A_1^2}{2}+\eta^{-\frac12}A_2+\frac{I^2}{2}+g^2(\cos\varphi-1)+\varepsilon f(\varphi,\alpha_1,\alpha_2) \tag{5.7}$$

is studied, with f an even trigonometric polynomial of degree N,N_0 in $\underline{\alpha}\,,\varphi$ respectively, and the homoclinic splitting of the invariant tori whiskers is found to be of size of order $C(\eta)|_{s=\pi/2g}$. Note that (5.7) with (I,φ) near $(0,0)$ is a particular case of (1.1). One just uses "Jacobi's coordinates" (p,q) instead of (I,φ), see Appendix A9 of [CG].

In [GGM1] the slow frequency is considered to be η^a, with $a\ge0$, so that $a=1/2$ is only a particular case. Of course in what follows we could also consider an arbitrary value $a\ge0$, and essentially nothing would change, but we have preferred to confine ourselves to the case $a=1/2$ for definiteness.

(2) In general the functions $\underline{H}\,,\,\underline{h}$ in theorem 3 do not have zero average for $p'=q'=0$. The even symmetry of f and of the pendulum Hamiltonian in (5.7) imply that the variables $\underline{A}'$ *have a simple physical interpretation* for the points on the invariant tori: they are just the *time averages* of the unperturbed actions. In fact this "*absence of torsion*" is a very remarkable and useful property of the system (5.7) (*Thirring model*) which was pointed out in [G3,G4] (and called the property of the tori of being *twistless*). It is due to the special symmetries of the system (5.7) and to the separation of the energy into a quadratic part involving actions only and an angular part involving only the angles.

(3) A further consequence of the symmetries of (5.7) is that each invariant torus described by theorem 2 has stable and unstable manifolds that intersect along a trajectory, a *homoclinic trajectory*, that when $\varphi=\pi$ has angular coordinates $\underline{\alpha}=\underline{0}$.

The difference at $\underline{\alpha}\,,\varphi$, that we denote $\underline{Q}\,(\underline{\alpha}\,,\varphi;\eta^{\frac12}A_1')$, between the $\underline{A}$–coordinates of the stable and unstable manifolds of the torus $\mathcal{T}(A_1')$ with average action A_1', vanishes at $\varphi=\pi$, $\underline{\alpha}=\underline{0}$, see [CG], §9. Hence we say that $\underline{\alpha}=\underline{0},\varphi=\pi$ is a *homoclinic point* for $\mathcal{T}(A_1')$. Setting $\underline{Q}\,(\underline{\alpha}\,)\stackrel{def}{=}\underline{Q}\,(\underline{\alpha}\,,\pi;\eta^{\frac12}A_1')$ the matrix $D_{ij}=\partial_{\alpha_i}Q_j(\underline{\alpha}\,)|_{\underline{\alpha}=\underline{0}}$ is called the *intersection matrix* and in general has a determinant $\det D$ that does not vanish: its value is called the *homoclinic splitting*, see [GGM1].

(4) The same result as theorem 2 can be obtained, for the Hamiltonian (5.7), also from the Lindstedt's series expansion for the stable and unstable manifolds of the invariant tori, by using the methods developed in [E,G3,Ge1,Ge2] and an (improved) version of Siegel–Bryuno's lemma; but this is a completely different approach to the problem and we expose it, in detail, in a forthcoming paper, [GGM2]. The new approach yields, as expected, a better bound on the convergence radius ε_0: namely $\varepsilon_0=O(\eta^2)$ compared to the $\sim O(\eta^{4.5+})$ that would be obtained by applying theorem 1 to (5.7) (see the remark 1 after theorem 1 and the comments after (5.5)).

(5) Theorem 3 means that there is a canonical system of coordinates in which the Hamiltonian takes the *normal form* $\overline{\mathcal{H}}_0$ given by an expression like (3.3) *on a set with very small complement, in general not empty and open* (we stress that this is a statement different from the one that would hold if the Hamiltonian could have been really put in the form (3.3), *i.e.* in the whole vicinity of the real domain of (1.1)). The reason $\overline{\mathcal{H}}_0$ is called a "normal form" is that the data with $\underline{\omega}'$ verifying (3.5) evolve in a very simple fashion *i.e.* as $\underline{A}'=$ const., $\underline{\alpha}'\to\underline{\alpha}'+\underline{\omega}'t$, $p'\to p'e^{-\overline{g}'t}$ and $q'\to q'e^{\overline{g}'t}$ (see (3.3) and (3.4)).

(6) Another consequence of theorem 3 is that fixed $\underline{A}'$ so that $\underline{\omega}'=\partial_{\underline{A}'}\overline{\mathcal{H}}_0(\underline{A}',0)$ verifies (3.5) it is possible to fix $A_1(x')$ so that $\partial_{\underline{A}'}\overline{\mathcal{H}}_0(\underline{A}'(x'),x')\equiv\underline{\omega}'$ for, say, $|x'|<\kappa_0/4$. And, at the same time, the energy of the motions that start in $\mathcal{D}_{\rho_0/2,\xi_0/2,\kappa_0/2}$ with $\underline{A}'=\underline{A}'(x')$ and p',q' with $p'q'=x'$ and with $\underline{\alpha}'\in\mathbf{T}^2$

arbitrary is x'-independent. Here the linearity in A_2' is used to adjust the energy once $A_1'(x')$ is determined: so that the apparently useless role of the variable A_2', "reservoir energy" for keeping the second rotator in a constant speed rotation, can be well appreciated here. This is a key property for the theory of diffusion in phase space developed in [CG].

§6. Existence of heteroclinic chains.

The novelty of the results in §5 is the possibility of using the "large angles theorem" of [GGM1] in the same way as the erroneous result (with the same name) in §10 of [CG] was used to discuss heteroclinic chains.

In this section we restrict considerations to the system described by the Hamiltonian (5.7). Given an invariant torus with average action $\underline{A}'$ such that the corresponding rotation vector $\underline{\omega}$ verifies (5.2) with $C(\eta) = \Omega e^{-s\eta^{-\frac{1}{2}}}$, $\Omega, s > 0$, one has that the torus and its stable and unstable manifolds (whiskers) are analytic for $|\varepsilon| < 2\varepsilon_0$, for a suitable ε_0. The discussion in §5 gives $\varepsilon_0 = O(\eta^{\frac{9}{2}+})$ (by taking $\mathcal{N} = \gamma\eta^{-\frac{1}{2}}$ with γ large enough, see (5.5)), by the argument leading to theorem 2; as mentioned in §5, remark 4, a better value $\varepsilon_0 = O(\eta^2)$ can be obtained with a different method.

Then we want to infer that the homoclinic splitting, at the homoclinic point $(\varphi, \alpha_1, \alpha_2) = (\pi, 0, 0)$ (see §5, remark 3), of such a torus is generically given by the first order (*Melnikov's integral*), *i.e.* it is of the form:

$$\sigma\varepsilon^2\eta^{-b}e^{-\frac{\pi}{2}|\omega_2|g^{-1}\eta^{-\frac{1}{2}}} \tag{6.1}$$

with σ a non vanishing constant and b a positive constant linearly depending on the degree N_0 (in φ) of the perturbation f. The finiteness of N_0 is thus a key assumption for the validity of the following analysis.

The theorems of [GGM1] do not apply directly because in the latter paper $C(\eta)$ was taken proportional to η^d for some $d > 0$. On the other hand, by the considerations in §5, the results of [GGM1] can be immediately extended to cover also this case, *i.e.* to conclude that the splitting is given asymptotically by Melnikov's integral, as we shall see below.

By theorems 1÷3 above, the invariant tori and their stable and unstable manifolds are analytic in ε for $|\varepsilon| < 2\varepsilon_0$, with $\varepsilon_0 = O(\eta^{c'})$, where $c' > 9/2$ or $c' = 2$ (according to which bound for ε_0 is chosen, see the remark 4 in §5). This implies that, to any order h, the parametric equations of the manifolds (*i.e.* $\underline{h}$, $\underline{H}$ as functions of $\underline{\alpha}, \varepsilon$ at $x' = p'q' = 0$, A_1' fixed) evaluated at $\varphi = \pi$ are analytic in ε and their Taylor series coefficients can be bounded to order h by:

$$D_2B_2^{h-1}(\varepsilon 2^{-1}\varepsilon_0^{-1})^h \tag{6.2}$$

for some positive constants D_2, B_2.

Moreover we know, from the analysis in [G3] and [GGM1], that, to any order $h \le \mathcal{N} = \gamma\eta^{-\frac{1}{2}}$, the Taylor coefficients for the splitting are bounded, to order h, by

$$D_3h!^4B_3^{h-1}(\varepsilon\eta^{-\beta'})^h e^{-\frac{\pi}{2}|\omega_2|g^{-1}\eta^{-\frac{1}{2}}} \tag{6.3}$$

with $\beta' = 2(N_0+1)+2$, for some positive constants D_3, B_3 and with N_0 equal to the degree in φ of the perturbation f (hence $N_0 \le N$: the convenience in distinguishing between N and N_0 is that it is possible to extend the present work to cover cases in which f is analytic in $\underline{\alpha}$, *i.e.* $N = \infty$, provided one keeps it polynomial in the angles φ).

No factor $C(\eta)$ appears in (6.3) as no resonances occur in the bounds of the divisors that appear in the perturbation expansions (of [G3,GGM1]) contributions to the splitting at orders $\leq \mathcal{N}$, by the very choice of $\mathcal{N}$. In fact as long as only orders $\leq \mathcal{N}$ are considered one has $|\underline{\omega}' \cdot \underline{\nu}| \geq \eta^{\frac{1}{2}}$ as $|\underline{\nu}| \leq N\gamma\eta^{-\frac{1}{2}} = N\mathcal{N}$.

Then if $\varepsilon \leq O(\eta^c)$, $c > \max\{c', \beta' + 2\} = \beta' + 2$, the contribution to the splitting of the orders $> \mathcal{N}$ can be estimated from (6.2) by $O((\varepsilon 2^{-1}\varepsilon_0^{-1})^{\mathcal{N}})$: which is much smaller than the generic homoclinic splitting, (6.1), evaluated from Melnikov's integral.

The contribution arising from the orders $3 \leq h \leq \mathcal{N}$ is bounded, via (6.3), by $O((\varepsilon\eta^{-\beta})^3 e^{-\frac{\pi}{2}|\omega_2|g^{-1}\eta^{-\frac{1}{2}}})$, with $\beta = \beta' + 2$, see [G3,GGM1], also much smaller than (6.1) if ε is smaller than a high enough power of η (*i.e. without a condition of exponential smallness of* ε).

Hence we conclude that under a condition like $|\varepsilon| < O(\eta^c)$, with η small enough, tori with average action $\underline{A}'$, arbitrarily prefixed energy *and* verifying (5.2) with $C(\eta) = \Omega e^{-s\eta^{-\frac{1}{2}}}$, Ω, s preassigned) exist and their homoclinic splitting has size $O(\varepsilon^2\eta^{-b}e^{-\frac{\pi}{2}|\omega_2|g^{-1}\eta^{-\frac{1}{2}}})$ generically in perturbations f which are trigonometric polynomials in φ (and the constant b depends on the degree N_0 in φ of f).

For instance the simplest case (5.7) with $f = \cos(\alpha_1 + \varphi) + \cos(\alpha_2 + \varphi)$ and $g = 1$ gives a leading splitting, see [GGM1] eq. (6.14) and (7.19), as $\eta \to 0$:

$$32\pi\eta^{-\frac{1}{2}}\varepsilon^2 e^{-\frac{\pi}{2}\eta^{-\frac{1}{2}}} \tag{6.4}$$

We also conclude the (generic) existence, for $\varepsilon < O(\eta^c)$ and η small enough, of chains of isoenergetic invariant tori ("*heteroclinic chains*") with average actions $\underline{A}'_j$ such that:

(i) $\underline{\omega}'(0, \eta^{-\frac{1}{2}}A'_{j1})$ verify (5.2) with $C(\eta) = \Omega e^{-s\eta^{-\frac{1}{2}}}$ with s, Ω that can be fixed *a priori*, and $|\underline{A}'_j - \underline{A}'_{j+1}| < O(\eta^{-c''}e^{-s\eta^{-\frac{1}{2}}})$ for a suitable c'';

(ii) the homoclinic splitting has size $O(\varepsilon^2\eta^{-b}e^{-\frac{\pi}{2}|\omega_2|g^{-1}\eta^{-\frac{1}{2}}})$ (*i.e.* far larger than $C(\eta)$ if s is fixed *a priori* to be $s > \frac{\pi}{2}g^{-1}$);

(iii) the unstable manifold of the torus with average action $\underline{A}'_j$ intersects the stable manifold of the torus with average $\underline{A}'_{j+1}$ in a *heteroclinic trajectory* with splitting at $\varphi = \pi$ of order $O(\varepsilon^2\eta^{-\beta}e^{-\frac{\pi}{2}|\omega_2|g^{-1}\eta^{-\frac{1}{2}}})$: this follows, as usual, from an application of the implicit functions theorem because the equation to be solved for finding a heteroclinic point is an implicit equation with Jacobian determinant, at the trivial solution $\underline{\alpha} = \underline{0}$ (corresponding to the homoclinic point at $\varphi = \pi$ of $\underline{A}'_j$), given by the splitting $\det D$, see also [CG,GGM1];

(iv) the "genericity" is a very explicit condition because it simply requires the lowest order value of the splitting to be non vanishing: this holds generically in f (picked up inside the class of functions we are considering);

(v) furthermore, since the set of points verifying (5.2) is abundant in the sense of the above theorems 2 and 3, we also obtain (at no extra cost, *i.e.* by "abstract reasoning" on measure theory) that the values of A'_{j1} are density points for the set of values $A'_{j1} \in [-\eta^{-\frac{1}{2}}R, \eta^{-\frac{1}{2}}R]$ whose corresponding rotation vectors verify (5.2) with the chosen value of $C(\eta)$; and we can suppose that the values of A'_{j1} at the extremes of the chain are close to the extremes of the interval $[-\eta^{-\frac{1}{2}}R, \eta^{-\frac{1}{2}}R]$ within ε (or even closer).

We can summarize the above discussion in the following result.

THEOREM 4. *Given the system described by the Hamiltonian (5.7), there is a sequence of invariant tori with average actions* $\underline{A}'_1, \ldots, \underline{A}'_{\mathcal{N}}$, *with* $|\underline{A}'_1 - \underline{A}'_{\mathcal{N}}| \simeq 2R\eta^{-\frac{1}{2}}$ *(within* $O(\varepsilon)$*) and* $|\underline{A}'_i - \underline{A}'_{i+1}| \leq Re^{-s\eta^{-\frac{1}{2}}}$ *for* $i = 1, \ldots, \mathcal{N} - 1$ *and* $s > 0$, *such that the unstable manifold of each invariant torus intersects the stable manifold of the torus following it along the chain. At the heteroclinic point at* $\varphi = \pi$ *the* *splitting between the two manifolds is given by* $\sigma'\,\varepsilon^2\eta^{-b}e^{-\frac{\pi}{2}|\omega_2|g^{-1}\eta^{-\frac{1}{2}}}$, *where* σ', b *are constants depending on the perturbation* f, *with* σ' *generically not vanishing.*

This extends the theorem of existence of heteroclinic chains discussed in [GGM1], §8, to the anisochronous case.

§7. Fast averaging theorem and abundance.

A second remarkable application is a fast averaging theorem which follows from the proof of theorem 1 and concerns a system with Hamiltonian:

$$\mathcal{H} = h(\eta^{\frac{1}{2}}A_1) + \eta^{-\frac{1}{2}}\omega_2 A_2 + \frac{I^2}{2J_0} + J_0 g^2(\cos\varphi - 1) + \varepsilon_1 f_1(\underline{\alpha}, \varphi) + \varepsilon_2 f_2(\underline{\alpha}, \varphi) \quad (7.1)$$

where I, φ are conjugate variables, J_0, g are strictly positive constants and $\varepsilon_1, \varepsilon_2$ are perturbation parameters. The function f_1 is *monochromatic* (*i.e.* with Fourier transform not vanishing only on modes different from zero and multiples of a fixed $\underline{\nu}_0 = (\nu_1, \nu_2)$ with $\nu_2 \neq 0$), while f_2 is only required to be a trigonometric polynomial.

On the basis of the results of the previous sections we can expect that in order to have "long" heteroclinic chains of invariant tori we need to require ε_1 and ε_2 to be *both* small (of the order of ε_0 of theorems $1\div4$).

But this is not the case because of the monochromatic nature of f_1 and we shall show that the results of theorem $1\div3$ hold essentially under the assumptions that ε_2 is small of order ε_0 while ε_1 can be quite large, up to almost $\eta^{-\frac{1}{2}}$: in particular the value $\varepsilon_1 = 1$ is amply allowed. In a sense also theorem 4 could hold; however we can only show (see below) that verification of its validity can be reduced, at least in principle, to a finite computation if η, ε_2 and $\varepsilon_1 = 1$ are fixed with $\varepsilon_2 < \varepsilon_0$.

We consider first the case $\varepsilon_2 = 0$. In this case, by performing the canonical transformation with generating function $S = \underline{A}'\cdot\underline{\alpha} + I'\varphi - \varepsilon_1\eta^{\frac{1}{2}}\omega_2^{-1}\partial_2^{-1}f_0(\underline{\alpha},\varphi)$, one sees that the size of the perturbation can be reduced to $O(\eta^{\frac{1}{2}}\varepsilon_1)$ under the condition that $\eta^{\frac{1}{2}}\varepsilon_1$ is small enough (so that the canonical map generated by S makes sense in a domain slightly smaller than the domain of h).

This is *remarkable* because the change of coordinates is *globally defined* and it is not restricted to the vicinity of $I = \varphi = 0$, and it leaves the perturbation still a monochromatic trigonometric polynomial with zero average over $\underline{\alpha}$. This cannot be pursued to higher orders without losing the trigonometric polynomial nature of the perturbation.

Then we can then use Jacobi's coordinates near $I = \varphi = 0$, see [G2], to put the Hamiltonian in the form (1.1) with εf of order $O(\eta^{\frac{1}{2}}\varepsilon_1)$.

It follows from the proof of theorem 1 (see the remark 2 in §4) that if the perturbation is monochromatic the condition for being able to cast the Hamiltonian in the form (4.4), with a prefixed and η independent $\mathcal{N}$, say $\mathcal{N}_0$, is that $|\varepsilon_1| < O(1)$, so that we see that (7.1) can be cast in the form (4.4) with a prefixed $\mathcal{N}_0$ for $|\varepsilon_1|\eta^{\frac{1}{2}}$ small enough.

We consider now the case $\varepsilon_2 \neq 0$ and we imagine performing the change of variables that puts the part of (7.1) *not containing the last term* into the form (4.4) for some $\mathcal{N}_0$.

This casts the full Hamiltonian (7.1) into the form (1.1) *except for the fact that the perturbation is no longer* a trigonometric polynomial in the $\underline{\alpha}$ variables: it is just an analytic function in a domain of size $\xi_0 > 0$ in the complex planes for the $\underline{\alpha}$ variables.

We truncate its Fourier expansion in the $\underline{\alpha}$ at $|\underline{\nu}| < N$, where N is left as a parameter. The remainder can be bounded in its domain by $ae^{-\xi_0 N}$ for some $a > 0$ of order $O(|\varepsilon_1\eta^{\frac{1}{2}}| + |\varepsilon_2|)$.

We are, therefore, in a position to apply again theorem 1 and to cast the full (7.1) into the form (4.4). We make the following choices: $\mathcal{N}=\eta^{-\frac{1}{2}}\sqrt{(\log\eta^{-1})^{-1}}$, $N=\frac{1}{2}\eta^{-\frac{1}{2}}\sqrt{\log\eta^{-1}}$, (assuming $\eta<\frac{1}{2}$, which is not restrictive as we are interested in properties holding as $\eta\to0$).

Combined again with theorem 1 above, this means that we can put the Hamiltonian in the form (4.4) with the last term replaced by an "effective interaction" of order:

$$\Big(\frac{((\varepsilon_1\eta^{\frac{1}{2}})^{\mathcal{N}_0}+\varepsilon_2)(\log\eta^{-1})^{-\frac{3}{2}}}{\eta^{9/2}\overline{\varepsilon}_0}\Big)^{\eta^{-\frac{1}{2}}(\log\eta^{-1})^{-1/2}}+ae^{-\frac{\xi_0}{2}\eta^{-\frac{1}{2}}(\log\eta^{-1})^{1/2}}\ll e^{-z\eta^{-\frac{1}{2}}} \tag{7.2}$$

provided $\varepsilon_2=O(\eta^c)$, with c large enough and $\mathcal{N}_0$ so prefixed that $\mathcal{N}_0/2\ge c$; $a,\overline{\varepsilon}_0$ are constants that (if wished) can be explicitly estimated from the above argument. The inequality holds for *any* prefixed $z>0$ provided η is correspondingly suitably small.

This means that, in some sense, the size of the monochromatic term in (7.1) does not matter too much and $\varepsilon_1=1$ (or even almost $\varepsilon_1\simeq\eta^{-\frac{1}{2}}$) is a sufficient condition, together with $|\varepsilon_2|<\varepsilon_0$ and η small enough, to guarantee the existence of an invariant torus which is run quasi-periodically with a rotation $\underline{\omega}'$ verifying a (very weak) Diophantine condition ($C(\eta)=\Omega e^{-s\eta^{-\frac{1}{2}}}$) and whose equations can be constructed by a convergent perturbation series.

A further consequence is that the Hamiltonian (7.1) has many invariant tori whose average actions fill the action space as described in theorem 2 above, so that we can apply the results of [GGM1]. We summarize the above discussion in the following averaging theorem.

THEOREM 5. *Given $s>0$ consider the Hamiltonian (7.1) with f_j verifying the assumptions following (7.1). If $|\varepsilon_1|<\eta^{-d}$, $|\varepsilon_2|<\eta^{+d'}$ with prefixed $0\le d<\frac{1}{2}$ and $d'>\frac{9}{2}$ there is a family of invariant tori $\mathcal{T}(\underline{A}')$ with average actions $\underline{A}'$ filling action space within $O(e^{-s\eta^{-\frac{1}{2}}})$. The parametric equations of such tori can be computed by perturbation theory (i.e. by convergent power series in $\varepsilon_1,\varepsilon_2$) and they are, together with the homoclinic splitting, analytic functions of $\varepsilon_1,\varepsilon_2$ provided η is small enough (depending on the size of s,d,d').*

Note that the "torsion free" property allows us to fix the rotation vector of the motions on an invariant torus by fixing the average value A'_1 of the action, no matter which is the perturbation size ε: in fact in the model (7.1) that we are considering it is $\underline{\omega}'=(\eta^{\frac{1}{2}}+\eta A'_1,\eta^{-\frac{1}{2}}\omega_2)$ which is ε–independent (remarkably enough).

Apart from the intrinsic interest of theorem 3 it has some relevance for the theory of drift and diffusion in *a priori* stable systems, like the precession problem whose analysis was attempted in [CG], §12. We discuss this aspect in the next section.

§8. Drift and diffusion under large monochromatic forcing and small quasi periodic perturbation, or in a priori stable systems

The interest of the above theorem 5 is, again, that it *might* be used, by repeating the argument in §6 to show the existence of heteroclinic chains with the same properties of the ones discussed in that comment even if $\varepsilon_1=O(1)$.

This time *there is, however, an extra difficulty if ε_1 is large, i.e. really of order* 1: the argument of §6 provides us with a tool to bound the remainders, but *not* to find out which is the dominant term among those of "small" order (*i.e.* smaller than $\mathcal{N}$ in §6).

The reason is simply that the contributions of higher order in ε_1 not only are not smaller than those of order 1 but in fact are checked to be *a priori* larger. Meaning that one can find *some* contributions to them coming from suitable graphs, in the sense of the representation in [GGM1], and *larger* than the lowest order contribution: but in a subject where cancellations are quite common this does not seem to mean much.

However the results in [GGM1] and theorem 4 above show that the problem of proving the existence of a heteroclinic chain joining the extremes of the action A'_1 *interval* $[-R\eta^{-\frac{1}{2}}, R\eta^{-\frac{1}{2}}]$ *can in principle be solved by a finite calculation*, if ε_1, ε_2 and η are fixed. The program is illustrated in the following.

In fact such chains can be proved to exist and to consist of finitely many elements for $|\varepsilon_1|$ small enough (by theorem 3) and the above discussion shows that the series in $\varepsilon_1, \varepsilon_2$ for the splitting is analytic, and the term of order h_1 in ε_1 and h_2 in ε_2 is bounded by

$$B^{h_1+h_2}(\varepsilon_1\eta^{\frac{1}{2}})^{h_1}(\varepsilon_2\eta^{-c})^{h_2} \tag{8.1}$$

where B is a suitable constant.

Assume that we can show that the sum of the first $h_1+h_2 \le \gamma\eta^{-\frac{1}{2}}$ terms in the series for the splitting is bounded from above and below by a quantity $O(e^{-\frac{\pi}{2}|\omega_2|g^{-1}\eta^{-\frac{1}{2}}})$; then the remainder is surely smaller than this quantity. Hence we can approximate the heteroclinic points for an allowed value of ε_1: keeping fixed the average action A'_1 we can increase it following the evolution of the heteroclinic point. This is a finite calculation, in principle, because we start from $\varepsilon_1 > 0$ and have convergent expansions up to $\varepsilon_1 = 1$ (and beyond up to almost $\eta^{-\frac{1}{2}}$) thanks to the above fast averaging observations, valid close enough to the tori (where the changes of variables discussed to reach (7.2) make sense). Thus with a finite calculation one can get controlled approximations of the values of the heteroclinic angles (at fixed $\varepsilon_1, \varepsilon_2, \eta$).

What is not *a priori* guaranteed is that such angles, *which start positive* and "large" compared to the separation between the tori, do not become small while ε_1 grows.

However there is no reason for this to happen: one expects the angles to become wider, not smaller, or to vanish only for finitely many values of ε_1, at least generically.

Hence a computer assisted analysis of the problem is likely to succeed and it might even be used to reach values of η and $\varepsilon_1, \varepsilon_2$ where the angles are large and the spacing between the tori can be widened (thus diminishing the number of invariant tori of the chains). As discussed in [CG] the drift along a heteroclinic chain with non zero splitting is always present and one would obtain models with Arnol'd diffusion in systems like (7.1) with $\varepsilon_1 = 1$. This would correct even the applications of the incorrect version of the large angle theorem of [CG] (see §12 of [CG]) and make it rest entirely on the corrected version in [GGM1].

In fact a problem like (7.1) with ε_1 large ($\varepsilon_1 = O(1)$) arises in the theory of Arnol'd diffusion in *a priori* stable systems, like the precession problem treated in §12 of [CG]. From the analysis in [CG] it seems that such large (but monochromatic) coupling Hamiltonian systems might arise often in the reduction of a motion near a multiple resonance of a *a priori* stable system.

Remark. It is interesting to note that the above program with $h(a) = \omega_1 a$, *i.e. (7.1) in the isochronous case, can be carried out without need of a computer as remarked in [GGM1] (the diffusion then follows via the results in [G2]) at least for all* $\varepsilon_1 < O(\eta^{-\frac{1}{2}})$ *except possibly a finite number of them.*

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Appendix A1. Proof of theorem 1.

Suppose that for $h \le k-1$ one has:

$$||\Phi^{(h)}|| \le DB^{h-1}(h-1)!, \qquad ||\tilde G^{(h)}|| \le E_0 B^{h-1}(h-1)! \tag{A1.1}$$

where D is such that the bound holds for $h=1$. Here $||\cdot||$ denotes the maximum in $\mathcal{D}_{\rho_h,\xi_h,\kappa_h}$ with $\xi_h = \xi_0 - h\delta, \rho_h = \rho_0 e^{-\delta h}, \kappa_h = \kappa_0 e^{-h\delta}$. Furthermore the Fourier transform of $\Phi^{(h)}$ with respect to the $\underline{\alpha}$–variables has only modes $\underline{\nu}$ with $|\underline{\nu}| < Nh$.

Therefore for $h < \zeta\eta^{-1}N^{-1}$, with $\zeta = |\omega_2|(2\tilde\omega_b)^{-1}$, we can bound the various terms in the difference between the two sides of (3.1) as follows.

We expand in Taylor's series the functions in (3.1) with respect to the point obtained by setting $\Phi = 0$. We define $\underline{\omega} \stackrel{def}{=} (\eta^{\frac12}\partial h(\eta^{\frac12}A_1') + \eta^{\frac12}\partial G(p'q,\eta^{\frac12}A_1'), \eta^{-\frac12}\omega_2)$, $\overline{g}(p'q,\eta^{\frac12}A_1') \stackrel{def}{=} \partial_x G(x,\eta^{\frac12}A_1')|_{x=p'q}$, $\not\partial \stackrel{def}{=} q\partial_q - p'\partial_{p'}$ and $\Delta \stackrel{def}{=} \underline{\omega}\cdot\partial_{\underline{\alpha}} + \overline{g}\not\partial$. Then the *linear term* in $\Phi^{(k)}$ is simply $\Delta\Phi^{(k)}$.

The nonlinear terms arising from $h(\eta^{\frac12}(A_1' + \partial_{\alpha_1}\Phi))$ contribute to $\Phi^{(k)}$ an amount $X_1^{(k)}$ such that:

$$\Delta X_1^{(k)} = \Big(\sum_{a\ge2} h^{(a)}\eta^{\frac12 a} \sum_{\substack{k_1+\ldots+k_a=k\\ k_j\ge1}} \prod_{j=1}^{a} \partial_{\alpha_1}\Phi^{(k_j)}\Big)^* \tag{A1.2}$$

where the $*$ means that from the Taylor expansion in p', q and Fourier expansion in $\underline{\alpha}$ of the term in parentheses one has to subtract the terms which have $\underline{0}$ Fourier mode and that have an equal power of p' and q. In fact the latter terms make up the contribution to $\tilde G^{(k)}$ of the nonlinear terms arising from $h(\eta^{\frac12}(A_1' + \partial_{\alpha_1}\Phi))$. After this subtraction the operator Δ is invertible, *i.e.* $X_1^{(k)}$ is well defined and the action of Δ^{-1} can be bounded by $\Gamma_0^{-1}C(\eta,\delta)$ because as long as $|\underline{\nu}| < \zeta\eta^{-1}$ one has:

$$\sum_{\underline{\nu},n,m}^{+} \frac{e^{-\frac{\delta}{2}(|\nu_1|+|\nu_2|+|m|+|n|)}}{(|\nu_1|\eta^{\frac12} + |\nu_2|\eta^{-\frac12} + |m-n|)\Gamma_0} \le \Gamma_0^{-1}C(\eta,\delta) \tag{A1.3}$$

where the $+$ means that the sum runs over the $\underline{\nu}, m, n$ for which the divisor does not vanish and, furthermore, $|\underline{\nu}| < \zeta\eta^{-1}$, so that no "resonances" occur. The quantity $C(\eta,\delta)$ can be taken

$$C(\eta,\delta) = c\left(\eta^{\frac12}\delta^{-3} + \eta^{-\frac12}\delta^{-2} + \delta^{-1}\right)\log\delta^{-1} > 1 \tag{A1.4}$$

where c is a suitable constant (recalling also that $\varepsilon, \eta < 1$, see §2).

Making use of the analyticity assumption, which implies that $|h^{(a)}| < E_0\rho_0^{-a}$ in $D_{\rho_0,\xi_0,\kappa_0}$, one has in $D_{\rho_k,\xi_k,\kappa_k}$:

$$|X_1^{(k)}| \le \frac{E_0 C(\eta,\delta)}{\Gamma_0} \sum_{a\ge2} \Big(\frac{2\eta^{\frac12}D}{\rho_0\delta}\Big)^a B^{k-a} \sum_{\substack{k_1+\ldots+k_a=k\\ k_j\ge1}} \prod_{j=1}^{a}(k_j-1)! \tag{A1.5}$$

where the factor 2 above comes from the estimates of the Fourier transform of the factors $\partial_{\alpha_1}\Phi^{(k_j)}$ in (A1.2) inside the domains $D_{\rho_{k_j+1/2},\xi_{k_j+1/2},\kappa_{k_j+1/2}}$.

We shall repeatedly use below the inequality: $\sum_{\substack{k_1+\ldots+k_a=k\\ k_j\ge1}} \prod_{j=1}^{a}(k_j-1)! \le (k-1)!$ (to prove it, simply bound the product of the factorials by the $(k-a)!$, and the number of addends by $(k-a+1)^{a-1}/a!$). Then, from (A1.5),

$$|X_1^{(k)}| \le \frac{E_0 C(\eta,\delta)}{\Gamma_0} B^{k-1}(k-1)!\,B\,\Big(\frac{2\eta^{\frac12}D}{\rho_0 B\delta}\Big)^2 2 \qquad \text{if} \quad \frac{2\eta^{\frac12}D}{\rho_0 B\delta} < \frac12 \tag{A1.6}$$

Hence in $D_{\rho_k,\xi_k,\kappa_k}$:

$$|X_1^{(k)}| < \frac{1}{5} D B^{k-1}(k-1)! \qquad \text{if} \quad \frac{2\eta^{\frac{1}{2}} D}{\rho_0 B\delta} < \frac{1}{2}, \qquad 2\frac{E_0 C(\eta,\delta)}{\Gamma_0}\frac{4\eta D}{\rho_0^2 B\delta^2} < \frac{1}{5} \tag{A1.7}$$

The higher order terms contribution $X_2^{(k)}$ from the third term in (3.1) is:

$$\Delta X_2^{(k)} = \Big(\sum_{\substack{n,a \\ a+n\geq 2}} G^{(a,n)} \sum_{\substack{k_1+\ldots+k_a+ \\ h_1+\ldots+h_n=k}} \eta^{\frac{1}{2}n} \prod_{j=1}^{n} \partial_{\alpha_1}\Phi^{(h_j)} \prod_{i=1}^{a} q\partial_q \Phi^{(k_i)} \Big)^* \tag{A1.8}$$

with $h_j, k_i \geq 1$. So that in $D_{\rho_k,\xi_k,\kappa_k}$:

$$\begin{aligned} |X_2^{(k)}| &\leq \frac{E_0 C(\eta,\delta)}{\Gamma_0} \sum_{\substack{n,a \\ a+n\geq 2}} \frac{2^n \eta^{\frac{1}{2}n}}{\rho_0^n \delta^n} D^n \kappa_0^{-a} B^{k-n-a} \frac{2^a D^a}{\delta^a}(k-1)! \leq \\ &\leq \frac{E_0 C(\eta,\delta)}{\Gamma_0} \sum_{\substack{n,a \\ a+n\geq 2}} \Big(\frac{2D}{\lambda_0 B\delta}\Big)^{n+a} \eta^{\frac{1}{2}n} B^k (k-1)! \leq \frac{1}{5} D B^{k-1}(k-1)! \end{aligned} \tag{A1.9}$$

provided:

$$\frac{2D}{\delta\lambda_0 B} < \frac{1}{2}, \qquad 4\frac{E_0 C(\eta,\delta)}{\Gamma_0}\frac{4D}{\lambda_0^2 B\delta^2} < \frac{1}{5} \tag{A1.10}$$

and note that (A1.10) implies the conditions in (A1.7).

Likewise the fourth term in (3.1) yields a higher order contribution $X_3^{(k)}$ bounded as:

$$\begin{aligned} |X_3^{(k)}| &\leq \frac{E_0 C(\eta,\delta)}{\Gamma_0} \sum_{\substack{a,n \\ a+n\geq 1}} \frac{\eta^{\frac{1}{2}a}}{\rho_0^a \kappa_0^{\frac{1}{2}n}} \frac{2^{a+n} D^{a+n} B^{k-1-a-n}}{\delta^{a+n}\kappa_0^{\frac{n}{2}}}(k-1)! \leq \\ &\leq 4B^{k-1}(k-1)! \frac{E_0 C(\eta,\delta)}{\Gamma_0}\frac{2D}{\lambda_0 B\delta} \qquad \text{if} \quad \frac{2D}{\lambda_0 B\delta} < \frac{1}{2} \end{aligned} \tag{A1.11}$$

so that one has in $D_{\rho_k,\xi_k,\kappa_k}$:

$$|X_3^{(k)}| \leq \frac{1}{5} D B^{k-1}(k-1)!, \qquad \text{if} \quad \frac{2D}{\lambda_0 B\delta} < \frac{1}{2}, \qquad 4\frac{E_0 C(\eta,\delta)}{\Gamma_0}\frac{2}{\lambda_0 B\delta} < \frac{1}{5} \tag{A1.12}$$

The fourth higher order contribution comes from the third term in the r.h.s. of (3.1) and is bounded in the same way as $X_2^{(k)}$, leading to the same bound under the same conditions. The fifth higher contribution comes from the last term in the r.h.s. of (3.1), which, up to the term $\tilde{G}^{(k)}$, is bounded in $D_{\rho_k,\xi_k,\kappa_k}$ as:

$$\begin{aligned} |X_5^{(k)}| &\leq \frac{E_0 C(\eta,\delta)}{\Gamma_0} \sum_{n\geq 1} \kappa_0^{-n} B^{k-n} 2^n D^n \frac{1}{\delta^n}(k-1)! \leq \\ &\leq 2\frac{E_0 C(\eta,\delta)}{\Gamma_0}\frac{2D}{\lambda_0 B\delta} B^{k-1}(k-1)! \leq \frac{1}{5} D B^{k-1}(k-1)! \end{aligned} \tag{A1.13}$$

provided:

$$\frac{2D}{\lambda_0 B\delta} < \frac{1}{2} \qquad 2\frac{E_0 C(\eta,\delta)}{\Gamma_0}\frac{2}{\lambda_0 B\delta} < \frac{1}{5} \tag{A1.14}$$

It remains to estimate $\tilde{G}^{(k)}$ itself. Noting that it is the collection of the terms that are subtracted by the above $*$ operations it is clear that it is bounded by

$DB^{k-1}(k-1)!\Gamma_0 C(\eta,\delta)^{-1}$ because one does not have to invert Δ to find it from the bounds on the higher order terms of (3.1), and one thus saves a division by $\Gamma_0 C(\eta,\delta)^{-1}$. Hence we fix $D = E_0\Gamma_0^{-1}C(\eta,\delta)$ and, with this choice, the conditions imposed to get the above bounds, are all implied by:

$$\frac{E_0 C(\eta,\delta)}{\Gamma_0\lambda_0 B\delta},\ \Big(\frac{E_0 C(\eta,\delta)}{\Gamma_0\lambda_0\delta}\Big)^2\frac{1}{B} < c' \tag{A1.15}$$

for some small enough c'. Hence fixing an order $\mathcal{N}$ the induction works for $k \le \mathcal{N}$ provided we take $\delta = \xi_0/4\mathcal{N}$ and we find (since $4E_0C(\eta,\delta)/(\Gamma_0\lambda_0\xi_0) > 1$, for η small enough):

$$b\Big(\frac{E_0 C(\eta,\delta)\mathcal{N}}{\Gamma_0\lambda_0\xi_0}\Big)^2 = B, \qquad D = \frac{E_0C(\eta,\delta)}{\Gamma_0} \tag{A1.16}$$

The condition on ε has to be such that the map (3.2) can be defined in the domain $\mathcal{D}_{\xi_0/2,\rho_0 e^{-2\xi_0},\kappa_0 e^{-2\xi_0}}$, which means that:

$$|\partial_{\underline{\alpha}}\Phi| \ll \xi_0\rho_0, \quad |\partial_{\underline{A}}\Phi| \ll \xi_0, \quad |\partial_{p'}\Phi| \ll \kappa_0^{\frac12}\xi_0, \quad |\partial_q\Phi| \ll \kappa_0^{\frac12}\xi_0 \tag{A1.17}$$

Bounding the sum $\Phi = \sum_{j=1}^{\mathcal{N}}\varepsilon^j\Phi^{(j)}$ and $\sum_{j=1}^{\mathcal{N}}\varepsilon^j\tilde G^{(j)}$ via the bound (A1.1), (A1.16) this means that the corresponding conditions for the existence of the canonical maps become: $|\varepsilon B\mathcal{N}| < 1/2$, $|\varepsilon D| \le \xi_0^2\lambda_0$, which requires ε to be smaller than the minimum ε_0 between $(\Gamma_0\lambda_0\xi_0E_0^{-1}C(\eta,\delta)^{-1})^2\mathcal{N}^{-3}b_0$, with b_0 suitably small, and $\Gamma_0\lambda_0\xi_0^2E_0^{-1}C(\eta,\delta)^{-1}b_0$:

$$|\varepsilon| < \varepsilon_0 = b_0\Big(\frac{\Gamma_0\lambda_0\xi_0}{E_0}\Big)^2\frac{1}{\mathcal{N}^3C(\eta,\delta)^2} \tag{A1.18}$$

where one power of $\mathcal{N}$ in the factor $\mathcal{N}^{-3}$ comes from $\mathcal{N}! < \mathcal{N}^{\mathcal{N}}$ (see (A1.1)) and the other two from the $\mathcal{N}$ in (A1.16); and we use also $\xi_0 < 1$ and $\Gamma_0\lambda_0C(\eta,\delta)^{-1}E_0^{-1} < 1$ (see (2.2)), so that the minimum in (A1.18) is reached in the first term.

It remains to study the remainders of order $> \mathcal{N}$ in the Taylor expansion: this can easily done in terms of $\mathcal{N}$, B and D, and a bound $2(\varepsilon/\varepsilon_0)^{\mathcal{N}}E_0$ is obtained. Then from the fact that $C(\eta,\delta) \le c_0\eta^{-\frac12}\xi_0^{-1}(\mathcal{N}\xi_0^{-1})^2\log(\mathcal{N}\xi_0^{-1})$, for $\mathcal{N} \le \zeta(\eta N)^{-1}$, theorem 1 follows, with $b' = b_0c_0^{-2}$ small enough.

A *corollary* of theorem 1 is a *fast averaging* result: namely if there are *no slow frequencies* (*i.e.* if the Fourier transform in $\underline{\alpha}$ contains only Fourier modes which are non-zero multiples of a given $\underline{\nu}_0 = (0,\nu_2)$ or more generally of $\underline{\nu}_0 = (\nu_1,\nu_2)$ with $\nu_2 \ne 0$, *i.e.* if f is "*monochromatic*") the lower bound on Δ is $\Gamma_0\overline{C}(\eta,\delta)^{-1}$ with $\overline{C}(\eta,\delta) = (\eta^{\frac12}\delta^{-2}+\delta^{-1})\log\delta^{-1}$. Hence if $\mathcal{N}$ is *fixed arbitrarily but η–independent* the condition for casting the Hamiltonian in the form (4.4) is $|\varepsilon| < O(1)$.

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