GEOMETRIC MODULAR ACTION AND SPACETIME SYMMETRY GROUPS

Detlev Buchholz, Olaf Dreyer,
Martin Florig and Stephen J. Summers

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Abstract. A condition of geometric modular action is proposed as a selection principle for physically interesting states on general space-times. This condition is naturally associated with transformation groups of partially ordered sets and provides these groups with projective representations. Under suitable additional conditions, these groups induce groups of point transformations on these space-times, which may be interpreted as symmetry groups. The consequences of this condition are studied in detail in application to two concrete space-times – four-dimensional Minkowski and three-dimensional de Sitter spaces – for which it is shown how this condition characterizes the states invariant under the respective isometry group. An intriguing new algebraic characterization of vacuum states is given. In addition, the logical relations between the condition proposed in this paper and the condition of modular covariance, widely used in the literature, are completely illuminated.

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I. Introduction

In [9][10], Bisognano and Wichmann showed that for quantum field theories satisfying the Wightman axioms the modular objects associated by Tomita-Takesaki theory to the vacuum state and local algebras in wedgelike regions in Minkowski space have geometrical interpretation. This fundamental insight has opened up a number of fascinating lines of research for algebraic quantum field theory. To better appreciate the ramifications of their result, it is important to realize that the modular objects of the Tomita-Takesaki theory are completely determined by the choice of physical state and algebra of observables. That these modular objects can also have geometrical and dynamical significance thus allows the conceptually important possibility of *deriving* geometrical and dynamical information from the latter physical data.

For example, it has become possible to characterize physically distinguished states by the geometric action of the modular objects associated with suitably chosen local algebras. This approach was taken in [24] (cf. also [13]), where it was shown how the vacuum state on Minkowski space can be characterized by the action of the modular objects associated with wedge algebras and how the dynamics of the theory can be derived from the modular involutions. The present paper is in several respects a refinement and generalization of [24].

Another program which has grown out of Bisognano and Wichmann's insight is the construction of nets of local algebras and representations of a group acting covariantly upon the net, starting from a state, a small number of algebras, and a suitable "geometric" action of the associated modular objects upon these algebras. This line was first addressed in [12], cf. also [58]. The most complete results in this direction have been, on the one hand, the construction of conformally covariant nets of local algebras in two spacetime dimensions in [70][72] and, on the other, of Poincaré covariant nets in three spacetime dimensions in [74] (see also [73][15]).

Yet another closely related research program is the generation of unitary representations of spacetime symmetry groups by modular objects which are assumed to implement the action of subgroups of these symmetry groups upon a given net of algebras. This course of study using the unitary modular groups was also opened up by Borchers [12] and followed in [69][21][22][36][35], whereas the derivation of such representations from the modular involutions was initiated in [24]. This aspect we also generalize in this paper. Moreover, we shall clarify the relations between these two different approaches to geometric action of modular objects. For a more detailed review of the prior literature, see [14].

As explained in our first paper on the subject [24], a further interesting step is the *derivation* of spacetime symmetry groups from the underlying algebraic structure and the given state. By "space-time" we here mean some smooth manifold without *a priori* given metric or conformal structure. From our point of view, if a given net of observable algebras happens to be covariant under the action of a unitary representation of some group of point transformations of the underlying manifold, then these point transformations should be regarded as the isometries of a metric structure to be imposed upon the space-time. We mention in this context the papers [43][76], in which the causal (*i.e.* conformal) metric structure of the space-time is derived from the states and algebras of observables, under certain conditions.

It is the essential lesson of the present paper that the various goals mentioned above – the derivation of spacetime symmetry groups, the generation of corresponding unitary representations, and the characterization of physically distinguished states from the algebraic data – can all be accomplished in physically interesting examples by a Condition of Geometric Modular Action proposed in [24]. This fact sheds new light on the results mentioned above and poses some new and intriguing questions.

We shall present this condition somewhat imprecisely in this introduction – further details will be given in the main text. Let $\mathcal{W}$ be a suitable collection of open sets on a space-time $\mathcal{M}$ and $\{\mathcal{A}(W)\}_{W\in\mathcal{W}}$ be a net of C^*-algebras indexed by $\mathcal{W}$, each of which is a subalgebra of the C^*-algebra $\mathcal{A}$. A state on $\mathcal{A}$ will be denoted by ω and the corresponding GNS representation of $\mathcal{A}$ will be signified by $(\mathcal{H},\pi,\Omega)$. For each $W\in\mathcal{W}$ the von Neumann algebra $\pi(\mathcal{A}(W))''$ will be denoted by $\mathcal{R}(W)$. The modular involution associated to the pair $(\mathcal{R}(W),\Omega)$ will be represented by J_W, while the modular group associated to the same pair will be written as $\{\Delta_W^{it}\}_{t\in\mathbb{R}}$.

Condition of Geometric Modular Action. *Given the structures indicated above, then the pair $(\{\mathcal{R}(W)\}_{W\in\mathcal{W}},\omega)$ satisfies the Condition of Geometric Modular Action if the collection of algebras $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ is stable under the adjoint action of the modular involution J_W associated with the pair $(\mathcal{R}(W),\Omega)$, for all $W\in\mathcal{W}$. In other words, for every pair of regions $W_1,W_2\in\mathcal{W}$ there is some region $W_1\circ W_2\in\mathcal{W}$ such that*

$$J_{W_1}\mathcal{R}(W_2)J_{W_1}=\mathcal{R}(W_1\circ W_2) \quad . \tag{1.1}$$

This condition was initially motivated by a number of examples in Minkowski space-time in which modular objects have a geometric action implying the above condition (see [9][10][23][39][29]). We emphasize that this condition does not assume that the adjoint action of the modular involutions upon the net acts in the detailed manner of the cited examples – indeed, it is not even assumed that this action can be realized as a point transformation on the space-time. In fact, we imagine that there will be situations of physical interest in which this geometric action is *not* implemented by point transformations, but where this condition will still serve as a useful selection criterion.

Note that this condition can be stated sensibly for arbitrary space-time, indeed for arbitrary topological space $\mathcal{M}$. This enables us to propose this Condition of Geometric Modular Action as a criterion for selecting physically interesting states on *general* space-times. We anticipate that in some applications this condition will have to be weakened in evident ways. In particular, there are circumstances where only (even) *products* of modular involutions will act "geometrically" in this manner – here we think, for example, of the Rindler wedge [41]. We expect that also these weakened versions should select states of notable physical interest.

We emphasize that our selection criterion is one for a *state* and not an entire folium. In particular, previously suggested criteria, such as the Hadamard condition [42] and the microlocal spectrum condition [56], are valid for an entire folium of states. Though these criteria are valuable, they beg the question of which state (or states) of the respective folium is to be regarded as fundamental, *i.e.* as a reference state.

In Chapter II we shall state and study our Condition of Geometric Modular Action in a very general form, which will enable us to explicate more clearly how it selects an intriguing class of transformation groups on the index sets of nets of von Neumann algebras and supplies them with projective representations. Returning to the original situation of nets indexed by open subsets of a space-time $\mathcal{M}$ in Chapter III, we explain how to choose a suitable family $\mathcal{W}$ depending only on the space-time itself and present some results of conceptual importance for our framework. There we also outline the program opened up by our framework – a program we carry out explicitly in two examples in Chapters IV and VI.

In Chapter IV we shall illustrate the power of our condition by choosing $\mathcal{M}$ to be topological $\mathbb{R}^4$ and $\mathcal{W}$ to be the set of wedgelike regions in $\mathbb{R}^4$. It will be shown that with a few additional assumptions – all expressible in terms of the state, the net of algebras and the associated modular involutions – the transformations induced upon the index set $\mathcal{W}$ by (1.1) are implemented by point transformations – in fact, by the proper Poincaré group $\mathcal{P}_+$. We obtain after a series of steps a representation of $\mathcal{P}_+$ which acts covariantly upon the net. Therefore, we have an algebraic characterization of Poincaré invariant states on nets of algebras indexed by open subsets of $\mathbb{R}^4$, which induce Poincaré covariant representations of these nets. A more detailed overview of Chapter IV may be found at its beginning. Yet another example is worked out in Chapter VI, where it is shown how similar results for the de Sitter group in three dimensions may be obtained with suitable choices of $\mathcal{M}$ and $\mathcal{W}$.

Continuing the development presented in Chapter IV, Chapter V harbors a discussion of how also the spectrum condition can be characterized in terms of the modular objects, which then leads to how to derive algebraic PCT- and Spin & Statistics Theorems in our setting. We present a striking new algebraic characterization of vacuum states on Minkowski space in terms of quantities which have meaning for arbitrary space-times. This condition may prove to be useful as a criterion for "stability" for quantum states on general space-times. Moreover, we show that if the adjoint action of the modular *groups* associated to the wedge algebras in $\mathbb{R}^4$ leaves the set $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ invariant, then these modular groups satisfy *modular covariance*, and all of the results in Chapter IV hold once again, along with either the positive or negative spectrum condition. We provide further details which clarify the relation between our condition and the widely-used condition of modular covariance. Finally, in Chapter VII we collect some further comments and speculations.

An overview of an earlier version of the results of this paper has appeared in [62]. In addition to the detailed proofs, most of which were suppressed in [62], the present paper contains somewhat more transparent arguments, as well as many additional or strengthened results.

II. Nets of Operator Algebras and Modular Transformation Groups

We begin the main text of this paper with a more abstract setting of our Condition of Geometric Modular Action, since then its connection with transformation groups on partially ordered sets and projective representations of these groups emerges particularly clearly. We shall return to the original situation with further precisions in the next chapter.

Let $\{\mathcal{A}_i\}_{i\in I}$ be a collection of C^*-algebras labeled by the elements of some index set I. If $(I,\leq)$ is a directed set and the property of isotony holds, *i.e.* if for any $i_1, i_2 \in I$ such that $i_1 \leq i_2$ one has $\mathcal{A}_{i_1} \subset \mathcal{A}_{i_2}$, then $\{\mathcal{A}_i\}_{i\in I}$ is said to be a net. However, for our purposes it will suffice that $(I,\leq)$ be only a partially ordered set and that $\{\mathcal{A}_i\}_{i\in I}$ satisfies isotony. We are therefore working with two partially ordered sets, $(I,\leq)$ and $(\{\mathcal{A}_i\}_{i\in I},\subseteq)$, and we require that the assignment $i \mapsto \mathcal{A}_i$ be an order-preserving bijection (*i.e.* it is an isomorphism in the structure class of partially ordered sets). We note that any such assignment which is not an isomorphism in this sense would involve some kind of redundancy in the description. In algebraic quantum field theory the index set I is usually a collection of open causally closed subsets of an appropriate metric space-time $(\mathcal{M}, g)$. In such a case the algebra $\mathcal{A}_i$ is interpreted as the C^*-algebra generated by all the observables measurable in the space-time region i. Hence, to different spacetime regions should correspond different algebras.

If $\{\mathcal{A}_i\}_{i\in I}$ is a net, then the inductive limit $\mathcal{A}$ of $\{\mathcal{A}_i\}_{i\in I}$ exists and may be used as a reference algebra. However, even if $\{\mathcal{A}_i\}_{i\in I}$ is not a net, it is still possible [32] to naturally embed the algebras $\mathcal{A}_i$ in a C^*-algebra $\mathcal{A}$ in such a way that the inclusion relations are preserved. In the following we need therefore not distinguish these two cases and refer, somewhat loosely, to any collection $\{\mathcal{A}_i\}_{i\in I}$ of algebras, as specified, as a net. Any state on $\mathcal{A}$ restricts to a state on $\mathcal{A}_i$, for each $i \in I$. For that reason, we shall speak of a state on $\mathcal{A}$ as being a state on the net $\{\mathcal{A}_i\}_{i\in I}$.

A net automorphism is an automorphism α of the global algebra $\mathcal{A}$ such that there exists an order-preserving bijection $\hat{\alpha}$ on I satisfying $\alpha(\mathcal{A}_i) = \mathcal{A}_{\hat{\alpha}(i)}$. Symmetries, whether dynamical or otherwise, are generally expressed in terms of net automorphisms (or antiautomorphisms) [57]. An internal symmetry of the net is represented by an automorphism α such that $\alpha(\mathcal{A}_i) = \mathcal{A}_i$ for every $i \in I$, *i.e.* the corresponding order-preserving bijection $\hat{\alpha}$ is just the identity on I.

Given a state ω on the algebra $\mathcal{A}$, one can consider the corresponding GNS representation $(\mathcal{H}_\omega, \pi_\omega, \Omega)$ and the von Neumann algebras $\mathcal{R}_i \equiv \pi_\omega(\mathcal{A}_i)''$, $i \in I$. We shall assume that the representation space $\mathcal{H}_\omega$ is separable. We extend the assumption of nonredundancy of indexing to the net $\{\mathcal{R}_i\}_{i\in I}$, *i.e.* we assume that also the map $i \mapsto \mathcal{R}_i$ is an order-preserving bijection.[1] If the GNS vector Ω is cyclic and separating for each algebra $\mathcal{R}_i$, $i \in I$, then from the modular theory of Tomita-Takesaki, we are presented with a collection $\{J_i\}_{i\in I}$

[1] This is automatically the case if the algebras $\mathcal{A}_i$ are von Neumann algebras and ω induces a faithful representation of $\bigcup_{i\in I} \mathcal{A}_i$.

of modular involutions (and a collection $\{\Delta_i\}_{i\in I}$ of modular operators), directly derivable from the state and the algebras. This collection $\{J_i\}_{i\in I}$ of operators on $\mathcal{H}_\omega$ generates a group $\mathcal{J}$, which becomes a topological group in the strong operator topology on $\mathcal{B}(\mathcal{H}_\omega)$, the algebra of all bounded operators on $\mathcal{H}_\omega$. Note that $J\Omega = \Omega$ for $J \in \mathcal{J}$.

In the following we shall denote the adjoint action of J_i upon the elements of the net $\{\mathcal{R}_i\}_{i\in I}$ by $\mathrm{ad}J_i$, *i.e.* $\mathrm{ad}J_i(\mathcal{R}_j) \equiv J_i\mathcal{R}_jJ_i = \{J_iAJ_i \mid A \in \mathcal{R}_j\}$. Note that if $\mathcal{R}_1 \subset \mathcal{R}_2$, then one necessarily has $\mathrm{ad}J_i(\mathcal{R}_1) \subset \mathrm{ad}J_i(\mathcal{R}_2)$, in other words the map $\mathrm{ad}J_i$ is order-preserving. Hence, the content of the Condition of Geometric Modular Action in this abstract setting is that each $\mathrm{ad}J_i$ is a net automorphism. Thus, for each $i \in I$, there is an order-preserving bijection (an automorphism) τ_i on I $((I, \leq))$ such that $J_i\mathcal{R}_jJ_i = \mathcal{R}_{\tau_i(j)}$, $j \in I$. The group generated by the τ_i, $i \in I$, is denoted by $\mathcal{T}$ and forms a subgroup of the transformations on the index set I. For the convenience of the reader, we summarize our standing assumptions.

Standing Assumptions. *For the net $\{\mathcal{A}_i\}_{i\in I}$ and the state ω on $\mathcal{A}$ we assume*
(i) $i \mapsto \mathcal{R}_i$ is an order-preserving bijection;
(ii) Ω is cyclic and separating for each algebra $\mathcal{R}_i$, $i \in I$;
(iii) each $\mathrm{ad}J_i$ leaves the set $\{\mathcal{R}_i\}_{i\in I}$ invariant.[2]

We collect some basic properties of the group $\mathcal{T}$ in the following lemma.

Lemma 2.1. *The group $\mathcal{T}$ defined above has the following properties.*
(1) For each $i \in I$, $\tau_i^2 = \iota$, where ι is the identity map on I.
(2) For every $\tau \in \mathcal{T}$ one has $\tau\tau_i\tau^{-1} = \tau_{\tau(i)}$.
(3) If $\tau(k) = k$ for some $\tau \in \mathcal{T}$ and some $k \in I$, then $\tau\tau_k = \tau_k\tau$.
(4) One has $\tau_i(i) = i$, for some $i \in I$, if and only if the algebra $\mathcal{R}_i$ is maximally abelian. If $\mathcal{T}$ acts transitively on I, then $\tau_i(i) = i$, for some $i \in I$, if and only if $\tau_i(i) = i$, for all $i \in I$. Moreover, if $\tau_i(i) = i$ for some $i \in I$, then i is an atom in $(I, \leq)$, i.e. if $j \in I$ and $j \leq i$, then $j = i$.
(5) If $i \leq j \leq k \leq l$, then $\tau_i(j) \geq \tau_l(k)$.

Proof. 1. The first assertion is immediate since $J_i^2 = \mathbb{I}$, the identity operator on $\mathcal{H}_\omega$, hence for each $j \in I$ one has $\mathcal{R}_j = J_iJ_i\mathcal{R}_jJ_iJ_i = J_i\mathcal{R}_{\tau_i(j)}J_i = \mathcal{R}_{\tau_i(\tau_i(j))}$. Standing Assumption (i) then yields $\tau_i^2 = \iota$.

2. Since every element of $\mathcal{J}$ leaves Ω invariant, standard arguments in modular theory show that the basic assumption $J_i\mathcal{R}_jJ_i = \mathcal{R}_{\tau_i(j)}$ implies the relation $J_iJ_jJ_i = J_{\tau_i(j)}$. Therefore one has the equalities

$$\mathcal{R}_{(\tau_i\tau_j\tau_i)(k)} = J_iJ_jJ_i\mathcal{R}_kJ_iJ_jJ_i = J_{\tau_i(j)}\mathcal{R}_kJ_{\tau_i(j)} = \mathcal{R}_{\tau_{\tau_i(j)}(k)} \quad ,$$

for every $k \in I$. Once again, the nonredundancy assumption yields the assertion $\tau_i\tau_j\tau_i = \tau_{\tau_i(j)}$, for each $i, j \in I$. Since $\mathcal{T}$ is generated by the set $\{\tau_i \mid i \in I\}$, this entails assertion (2).

3. Assume one has $J_{i_1}\cdots J_{i_n}\mathcal{R}_kJ_{i_n}\cdots J_{i_1} = \mathcal{R}_k$ for some $i_1, \ldots, i_n, k \in I$. Then the (anti)unitary operator $J_{i_1}\cdots J_{i_n}$ induces an (anti)automorphism of $\mathcal{R}_k$

[2] and is *a fortiori* a net automorphism

and leaves Ω invariant. It must therefore commute with the modular objects associated with the pair $(\mathcal{R}_k, \Omega)$ (see Theorem 3.2.18 in [18]). But this implies that $\tau_{i_1} \cdots \tau_{i_n} \tau_k = \tau_k \tau_{i_1} \cdots \tau_{i_n}$.

4. If $\tau_i(i) = i$ for some $i \in I$, then one has $\mathcal{R}_i' = J_i \mathcal{R}_i J_i = \mathcal{R}_{\tau_i(i)} = \mathcal{R}_i$, so that $\mathcal{R}_i$ is abelian. Moreover, since Ω is cyclic for this abelian von Neumann algebra, it must be maximally abelian. If $\mathcal{T}$ acts transitively on I, then since the modular involutions are (anti)unitary, every $\mathcal{R}_i$ must be maximally abelian. On the other hand, if $\mathcal{R}_i$ is maximally abelian, one has $\mathcal{R}_i = \mathcal{R}_i' = J_i \mathcal{R}_i J_i = \mathcal{R}_{\tau_i(i)}$. Hence, by the nonredundancy assumption, one has $\tau_i(i) = i$. It follows that if every algebra $\mathcal{R}_k$ is maximally abelian, then $\tau_k(k) = k$ for every $k \in I$.

As already pointed out, under Standing Assumption (ii), any abelian $\mathcal{R}_i$ must be maximally abelian. Hence, if there exist $i_1 < i_2$ with $\mathcal{R}_{i_1}$ and $\mathcal{R}_{i_2}$ both abelian, then $\mathcal{R}_{i_1} \subset \mathcal{R}_{i_2}$, which yields $\mathcal{R}_{i_1} = \mathcal{R}_{i_2}$, since both algebras are maximally abelian. This would violate Standing Assumption (i).

5. If $i \leq j \leq k \leq l$, then one observes that

$$J_i \mathcal{R}_j J_i \supset J_i \mathcal{R}_i J_i = \mathcal{R}_i' \supset \mathcal{R}_l' = J_l \mathcal{R}_l J_l \supset J_l \mathcal{R}_k J_l$$

implies $\tau_i(j) \geq \tau_l(k)$. □

For index sets without atoms, such as the index set $\mathcal{W}$ used as an example in Chapter IV (however, not the example used in Chapter VI), Lemma 2.1 (4) implies that $\mathcal{R}_i$ must be nonabelian for every $i \in I$.

Certain aspects of Lemma 2.1 may be interpreted as follows: given the set I, we consider functions $\underline{\tau} : I \mapsto \mathcal{T}$, where $\mathcal{T}$ is some subgroup of the symmetric group on I. There exist two natural automorphisms on these functions. The first one is given by the adjoint action on $\mathcal{T}$ - namely, $\mathrm{ad}\tau_0(\underline{\tau})(\cdot) = \tau_0 \underline{\tau}(\cdot) \tau_0^{-1}$ for each $\tau_0 \in \mathcal{T}$, and the second one is induced by the action of $\mathcal{T}$ on I: $(\underline{\tau} \circ \tau_0)(\cdot) = \underline{\tau}(\tau_0(\cdot))$. If, for a given function $\underline{\tau}$, these two actions coincide for all $\tau_0 \in \mathcal{T}$, we say that $\underline{\tau}$ is $\mathcal{T}$-covariant. Note that the $\mathcal{T}$-covariant functions form a group under pointwise multiplication, the identity being the constant function on I with value ι. A particularly interesting case arises if the range of a function $\underline{\tau}$ generates $\mathcal{T}$; we then say that $\underline{\tau}$ is a generating function. The preceding proposition thus shows that the condition of geometric modular action provides us with subgroups $\mathcal{T}$ of the symmetric group on I which admit an idempotent, $\mathcal{T}$-covariant generating function. This is a rather strong consistency condition on $\mathcal{T}$. For example, the full symmetric groups of index sets do not in general admit such functions. What is of interest here is the fact that the structure is fixed once the index set I is given.

We feel it is useful to elaborate further the relation between the groups $\mathcal{J}$ and $\mathcal{T}$. Recall that an operator $Z \in \mathcal{J}$ is said to be an *internal symmetry* of the net $\{\mathcal{R}_i\}_{i \in I}$, if $Z \mathcal{R}_k Z^{-1} = \mathcal{R}_k$ for all $k \in I$.

Proposition 2.2. *The surjective map* $\xi : \mathcal{J} \mapsto \mathcal{T}$ *given by*

$$\xi(J_{i_1} \cdots J_{i_m}) = \tau_{i_1} \cdots \tau_{i_m}, \qquad i_1, \ldots, i_m \in I, \quad m \in \mathbb{N},$$

is a group homomorphism. Its kernel is a subgroup $\mathcal{Z}$ *of internal symmetries of the net* $\{\mathcal{R}_i\}_{i \in I}$ *which is contained in the center of* $\mathcal{J}$.

Proof. If $J_{i_1} \cdots J_{i_m} = J_{j_1} \cdots J_{j_n}$, then one has

$$\mathcal{R}_{\tau_{i_1} \cdots \tau_{i_m}(k)} = J_{i_1} \cdots J_{i_m} \mathcal{R}_k J_{i_m} \cdots J_{i_1} = J_{j_1} \cdots J_{j_n} \mathcal{R}_k J_{j_n} \cdots J_{j_1} = \mathcal{R}_{\tau_{j_1} \cdots \tau_{j_n}(k)} \quad ,$$

for all $k \in I$. Thus the equality $\tau_{i_1} \cdots \tau_{i_m} = \tau_{j_1} \cdots \tau_{j_n}$ follows. It is therefore clear that the map ξ is well-defined. Moreover,

$$\xi(J_{i_1} \cdots J_{i_m}) \xi(J_{j_1} \cdots J_{j_n}) = \tau_{i_1} \cdots \tau_{i_m} \tau_{j_1} \cdots \tau_{j_n} = \xi(J_{i_1} \cdots J_{i_m} J_{j_1} \cdots J_{j_n}) \quad ,$$

and by Lemma 2.1 (1), it follows that

$$\xi(J_{i_1} \cdots J_{i_m})^{-1} = \tau_{i_m}^{-1} \cdots \tau_{i_1}^{-1} = \tau_{i_m} \cdots \tau_{i_1} = \xi(J_{i_m} \cdots J_{i_1}) = \xi((J_{i_1} \cdots J_{i_m})^{-1}) \quad .$$

Hence ξ is a group homomorphism.

If $\xi(J_{i_1} \cdots J_{i_m}) = \iota$, then the operator $Z = J_{i_1} \cdots J_{i_m}$ is an internal symmetry, by definition. It remains to be shown that the set $\mathcal{Z}$ of internal symmetries is contained in the center of $\mathcal{J}$. But as argued before, since $Z\Omega = \Omega$ and $Z\mathcal{R}_i Z^{-1} = \mathcal{R}_i$, for all $i \in I$, it follows from standard arguments in modular theory (see Theorem 3.2.18 in [18]) that Z commutes with the modular involutions J_i, $i \in I$. But $\mathcal{J}$ is generated by these operators and Z is an element of $\mathcal{J}$, so the proof of the statement is complete. □

This proposition may be reformulated as the assertion that there exists a short exact sequence

$$\mathbb{1} \to \mathcal{Z} \xrightarrow{\imath} \mathcal{J} \xrightarrow{\xi} \mathcal{T} \to \iota,$$

where $\imath$ denotes the natural identification map. In other words, $\mathcal{J}$ is a central extension of the group $\mathcal{T}$ by $\mathcal{Z}$, a situation for which the mathematics has reached a certain maturity.

It is an immediate consequence of the preceding that there exists an (anti)unitary projective representation of the group $\mathcal{T}$ on $\mathcal{H}_\omega$ by operators in $\mathcal{J}$. For an arbitrary $\tau \in \mathcal{T}$ there may be many ways of writing τ as a product of the elementary $\{\tau_i \mid i \in I\}$. For each $\tau \in \mathcal{T}$ choose some product $\tau = \prod_{j=1}^{n(\tau)} \tau_{i_j}$; which choice one makes is irrelevant for our immediate purposes. Having made such a choice for each $\tau \in \mathcal{T}$, define $J(\tau) \equiv \prod_{j=1}^{n(\tau)} J_{i_j}$.

Corollary 2.3. *The above construction provides an (anti)unitary projective representation of $\mathcal{T}$ on $\mathcal{H}_\omega$ with coefficients in an abelian group $\mathcal{Z}$ of internal symmetries in the center of $\mathcal{J}$. Moreover, one has $J(\tau)\Omega = \Omega$, for all $\tau \in \mathcal{T}$, as well as $\mathcal{Z}\Omega = \Omega$.*

Proof. Consider $\tau, \tau' \in \mathcal{T}$ and the corresponding (anti)unitary operators $J(\tau)$, $J(\tau')$ and $J(\tau\tau')$. If $\xi : \mathcal{J} \mapsto \mathcal{T}$ is the group homomorphism established in Proposition 2.2, one has $\xi(J(\tau\tau')^{-1} J(\tau) J(\tau')) = \iota$, and the initial assertion thus follows from that Proposition. The final assertions are trivial, since the modular conjugations J_i leave Ω invariant. □

It is an interesting mathematical question which groups and corresponding representations can arise in this manner. As we shall see, both finite and continuous groups can be obtained with appropriate choices of net and index set. Before dealing with infinite groups, let us briefly discuss the finite case and consider a family $\{\mathcal{R}_i \mid i = 1, \dots, n\}$ of von Neumann algebras with a common cyclic and separating vector Ω such that the corresponding modular conjugations J_i leave this family invariant, *i.e.* $J_i \mathcal{R}_k J_i = \mathcal{R}_{\tau_i(k)}$ for $i, k = 1, \dots, n$. The maps τ_i are in this case permutations on the set $I \equiv \{1, \dots, n\}$ which are also involutions. Hence, the group $\mathcal{T}$ is a subgroup of the symmetric group S_n which is generated by involutions – a Coxeter group. Here we shall only consider the case where $\mathcal{T}$ acts transitively upon the set I.

If the algebras $\mathcal{R}_i$ are nonabelian, then it is clear that $\mathcal{T}$ cannot be primitive. This is because $J_i \mathcal{R}_i J_i = \mathcal{R}_i{}'$, so that if $\mathcal{R}_i$ is not maximally abelian, one must have, by hypothesis, $\mathcal{R}_i{}' = \mathcal{R}_{i'}$ for some $i' \neq i$. But, since from $J(\tau)\mathcal{R}_i J(\tau)^{-1} = \mathcal{R}_i{}'$ follows $J(\tau)\mathcal{R}_i{}' J(\tau)^{-1} = \mathcal{R}_i$, the index pair (i, i') is either transformed by the elements of $\mathcal{T}$ onto itself or onto a disjoint pair. In other words, (i, i') is a set of imprimitivity of $\mathcal{T}$.

Since $\mathcal{T}$ is not primitive, it also is not 2-transitive (Satz II.1.9 in [40]). Moreover, since the magnitude of every set of imprimitivity in I must be a divisor of the magnitude of I, (see, *e.g.* Satz II.1.2 in [40]), the magnitude n of I is then necessarily even. Hence, if n is odd, then all the algebras must be maximally abelian (the converse is false).

It is easy to compute explicitly the possible groups $\mathcal{T}$ which arise in this manner for small values of n. In the case $n = 2$ one clearly obtains S_2; for $n = 3$ one finds as the only possibility the symmetric group S_3. (And one can give corresponding examples of states and algebras which yield S_3.) The case $n = 4$ is not possible for a family of nonabelian algebras, since then the mentioned sets of imprimitivity are stable under the action of the group $\mathcal{T}$; in other words, $\mathcal{T}$ cannot act transitively on I when $n = 4$. This list can be continued without great effort, but a complete classification of the finite groups $\mathcal{T}$ which can be obtained in this manner is yet an open problem.

III. Geometric Modular Action in Quantum Field Theory

We turn now to the physically interesting case of nets on a space-time manifold $(\mathcal{M}, g)$. The index set I appearing in the abstract formulation of our Condition of Geometric Modular Action in the previous chapter will be denoted henceforth by $\mathcal{W}$ and will consist of certain open subsets $W \subset \mathcal{M}$. A natural question is: for the given manifold $(\mathcal{M}, g)$, how should the index set of the net $\{\mathcal{A}(W)\}_{W \in \mathcal{W}}$ of algebras be chosen so that any state on that net satisfying the Standing Assumptions of Chapter II yields a group $\mathcal{T}$ which can be identified with a subgroup of isometries of the space-time? Evidently, not every choice of such regions will be appropriate. One purpose of this chapter is to explain which considerations should be made when choosing $\mathcal{W}$, once the underlying space-time has been fixed. After this is done, we specify in detail the technical assumptions which constitute our Condition of Geometric Modular Action, which was heuristically presented in the introduction.

We emphasize that in this chapter the starting point is a smooth manifold $\mathcal{M}$ and that some target space-time $(\mathcal{M}, g)$ has already been fixed. In other words, we have in mind a particular metric structure on $\mathcal{M}$ for which we are looking. If one does not have a specific target, that is to say if one just has a net $\{\mathcal{A}(R)\}_{R \in \mathcal{R}}$ indexed by open subregions R of the manifold without any further clue to the metric structure on the manifold $\mathcal{M}$, then, in principle, one would have to test the Condition of Geometric Modular Action for various states and for various subnets $\{\mathcal{A}(W)\}_{W \in \mathcal{W}}$ of $\{\mathcal{A}(R)\}_{R \in \mathcal{R}}$. If the Condition of Geometric Modular Action would hold for one of these, then the program outlined below in this section would be applicable to that state and subnet.

As the Condition of Geometric Modular Action is to be an *a priori* criterion for a characterization of elementary states on $(\mathcal{M}, g)$, the set $\mathcal{W}$ should depend only on the space-time manifold $(\mathcal{M}, g)$. Moreover, it should be sufficiently large to express all desired features of nets on $(\mathcal{M}, g)$ such as locality, covariance (in the presence of spacetime symmetries), *etc.* On the other hand, it should be as small as possible in order to subsume a large class of theories (on the target space-time).

In light of these requirements, it is natural to assume that $\mathcal{W}$ has the following properties.

(a) For each $W \in \mathcal{W}$ the causal (spacelike) complement W' of W (*i.e.* the interior of the set of all points in $\mathcal{M}$ which cannot be connected with any point in the closure $\overline{W}$ of W by a causal curve) is also contained in $\mathcal{W}$. It is convenient to require each $W \in \mathcal{W}$ to be causally closed, that is to say $W = (W')' \equiv W''$. Moreover, the collection $\mathcal{W}$ should be large enough to separate spacelike separated points in $\mathcal{M}$.

(b) The set $\mathcal{W}$ is stable under the action of the group of isometries (spacetime symmetries) of $(\mathcal{M}, g)$.

The latter constraint is consistent with the idea that the Condition of Geometric Modular Action should characterize the most elementary states on $(\mathcal{M}, g)$ with the highest symmetry properties.

We append to the preceding conditions another constraint of a topological nature. In order to motivate it, let us assume for a moment that the transformations τ_W, $W \in \mathcal{W}$, on the index set $\mathcal{W}$ arising from a given net and state satisfying the Condition of Geometric Modular Action are induced by diffeomorphisms (or even just homeomorphisms) of $\mathcal{M}$ and together act transitively on $\mathcal{W}$. This is only possible if all regions in $\mathcal{W}$ belong to the same homotopy class. We therefore assume the following additional condition.

(c) All regions $W \in \mathcal{W}$ are contractible.

Condition (c) excludes, for example, the appearance of double cones in $\mathcal{W}$ when $(\mathcal{M}, g)$ is asymptotically flat (such as Minkowski space), since their causal complements, which are to be elements of $\mathcal{W}$ by condition (a), are not contractible. But double cones *would* be admissible in space-times such as the Einstein universe. We shall call families $\mathcal{W}$ of open regions $W \subset \mathcal{M}$ satisfying (a)-(c) *admissible*.

Given an admissible family $\mathcal{W}$ of regions, it may contain proper subfamilies $\mathcal{W}_0 \subset \mathcal{W}$ which are also admissible. One could then base the Condition of

Geometric Modular Action on the subnet indexed by $\mathcal{W}_0$, instead. It should be noticed that there may exist nets which satisfy our condition with respect to $\mathcal{W}$ but not for $\mathcal{W}_0$. In other words, the subgroup $\mathcal{T}_0 \subset \mathcal{T}$ induced by the underlying modular involutions corresponding to $W_0 \in \mathcal{W}_0$ may not be a stability group of $\mathcal{W}_0$ in certain cases. However, it seems plausible that there exists a larger class of theories (nets and states) satisfying the condition based on $\mathcal{W}_0$ than that based on $\mathcal{W}$, since there are fewer constraints imposed on the nets in the former case. So from this point of view, it appears to be natural to select sets $\mathcal{W}$ which, heuristically speaking, are small.

It is of interest in this context that for certain space-times $(\mathcal{M}, g)$ with large isometry groups, there exist distinguished families $\mathcal{W}$ which are generated by applying the isometry group to a single region W, which itself has a maximal stability group, (*i.e.* a group which cannot be extended to the stability group of some other region which is still a member of the admissible family). Identifying $\mathcal{W}$ with the collection of corresponding coset spaces, it is then meaningful to say that these families are minimal and thus very natural candidates for a concrete formulation of the Condition of Geometric Modular Action. We shall consider certain examples of this type in the subsequent chapters.

As was explained in the introduction, it is one of the aims of the Condition of Geometric Modular Action to distinguish, for any given net $\{\mathcal{A}(W)\}_{W\in\mathcal{W}}$ of C^*-algebras on the manifold $\mathcal{M}$, states ω on the net which can be attributed to the most symmetric physical systems in the space-time $(\mathcal{M}, g)$. Fix an admissible family $\mathcal{W}$ of regions in $(\mathcal{M}, g)$ and consider the von Neumann algebras $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ associated to $(\{\mathcal{A}(W)\}_{W\in\mathcal{W}}, \omega)$ as before. We state our Condition of Geometric Modular Action (henceforth, CGMA) for this structure.

Condition of Geometric Modular Action. *Let $\mathcal{W}$ be an admissible family of open regions in the space-time $(\mathcal{M}, g)$, let $\{\mathcal{A}(W)\}_{W\in\mathcal{W}}$ be a net of C^*-algebras indexed by $\mathcal{W}$, and let ω be a state on $\{\mathcal{A}(W)\}_{W\in\mathcal{W}}$. The CGMA is fulfilled if the corresponding net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ satisfies*

(i) $W \mapsto \mathcal{R}(W)$ is an order-preserving bijection,

(ii) for $W_1, W_2 \in \mathcal{W}$, if $W_1 \cap W_2 \neq \emptyset$, then Ω is cyclic and separating for $\mathcal{R}(W_1) \cap \mathcal{R}(W_2)$,

(iii) for $W_1, W_2 \in \mathcal{W}$, if Ω is cyclic and separating for $\mathcal{R}(W_1) \cap \mathcal{R}(W_2)$, then $\overline{W_1} \cap \overline{W_2} \neq \emptyset$,

and

(iv) for each $W \in \mathcal{W}$, the adjoint action of J_W leaves the set $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ invariant.

The somewhat curious lack of symmetry in conditions (ii) and (iii) is introduced in order to admit theories for which $W_1 \cap W_2 = \emptyset$, but nonetheless the vector Ω is cyclic and separating for the intersection $\mathcal{R}(W_1) \cap \mathcal{R}(W_2)$. This can occur, for example, in certain massless models in Minkowski space, when W_1 and W_2 are disjoint wedgelike regions but where $\overline{W_1} \cap \overline{W_2}$ contains an unbounded lower-dimensional set.

We would like to emphasize that this condition is to be viewed as a selection criterion for states of *particular* physical interest. We do not assert that *every* state of physical interest will satisfy this condition. We observe that its formulation does not require any specific structure of the net $\{\mathcal{A}(W)\}_{W\in\mathcal{W}}$ such as local commutativity, existence of spacetime symmetries, and so forth. As a matter of fact, $\{\mathcal{A}(W)\}_{W\in\mathcal{W}}$ could be a free net on the manifold $\mathcal{M}$ satisfying no other relations but isotony. The above assumptions (i)-(iv) imply the Standing Assumptions of Chapter II, so that all the results from that chapter will be available to us. In particular, we have a group $\mathcal{T}$ of bijections acting on $\mathcal{W}$. The corresponding maps τ_W on $\mathcal{W}$ have additional convenient properties.

Proposition 3.1. *Let $\mathcal{W}$ be an admissible family of open regions in the space-time $(\mathcal{M}, g)$, and let $\{\mathcal{A}(W)\}_{W\in\mathcal{W}}$ be a net of C^*-algebras indexed by $\mathcal{W}$. If ω is a state on $\{\mathcal{A}(W)\}_{W\in\mathcal{W}}$ such that the CGMA is satisfied, then the involutions $\tau_W : \mathcal{W} \mapsto \mathcal{W}$, $W \in \mathcal{W}$, satisfy the following conditions:*

$$\overline{W_1} \cap \overline{W_2} = \emptyset \quad \textit{implies} \quad \tau_W(W_1) \cap \tau_W(W_2) = \emptyset \quad , \tag{3.1}$$

and

$$W_1 \subset W_2 \quad \textit{if and only if} \quad \tau_W(W_1) \subset \tau_W(W_2) \quad , \tag{3.2}$$

with $W_1, W_2 \in \mathcal{W}$.

Proof. Since each J_W is antiunitary and leaves Ω invariant, it is evident that the set $(\mathcal{R}(W_1) \cap \mathcal{R}(W_2))\Omega$ is dense if and only if the set

$$\begin{aligned} J_W(\mathcal{R}(W_1) \cap \mathcal{R}(W_2))\Omega &= (J_W\mathcal{R}(W_1)J_W \cap J_W\mathcal{R}(W_2)J_W)\Omega \\ &= (\mathcal{R}(\tau_W(W_1)) \cap \mathcal{R}(\tau_W(W_2)))\Omega \end{aligned}$$

is dense. Hence (3.1) follows from (ii) and (iii). The assertion (3.2) is a consequence of (i). □

The lack of symmetry in conditions (ii) and (iii) above entails the lack of symmetry in (3.1). If the map τ_W were continuous in the obvious sense, then (3.1) would imply

$$\tau_W(W_1) \cap \tau_W(W_2) = \emptyset \quad \text{if and only if} \quad W_1 \cap W_2 = \emptyset \quad . \tag{3.3}$$

For the two examples worked out in the present paper, it will be seen that in Minkowski space the maps on the index sets $\mathcal{W}$ do indeed satisfy (3.3). In de Sitter space, condition (iii) is trivial and will be supplemented by an algebraic condition yielding (3.3).

Having thus fixed the framework in detail, there arises the interesting question: which transformation groups $\mathcal{T}$ are associated with states fulfilling this criterion and how do they act on the corresponding nets? In particular, are they implemented by point transformations on the manifold $\mathcal{M}$, and are these isometries of the space-time $(\mathcal{M}, g)$? A comprehensive answer to this question does not seem to be an easy problem, but there are some engaging facts of a quite general nature which we wish to explain.

Let us first consider the question of whether the elements of $\mathcal{T}$ could be implemented by point transformations on $\mathcal{M}$. *If* we knew from the outset that the maps τ_W also leave stable a larger net $\{\mathcal{R}(\mathcal{O})\}_{\mathcal{O}\in\mathfrak{S}}$ containing $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ and indexed by a *base* for the topology on $\mathcal{M}$, we could rely upon an approach initiated by Araki [4] (building upon [7][6]) and further developed by Keyl [43] in order to prove under certain conditions that the maps τ_W are induced by point transformations of $\mathcal{M}$ which generate a group $\mathcal{G}$. If these maps also preserved the causal structure on $(\mathcal{M}, g)$ in the sense of

$$\tau_W(W_0)' = \tau_W({W_0}') \quad , \quad \text{for all} \quad W_0 \in \mathcal{W} \quad , \tag{3.4}$$

for each $W \in \mathcal{W}$,[3] then since the regions in $\mathcal{W}$ separate spacelike separated points, we could, for a significant class of spacetimes, appeal to the well-known result of Alexandrov [2][3], (see also Zeeman [77], Borchers and Hegerfeldt [11], Lester [48], Benz [8]) and conclude that the group $\mathcal{G}$ is a subgroup of the conformal group of $(\mathcal{M}, g)$.[4] Moreover, as was shown in the preceding chapter, there exists an (anti)unitary projective representation of $\mathcal{T}$ (and thus of $\mathcal{G}$) on the Hilbert space $\mathcal{H}_\omega$. Well-known examples which nicely illustrate this scenario are conformal quantum field theories on compactified Minkowski space (see [21]).

However, in order to cover a larger class of spacetimes, we would like to avoid the initial strong assumption that the adjoint action of the modular involutions $\{J_W \mid W \in \mathcal{W}\}$ leaves the net $\{\mathcal{R}(\mathcal{O})\}_{\mathcal{O}\in\mathfrak{S}}$ invariant. In particular, the CGMA can obtain without the maps τ_W being induced by point transformations of $\mathcal{M}$. In order to indicate what can occur, let us consider any decreasing net $\{\bigcap_i W_{i,n}\}_{n\in\mathbb{N}}$ which converges to some point $x \in \mathcal{M}$. Because τ_W is order-preserving, the images $\{\bigcap_i \tau_W(W_{i,n})\}_{n\in\mathbb{N}}$ also form a decreasing net, and if the limit set is nonempty, it is straightforward to show that it consists of a single point (see [4]). But the net may have no limit for certain points $x \in \mathcal{M}$. Hence, loosely speaking, our CGMA admits the possibility of singular point transformations which are *not* contained in the conformal group of $(\mathcal{M}, g)$ but which nonetheless preserve the causal structure.[5] This flexibility is actually very advantageous for our purposes, since the conformal group is rather small for certain space-times and thus not suitable for the characterization of elementary physical states. Hence, the CGMA may still be a useful selection criterion for physically interesting states even in these cases, where the point transformation group $\mathcal{G}$ has very little indeed to say about the underlying space-time.

We conclude this chapter with a list of mathematical problems which naturally arise if one wants to use our principle of geometric modular action for the determination of the possible symmetry groups $\mathcal{T}$ and their action on nets for a given space-time $(\mathcal{M}, g)$. The first step is to pick an admissible family $\mathcal{W}$ of regions $W \subset \mathcal{M}$. We do not have a general algorithm for the choice of $\mathcal{W}$,

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[3]It is of interest to note that we shall *derive*, not assume, (3.4) in our examples, hence deduce, not postulate, locality and Haag duality for wedge algebras.

[4]Some of the details of the argument which would be involved here may be gleaned from the proofs presented in Section 4.1. The basic ideas are sketched in Section 3 of [62].

[5]See also the example discussed at the end of Section 4.1.

but, as previously mentioned, there do exist space-times for which the family $\mathcal{W}$ is uniquely fixed by our general requirements. One then has to solve, step by step, each of the following problems.

1) Are the transformations on $\mathcal{W}$ satisfying the conditions (3.1) and (3.2) induced by (singular) point transformations on $(\mathcal{M}, g)$ (forming a group $\mathcal{G}$)?

2) Which subgroups $\mathcal{T}$ of the symmetric group on $\mathcal{W}$ can appear? More precisely, which groups are generated by families $\{\tau_W\}_{W \in \mathcal{W}}$ of such automorphisms for which

$$\tau_{W_1} \tau_{W_2} \tau_{W_1} = \tau_{\tau_{W_1}(W_2)} \quad , \quad \text{for} \quad W_1, W_2 \in \mathcal{W} \quad ?$$

Of special interest are cases where $\mathcal{T}$ is large and acts transitively on $\mathcal{W}$.

3) Do $\mathcal{W}$ and $\mathcal{T}$ (as an abstract group) determine the action of the automorphisms $\{\tau_W\}_{W \in \mathcal{W}}$?

4) If the group $\mathcal{G}$ of point transformations is a continuous group or contains a continuous subgroup, (when) do the underlying modular involutions induce a continuous unitary projective representation of $\mathcal{G}$, respectively of its continuous subgroup?

5) Can this projective representation be lifted to a continuous unitary representation of $\mathcal{G}$?

6) If there exists a one-parameter subgroup in $\mathcal{G}$ which can be interpreted as time evolution on $(\mathcal{M}, g)$, what are the spectral properties of the generator of the corresponding unitary representation? In particular, when is the spectrum bounded from below (as one would expect in the case of elementary physical states such as the vacuum)?

Whereas the latter three problems are standard in the representation theory of groups, the first three are problems in the theory of transformation groups of subsets of topological spaces, which apparently have not received the attention they seem to deserve. We discuss in the subsequent chapters the physically interesting examples of Minkowski space and de Sitter space, for which the preceding program can be completely carried out. Our proofs are largely based on explicit calculations which do not yet provide the basis for a more general argument. But as our results are promising, we believe that a more systematic study of these mathematical problems would be worthwhile.

IV. Geometric Modular Action Associated With Wedges in $\mathbb{R}^4$

We now carry out the program outlined at the end of the preceding chapter for the case of four-dimensional Minkowski space with the standard metric

$$g = \mathrm{diag}(1, -1, -1, -1) \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad . \tag{4.1}$$

in proper coordinates as the target space. The isometry group of this space is the Poincaré group $\mathcal{P}$ and an admissible family $\mathcal{W}$ of regions is obtained by applying the elements of $\mathcal{P}$ to a single wedge-shaped region of the form

$$W_R \equiv \{x \in \mathbb{R}^4 \mid x_1 > |x_0|\} \quad , \tag{4.2}$$

i.e. $\mathcal{W} = \{\lambda W_R \mid \lambda \in \mathcal{P}\}$, where $\lambda W_R = \{\lambda(x) \mid x \in W_R\}$. It is easy to show that $\mathcal{W}$ is an admissible family in four-dimensional Minkowski space. Because of the requirement that the admissible family be mapped onto itself by the isometry group of the space-time, an admissible family $\mathcal{W}$ in the case of Minkowski space must contain the orbit of each of its elements under the action of the Poincaré group. Recall that an admissible family $\mathcal{W}$ is called minimal if it coincides with the orbit under the action of the isometry group of a single region with a maximal stability group. As the only open, causally closed regions which are invariant under the stability group $\mathrm{InvP}(W_R)$ of W_R are W_R itself, its causal complement W_R' and the entire space $\mathbb{R}^4$, one concludes that $\mathbb{R}^4$ is the only open, causally closed region which is stable under the action of any proper extension of $\mathrm{InvP}(W_R)$. Hence, $\mathcal{W}$ is a minimal admissible family for four-dimensional Minkowski space. We therefore base the analysis in this chapter on this canonical choice of regions. We remark that, in fact, one has $\mathcal{W} = \{\lambda W_R \mid \lambda \in \mathcal{P}_+^\uparrow\}$, where $\mathcal{P}_+^\uparrow$ is the identity component of the Poincaré group.

Note that the metric is introduced because a specific target space is envisioned. The wedges in the smooth manifold $\mathbb{R}^4$ can be defined without reference to the Minkowski metric by introducing coordinates. Then the set $\mathcal{W}$ of wedges is determined only up to diffeomorphism, which is all we shall require. Nonetheless, it is clear that there is nothing intrinsic about such a definition of wedges. For a discussion of a possible means to determine an *intrinsic* algebraic characterization of “wedges” for our purpose, see Chapter VII.

We commence with a state on an initial net $\{\mathcal{A}(W)\}_{W \in \mathcal{W}}$ which satisfies the CGMA discussed in the previous chapter. In Section 4.1 we consider the elements of the transformation group $\mathcal{T}$ associated with any such state and establish a considerable extension of the Alexandrov-Zeeman-Borchers-Hegerfeldt theorems by showing that these maps are induced by point transformations which form a subgroup $\mathcal{G}$ of the Poincaré group. This section also contains a simple example of a space-time manifold and well-behaved transformations of a corresponding family of regions which are *not* induced by point transformations.

In Section 4.2 and the subsequent sections, we restrict attention to those cases where the transformation group $\mathcal{T}$ is large enough to act transitively upon the set $\mathcal{W}$. It turns out that $\mathcal{G}$ then contains the full identity component $\mathcal{P}_+^\uparrow$ of the Poincaré group $\mathcal{P}$. The specific form of the Poincaré elements corresponding to the generating involutions in $\mathcal{T}$, which themselves arise from the adjoint action of the initial modular conjugations upon the net $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$, is also identified in Section 4.2, and it is found that this form is uniquely fixed and agrees with the one first determined by Bisognano and Wichmann for the case of the vacuum state on Minkowski space and any net of von Neumann algebras locally associated with a quantum field satisfying Wightman's axioms [9][10]. It then follows from this explicit knowledge of the form of the implementing

Poincaré elements that $\mathcal{G}$ is exactly equal to the proper Poincaré group $\mathcal{P}_+$. Thus, starting with the CGMA, we find a unique and familiar solution for the possible symmetry groups and their respective actions.

In the remaining portion of Chapter IV we discuss the properties of the representations of $\mathcal{T}$ – and hence of $\mathcal{G} = \mathcal{P}_+$ – which are induced by the modular conjugations. In Section 4.3 we shall identify a natural continuity condition on the net $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$ which implies that there exists a strongly continuous (anti)unitary projective representation of $\mathcal{G} = \mathcal{P}_+$. This requires a certain choice of product decomposition in the definition of the projective representation (cf. the discussion before Corollary 2.3). These results are used in Section 4.3 for the proof, first of all, that one can always lift this projective representation to a continuous unitary representation of the covering group of $\mathcal{P}_+^\uparrow$. Our analysis, which is based on results in Borel measurable group cohomology theory and is carried out in the Appendix, parallels to some extent the discussion in [22]; but our more global point of view and our explicit construction of the projective representation provide certain simplifications. In particular, we shall not need to argue *via* the Lie algebra, since the results of Section 4.2 and modular theory give us sufficient control over our explicit representation. And then we show that, after all, this representation of the covering group provides a strongly continuous representation of $\mathcal{P}_+^\uparrow$ and *coincides* with the initially and explicitly constructed projective representation.

It is worth emphasizing that we explicitly construct a strongly continuous unitary representation of the translation subgroup (using ideas of [24]), thereby determining the generator of the timelike translations, which has the physical interpretation of the Hamiltonian, or total energy operator, of the theory. In other words, we derive the dynamics of the theory from the physical data of the state and net of observable algebras.

We recall that it is the main purpose of this chapter to illustrate the steps which are necessary to apply the CGMA in our program. As already mentioned at the end of Chapter III, the mathematics relevant to the first three group theoretical problems does not seem to be sufficiently well developed for our purposes, and we must therefore rely on explicit and sometimes tedious computations to carry out our program. But our results demonstrate that the CGMA, which at first glance appears very general and diaphanous, actually imposes strong constraints on the admissible states and allows one to characterize the vacuum states in the case of Minkowski space.

4.1. Wedge Transformations Are Induced By Elements of the Poincaré Group

The aim of this section is to show that the elements of the transformation group $\mathcal{T}$ acting upon the wedges $\mathcal{W}$, which arises when one assumes the CGMA discussed in the previous chapter, are induced by point transformations on Minkowski space, indeed, by elements of the Poincaré group. In other words, we wish to show that $\mathcal{T}$ can be identified with a subgroup of the Poincaré group. Since one can define points as intersections of edges of suitable wedges, it is an intuitively appealing possibility that transformations of wedges could lead to point transformations. The assumptions made in this section are slightly more general than actually needed for our primary purpose, but these somewhat more general results have interest going beyond the immediate problem we are addressing. In particular, we shall also employ these results in Chapter V, where we consider the consequences of the geometric action of modular groups.

In the remainder of this section, we shall assume that we have a bijective map $\tau : \mathcal{W} \mapsto \mathcal{W}$ with the following properties:

(A) If $W_1, W_2 \in \mathcal{W}$ satisfy $\overline{W_1} \cap \overline{W_2} = \emptyset$, then $\tau(W_1) \cap \tau(W_2) = \emptyset$ and $\tau^{-1}(W_1) \cap \tau^{-1}(W_2) = \emptyset$;

(B) $W_1, W_2 \in \mathcal{W}$ satisfy $W_1 \subset W_2$ if and only if $\tau(W_1) \subset \tau(W_2)$.

By Prop. 3.1, these are properties shared by the maps τ_W, $W \in \mathcal{W}$, arising from states complying with the CGMA. We do not assume in this section that the map τ is an involution or that (3.3) holds. We shall show that conditions (A) and (B) *imply* (3.3).

We introduce the following notation: $\ell \in \mathrm{IR}^4$ denotes a future-directed lightlike vector and $p \in \mathrm{IR}$ a real parameter. For given ℓ, p we define the characteristic half-spaces

$$H_p[\ell]^\pm \equiv \{x \in \mathrm{IR}^4 \mid \pm(x \cdot \ell - p) > 0\} \quad . \tag{4.1.1}$$

Note that the boundary of such a half-space, $H_p[\ell] = \partial H_p[\ell]^\pm =$ $= \{x \in \mathrm{IR}^4 \mid x \cdot \ell = p\}$, is a characteristic hyperplane with the properties that all lightlike vectors parallel to this hyperplane are parallel to ℓ and all other vectors parallel to $H_p[\ell]$ are spacelike. Given two such pairs, $\{\ell_i, p_i\}$, $i = 1, 2$, where ℓ_1 and ℓ_2 are not parallel, then $W = H_{p_1}[\ell_1]^+ \cap H_{p_2}[\ell_2]^-$ is a wedge. All wedges can be obtained in this manner. In particular, for any wedge $W \in \mathcal{W}$ there exist two future-directed lightlike vectors $\ell_\pm$ such that $W \pm \ell_\pm \subset W$. These vectors are unique up to a positive scaling factor. The half-spaces $H^\pm$ generating W as above are given by

$$H^\pm = \bigcup_{\lambda \in \mathrm{IR}} (W + \lambda \ell_\mp) \quad . \tag{4.1.2}$$

In the sequel, we shall denote by $\mathcal{F}^\pm$ the following family of wedges:

$$\mathcal{F}^\pm \equiv \{W + \lambda \ell_\mp \mid \lambda \in \mathrm{IR}\} \quad .$$

We shall say that $\mathcal{F}^\pm$ generates $H^\pm$ via (4.1.2). Note that every such family $\mathcal{F}^\pm$ has the following properties:

(i) $\mathcal{F}^\pm$ is linearly ordered, *i.e.* if $W_1, W_2 \in \mathcal{F}^\pm$, then either $W_1 \subset W_2$ or $W_2 \subset W_1$.

(ii) $\mathcal{F}^\pm$ is maximal in the sense that if $W_1, W_2 \in \mathcal{F}^\pm$ satisfy $W_1 \subset W_2$ and there exists a wedge $W \in \mathcal{W}$ such that $W_1 \subset W \subset W_2$, then $W \in \mathcal{F}^\pm$.

(iii) $\mathcal{F}$ has no upper or lower bound in $(\mathcal{W}, \subset)$, *i.e.* there exists no element $W_< \in \mathcal{W}$ such that $W_< \subset W$ for all $W \in \mathcal{F}$ and also no element $W_> \in \mathcal{W}$ such that $W_> \supset W$ for all $W \in \mathcal{F}$.

We shall call a collection of wedges $\mathcal{F} \subset \mathcal{W}$ with the properties (i)-(iii) a characteristic family of wedges. Every characteristic family of wedges is, in fact, of the form of $\mathcal{F}^{\pm}$. The proof of this assertion rests upon the following well-known properties of wedges. For wedges $W, W_0 \in \mathcal{W}$ with $W_0 \subset W$ and $W_0 \neq W$, there exists a space- or lightlike translation $a \in \mathrm{IR}^4$ such that

$$W_0 = W + a \subset W + \lambda a \subset W \qquad \text{for all} \quad 0 \leq \lambda \leq 1 \quad .$$

If the edge of W_0 lies on the boundary of W, then the translation a can be chosen to be lightlike (and is therefore a multiple of one of the lightlike vectors $\ell_{\pm}$ determining W). On the other hand, if the edge of W_0 lies in the interior of W, then there exists an open set $\mathcal{N} \subset \mathrm{IR}^4$ such that $W_0 \subset W + a \subset W$, for all $a \in \mathcal{N}$. As in [24], we shall say that two wedges $W_1, W_2 \in \mathcal{W}$ are coherent if one is obtained from the other by a translation, or, equivalently, if there exists another wedge W_3 such that $W_1 \subset W_3$ and $W_2 \subset W_3$. Hence, all wedges in a characteristic family are mutually coherent. We now prove the initial assertion.

Lemma 4.1.1. *Every characteristic family of wedges $\mathcal{F}$ has the form $\mathcal{F} = \{W + \lambda\ell \mid \lambda \in \mathrm{IR}\}$, for some wedge $W \in \mathcal{W}$ and some future-directed lightlike vector ℓ with the property that $W + \ell \subset W$ or $W - \ell \subset W$.*

Proof. Let $W_0, W \in \mathcal{F}$. By the linear ordering of $\mathcal{F}$, one may assume without loss of generality that $W_0 \subset W$. If the edge of W_0 would lie in the interior of W, then, as mentioned above, there exists an open set $\mathcal{N}$ in IR^4 such that $W_0 \subset W + a \subset W$, for all $a \in \mathcal{N}$. By the maximality of $\mathcal{F}$ in $\mathcal{W}$, this would entail that $W + a \in \mathcal{F}$, for all $a \in \mathcal{N}$. However, the elements of $\{W + a \mid a \in \mathcal{N}\}$ clearly violate the linear ordering of $\mathcal{F}$. Hence, the edge of W_0 must lie on the boundary of W, so there exists a lightlike translation $a \in \mathrm{IR}^4$ such that $W_0 = W + a \subset W$.

Let now $W, W + a, W + b \in \mathcal{F}$ be chosen such that a and b are lightlike and $W + a \subset W \subset W + b$. As in the preceding paragraph one shows that the edge of $W + a$ lies on the boundary of $W + b$. The assumed inclusion then implies that the edge of W lies on the same characteristic hyperplane. This entails that a and b are proportional, *i.e.* the elements of $\mathcal{F}$ are all of the form $W + \lambda\ell$ with real λ and future-directed lightlike vector $\ell \in \mathrm{IR}^4$. That every $\lambda \in \mathrm{IR}$ must occur follows at once from properties (ii) and (iii) of characteristic families. □

In the next lemma we show that order-preserving bijections $\tau : \mathcal{W} \mapsto \mathcal{W}$ map characteristic families onto characteristic families.

Lemma 4.1.2. *Let $\tau : \mathcal{W} \mapsto \mathcal{W}$ be a bijective map with the property (B). Then τ maps every characteristic family $\mathcal{F}$ of wedges onto a characteristic family $\tau(\mathcal{F}) \equiv \{\tau(W) \mid W \in \mathcal{F}\}$. In fact, if $\mathcal{F}_1 = \{W_1 + \lambda\ell_1 \mid \lambda \in \mathrm{IR}\}$, for some wedge $W_1 \in \mathcal{W}$ and some future-directed lightlike vector ℓ_1 with the property that $W_1 + \ell_1 \subset W_1$ or $W_1 - \ell_1 \subset W_1$, and if $\tau(W_1) = W_2$, then $\tau(W_1 + \lambda\ell_1) = W_2 + f(\lambda)\ell_2$, where $f : \mathrm{IR} \mapsto \mathrm{IR}$ is a continuous monotonic bijection, $f(0) = 0$, and ℓ_2 is a future-directed lightlike vector with the property that $W_2 + \ell_2 \subset W_2$ or $W_2 - \ell_2 \subset W_2$.*

Proof. Since τ is an order isomorphism, the linear ordering of $\tau(\mathcal{F})$, property (i), follows at once. If one has for some $W \in \mathcal{W}$ and $W_1, W_2 \in \mathcal{F}$ the inclusions $\tau(W_1) \subset W \subset \tau(W_2)$, one must also have the inclusions $W_1 \subset \tau^{-1}(W) \subset W_2$, since τ^{-1} is also an order isomorphism. Hence, by the maximality of $\mathcal{F}$ it follows that $\tau^{-1}(W) \in \mathcal{F}$, so that $W \in \tau(\mathcal{F})$, establishing the maximality of $\tau(\mathcal{F})$.

Finally, if there were to exist a lower bound $W_< \in \mathcal{W}$ to $\tau(\mathcal{F})$, then since τ is an order isomorphism, the wedge $\tau^{-1}(W_<)$ would be a lower bound for $\mathcal{F}$, a contradiction. Similarly, one can exclude the existence of an upper bound in $\mathcal{W}$ for $\tau(\mathcal{F})$.

Let $\mathcal{F}_1$, W_1, ℓ_1, and W_2 be as indicated in the hypothesis. Since it has just been established that inclusion-preserving bijections on $\mathcal{W}$ map characteristic families of wedges onto characteristic families, one sees from Lemma 4.1.1 that there exist future-directed lightlike vectors k_1, k_2, such that $W_2 + k_1 \subset W_2$ and $W_2 - k_2 \subset W_2$, and a function $f : \mathbb{R} \mapsto \mathbb{R}$ such that for all $\lambda \in \mathbb{R}$ either

$$\tau(W_1 + \lambda\ell_1) = W_2 + f(\lambda)k_1 \quad \text{or} \quad \tau(W_1 + \lambda\ell_1) = W_2 - f(\lambda)k_2 \quad .$$

Since τ is an inclusion-preserving bijection, f is bijective and monotone; hence f is continuous. $\square$

We wish now to show that the apparent asymmetry in condition (A) can be removed without loss of generality; in other words, condition (3.3) holds for the mappings considered in this section.

Corollary 4.1.3. *Let $\tau : \mathcal{W} \mapsto \mathcal{W}$ be a bijection which satisfies conditions (A) and (B). Then τ also satisfies*

$$W_1 \cap W_2 = \emptyset \quad \textit{if and only if} \quad \tau(W_1) \cap \tau(W_2) = \emptyset \quad . \tag{4.1.3}$$

Relation (4.1.3) is also true for the mapping τ^{-1}.

Proof. Let $W_1, W_2 \in \mathcal{W}$ such that $W_1 \cap W_2 = \emptyset$ but $\overline{W_1} \cap \overline{W_2} \neq \emptyset$. It suffices to show that in this case one has $\tau(W_1) \cap \tau(W_2) = \emptyset$.

First note that if N is a convex subset of the boundary $\overline{W} \setminus W$ of the wedge W, it is contained in one of the two characteristic hyperplanes $H_p[\ell_\pm]$ determined by W, and thus it is easy to see that either

$$N \cap \overline{W + \lambda\ell_+} = \emptyset \quad \text{or} \quad N \cap \overline{W - \lambda\ell_-} = \emptyset \quad ,$$

for all $\lambda > 0$. Since both $\overline{W_1}$ and $\overline{W_2}$ are convex, so is their intersection $\overline{W_1} \cap \overline{W_2} \subset \overline{W_1} \setminus W_1$; hence, with ℓ_1, ℓ_2 future-directed lightlike vectors with $W_1 + \ell_1 \subset W_1$ and $W_1 - \ell_2 \subset W_1$, it follows that

$$\emptyset = \overline{W_1 + \lambda\ell_1} \cap (\overline{W_1} \cap \overline{W_2}) = \overline{W_1 + \lambda\ell_1} \cap \overline{W_2}$$

or

$$\emptyset = \overline{W_1 - \lambda\ell_2} \cap (\overline{W_1} \cap \overline{W_2}) = \overline{W_1 - \lambda\ell_2} \cap \overline{W_2} \quad ,$$

for all $\lambda > 0$. Consider the first case and note that Lemma 4.1.2 entails that $\tau(W_1 + \lambda\ell_1) = \tau(W_1) + f(\lambda)\ell$, with $\tau(W_1) + \ell \subset \tau(W_1)$ or $\tau(W_1) - \ell \subset \tau(W_1)$

and $f : \mathrm{IR} \mapsto \mathrm{IR}$ a continuous bijection which is either monotone increasing or monotone decreasing. Consider the subcase where f is monotone increasing and $\tau(W_1) + \ell \subset \tau(W_1)$. Then by the continuity of f, one has

$$\begin{aligned}\tau(W_1) \cap \tau(W_2) &= (\tau(W_1) + f(0)\ell) \cap \tau(W_2) \\ &= (\bigcup_{\lambda>0} (\tau(W_1) + f(\lambda)\ell)) \cap \tau(W_2) \\ &= \bigcup_{\lambda>0} (\tau(W_1 + \lambda\ell_1) \cap \tau(W_2)) \\ &= \emptyset \quad ,\end{aligned}$$

using assumption (A). On the other hand, the subcase f monotone decreasing and $\tau(W_1) + \ell \subset \tau(W_1)$ cannot arise, since τ is inclusion-preserving. Similarly, the subcase f monotone increasing and $\tau(W_1) - \ell \subset \tau(W_1)$ cannot occur. Finally, in the subcase f monotone decreasing and $\tau(W_1) - \ell \subset \tau(W_1)$ one finds the same chain of equalities as above.

In the second case, namely $\emptyset = \overline{W_1 - \lambda\ell_2} \cap \overline{W_2}$, for all $\lambda > 0$, one similarly sees that the subcases $\tau(W_1) + \ell \subset \tau(W_1)$ with f increasing, and $\tau(W_1) - \ell \subset \tau(W_1)$ with f decreasing are excluded by the inclusion-preserving property of τ. In the other two subcases, one has from Lemma 4.1.2 in a like manner

$$\begin{aligned}\tau(W_1) \cap \tau(W_2) &= (\tau(W_1) + f(0)\ell) \cap \tau(W_2) \\ &= (\bigcup_{\lambda>0} (\tau(W_1) + f(-\lambda)\ell)) \cap \tau(W_2) \\ &= \bigcup_{\lambda>0} (\tau(W_1 - \lambda\ell_2) \cap \tau(W_2)) \\ &= \emptyset \quad ,\end{aligned}$$

by assumption (A). Thus, one has proven that $W_1 \cap W_2 = \emptyset$ implies $\tau(W_1) \cap \tau(W_2) = \emptyset$. The argument for τ^{-1} is identical, completing the proof of the lemma.

□

To proceed further, it is convenient to use the following notation for wedges. For any linearly independent future-directed lightlike vectors $\ell_1, \ell_2 \in \mathrm{IR}^4$ and any $a \in \mathrm{IR}^4$, we define the wedge

$$\begin{aligned}W[\ell_1, \ell_2, a] &\equiv \{\alpha\ell_1 + \beta\ell_2 + \ell^\perp + a \mid \alpha > 0, \beta < 0, \ell^\perp \in \mathrm{IR}^4, \ell^\perp \cdot \ell_1 = \ell^\perp \cdot \ell_2 = 0\} \\ &= W[\ell_1, \ell_2, 0] + a \quad ,\end{aligned}$$

where the dot product here represents the Minkowski scalar product. Then with

$$\ell_{1\pm} = (1, \pm1, 0, 0)\,,\ \ell_{2\pm} = (1, 0, \pm1, 0)\,,\ \ell_{3\pm} = (1, 0, 0, \pm1)\,,$$

one sees that $W_R = W[\ell_{1+}, \ell_{1-}, 0]$. Note that with this notation, one has $W[\ell_1, \ell_2, a] + \ell_1 \subset W[\ell_1, \ell_2, a]$ and $W[\ell_1, \ell_2, a] - \ell_2 \subset W[\ell_1, \ell_2, a]$, *i.e.* for this wedge ℓ_+ is a positive multiple of ℓ_1 and ℓ_- is a positive multiple of ℓ_2. Moreover, the half-spaces $H^\pm$ generating $W[\ell_1, \ell_2, a]$ as above are given by $H^+ = H_{a\cdot\ell_2}[\ell_2]^+$ and $H^- = H_{a\cdot\ell_1}[\ell_1]^-$, and the associated characteristic families are given by

$$\mathcal{F}^+ = \{W[\ell_1, \ell_2, a + \lambda\ell_2] \mid \lambda \in \mathrm{IR}\} \quad \text{and} \quad \mathcal{F}^- = \{W[\ell_1, \ell_2, a + \lambda\ell_1] \mid \lambda \in \mathrm{IR}\} \quad .$$

We next show a useful characterization of pairs of spacelike separated wedges.

Lemma 4.1.4. *Let W_1, W_2 be wedges. $W_1 \subset W_2'$ if and only if the two characteristic families $\mathcal{F}_2^+$ and $\mathcal{F}_2^-$ containing W_2 satisfy $W_1 \cap W = \emptyset$ for every $W \in \mathcal{F}_2^+ \cup \mathcal{F}_2^-$.*

Proof. Let $H_2^\pm$ be the characteristic half-spaces generated by the families $\mathcal{F}_2^\pm$, so that one has $W_2 = H_2^+ \cap H_2^-$ and $W_2' = H_2^{+c} \cap H_2^{-c}$, where the superscript c signifies that one takes the complementary half-space. From $W_1 \subset W_2'$ follows therefore the containment $W_1 \subset H_2^{\pm c}$ and hence also $W_1 \cap H_2^\pm = \emptyset$. Conversely, the last equality follows from the disjointness of W_1 from each member of the set $\mathcal{F}_2^+ \cup \mathcal{F}_2^-$, so that one must have $W_1 \subset H_2^{+c} \cap H_2^{-c} = W_2'$. □

It is next established that bijections on $\mathcal{W}$ satisfying conditions (A) and (B) preserve causal complements and thus causal structure.

Corollary 4.1.5. *A bijection $\tau : \mathcal{W} \mapsto \mathcal{W}$ which fulfills conditions (A) and (B) also satisfies the following condition:*

$$\tau(W') = \tau(W)' \quad , \quad \textit{for any} \quad W \in \mathcal{W} \quad . \tag{4.1.4}$$

Proof. Consider an arbitrary wedge $W \in \mathcal{W}$, and let $\mathcal{F}^+$ and $\mathcal{F}^-$ be the characteristic families of wedges containing W'. By Lemma 4.1.2, τ maps $\mathcal{F}^+$ and $\mathcal{F}^-$ onto two characteristic families $\tau(\mathcal{F}^+)$ and $\tau(\mathcal{F}^-)$ containing $\tau(W')$. Lemma 4.1.4 entails that W is disjoint from every element of $\mathcal{F}^+ \cup \mathcal{F}^-$, and hence Corollary 4.1.3 implies that $\tau(W)$ is disjoint from every element of $\tau(\mathcal{F}^+) \cup \tau(\mathcal{F}^-)$. Thus, Lemma 4.1.4 yields the containment $\tau(W') \subset \tau(W)'$. The reverse containment follows by applying the same argument to τ^{-1}. □

We continue now with our development of point transformations. A pair (W_1, W_2) of disjoint wedges will be called maximal if there is no wedge W properly containing W_1, resp. W_2, such that $W \cap W_2 = \emptyset$, resp. $W \cap W_1 = \emptyset$. Note that a bijection $\tau : \mathcal{W} \mapsto \mathcal{W}$ fulfilling conditions (A) and (B) maps maximal pairs of wedges onto maximal pairs of wedges. We need a computational characterization of a maximal pair of wedges. To this end, we remark that given a pair (W_1, W_2) such that W_2 is not a translate of W_1 or W_1', there exists a Poincaré transformation (Λ, x) mapping W_1 onto W_R and W_2 onto either the wedge $W[\ell_{2+}, \ell, d]$ or its causal complement $W[\ell_{2+}, \ell, d]'$, where ℓ is some positive lightlike vector which is not parallel to ℓ_{2+} and $d \in \mathbb{R}^4$. This follows from the observations that there always exists a Lorentz transformation Λ_1 such that $\Lambda_1 W_1 = W_R$ and that every positive lightlike vector not parallel to $\ell_{1\pm}$ is mapped by some element of the invariance group of W_R to ℓ_{2+}. We shall therefore consider the pair $(W_R, W[\ell_{2+}, \ell, d])$ – indeed, without loss of generality, the pair $(W_R, W[\ell_{2+}, \ell, d])$, for suitable $\ell = (1, a, b, c)$ with $a^2 + b^2 + c^2 = 1$, $b \neq 1$, and $d \in \mathbb{R}^4$ – and determine under which conditions this pair is maximal. In preparation, we prove the following simple lemma.

Lemma 4.1.6. *Let $P : \mathbb{R}^4 \mapsto \mathbb{R}^2$ be given by $P(x_0, x_1, x_2, x_3) = (x_0, x_1)$ and let $W = W[\ell_{2+}, \ell, d]$ with $\ell = (1, a, b, c)$, where $a, b, c \in \mathbb{R}$ satisfy $a^2 + b^2 + c^2 = 1$, $b \neq 1$, and $d \in \mathbb{R}^4$. Then $PW = \mathbb{R}^2$ for $b < 0$ or $c \neq 0$. On the other hand, if $0 \leq b < 1$ and $c = 0$, one has*

$$PW = \{x \in \mathbb{R}^2 \mid (x - Pd) \cdot (1-b, -a) > 0\} \quad ,$$

where here the dot product represents the Euclidean scalar product on $\mathbb{R}^2$.

Proof. Without loss of generality, one may assume $d = 0$. One has

$$\begin{aligned} PW &= P\{\alpha\ell_{2+} + \beta(1,a,b,c) + s(c,0,c,1-b) + t(a,1-b,a,0) \mid \\ &\qquad \alpha > 0, \beta < 0, s,t \in \mathbb{R}\} \\ &= \{\alpha(1,0) + \beta(1,a) + s(c,0) + t(a,1-b) \mid \alpha > 0, \beta < 0, s,t \in \mathbb{R}\} \quad . \end{aligned}$$

And since $1 - b \neq 0$, this shows that $PW = \mathbb{R}^2$ for $c \neq 0$. Hence, one may restrict one's attention to $c = 0$. Since $(1-b,-a)$ is a normal vector for the line $\{t(a, 1-b) \mid t \in \mathbb{R}\}$, the remaining assertions readily follow from

$$\alpha(1,0) \cdot (1-b,-a) = \alpha(1-b) > 0 \quad ,$$

for $\alpha > 0$, and

$$\beta(1,a) \cdot (1-b,-a) = \beta(1-b-a^2) = \beta(b^2 - b) \quad ,$$

which is nonnegative for $0 \leq b < 1$ and negative for $b < 0$. □

This straightforward observation leads to the following characterization of maximal pairs of wedges.

Lemma 4.1.7. *The wedges* $W_R = W[\ell_{1+}, \ell_{1-}, 0]$ *and* $W = W[\ell_{2+}, \ell, d]$, *where* $\ell = (1,a,b,c)$ *and* $a,b,c \in \mathbb{R}$ *satisfy* $a^2 + b^2 + c^2 = 1$, $b \neq 1$, *and* $d \in \mathbb{R}^4$, *form a maximal pair of wedges if and only if* $0 < a < 1$, $0 < b < 1$, $c = 0$, *and the vector* d *is a linear combination of vectors whose associated translations leave either* W_R *or* W *fixed. The statement is true if* W *is replaced by* W' *and the condition* $0 < a < 1$ *is replaced by* $-1 < a < 0$ *or also if* ℓ_{2+} *is replaced by* ℓ_{2-} *and* $0 < b < 1$ *by* $-1 < b < 0$.

Proof. Using the projection P from Lemma 4.1.6, note that $x \in PW_R$ if and only if

$$x = \alpha(1,1) - \beta(1,-1) \quad \text{for suitable} \quad \alpha, \beta > 0 \quad . \tag{4.1.5}$$

W_R is invariant with respect to translations by vectors in the subspace generated by $(0,0,1,0)$ and $(0,0,0,1)$, so one has $W_R \cap W = \emptyset$ if and only if $PW_R \cap PW = \emptyset$. By Lemma 4.1.6, the condition $PW_R \cap PW = \emptyset$ is equivalent to $c = 0$, $0 \leq b < 1$ and (by (4.1.5))

$$\begin{aligned} 0 &\geq (\alpha(1,1) - \beta(1,-1) - Pd) \cdot (1-b,-a) \\ &= (\alpha - \beta)(1-b) - (\alpha+\beta)a - Pd \cdot (1-b,-a) \quad , \end{aligned}$$

for all $\alpha, \beta > 0$. This clearly entails that $a \geq 0$. Note also that $a, b \geq 0$ and $c = 0$ imply $a > 0$, since $b \neq 1$. It is then easy to check that this implies

$$1 - b - a = (1,1) \cdot (1-b,-a) \leq 0 \tag{4.1.6}$$

and

$$1 - b + a = (1, -1) \cdot (1 - b, -a) > 0 \quad . \tag{4.1.7}$$

Hence, $W_R \cap W = \emptyset$ is equivalent to the conditions $c = 0$, $0 \leq b < 1$, $a > 0$, and $-Pd \cdot (1 - b, -a) \leq 0$.

Assume first the maximality of the pair (W_R, W). Then $-Pd \cdot (1-b, -a) \leq 0$ and the conditions just established entail

$$(x - Pd) \cdot (1 - b, -a) \leq -Pd \cdot (1 - b, -a) \leq 0 \quad , \tag{4.1.8}$$

for all $x \in PW_R$. The maximality then implies the equality

$$-Pd \cdot (1 - b, -a) = 0 \quad , \tag{4.1.9}$$

since, if not, one could obtain a wedge which properly contains W and yet is still disjoint from W_R by choosing a different d such that (4.1.8) is still satisfied. Thus, one concludes that Pd is a multiple of $(a, 1 - b) = P(a, 1 - b, a, 0)$. Therefore, d is a linear combination of the vectors $(a, 1 - b, a, 0)$, $(0, 0, 0, 1)$ and $(0, 0, 1, 0)$, where translations by the former two leave W invariant and translations by the latter two leave W_R fixed.

The possibility that $b = 0$ still remains to be excluded. But $b = 0$ entails $a = 1$, so W, resp. PW, is invariant with respect to translations by multiples of $(1, 1, 1, 0)$, resp. $(1, 1)$. Translating the disjoint pair (W, W_R) by $d = -(1, 1, 1, 0)$, one would therefore obtain another disjoint pair (W, W_2) such that $W_2 = W[\ell_{1+}, \ell_{1-}, d]$ properly contains W_R, contradicting the assumed maximality of (W, W_R).

For the converse, assume that W has the stated form. By the first part of this proof, one already knows that W and W_R are then disjoint. Only the proof of maximality remains. By hypothesis, (4.1.9) holds in this direction, as well. Furthermore, (4.1.6) and (4.1.7) are fulfilled. Note that if $b \neq 0$, then (4.1.6) holds with strict inequality. A wedge W_3 which contains W_R must be coherent with W_R and is thus obtained by translating W_R by a vector of the form $-\alpha_0 \ell_{1+} + \beta_0 \ell_{1-}$, with $\alpha_0, \beta_0 \geq 0$. For $W_3 \neq W_R$, *i.e.* for $\alpha_0 \neq 0$ or $\beta_0 \neq 0$, (4.1.6) and (4.1.7) imply

$$P(-\alpha_0 \ell_{1+} + \beta_0 \ell_{1-}) \cdot (1 - b, -a) > 0 \quad .$$

The vertex of PW_3 lies in PW (Lemma 4.1.6 and (4.1.9)), hence $PW_3 \cap PW \neq \emptyset$ and so $W_3 \cap W \neq \emptyset$. One can argue similarly to eliminate the possibility that there does not exist a wedge properly containing W and yet being disjoint from W_R.

To establish the final assertions of the lemma, one need but consider the wedges transformed by suitable reflections. □

Since the union of the elements of a characteristic family of wedges yields a characteristic half-space, it is natural to use Lemma 4.1.2 to extend the map τ to the set $\mathcal{H}$ of all characteristic half-spaces in IR^4. In order to establish that this extension is well-defined, it is necessary to consider the possibility that two characteristic families generate the same half-space.

According to Lemma 4.1.1, every characteristic family $\mathcal{F}$ can be represented in the form $\mathcal{F} = \{W + \lambda\ell \mid \lambda \in \mathrm{IR}\}$. We define the complementary characteristic family $\mathcal{F}^c \equiv \{(W+\lambda\ell)' \mid \lambda \in \mathrm{IR}\}$. The families $\mathcal{F}$ and $\mathcal{F}^c$ generate complementary characteristic half-spaces H and H^c, respectively, *i.e.* $H^c = \mathrm{IR}^4 \setminus \overline{H}$. In order to simplify notation, we shall write $\mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$ for two characteristic families to mean $W_1 \cap W_2 = \emptyset$ for all $W_1 \in \mathcal{F}_1$ and all $W_2 \in \mathcal{F}_2$. Hence, one has $\mathcal{F} \cap \mathcal{F}^c = \emptyset$, for any characteristic family $\mathcal{F}$.

Lemma 4.1.8. *Let $\tau : \mathcal{W} \mapsto \mathcal{W}$ be a bijection with properties (A) and (B). Moreover, let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two characteristic families of wedges generating the same half-space, i.e. $\underset{W_1 \in \mathcal{F}_1}{\cup} W_1 = \underset{W_2 \in \mathcal{F}_2}{\cup} W_2$. Then one has $\underset{W_1 \in \mathcal{F}_1}{\cup} \tau(W_1) = \underset{W_2 \in \mathcal{F}_2}{\cup} \tau(W_2)$.*

Proof. Since $\mathcal{F}_1$ and $\mathcal{F}_2$ generate the same half-space, one must have $\mathcal{F}_1 \cap \mathcal{F}_2^c = \emptyset$. Hence, Corollary 4.1.3 entails $\tau(\mathcal{F}_1) \cap \tau(\mathcal{F}_2^c) = \emptyset$. Similarly, one derives $\tau(\mathcal{F}_1^c) \cap \tau(\mathcal{F}_2) = \emptyset$. From (4.1.4) it also follows that $\tau(\mathcal{F}^c) = \tau(\mathcal{F})^c$, so that one finds $\tau(\mathcal{F}_1) \cap \tau(\mathcal{F}_2)^c = \emptyset$ and $\tau(\mathcal{F}_1)^c \cap \tau(\mathcal{F}_2) = \emptyset$. By Lemma 4.1.2, $\tau(\mathcal{F}_1)$ and $\tau(\mathcal{F}_2)$ generate half-spaces H_1 and H_2, respectively, for which the following relations must therefore hold: $H_1 \cap H_2^c = \emptyset$ and $H_1^c \cap H_2 = \emptyset$. It follows that $H_1 = H_2$. □

Lemmas 4.1.2 and 4.1.8 ensure that the following map is well-defined:

Definition. *Let $\tau : \mathcal{W} \mapsto \mathcal{W}$ be a bijection satisfying the properties (A) and (B). Then an associated map $\tau : \mathcal{H} \mapsto \mathcal{H}$ is obtained by setting for $H \in \mathcal{H}$*

$$\tau(H) \equiv \underset{W \in \mathcal{F}}{\cup} \tau(W) \quad ,$$

where $\mathcal{F}$ is any characteristic family generating H.

We permit ourselves this abuse of notation in order to keep the notation as simple as possible, and because there will be no possibility of confusion of context. We next collect some useful properties of this map. We let $\mathcal{H}^\pm \subset \mathcal{H}$ denote the set of all future-directed (resp. past-directed) characteristic half-spaces $H^\pm$.

Lemma 4.1.9. *Let $\tau : \mathcal{W} \mapsto \mathcal{W}$ be a bijection satisfying the properties (A) and (B), and let $\tau : \mathcal{H} \mapsto \mathcal{H}$ be the associated mapping of characteristic half-spaces.*

(1) τ is bijective on $\mathcal{H}$;

(2) $\tau(H^c) = \tau(H)^c$, for all $H \in \mathcal{H}$;

(3) for $H_1, H_2 \in \mathcal{H}$, $H_1 \cap H_2 = \emptyset$ if and only if $\tau(H_1) \cap \tau(H_2) = \emptyset$; moreover, $H_1 \subset H_2$ if and only if $\tau(H_1) \subset \tau(H_2)$;

(4) for given $H \in \mathcal{H}$ and every element $a \in \mathrm{IR}^4$ there exists an element $b \in \mathrm{IR}^4$ (and vice versa) such that $\tau(H + a) = \tau(H) + b$;

(5) for any $W \in \mathcal{W}$, $W = H_+ \cap H_-$ if and only if $\tau(W) = \tau(H_+) \cap \tau(H_-)$;

(6) either $\tau(\mathcal{H}^\pm) = \mathcal{H}^\pm$ or $\tau(\mathcal{H}^\pm) = \mathcal{H}^\mp$.

Proof. 1. Let $\mathcal{F}_1, \mathcal{F}_2$ be characteristic families such that $\underset{W_1 \in \mathcal{F}_1}{\cup} \tau(W_1) = \underset{W_2 \in \mathcal{F}_2}{\cup} \tau(W_2)$. Since τ^{-1} has the same properties as τ does, Lemma 4.1.8 entails that $\underset{W_1 \in \mathcal{F}_1}{\cup} W_1 = \underset{W_2 \in \mathcal{F}_2}{\cup} W_2$, *i.e.* τ is injective on $\mathcal{H}$. Let now $H \in \mathcal{H}$ be generated by a characteristic family $\mathcal{F}$: $H = \underset{W \in \mathcal{F}}{\cup} W$. Then defining $H_0 = \underset{W \in \mathcal{F}}{\cup} \tau^{-1}(W)$, one has $\tau(H_0) = H$. *i.e.* τ is surjective on $\mathcal{H}$.

2. Assertion (2) is an immediate consequence of the property (4.1.4) of the map τ on $\mathcal{W}$.

3. Let $\mathcal{F}_1, \mathcal{F}_2$ be characteristic families which generate the characteristic half-spaces H_1, H_2, respectively. If $H_1 \cap H_2 = \emptyset$, then $\mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$, which implies $\tau(\mathcal{F}_1) \cap \tau(\mathcal{F}_2) = \emptyset$, by property (4.1.3) of the transformation τ. Hence one has $\tau(H_1) \cap \tau(H_2) = \emptyset$. The converse is proven using the fact that the map τ^{-1} also has the stated properties.

If one has instead the inclusion $H_1 \subset H_2$, then by Lemma 4.1.1 there exist wedges W_1, W_2 such that $H_i = \cup\{W_i + \lambda\ell \mid \lambda \in \mathrm{I\!R}\}$, $i = 1, 2$, for a fixed future-directed lightlike vector ℓ (one characteristic half-space is contained in another only if their boundaries are parallel hyperplanes). One can choose W_1, W_2 such that $W_1 \subset W_2$. From condition (B) it then follows that $\tau(W_1 + \lambda\ell) \subset \tau(W_2 + \lambda\ell)$ for all $\lambda \in \mathrm{I\!R}$, so that one must have the inclusion $\tau(H_1) \subset \tau(H_2)$.

4. One first notes some general properties of characteristic half-spaces: if H_1, H_2 are half-spaces with $H_1 \subset H_2$, then there exists a translation $c \in \mathrm{I\!R}^4$ such that $H_2 = H_1 + c$. If, on the other hand, the latter relation holds, then one must have either $H_1 \subset H_2$ or $H_2 \subset H_1$.

Let now $H \in \mathcal{H}$ and $a \in \mathrm{I\!R}^4$ be given. Then either $H \subset H + a$ or $H + a \subset H$. In the former case, part (3) of this lemma entails the inclusion $\tau(H) \subset \tau(H + a)$, so that $\tau(H + a) = \tau(H) + b$ for some $b \in \mathrm{I\!R}^4$. The second case is handled analogously. Since the map τ^{-1} on $\mathcal{H}$ satisfies assertions (1)-(3) of this lemma, the assertion (4) also follows when the roles of a and b are exchanged.

5. Given a wedge $W \in \mathcal{W}$ there exist unique characteristic half-spaces $H^\pm$ such that $W = H^+ \cap H^-$. They are determined by the characteristic families $\mathcal{F}^\pm = \{W + \lambda\ell_\mp \mid \lambda \in \mathrm{I\!R}\}$, where $\ell^\pm$ are future-directed lightlike vectors such that $W \pm \ell_\pm \subset W$. Clearly one has $\tau(W) \in \tau(\mathcal{F}^\pm)$. Since $\mathcal{F}^\pm$ are characteristic families, by Lemma 4.1.1 there exist future-directed lightlike vectors $\ell_\tau^\pm$ such that $\tau(\mathcal{F}^\pm) = \{\tau(W) + \lambda\ell_\tau^\pm \mid \lambda \in \mathrm{I\!R}\}$. Since the set $\mathcal{F}^+ \cup \mathcal{F}^-$ is not linearly ordered, condition (B) entails that also the set $\tau(\mathcal{F}^+) \cup \tau(\mathcal{F}^-)$ is not linearly ordered, in other words, $\tau(\mathcal{F}^+) \neq \tau(\mathcal{F}^-)$. Hence the vectors ℓ_τ^+ and ℓ_τ^- are not parallel. Therefore, the intersection of the half-spaces $\tau(H^\pm)$ generated by $\tau(\mathcal{F}^\pm)$ must coincide with $\tau(W)$.

6. Let $H^\pm \in \mathcal{H}^\pm$. If the hyperplanes which form the boundaries of $H^\pm$ are parallel, then one must have either $H^+ \cap H^- = \emptyset$ or $H^{+c} \cap H^{-c} = \emptyset$. Parts (2) and (3) of this lemma then entail that either $\tau(H^+) \cap \tau(H^-) = \emptyset$ or $\tau(H^+)^c \cap \tau(H^-)^c = \emptyset$ must hold. Hence the boundary hyperplanes of the characteristic half-spaces $\tau(H^\pm)$ are parallel, and the time-like orientations of these half-spaces are oppositely directed. On the other hand, if the boundary hyperplanes of $H^\pm$ are not parallel, then their intersection $H^+ \cap H^- = W$ is a wedge, and it follows from part (4) that $\tau(H^+) \cap \tau(H^-) = \tau(W) \in \mathcal{W}$. Hence,

also in this situation the time-like orientations of the half-spaces $\tau(H^\pm)$ are oppositely directed.

Fixing H^- and letting H^+ range through $\mathcal{H}^+$, one concludes that either $\tau(\mathcal{H}^+) \subset \mathcal{H}^+$ and $\tau(H^-) \in \mathcal{H}^-$ or $\tau(\mathcal{H}^+) \subset \mathcal{H}^-$ and $\tau(H^-) \in \mathcal{H}^+$. Varying H^- while holding H^+ fixed completes the proof of assertion (6), when one recalls the result of part (1). $\square$

Each characteristic half-space $H_p[\ell]^\pm$ determines uniquely a characteristic hyperplane $H_p[\ell] = \overline{H_p[\ell]^+} \cap \overline{H_p[\ell]^-}$, and so the map τ on $\mathcal{H}$ naturally induces a map on the set of characteristic hyperplanes.

Definition. *Let $\tau : \mathcal{W} \mapsto \mathcal{W}$ be a bijection satisfying properties (A) and (B) and $\tau : \mathcal{H} \mapsto \mathcal{H}$ the associated mapping of characteristic half-spaces. Then*

$$\tau(H_p[\ell]) \equiv \overline{\tau(H_p[\ell]^+)} \cap \overline{\tau(H_p[\ell]^-)}$$

defines a mapping of characteristic hyperplanes onto characteristic hyperplanes.

The following properties of this mapping of characteristic hyperplanes are an immediate consequence of Lemma 4.1.9.

Corollary 4.1.10. *Let $\tau : \mathcal{W} \mapsto \mathcal{W}$ be a bijection satisfying properties (A) and (B) and τ be the associated mapping of characteristic hyperplanes.*

(1) τ is bijective on the set of characteristic hyperplanes in IR^4;

(2) for a given hyperplane $H_p[\ell]$ and every element $a \in \mathrm{IR}^4$ there exists an element $b \in \mathrm{IR}^4$ (and vice versa) such that $\tau(H_p[\ell] + a) = \tau(H_p[\ell]) + b$;

(3) τ maps distinct parallel characteristic hyperplanes onto distinct parallel characteristic hyperplanes.

We next prove some further properties of this mapping τ which are not quite so obvious.

Lemma 4.1.11. *Let $\tau : \mathcal{W} \mapsto \mathcal{W}$ be a bijection satisfying properties (A) and (B) and τ be the associated mapping of characteristic hyperplanes. If $\ell_1, \ell_2, \ell_3, \ell_4$ are linearly dependent future-directed lightlike vectors such that any two of them are linearly independent, then*

$$\bigcap_{i=1}^{4} \tau(H_0[\ell_i]) = \bigcap_{i \neq k} \tau(H_0[\ell_i]) \quad \textit{for} \quad k = 1, 2, 3, 4 \quad .$$

Proof. As pointed out earlier, an arbitrary maximal pair $(W[\widetilde{\ell}_1, \widetilde{\ell}_2, d_1], W[\widetilde{\ell}_3, \widetilde{\ell}_4, d_2])$ with $\{\widetilde{\ell}_1, \widetilde{\ell}_2\} \neq \{\widetilde{\ell}_3, \widetilde{\ell}_4\}$ can be brought into the form $(W_R, W[\ell_{2+}, \widetilde{\ell}, d])$ (or $(W_R, W[\ell_{2+}, \widetilde{\ell}, d]')$ by a suitable Poincaré transformation, and by Lemma 4.1.7 it is no loss of generality to take $d = 0$. Hence, $H_0[\ell_{1+}]$, $H_0[\ell_{1-}]$, $H_0[\ell_{2+}]$ and $H_0[\widetilde{\ell}]$ are the characteristic hyperplanes determined by these wedges. Since Lemma 4.1.7 entails that $\widetilde{\ell} = (1, a, b, 0)$, with $0 < a < 1$ and $0 < b < 1$, one observes that any three of the four vectors ℓ_{1+}, ℓ_{1-}, ℓ_{2+} and $\widetilde{\ell}$ are linearly independent. Hence, the intersection of any three of the hyperplanes $H_0[\ell_{1+}]$,

$H_0[\ell_{1-}]$, $H_0[\ell_{2+}]$, $H_0[\widetilde{\ell}]$ is one-dimensional. But, on the other hand, one evidently has

$$\{c(0,0,0,1) \mid c \in \mathbb{R}\} \subset H_0[\ell_{1+}] \cap H_0[\ell_{1-}] \cap H_0[\ell_{2+}] \cap H_0[\widetilde{\ell}] \quad . \tag{4.1.10}$$

Therefore, one may conclude that the right-hand side of (4.1.10) is equal to the one-dimensional intersection of any three of the hyperplanes in that expression. Employing the suitable Poincaré transformation, one sees that

$$\bigcap_{i=1}^{4} H_{c_i}[\widetilde{\ell}_i] = \bigcap_{i \neq j} H_{c_i}[\widetilde{\ell}_i] \quad \text{for} \quad j = 1,2,3,4 \quad , \tag{4.1.11}$$

where $\{H_{c_i}[\widetilde{\ell}_i]\}_{i=1}^4$ are the hyperplanes determined by the maximal pair $(W[\widetilde{\ell}_1, \widetilde{\ell}_2, d], W[\widetilde{\ell}_3, \widetilde{\ell}_4, d'])$.

Returning to the vectors $\{\ell_1, \dots, \ell_4\}$ of the hypothesis, there exists a Lorentz transformation Λ with $\Lambda\ell_1 = a_1\ell_{1+}$, $\Lambda\ell_2 = a_2\ell_{1-}$, $\Lambda\ell_3 = a_3\ell_{2+}$, and $\Lambda\ell_4 = a_4\ell$, where

$$\ell = (1,a,b,0), \quad a,b \in \mathbb{R} \quad , \quad a^2 + b^2 = 1 \quad ,$$

and a_i, $i = 1, \dots, 4$, are positive constants. Hence, one may once again consider the pair $(\Lambda W[\ell_1, \ell_2, 0], \Lambda W[\ell_3, \ell_4, 0]) = (W_R, W[\ell_{2+}, \ell, 0])$ without loss of generality, since $\tau \circ \Lambda^{-1}$ maps maximal pairs onto maximal pairs. If this pair is maximal, then (4.1.11) yields the desired assertion. If this pair and $({W_R}', W[\ell_{2+}, \ell, 0])$ are not maximal, then Lemma 4.1.7 entails $b \leq 0$. But, in fact, $b = 0$ is excluded by the linear independence assumption. Set $\ell_0 = (1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$. Then using (4.1.11) for the maximal pairs $(\tau(W_R), \tau(W[\ell_{2+}, \ell_0, 0]))$ and $(\tau(W[\ell_{2+}, \ell_{2-}, 0]), \tau(W[\ell_{1+}, \ell_0, 0]))$ as well as Lemma 4.1.7 and the fact that τ preserves the maximality of pairs of wedges, one finds

$$\begin{aligned}
\bigcap_{i=1}^{3} \tau(\Lambda H_0[\ell_i]) &= \tau(H_0[\ell_{1+}]) \cap \tau(H_0[\ell_{1-}]) \cap \tau(H_0[\ell_{2+}]) \\
&= \tau(H_0[\ell_{1+}]) \cap \tau(H_0[\ell_{1-}]) \cap \tau(H_0[\ell_{2+}]) \cap \tau(H_0[\ell]) \\
&= \tau(H_0[\ell_{1+}]) \cap \tau(H_0[\ell_0]) \cap \tau(H_0[\ell_{2+}]) \\
&= \tau(H_0[\ell_{1+}]) \cap \tau(H_0[\ell_0]) \cap \tau(H_0[\ell_{2+}]) \cap \tau(H_0[\ell_{2-}]) \\
&\subset \tau(H_0[\ell_{2-}]) \quad .
\end{aligned} \tag{4.1.12}$$

If $a \neq 0$, then either $(W_R, W[\ell_{2-}, \ell, 0])$ or $(W_R, W[\ell_{2-}, \ell, 0]')$ is maximal, by Lemma 4.1.7. Hence, (4.1.11) and (4.1.12) yield

$$\begin{aligned}
\bigcap_{i=1}^{3} \tau(\Lambda H_0[\ell_i]) &\subset \tau(H_0[\ell_{1+}]) \cap \tau(H_0[\ell_{1-}]) \cap \tau(H_0[\ell_{2-}]) \\
&= \tau(H_0[\ell_{1+}]) \cap \tau(H_0[\ell_{1-}]) \cap \tau(H_0[\ell_{2-}]) \cap \tau(H_0[\ell]) \\
&\subset \tau(H_0[\ell]) \quad ,
\end{aligned}$$

implying the desired assertion. If, on the other hand, $a = 0$, then ℓ is a positive multiple of ℓ_{2-}, so that $H_0[\ell] = H_0[\ell_{2-}]$ and use of (4.1.12) completes the proof.

□

It is evident that the intersection of four hyperplanes $H_{c_i}[\ell_i]$ corresponding to a linearly independent set of four future-directed lightlike vectors ℓ_i and four real numbers c_i is a set containing a single point. We now have established sufficient background to prove that the map τ preserves this property.

Lemma 4.1.12. *Let $\tau : \mathcal{W} \mapsto \mathcal{W}$ be a bijection satisfying properties (A) and (B) and τ be the associated mapping of characteristic hyperplanes. Then the intersection $\cap_\ell \, \tau(H_0[\ell])$ taken over all future-directed lightlike vectors is a singleton set (a point) in IR^4.*

Proof. Since, from Corollary 4.1.10, τ maps parallel characteristic hyperplanes onto parallel characteristic hyperplanes, there exist suitable pairwise linearly independent lightlike vectors $\widetilde{\ell}_1, \widetilde{\ell}_2, \widetilde{\ell}_3, \widetilde{\ell}_4$ and $c_1, c_2, c_3, c_4 \in \mathrm{IR}$ such that $\tau(H_0[\ell_{1+}]) = H_{c_1}[\widetilde{\ell}_1]$, $\tau(H_0[\ell_{1-}]) = H_{c_2}[\widetilde{\ell}_2]$, $\tau(H_0[\ell_{2+}]) = H_{c_3}[\widetilde{\ell}_3]$, and $\tau(H_0[\ell_{3+}]) = H_{c_4}[\widetilde{\ell}_4]$. By part (2) of Corollary 4.1.10, there exist real numbers $b_1, b_2, b_3, b_4 \in \mathrm{IR}$ such that $\tau(H_{b_1}[\ell_{1+}]) = H_0[\widetilde{\ell}_1]$, $\tau(H_{b_2}[\ell_{1-}]) = H_0[\widetilde{\ell}_2]$, $\tau(H_{b_3}[\ell_{2+}]) = H_0[\widetilde{\ell}_3]$, and $\tau(H_{b_4}[\ell_{3+}]) = H_0[\widetilde{\ell}_4]$. If $\{\widetilde{\ell}_i\}_{i=1,\dots,4}$ is a linearly dependent set, then Lemma 4.1.11 applied to τ^{-1} as a mapping on the set of characteristic hyperplanes would entail that $\{\ell_{1+}, \ell_{1-}, \ell_{2+}, \ell_{3+}\}$ is linearly dependent, a contradiction. Hence, $\{\widetilde{\ell}_i\}_{i=1,\dots,4}$ is a linearly independent set and so the intersection $\cap_{i=1}^4 H_{c_i}[\widetilde{\ell}_i]$ is a singleton set.

An arbitrary lightlike vector $\ell \neq 0$ is a linear combination of ℓ_{3+} and two linearly independent lightlike vectors ℓ_1, ℓ_2 with zero x_3-component. By Lemma 4.1.11, it follows that

$$\tau(H_0[\ell_1]) \cap \tau(H_0[\ell_2]) \cap \tau(H_0[\ell_{3+}]) \subset \tau(H_0[\ell])$$

(note that if $\ell_1, \ell_2, \ell_{3+}, \ell$ are not pairwise linearly independent, then ℓ is a positive multiple of one of the others and determines the same hyperplane as the latter) and also

$$\tau(H_0[\ell_{1+}]) \cap \tau(H_0[\ell_{1-}]) \cap \tau(H_0[\ell_{2+}]) \subset \tau(H_0[\ell_j]) \quad \text{for} \quad j = 1, 2 \quad .$$

This proves the claim, since

$$\bigcap_{i=1}^{4} H_{c_i}[\widetilde{\ell}_i] = \tau(H_0[\ell_{1+}]) \cap \tau(H_0[\ell_{1-}]) \cap \tau(H_0[\ell_{2+}]) \cap \tau(H_0[\ell_{3+}]) \subset \tau(H_0[\ell]) \quad ,$$

for arbitrary future-directed lightlike $\ell \neq 0$. □

This result entails that τ induces a point transformation on IR^4.

Definition. *For each $x \in \mathrm{IR}^4$, $W \in \mathcal{W}$, and each characteristic hyperplane H, let $T_x(H) \equiv H + x$ and $T_x(W) \equiv W + x$. Let $\tau : \mathcal{W} \mapsto \mathcal{W}$ be a bijection satisfying properties (A) and (B) and τ be the associated mapping of characteristic hyperplanes. Then define $\delta : \mathrm{IR}^4 \mapsto \mathrm{IR}^4$ by*

$$\{\delta(x)\} \equiv \bigcap_{\ell} \tau(T_x H_0[\ell]) \quad \textit{for} \quad x \in \mathrm{IR}^4 \quad ,$$

where the intersection is taken over all non-zero future-directed lightlike vectors $\ell \in \mathrm{IR}^4$.

Note that the mapping $\tau \circ T_x$ has the same properties as τ; applying Lemma 4.1.12 to this mapping implies that δ is well-defined. We next need to show that this point transformation is consistent with the mapping τ.

Proposition 4.1.13. *Let $\tau : \mathcal{W} \mapsto \mathcal{W}$ be a bijection satisfying properties (A) and (B) and δ be the associated point transformation. Then δ is a bijection and*

$$\tau(W) = \{\delta(x) \mid x \in W\} \quad \textit{for all} \quad W \in \mathcal{W} \quad .$$

Proof. Define a mapping $\gamma : \mathbb{R}^4 \mapsto \mathbb{R}^4$ by

$$\{\gamma(y)\} \equiv \bigcap_{\ell} \tau^{-1}(T_y H_0[\ell]) \quad .$$

For a fixed $x \in \mathbb{R}^4$, consider $y \equiv \delta(x)$, so that $y \in \tau(T_x H_0[\ell])$ for all non-zero positive lightlike vectors ℓ. But, by Corollary 4.1.10, for each such ℓ there exists a non-zero positive lightlike vector ℓ' such that $\tau(T_x H_0[\ell]) = T_y H_0[\ell']$. Since τ is bijective on the set of characteristic hyperplanes in $\mathbb{R}^4$, it follows that

$$\{\gamma(\delta(x))\} = \bigcap_{\ell'} \tau^{-1}(T_y H_0[\ell']) = \bigcap_{\ell} \tau^{-1}(\tau(T_x H_0[\ell]) = \{x\} \quad ;$$

hence, one has $\gamma = \delta^{-1}$ and δ is a bijection.

For arbitrary $W_0 \in \mathcal{W}$ and $y \in W_0$, there exists a wedge $W_1 \subset W_0$ such that y lies in the edge of W_1 and such that the characteristic hyperplanes determined by W_0 are different from (though parallel to) those determined by W_1. By Corollary 4.1.10, the same must be true of the hyperplanes determined by the wedges $\tau(W_1) \subset \tau(W_0)$. Thus, one has $\overline{\tau(W_1)} \subset \tau(W_0)$. Let H_1 and H_2 be the characteristic hyperplanes determined by W_1. There are two characteristic families $\mathcal{F}_1$ and $\mathcal{F}_2$, containing W_1, with $H_1 = \partial(\cup_{W \in \mathcal{F}_1} W)$ and $H_2 = \partial(\cup_{W \in \mathcal{F}_2} W)$. The wedge $\tau(W_1)$ is contained in both $\tau(\mathcal{F}_1)$ and $\tau(\mathcal{F}_2)$, so that $\tau(H_1) = \partial(\cup_{W \in \tau(\mathcal{F}_1)} W)$ and $\tau(H_2) = \partial(\cup_{W \in \tau(\mathcal{F}_2)} W)$ are the characteristic hyperplanes determined by $\tau(W_1)$. The characteristic hyperplanes containing the point y (H_1 and H_2 belong to this set) are mapped by τ into the set of characteristic hyperplanes containing $\delta(y)$, *i.e.* $\delta(y) \in \tau(H_1)$ and $\delta(y) \in \tau(H_2)$. This shows that $\delta(y)$ lies in the two characteristic hyperplanes determined by $\tau(W_1)$. But this entails $\delta(y) \in \overline{\tau(W_1)} \subset \tau(W_0)$, which yields

$$\{\delta(x) \mid x \in W\} \subset \tau(W) \quad \text{for every} \quad W \in \mathcal{W} \quad . \tag{4.1.13}$$

Since, by Corollary 4.1.3, τ^{-1} has the same properties as τ, one has similarly

$$\{\delta^{-1}(x) \mid x \in W\} \subset \tau^{-1}(W) \quad \text{for every} \quad W \in \mathcal{W} \quad .$$

Now let $y \in \tau(W)$. Then one has $x \equiv \delta^{-1}(y) \in \tau^{-1}(\tau(W)) = W$, and since $\delta(x) = y$, it follows that $\tau(W) \subset \{\delta(x) \mid x \in W\}$. The containment (4.1.13) completes the proof. □

We recall the well-known result of Alexandrov [2][3] (see also Zeeman [77], Borchers and Hegerfeldt [11]) to the effect that bijections on $\mathbb{R}^4$ mapping light cones to light cones must be elements of the extended Poincaré group, $\mathcal{DP}$, generated by the Poincaré group and the dilatation group. The above-established results can be used to show that the bijection $\delta : \mathbb{R}^4 \mapsto \mathbb{R}^4$ constructed above does indeed map light cones onto light cones. However, a more concise argument can be obtained by appealing to a related result of Alexandrov [3], to wit: a bijection on $\mathbb{R}^4$, who along with its inverse maps spacelike separated points onto spacelike separated points, is an element of the extended Poincaré group.

Lemma 4.1.14. *Let $\tau : \mathcal{W} \mapsto \mathcal{W}$ be a bijection satisfying properties (A) and (B) and $\delta : \mathrm{IR}^4 \mapsto \mathrm{IR}^4$ be the associated point transformation. Then δ is an element of $\mathcal{DP}$.*

Proof. Note that two points $x, y \in \mathrm{IR}^4$ are spacelike separated if and only if there exists a wedge $W \in \mathcal{W}$ such that $x \in W$ and $y \in W'$. But by Prop. 4.1.13 and Corollary 4.1.5, one sees that $x \in \mathcal{W}$ and $y \in W'$ if and only if $\delta(x) \in \tau(W)$ and $\delta(y) \in \tau(W') = \tau(W)'$, *i.e.* $\delta(x)$ and $\delta(y)$ are spacelike separated. It is therefore evident that both δ and δ^{-1} preserve spacelike separation. The desired assertion then follows from Theorem 1 of [3]. □

We have therefore established the following result, which we regard as a considerable extension of the theorems of Alexandrov *et alia* just cited.

Theorem 4.1.15. *Let $\tau : \mathcal{W} \mapsto \mathcal{W}$ be a bijection with the properties (A) and (B). Then there exists an element δ of the extended Poincaré group $\mathcal{DP}$ such that for all $W \in \mathcal{W}$ one has*

$$\tau(W) = \{\delta(x) \mid x \in W\} \quad .$$

Proof. This is an immediate consequence of Corollary 4.1.10, Prop. 4.1.13 and Lemma 4.1.14. □

Turning to the more special case of the transformations τ_W on $\mathcal{W}$ which arise when the CGMA holds, we know from Prop. 3.1 that they satisfy conditions (A) and (B) and are involutions. Each of those transformations will therefore fulfill the hypotheses of the next Corollary.

Corollary 4.1.16. *Let $\tau : \mathcal{W} \mapsto \mathcal{W}$ be an involutive bijection with the properties (A) and (B). Then there exists an element δ of the Poincaré group such that for all $W \in \mathcal{W}$*

$$\tau(W) = \{\delta(x) \mid x \in W\} \quad .$$

Proof. It follows from the preceding proposition that there exists an element δ of the extended Poincaré group such that the stated equality of sets holds. Since τ is an involution, one sees that $W = \tau^2(W) = \{\delta^2(x) \mid x \in W\}$, for each $W \in \mathcal{W}$. Hence, by taking suitable intersections one may conclude that $\delta^2(x) = x$, for all $x \in \mathrm{IR}^4$. Since δ is an affine map, it is then clear that it cannot contain a nontrivial dilatation. □

Since the group $\mathcal{T}$ is generated by elements satisfying the hypothesis of Corollary 4.1.16, and since Poincaré transformations are completely fixed by their action on the wedges $\mathcal{W}$, we conclude that $\mathcal{T}$ is isomorphic to a subgroup $\mathcal{G}$ of the Poincaré group. In order to indicate the strength of this result, we shall outline a closely related example, where the respective transformations are *not* induced by point transformations, even though properties (A) and (B) obtain.

We consider the manifold $\mathcal{M} \equiv \mathrm{IR}^4 \setminus \overline{V_+}$, which is the complement in Minkowski space of the closure of the forward light cone with apex at the origin, with the conformal structure inherited from Minkowski space, and we take as an admissible family $\mathcal{W}_+$ the set of all regions $W_+ = W \setminus \overline{V_+}$, where W ranges through the wedges in Minkowski space considered above. Note that W is uniquely determined once W_+ is given and that $(W_1)_+ \cap (W_2)_+ = \emptyset$ if and only if $W_1 \cap W_2 = \emptyset$. However, this latter implication fails to be true in general for the intersection of more than two regions. Moreover, we also note that the equality $(W_+)' = (W')_+$ holds for all wedges W. We pick now any Lorentz transformation which interchanges the forward and backward light cones in $\mathbb{R}^4$, such as time reversal T, and define on $\mathcal{W}_+$ the mapping

$$\tau(W_+) \equiv (TW)_+ \quad , \quad W_+ \in \mathcal{W}_+ \quad .$$

It follows from the preceding remarks that $\tau : \mathcal{W}_+ \mapsto \mathcal{W}_+$ is well-defined and has properties (A) and (B). But if the intersection of three (or more) partial wedges W_+ is contained in the backward light cone V_-, their images under the map τ have empty intersection. This shows that τ cannot be induced by a point transformation on $\mathcal{M}$.

4.2. Wedge Transformations Generate the Proper Poincaré Group

In the preceding section we have seen that for any theory on $\mathbb{R}^4$ satisfying the CGMA for the wedge regions $\mathcal{W}$, the corresponding transformation group $\mathcal{T}$ is isomorphic to a subgroup $\mathcal{G}$ of the Poincaré group $\mathcal{P}$. So the next question in our program is: which subgroups of $\mathcal{P}$ can appear in this way? We do not aim here at a complete answer to this question and restrict attention to those cases where the group $\mathcal{T}$ is "large". A natural way of expressing this mathematically is to assume that the group $\mathcal{T}$ acts transitively upon the set $\mathcal{W}$. It would be interesting to consider situations where this transitive action fails[6]. However, as our intention in this paper is to illustrate the application of our approach to just a few, albeit physically important cases, we make this additional asssumption and leave the other possibilities uninvestigated for the present. We remark that the condition that $\mathcal{T}$ acts transitively upon the set $\mathcal{W}$ is implied by the algebraic postulate that the adjoint action of the modular conjugations $\{J_W \mid W \in \mathcal{W}\}$ acts transitively upon the set $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$. Hence, this condition also is expressible in terms of algebraically determined quantities.

We have constructed a subgroup $\mathcal{G}$ of the Poincaré group, which is isomorphic to $\mathcal{T}$ and related to the group $\mathcal{T}$ as follows: For each $\tau \in \mathcal{T}$ there exists an element $g_\tau \in \mathcal{G}$ such that $\tau(W) = g_\tau W \equiv \{g_\tau(x) \mid x \in W\}$. To each of the defining involutions $\tau_W \in \mathcal{T}$, $W \in \mathcal{W}$, there exists a unique corresponding involution $g_W \in \mathcal{G} \subset \mathcal{P}$. The Poincaré group has four connected components, and the transitivity of the action of $\mathcal{G}$ upon the set $\mathcal{W}$, which implies that for every $W_1, W_2 \in \mathcal{W}$, there exists an element $g \in \mathcal{G}$ such that $g g_{W_1} g^{-1} = g_{gW_1} = g_{W_2}$, entails the relation

$$g_{W_1} g_{W_2} = g_{W_1} g g_{W_1} g^{-1} = g_{W_1} g g_{W_1}^{-1} g^{-1} \quad ,$$

[6]We shall return to this point in a subsequent publication.

since g_{W_1} is an involution. But the right-hand side is a group commutator, and in the Poincaré group such commutators are always contained in the identity component $\mathcal{P}_+^\uparrow$. Hence, for any wedges $W_1, W_2 \in \mathcal{W}$ the product of the corresponding group elements $g_{W_1} g_{W_2}$ must be contained in $\mathcal{P}_+^\uparrow$, and the same is true for products of an even number of the generating involutions of $\mathcal{G}$. Now pick a wedge $W \in \mathcal{W}$ and consider the corresponding involution $g_W \in \mathcal{G}$, which must lie in one of the four components of $\mathcal{P}$. One then notes that if $n \in \mathrm{I\!N}$ is odd, then it follows from $g_{W_1} \cdots g_{W_n} = g_W (g_W g_{W_1} \cdots g_{W_n})$ that $g_{W_1} \cdots g_{W_n}$ must lie in the same component of $\mathcal{P}$ as g_W. But this implies the following lemma.

Lemma 4.2.1. *The group $\mathcal{G}$ has nonempty intersection with at most one connected component of the Poincaré group $\mathcal{P}$ other than $\mathcal{P}_+^\uparrow$.*

Thus we are dealing with a subgroup $\mathcal{G}$ of $\mathcal{P}$ which is generated by involutions, intersects at most two of the four connected components of $\mathcal{P}$ and acts transitively on $\mathcal{W}$ in the obvious sense. Which subgroups can such $\mathcal{G}$ be? Answering this question turned out to be a somewhat laborious task. We begin by discussing an analogous problem for the Lorentz group.

Consider again the reference wedge $W_R = \{x \in \mathrm{I\!R}^4 \mid x_1 > |x_0|\}$, whose edge contains the origin, and let $\mathrm{InvL}(W_R) \equiv \{\Lambda \in \mathcal{L} \mid \Lambda W_R = W_R\}$ be its invariance subgroup in the full Lorentz group $\mathcal{L}$. The involutions in $\mathrm{InvL}(W_R)$ given by the identity $\mathrm{diag}(1,1,1,1) \in \mathcal{L}_+^\uparrow$, the temporal reflection $T = \mathrm{diag}(-1,1,1,1) \in \mathcal{L}_-^\downarrow$, the reflection through the 3-axis (in other words, about the $x_0 x_1 x_2$-hyperplane) $P_3 = \mathrm{diag}(1,1,1,-1) \in \mathcal{L}_-^\uparrow$, and their product $P_3 T = \mathrm{diag}(-1,1,1,-1) \in \mathcal{L}_+^\downarrow$ are distinguished, because all elements of $\mathrm{InvL}(W_R)$ can be obtained by multiplying elements of $\mathrm{InvL}_+^\uparrow(W_R) \equiv \mathrm{InvL}(W_R) \cap \mathcal{L}_+^\uparrow$ by these involutions. It is important in what is to come that $\mathrm{InvL}_+^\uparrow(W_R)$ is an abelian group, since it is generated by rotations about the 1-axis and velocity transformations (boosts) in the 0-1 direction, whereas $\mathrm{InvL}(W_R)$ is not abelian, precisely because of the mentioned involutions. The fact that $\mathrm{InvL}_+^\uparrow(W_R)$ is abelian is heavily used in our arguments, and for that reason our proof does not function in higher-dimensional Minkowski spaces. We wish to prove the following proposition.

Proposition 4.2.2. *Any subgroup G of the identity component $\mathcal{L}_+^\uparrow$ of the Lorentz group, which acts transitively upon the set $\mathcal{W}_0$ of wedges whose edges contain the origin of $\mathrm{I\!R}^4$, must equal $\mathcal{L}_+^\uparrow$. Furthermore, any subgroup G of the Lorentz group $\mathcal{L}$, which is generated by a collection of involutions, has nontrivial intersection with at most two connected components of $\mathcal{L}$ and acts transitively upon the set $\mathcal{W}_0$, must contain $\mathcal{L}_+^\uparrow$.*

The proof will proceed in a number of steps, since we find it convenient to consider the following alternatives: (i) $G \cap \mathrm{InvL}(W_R)$ is trivial, *i.e* consists only of the identity 1, (ii) $G \cap \mathrm{InvL}(W_R)$ is nontrivial but $G \cap \mathrm{InvL}_+^\uparrow(W_R)$ is trivial, or (iii) $G \cap \mathrm{InvL}_+^\uparrow(W_R)$ is nontrivial. We shall show that cases (i) and (ii) cannot obtain under our assumptions and that case (iii) implies the desired conclusion.

We shall exclude case (i) by proving the following claim.

Lemma 4.2.3. *Let G be a subgroup of $\mathcal{L}$ which acts transitively upon the set $\mathcal{W}_0$. Then one must have $G \cap InvL(W_R) \neq \{1\}$.*

If we knew from the outset that the group G in the statement of Prop. 4.2.2 has the property that also $G_+ \equiv G \cap \mathcal{L}_+^\uparrow$ acts transitively on $\mathcal{W}_0$, its proof would follow directly from a simplified version of this lemma and the fact that $\mathcal{L}_+^\uparrow$ is a simple group. For then the adjoint action of G_+ applied to the nontrivial element in $G_+ \cap \mathrm{InvL}(W_R)$ would generate all of $\mathcal{L}_+^\uparrow$. In particular, if G_+ acts transitively upon $\mathcal{W}_0$, then there exists for each $\Lambda \in \mathcal{L}_+^\uparrow$ a $g_\Lambda \in G_+$ and some $\widetilde{\Lambda} \in \mathrm{InvL}_+^\uparrow(W_R)$ such that $g_\Lambda = \Lambda\widetilde{\Lambda}$. Moreover, Lemma 4.2.3 would yield the existence of some nontrivial element $h_0 \in G_+ \cap \mathrm{InvL}(W_R)$. Since $G_+ \cap \mathrm{InvL}(W_R) \subset \mathrm{InvL}_+^\uparrow(W_R)$ and the latter group is abelian, we conclude $\Lambda h_0 \Lambda^{-1} = \Lambda\widetilde{\Lambda} h_0 \widetilde{\Lambda}^{-1}\Lambda^{-1} = g_\Lambda h_0 g_\Lambda^{-1} \in G_+$, for all $\Lambda \in \mathcal{L}_+^\uparrow$. But $\mathcal{L}_+^\uparrow$ is simple (see, *e.g.* Sect. I.2.8 in [37]), so it follows in this case that $G_+ = \mathcal{L}_+^\uparrow$. What makes the proofs somewhat cumbersome is the *a priori* possibility that for the transitivity of the action of G on $\mathcal{W}_0$ elements in $G \setminus G_+$ are essential. Note, however, given Lemma 4.2.3, the argument just given establishes the first assertion in Prop. 4.2.2.

As the proof of Lemma 4.2.3 is itself quite lengthy, we shall break it up into a series of sublemmas. The assumption that the intersection $G \cap \mathrm{InvL}(W_R)$ is trivial and that G acts transitively on $\mathcal{W}_0$ entail that for every $\Lambda \in \mathcal{L}_+^\uparrow$ there exists *exactly one* $g_\Lambda \in G$ and a *unique* $\widetilde{\Lambda} \in \mathrm{InvL}(W_R)$ such that $g_\Lambda = \Lambda\widetilde{\Lambda}$ (otherwise, one would have $\Lambda = g_1\widetilde{\Lambda}_1^{-1} = g_2\widetilde{\Lambda}_2^{-1}$, for $g_1, g_2 \in G$ and $\widetilde{\Lambda}_1, \widetilde{\Lambda}_2 \in \mathrm{InvL}(W_R)$, which entails $g_2^{-1}g_1 = \widetilde{\Lambda}_2^{-1}\widetilde{\Lambda}_1$, yielding a contradiction unless both sides are equal to the identity in $\mathcal{L}$). Thus, under the given assumption we have a map $m : \mathcal{L}_+^\uparrow \mapsto \mathrm{InvL}(W_R)$ with $m(\Lambda) = \widetilde{\Lambda} = \Lambda^{-1}g_\Lambda$. Note that, in view of the assumption $G \cap \mathrm{InvL}(W_R) = \{1\}$, the map $m : \mathcal{L}_+^\uparrow \mapsto \mathrm{InvL}(W_R)$ is the identity map when restricted to $\mathrm{InvL}_+^\uparrow(W_R)$. Moreover, for any $\Lambda \in \mathcal{L}_+^\uparrow$ the elements $m(\Lambda)$ and g_Λ lie in the same component of the Lorentz group.

Utilizing the fact that G is a group yields a strong condition on the map m. Consider any two elements $\Lambda_1, \Lambda_2 \in \mathcal{L}_+^\uparrow$ and the corresponding $g_{\Lambda_1}, g_{\Lambda_2} \in G$. Then since G is a group, we must have

$$g_{\Lambda_1}g_{\Lambda_2} = \Lambda_1\widetilde{\Lambda_1}\Lambda_2\widetilde{\Lambda_2} = \Lambda_1(\widetilde{\Lambda_1}\Lambda_2\widetilde{\Lambda_1}^{-1})\widetilde{\Lambda_1}\widetilde{\Lambda_2} \in G \quad .$$

Setting $\Lambda = \Lambda_1(\widetilde{\Lambda_1}\Lambda_2\widetilde{\Lambda_1}^{-1})$ we have on the other hand $g_\Lambda = \Lambda\widetilde{\Lambda}$ with $g_\Lambda \in G$ and consequently $g_\Lambda^{-1}g_{\Lambda_1}g_{\Lambda_2} = \widetilde{\Lambda}^{-1}\widetilde{\Lambda_1}\widetilde{\Lambda_2} \in G \cap \mathrm{InvL}(W_R) = \{1\}$. This yields the equation

$$m(\Lambda_1)m(\Lambda_2) = m(\Lambda_1 m(\Lambda_1)\Lambda_2 m(\Lambda_1)^{-1}) \quad , \tag{4.2.1}$$

for all $\Lambda_1, \Lambda_2 \in \mathcal{L}_+^\uparrow$.

For the solution of this equation it is convenient to proceed to the covering group $SL(2,\mathbb{C})$ of $\mathcal{L}_+^\uparrow$. One then has to consider the action of space and time reflections on $SL(2,\mathbb{C})$. Adopting standard conventions (see, *e.g.* [60]), one obtains by a straightforward computation the following result, which we state without proof.

Lemma 4.2.4. *Space and time reflections (P and T) acting on four-dimensional Minkowski spacetime induce the same automorphic action upon $SL(2,\mathbb{C})$, given by $\pi(A) = A^{*-1}$, whereas the reflection of the 3-axis P_3 induces the action $\pi_3(A) = -R\overline{A}R^*$, where $R = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ and the bar denotes complex conjugation.*

With $\rho : SL(2,\mathbb{C}) \mapsto \mathcal{L}_+^\uparrow$ the canonical homomorphism from the covering group, we proceed from m to the map $M : SL(2,\mathbb{C}) \mapsto \mathrm{InvL}(W_0)$ given by $M \equiv m \circ \rho$. Note that according to our assumptions on G, the set $M(SL(2,\mathbb{C}))$ is contained in at most two connected components of the Lorentz group. With $\Lambda_1 = \rho(A)$ and $\Lambda_2 = \rho(B)$, $A, B \in SL(2,\mathbb{C})$, and the fact that ρ is a homomorphism, equation (4.2.1) yields the following functional equation for M:

$$M(A)M(B) = M(A\gamma_A(B)) \quad , \quad A, B \in SL(2,\mathbb{C}) \quad . \tag{4.2.2}$$

γ_A is the unique automorphism of $SL(2,\mathbb{C})$ satisfying $\rho\circ\gamma_A(\cdot) = M(A)\rho(\cdot)M(A)^{-1}$. More concretely, for each $A \in SL(2,\mathbb{C})$, $M(A)$ can be written uniquely as a product of one of the reflections 1, T, P_3 or TP_3 and an element of $\mathrm{InvL}_+^\uparrow(W_R)$. The subgroup of $SL(2,\mathbb{C})$ corresponding to $\mathrm{InvL}_+^\uparrow(W_R)$ (with the appropriate choice of coordinates) is the maximally abelian subgroup $\mathcal{D}$ of matrices in $SL(2,\mathbb{C})$ of the form

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \quad , \quad \lambda \in \mathbb{C} \setminus \{0\} \quad .$$

Hence, any choice of $A \in SL(2,\mathbb{C})$ determines by the above decomposition of $M(A)$ such an element $D_\lambda \in \mathcal{D}$ (up to a sign). With this in mind, the action of γ_A on $SL(2,\mathbb{C})$ can be determined with the help of Lemma 4.2.4 and is given by

$$\gamma_A\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right) = \begin{cases} \begin{pmatrix} \alpha & \lambda^2\beta \\ \lambda^{-2}\gamma & \delta \end{pmatrix}, & \text{if } M(A) \in \mathcal{L}_+^\uparrow \quad , \qquad (a) \\ \begin{pmatrix} \overline{\delta} & -\lambda^2\overline{\gamma} \\ -\lambda^{-2}\overline{\beta} & \overline{\alpha} \end{pmatrix}, & \text{if } M(A) \in \mathcal{L}_-^\downarrow \quad , \qquad (b) \\ \begin{pmatrix} \overline{\alpha} & -\lambda^2\overline{\beta} \\ -\lambda^{-2}\overline{\gamma} & \overline{\delta} \end{pmatrix}, & \text{if } M(A) \in \mathcal{L}_-^\uparrow \quad , \qquad (c) \\ \begin{pmatrix} \delta & \lambda^2\gamma \\ \lambda^{-2}\beta & \alpha \end{pmatrix}, & \text{if } M(A) \in \mathcal{L}_+^\downarrow \quad , \qquad (d) \end{cases}$$

where $\alpha, \beta, \delta, \gamma \in \mathbb{C}$ with $\alpha\delta - \beta\gamma = 1$. We shall refer to these four possibilities in the following as cases (a), (b), (c) and (d).

After these preparations, we now turn to the solution of equation (4.2.2) and hence of equation (4.2.1). Let $\mathcal{U}_C \equiv \{\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \mid z \in \mathbb{C}\}$ be the subgroup of upper triangular matrices and $\mathcal{L}_C \equiv \{\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \mid z \in \mathbb{C}\}$ be the subgroup of lower triangular matrices in $SL(2,\mathbb{C})$. Note that in cases (a) and (c) γ_A leaves the sets $\mathcal{U}_C$ and $\mathcal{L}_C$ invariant, while in the other cases γ_A interchanges the two. Moreover, as long as A is in case (a) and γ_A is not the identity, one has for some $\lambda^2 \neq 1$

$$\gamma_A(\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}) \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ (\lambda^{-2}-1)z & 1 \end{pmatrix}$$

and

$$\gamma_A(\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}) \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & (\lambda^{2}-1)z \\ 0 & 1 \end{pmatrix} \quad ,$$

which entail $\{\gamma_A(X)X^{-1} \mid X \in \mathcal{L}_C\} = \mathcal{L}_C$, as well as $\{\gamma_A(X)X^{-1} \mid X \in \mathcal{U}_C\} = \mathcal{U}_C$. The following result is a simple consequence of the latter observation.

Lemma 4.2.5. *For any triangular matrix A in $SL(2,\mathbb{C})$ such that $M(A) \in \mathcal{L}_+^\uparrow$, one has $M(A) = 1$.*

Proof. Let A be contained in $\mathcal{U}_C$ or $\mathcal{L}_C$ and satisfy $M(A) \in \mathcal{L}_+^\uparrow$. If γ_A is not trivial, then from the above remarks there exists a matrix $X \in SL(2,\mathbb{C})$ such that $\gamma_A(X)X^{-1} = A^{-1}$. Therewith one has the equality $A\gamma_A(X) = X$, and equation (4.2.2) implies $M(A) = 1$. This is a contradiction, since then γ_A is trivial. Therefore γ_A must act as the identity map on $SL(2,\mathbb{C})$, and $M(A)$ has to lie in the center of $\mathcal{L}_+^\uparrow$, *i.e.* $M(A) = 1$. □

Some elementary properties of the elements of $SL(2,\mathbb{C})$ which are mapped by M to the identity are collected in the following lemma.

Lemma 4.2.6. *Let $\mathcal{E}$ consist of all $A \in SL(2,\mathbb{C})$ such that $M(A) = 1$. Then*
(1) $\mathcal{E}$ is a subgroup of $SL(2,\mathbb{C})$, and
(2) one has $M(AB) = M(B)$ for all $A \in \mathcal{E}$ and $B \in SL(2,\mathbb{C})$.

Proof. If $A \in \mathcal{E}$ and $B \in SL(2,\mathbb{C})$, then equation (4.2.2) and the triviality of γ_A entail that $M(AB) = M(A\gamma_A(B)) = M(A)M(B) = M(B)$, proving assertion (2). Clearly the identity element of $SL(2,\mathbb{C})$ is contained in $\mathcal{E}$, and if $A, B \in \mathcal{E}$, one has $M(AB) = M(B) = 1$. Thus, $\mathcal{E}$ is closed under products and taking inverses, hence assertion (1) follows. □

We exploit these results to show that, in fact, the image of any triangular matrix in $SL(2,\mathbb{C})$ under M is the identity.

Lemma 4.2.7. *For any triangular matrix A in $SL(2,\mathbb{C})$, one has $M(A) = 1$, i.e. $\mathcal{U}_C \cup \mathcal{L}_C \subset \mathcal{E}$.*

Proof. Since Lemma 4.2.5 has already established the claim for any triangular A in case (a) and since $M(A) \neq 1$ in the remaining cases, it is necessary to show that case (b), (c) and (d) cannot occur. Note that the set $\{M(A) \mid A \notin \mathcal{E}\}$ of Lorentz transformations lies in a single component of the Lorentz group $\mathcal{L}$ (unless, of course, it is empty), as a consequence of the assumption that the given group G intersects at most two components of $\mathcal{L}$ and here $M(A) \notin \mathcal{L}_+^\uparrow$. Hence if $A, B \notin \mathcal{E}$, it follows that $M(A)M(B) \in \mathcal{L}_+^\uparrow$ and consequently (4.2.2) yields $M(A\gamma_A(B)) \in \mathcal{L}_+^\uparrow$. The details of the exclusion of cases (b)-(d) will be given for such $A \in \mathcal{L}_C$ - the argument for $A \in \mathcal{U}_C$ is similar.

With $A \in \mathcal{L}_C$, one has $A\gamma_A(B) \in \mathcal{L}_C$ whenever (if A is in case (c)) $B \in \mathcal{L}_C$, respectively (if A is in case (b) or (d)) $B \in \mathcal{U}_C$. Moreover, the equation $A\gamma_A(X) = 1$ has, for any $A \in \mathcal{L}_C$, the solution $X = \gamma_A^{-1}(A^{-1})$ in $\mathcal{L}_C$ (in case (c)), respectively in $\mathcal{U}_C$ (in cases (b) and (d)), since γ_A^{-1} is an isomorphism between the respective groups. Given an element $A_0 \in \mathcal{L}_C$ which is not contained in $\mathcal{E}$, one can choose X_0 such that $A_0\gamma_{A_0}(X_0) = 1$ holds and get $M(A_0)M(X_0) = M(A_0\gamma_{A_0}(X_0)) = 1$. This shows that also X_0 does not lie in $\mathcal{E}$, hence, by the first paragraph, one finds that $M(A\gamma_A(X_0)) \in \mathcal{L}_+^\uparrow$ whenever $A \notin \mathcal{E}$. If, in addition, $A \in \mathcal{L}_C$, then by the second paragraph one has $A\gamma_A(X_0) \in \mathcal{L}_C$, and Lemma 4.2.5 implies that

$$M(A)M(X_0) = M(A\gamma_A(X_0)) = 1 \quad .$$

Thus one concludes that the Lorentz element $M(A)$ does not depend upon the choice of the element A contained in $\mathcal{L}_C$ but not contained in $\mathcal{E}$. The same is therefore true for the corresponding automorphisms γ_A.

Let $\mathcal{E}_0$ denote the subgroup $\mathcal{E} \cap \mathcal{L}_C$ of $\mathcal{E}$ and choose now $A, B \in \mathcal{L}_C \setminus \mathcal{E}_0$. By the preceding paragraph one also has $\gamma_B^{-1}(B^{-1}) \in \mathcal{L}_C \setminus \mathcal{E}_0$ (respectively $\gamma_B^{-1}(B^{-1}) \in \mathcal{U}_C \setminus \mathcal{E}_0$). Thus, taking into account the first paragraph and the fact that $\gamma_A = \gamma_B$ and $M(A) = M(B)$, one finds, using (4.2.2),

$$\begin{aligned} M(AB^{-1}) &= M(A\gamma_A(\gamma_B^{-1}(B^{-1}))) = M(A)M(\gamma_B^{-1}(B^{-1})) \\ &= M(B)M(\gamma_B^{-1}(B^{-1})) = M(B\gamma_B(\gamma_B^{-1}(B^{-1}))) = M(1) = 1 \quad . \end{aligned}$$

Therefore, $AB^{-1} \in \mathcal{E}_0$.

It has therefore been established that (1) $\mathcal{E}_0 \subset \mathcal{L}_C$ is a group, (2) if $A \in \mathcal{L}_C \setminus \mathcal{E}_0$, then $\mathcal{E}_0 \cdot A \subset \mathcal{L}_C \setminus \mathcal{E}_0$ (this is the content of Lemma 4.2.6 (2)), and (3) if $A, B \in \mathcal{L}_C \setminus \mathcal{E}_0$, then $AB^{-1} \in \mathcal{E}_0$. Hence, for each $A \in \mathcal{L}_C \setminus \mathcal{E}_0$ one has the disjoint decomposition $\mathcal{L}_C = \mathcal{E}_0 \cup (\mathcal{E}_0 \cdot A)$. But for each $A \in \mathcal{L}_C$ there exists an element $X \in \mathcal{L}_C$ such that $X^2 = A$. If $X \in \mathcal{E}_0$, then so is A, since $\mathcal{E}_0$ is a group. Thus for $A \in \mathcal{L}_C \setminus \mathcal{E}_0$ one must have $X \notin \mathcal{E}_0$. But on the other hand, if $X \in \mathcal{E}_0 \cdot A$, then $XA^{-1} \in \mathcal{E}_0$, so that $1 = X \cdot XA^{-1} \in X \cdot \mathcal{E}_0$, which implies $X^{-1} \in \mathcal{E}_0$. Then again one has $X \in \mathcal{E}_0$. This is a contradiction unless the set $\mathcal{L}_C \setminus \mathcal{E}_0$ is empty. □

We are now in the position to complete the proof of Lemma 4.2.3. Since $\mathcal{U}_C \cup \mathcal{L}_C$ generates all of $SL(2, \mathbb{C})$, we may conclude from Lemmas 4.2.6 and 4.2.7 that M maps $SL(2, \mathbb{C})$ onto $\{1\}$, and consequently m maps $\mathcal{L}_+^\uparrow$ onto $\{1\}$. But this contradicts the fact that m must be the identity map on $\mathrm{InvL}_+^\uparrow(W_R)$, so the assertion in Lemma 4.2.3 follows. Next we turn to case (ii), which is easily eliminated by pointing out the following simple consequence of Lemma 4.2.3.

Lemma 4.2.8. *If the group $G \subset \mathcal{L}$ intersects at most one connected component of $\mathcal{L}$ other than $\mathcal{L}_+^\uparrow$ and acts transitively upon the set $\mathcal{W}_0$, then the group $G \cap InvL_+^\uparrow(W_R)$ is nontrivial.*

Proof. From Lemma 4.2.3 there exists a nontrivial element g_0 in the group $G \cap \mathrm{InvL}(W_R)$. If one such g_0 happens to lie in $\mathcal{L}_+^\uparrow$, the proof is over. So assume that all such g_0 are not contained in $\mathcal{L}_+^\uparrow$. Since G intersects at most one other component of $\mathcal{L}$ besides $\mathcal{L}_+^\uparrow$, one must have $G = G_+ \cup G_+ g_0$, where $G_+ = G \cap \mathcal{L}_+^\uparrow$. Thus, the transitivity of the action of G upon $\mathcal{W}_0$ implies

$$\mathcal{W}_0 = G \cdot W_R = G_+ \cdot W_R \cup G_+ \cdot g_0 W_R = G_+ \cdot W_R \quad .$$

In other words, also the group G_+ acts transitively upon the set $\mathcal{W}_0$, even though $G_+ \cap \mathrm{InvL}(W_R) = \{1\}$. But this possibility has been excluded by Lemma 4.2.3. □

We are ready to show that in the only remaining case, case (iii), the identity component of the Lorentz group must be contained in G, which is the statement of Prop. 4.2.2. We begin by noting that for any involutive element $j \in \mathcal{L}$, there exists some wedge $W \in \mathcal{W}_0$ which is mapped by j either onto itself or onto its causal complement $W' = -W$. This follows from the fact that either j maps every lightlike vector ℓ onto ℓ, respectively $-\ell$, or there exists a lightlike vector ℓ_1 such that its (lightlike) image $\ell_2 = j\ell_1$ is not parallel to ℓ_1. In the latter case, the pair (ℓ_1, ℓ_2) is mapped onto itself by j, since j is an involution. As every wedge is determined by two lightlike vectors, the statement then follows after a moment's reflection.

Now, as above, let $G_+ = G \cap \mathcal{L}_+^\uparrow$ and let $G_- = G \setminus G_+$. We first consider the case where G_- is empty. Then G_+ acts transitively upon $\mathcal{W}_0$ and we can conclude from the simplicity of $\mathcal{L}_+^\uparrow$ that $G_+ = \mathcal{L}_+^\uparrow$ in this case (cf. the argument directly following the statement of Lemma 4.2.3).

Note that the assumption that G is generated by involutions has not been used in the preceding paragraph. This will be exploited now in the case where G_- is nonempty. For then there must be some involution $j \in G_-$ and a wedge $W \in \mathcal{W}_0$ such that either $jW = W$ or $jW = -W$. Without loss of generality, we may assume that $W = W_R$. Since $G = G_+ \cup G_+ j$, the relation $jW_R = W_R$ lets us conclude, as in the proof of Lemma 4.2.8, that G_+ acts transitively on $\mathcal{W}_0$ and hence $G_+ = \mathcal{L}_+^\uparrow$ by the preceding argument.

In the remaining case, where $jW_R = -W_R$, we have

$$\mathcal{W}_0 = G \cdot W_R = G_+ \cdot W_R \cup -G_+ W_R \quad .$$

In other words, for each $W \in \mathcal{W}_0$ there exists an element $g \in G_+$ such that either $gW_R = W$ or $gW_R = -W$.

Now consider the element $R_0 \in \mathcal{L}_+^\uparrow$ which implements the rotation of angle π about the x_2-axis: $R_0 = \mathrm{diag}(1, -1, 1, -1)$. This element maps W_R to its causal complement: $R_0 W_R = -W_R$. Moreover, conjugation by R_0 takes the elements of $\mathrm{InvL}_+^\uparrow(W_R)$ into their inverses, *i.e.* for each $\widetilde{\Lambda} \in \mathrm{InvL}_+^\uparrow(W_R)$ we find

$$R_0 \widetilde{\Lambda} R_0^{-1} = \widetilde{\Lambda}^{-1} \quad . \tag{4.2.3}$$

Since $\mathcal{W}_0 = \mathcal{L}_+^\uparrow W_R$, we may conclude from the above arguments that for every $\Lambda \in \mathcal{L}_+^\uparrow$ there exists an element $g_\Lambda \in G_+$ such that $\Lambda W_R = g_\Lambda W_R$ or

$\Lambda W_R = -g_\Lambda W_R = g_\Lambda R_0 W_R$. Hence, for every $\Lambda \in \mathcal{L}_+^\uparrow$ there exist elements $g_\Lambda \in G_+$ and $\widetilde{\Lambda} \in \mathrm{InvL}_+^\uparrow(W_R)$ so that either (1) $g_\Lambda = \Lambda\widetilde{\Lambda}$ or (2) $g_\Lambda = \Lambda R_0 \widetilde{\Lambda}$. Define therefore the subset ${\mathcal{L}_+^\uparrow}^{(1)}$, resp. ${\mathcal{L}_+^\uparrow}^{(2)}$, consisting of those elements Λ of $\mathcal{L}_+^\uparrow$ in case (1), resp. case (2). We have $\mathcal{L}_+^\uparrow = {\mathcal{L}_+^\uparrow}^{(1)} \cup {\mathcal{L}_+^\uparrow}^{(2)}$.

According to Lemma 4.2.8 there exists a nontrivial element $h_0 \in G \cap \mathrm{InvL}_+^\uparrow(W_R)$. Since $\mathrm{InvL}_+^\uparrow(W_R)$ is abelian, we find as before for any $\Lambda \in {\mathcal{L}_+^\uparrow}^{(1)}$ the relation $\Lambda h_0 \Lambda^{-1} = g_\Lambda h_0 g_\Lambda^{-1} \in G_+$. Moreover, since $G \cap \mathrm{InvL}_+^\uparrow(W_R)$ is a group, it also contains the element h_0^{-1}. It follows that for any $\Lambda \in {\mathcal{L}_+^\uparrow}^{(2)}$, we have

$$\Lambda h_0 \Lambda^{-1} = \Lambda R_0 \widetilde{\Lambda} h_0^{-1} \widetilde{\Lambda}^{-1} R_0^{-1} \Lambda^{-1} = g_\Lambda h_0^{-1} g_\Lambda^{-1} \in G_+ \quad ,$$

using (4.2.3). It has therefore been established that $\Lambda h_0 \Lambda^{-1} \in G_+$ for any element $\Lambda \in \mathcal{L}_+^\uparrow$. Once again, it then follows from the simplicity of $\mathcal{L}_+^\uparrow$ that $G_+ = \mathcal{L}_+^\uparrow$.

The proof of Proposition 4.2.2 is therewith completed. The next step is to show that a similar statement holds also for the Poincaré group.

Proposition 4.2.9. *Any subgroup G of the identity component $\mathcal{P}_+^\uparrow$ of the Poincaré group, which acts transitively upon the set $\mathcal{W}$ of wedges in $\mathrm{I\!R}^4$, must equal $\mathcal{P}_+^\uparrow$. Moreover, any subgroup G of the Poincaré group $\mathcal{P}$, which is generated by involutions, intersects at most two of the four connected components of $\mathcal{P}$ and which acts transitively upon the set $\mathcal{W}$ of wedges in $\mathrm{I\!R}^4$, must contain $\mathcal{P}_+^\uparrow$.*

Proof. As a first step, consider the canonical homomorphism $\sigma : G \mapsto \mathcal{L}$ which acts as $\sigma(\Lambda, a) = \Lambda$ for $(\Lambda, a) \in G$. Since G acts transitively on the set of wedges, it follows that $\sigma(G)$ acts transitively on the subset $\mathcal{W}_0$ of wedges whose edges contain the origin. For if $W \in \mathcal{W}_0$ there exists an element $(\Lambda, a) \in G$ such that $W = \Lambda W_R + a$, and since $\Lambda W_R \in \mathcal{W}_0$, it follows that $W = \Lambda W_R$. Since $\sigma(G) \subset \mathcal{L}$ is also generated by its involutions and intersects with at most two components of $\mathcal{L}$, one may apply Prop. 4.2.2 and conclude that $\mathcal{L}_+^\uparrow \subset \sigma(G)$. Consider now the following alternatives.

(1) There exist an element $\Lambda \in \mathcal{L}$ and $a, b \in \mathrm{I\!R}^4$ with $a \neq b$ such that *both* elements (Λ, a) and (Λ, b) are contained in G. Since G is a group, it follows that $(\Lambda, a)(\Lambda, b)^{-1} = (1, a - b) \in G$. As has already been seen, for every element $\Lambda \in \mathcal{L}_+^\uparrow$ there exists some element $(\Lambda, c) \in G$; hence it follows that

$$(\Lambda, c)(1, a - b)(\Lambda, c)^{-1} = (1, \Lambda(a - b)) \in G \qquad , \qquad \Lambda \in \mathcal{L}_+^\uparrow \quad .$$

Since $(1, c) \in G$ implies that $(1, -c) \in G$, one may conclude that G contains all translations $(1, x)$ with $x{\cdot}x$ equal to some fixed constant κ. Since $(1, x), (1, x') \in G$ imply that $(1, x + x') \in G$, and since every $y \in \mathrm{I\!R}^4$ can be written in the form $y = \sum_1^4 x_i$ with $x_i \cdot x_i = \kappa$, $i = 1, \ldots, 4$, it also follows that G contains *all* translations.

Consider now for given $\Lambda \in \mathcal{L}_+^\uparrow$ an element $c \in \mathrm{I\!R}^4$ for which $(\Lambda, c) \in G$. Then one has by the preceding result

$$(\Lambda, c)(1, -\Lambda^{-1} c) = (\Lambda, 0) \in G \quad ;$$

in other words, G also contains all the pure Lorentz transformations, as well. Thus, in this case one has $\mathcal{P}_+^\uparrow \subset G$.

(2) For every element $\Lambda \in \sigma(G)$ there exists *exactly one* $a(\Lambda) \in \mathbb{R}^4$ such that $(\Lambda, a(\Lambda)) \in G$. Since G is a group, this entails the following cocycle relation for the translations:

$$a(\Lambda\Lambda') = a(\Lambda) + \Lambda a(\Lambda') \quad , \quad \Lambda, \Lambda' \in \sigma(G) \quad . \tag{4.2.4}$$

Consider the subgroup $G_0 \subset G$ whose elements translate the wedge W_R without rotating it. The elements of G_0 have the form $(\Lambda, a(\Lambda))$ with $\Lambda \in \mathrm{InvL}(W_R)$. So it follows from the first paragraph of this proof that for $G_0^+ \equiv G_0 \cap \mathcal{P}_+^\uparrow$ the equality $\sigma(G_0^+) = \mathrm{InvL}(W_R) \cap \mathcal{L}_+^\uparrow$ holds. Since $\mathrm{InvL}(W_R) \cap \mathcal{L}_+^\uparrow$ is abelian, the cocycle equation (4.2.4) implies that

$$a(\Lambda) + \Lambda a(\Lambda') = a(\Lambda\Lambda') = a(\Lambda'\Lambda) = a(\Lambda') + \Lambda' a(\Lambda) \quad ,$$

for every $\Lambda, \Lambda' \in \mathrm{InvL}(W_R) \cap \mathcal{L}_+^\uparrow$, which itself entails that

$$(1 - \Lambda')a(\Lambda) = (1 - \Lambda)a(\Lambda') \quad .$$

Fixing an element $\Lambda' \in \mathrm{InvL}(W_R) \cap \mathcal{L}_+^\uparrow$ such that the matrix $(1-\Lambda')$ is invertible and setting $a \equiv (1 - \Lambda')^{-1} a(\Lambda')$, one obtains

$$a(\Lambda) = (1 - \Lambda)a \quad , \quad \Lambda \in \mathrm{InvL}(W_R) \cap \mathcal{L}_+^\uparrow \quad . \tag{4.2.5}$$

Hence G_0^+ is comprised of the elements $\{(\Lambda, (1-\Lambda)a) \mid \Lambda \in \mathrm{InvL}(W_R) \cap \mathcal{L}_+^\uparrow\}$ for some fixed $a \in \mathbb{R}^4$.

Now, if $G_0^- \equiv G_0 \setminus G_0^+$ is nonempty, there exists some $g_0 = (\Lambda_0, a_0) \in G_0^-$ such that $G_0^- = G_0^+ \cdot g_0$ (recall that G intersects at most two of the connected components of $\mathcal{P}$). Hence, without loss of generality, one may assume that $(1+\Lambda_0)$ is invertible. Since $g_0^2 = (\Lambda_0^2, a_0 + \Lambda_0 a_0) \in G_0^+$, it follows from equations (4.2.4) and (4.2.5) that $(1 - \Lambda_0^2)a = a(\Lambda_0^2) = a(\Lambda_0) + \Lambda_0 a(\Lambda_0) = (1 + \Lambda_0)a(\Lambda_0)$ and consequently $a(\Lambda_0) = (1 - \Lambda_0)a$. Applying equation (4.2.4) another time yields

$$a(\Lambda\Lambda_0) = a(\Lambda) + \Lambda a(\Lambda_0) = (1 - \Lambda\Lambda_0)a$$

for arbitrary $\Lambda \in \mathrm{InvL}(W_R) \cap \mathcal{L}_+^\uparrow$, which finally shows that $G_0 = \{(\Lambda, (1 - \Lambda)a) \mid \Lambda \in \sigma(G_0)\}$. Hence, G_0 induces solely translations of the edge of the wedge W_R along some (two-sheeted) hyperbola or light ray, contradicting the assumption that G acts transitively on $\mathcal{W}$. Therefore, only case (1) can arise and the proof of the proposition is complete. □

Summing up the results obtained so far in this section, we see that the symmetry groups $\mathcal{G}$ which arise by the CGMA in Minkowski space theories must contain the proper orthochronous Poincaré group $\mathcal{P}_+^\uparrow$ if they act transitively on the set of wedges $\mathcal{W}$. This result will enable us in the next step to determine $\mathcal{G}$ exactly, as well as the action of its generating involutions on Minkowski space.

Proposition 4.2.10. *Let the group $\mathcal{T}$ act transitively upon the set $\mathcal{W}$ of wedges in IR^4, and let $\mathcal{G}$ be the corresponding subgroup of the Poincaré group. Moreover, let $g_{W_R} = (\Lambda_{W_R}, a_{W_R})$ be the involutive element of the Poincaré group corresponding to the involution $\tau_{W_R} \in \mathcal{T}$. Then $a_{W_R} = 0$ and $\Lambda_{W_R} = P_1 T = diag(-1,-1,1,1)$, where P_1 is the reflection through the 1-axis and T is the time reflection. Since all wedges are transforms of W_R under $\mathcal{P}_+^\uparrow$, these assertions are also true, with the obvious modifications, for the involution g_W corresponding to any wedge $W \in \mathcal{W}$. In particular, one has $g_W W = W'$, for every $W \in \mathcal{W}$. In addition, $\mathcal{G}$ exactly equals the proper Poincaré group $\mathcal{P}_+$, and every element of $\mathcal{P}_+^\uparrow$ can be obtained as a product of an even number of involutions, g_W, $W \in \mathcal{W}$.*

Proof. If $\tau_0 \in \mathcal{T}$ leaves a given wedge $W \in \mathcal{W}_0$ fixed, then Lemma 2.1 (3) entails that $\tau_W \tau_0 = \tau_0 \tau_W$. Hence, if g_W and g_0 are the corresponding elements in the Poincaré group, one must have $g_0 g_W g_0^{-1} = g_W$. In light of Prop. 4.2.9, this implies that g_W must commute with every element of the invariance group $\mathrm{InvP}_+^\uparrow(W)$. With $g_W = (\Lambda_W, a_W)$, it follows that one must have

$$(\Lambda_0 \Lambda_W \Lambda_0^{-1}, a_0 + \Lambda_0 a_W - \Lambda_0 \Lambda_W \Lambda_0^{-1} a_0) = (\Lambda_W, a_W) \quad ,$$

for arbitrary $(\Lambda_0, a_0) \in \mathrm{InvP}_+^\uparrow(W)$. By setting $a_0 = 0$ and letting Λ_0 vary freely through $\mathrm{InvL}_+^\uparrow(W)$, this equation implies $a_W = 0$, and therefore $\Lambda_0 \Lambda_W \Lambda_0^{-1} = \Lambda_W$ *and*

$$(1 - \Lambda_W) a_0 = 0 \quad , \tag{4.2.6}$$

for all $(\Lambda_0, a_0) \in \mathrm{InvP}_+^\uparrow(W)$. Furthermore, one has $\Lambda_W^2 = 1$, since g_W is an involution.

Choosing $W = W_R$, one concludes from (4.2.6) that Λ_W must have the form

$$\Lambda_W = \begin{pmatrix} X & 0 \\ Y & 1 \end{pmatrix} \quad ,$$

for suitable 2×2-matrices X, Y. Since Λ_W is a Lorentz transformation, it is easy to see that $Y = 0$. The facts that Λ_W must commute with the Lorentz boosts in the 1-direction (leaving W_R invariant) and that $\Lambda_W^2 = 1$ lead then, after some elementary computation, to $X = \pm 1$. But in the case where the positive sign is taken, one would have $\tau_{W_R}(W_R) = W_R$, which is excluded by Lemma 2.1 (4) and the fact that there are no atoms in $\mathcal{W}$.

The remaining assertions are now easy to verify. □

4.3 From Wedge Transformations Back to the Net: Locality, Covariance and Continuity

Having established the geometrical features of the elements of the group $\mathcal{T}$, we turn now to the discussion of its representations induced by the modular conjugations. Proposition 4.2.10 implies that there exists a projective representation $J(\mathcal{P}_+)$ of the proper Poincaré group with coefficients in the internal symmetry group of the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$. The next step is to verify that this projective representation acts geometrically correctly upon the net, in other words that the net is Poincaré covariant under this projective representation.

Proposition 4.3.1. *Let the CGMA obtain with the choices $\mathcal{M} = \mathrm{IR}^4$ and $\mathcal{W}$ equal to the set of wedgelike regions in IR^4, and let the adjoint action of $\mathcal{J}$ upon the set $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ be transitive. Then the projective representation $J(\mathcal{P}_+)$ of the proper Poincaré group whose existence is entailed by Corollary 2.3 and Prop. 4.2.10 acts geometrically correctly upon the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$, i.e. for each $\Lambda \in \mathcal{P}_+$ and each $W \in \mathcal{W}$ one has*

$$J(\Lambda)\mathcal{R}(W)J(\Lambda)^{-1} = \mathcal{R}(\Lambda W) \quad .$$

Furthermore, Haag duality holds for $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$, hence the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ satisfies Einstein locality.

Proof. By construction, for each $g \in \mathcal{G} = \mathcal{P}_+$ there exists an element $\tau_g \in \mathcal{T}$ such that $\tau_g(W) = gW$ for all $W \in \mathcal{W}$. Hence, for all $W \in \mathcal{W}$, one has

$$J(g)\mathcal{R}(W)J(g)^{-1} = \mathcal{R}((\prod_{j=1}^{n(\tau_g)} \tau_{i_j})(W)) = \mathcal{R}(\tau_g(W)) = \mathcal{R}(gW) \quad ,$$

where the product indicated is taken over the chosen product for the element $\tau_g \in \mathcal{T}$ implicit in the definition of the projective representation $J(\mathcal{T})$.

By Lemma 4.2.10, one has

$$\mathcal{R}(W)' = J_W\mathcal{R}(W)J_W = \mathcal{R}(g_W W) = \mathcal{R}(W') \quad ,$$

for each $W \in \mathcal{W}$. So Haag duality holds; thus, for each $W_1 \subset W'$, one has $\mathcal{R}(W_1) \subset \mathcal{R}(W') = \mathcal{R}(W)'$. □

Note that these results do not depend upon the choice of projective representation $J(\mathcal{P}_+)$. We next provide conditions on the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ which imply that there exists a strongly continuous projective representation of $\mathcal{P}_+^\uparrow$. These conditions essentially involve a continuity property of the map $W \mapsto \mathcal{R}(W)$. First note that since $\mathcal{W} = \mathcal{P}_+^\uparrow W_R$, $\mathcal{W}$ is in 1-1 correspondence with the quotient space $\mathcal{P}_+^\uparrow/\mathrm{InvP}_+^\uparrow(W_R)$; the latter's topology induces thereby a topology on $\mathcal{W}$. Consider then a continuous collection $\{W_\epsilon\}_{\epsilon>0}$ of wedges in $\mathcal{W}$ such that $W_\epsilon \to W$ as $\epsilon \to 0$, for some fixed $W \in \mathcal{W}$. For $\delta > 0$, let $A_\delta \equiv \bigcup_{0\leq\epsilon<\delta} W_\epsilon$ and $I_\delta \equiv \bigcap_{0\leq\epsilon<\delta} W_\epsilon$, where $W_0 \equiv W$. Define $\mathcal{R}(I_\delta) \equiv \bigcap_{0\leq\epsilon<\delta} \mathcal{R}(W_\epsilon)$ and $\mathcal{R}(A_\delta) \equiv (\bigcup_{0\leq\epsilon<\delta} \mathcal{R}(W_\epsilon))''$ to be the indicated intersection and union of wedge algebras. Note that $\{\mathcal{R}(A_\delta)\}_{\delta>0}$, resp. $\{\mathcal{R}(I_\delta)\}_{\delta>0}$, is a monotone decreasing, resp. increasing, family of von Neumann algebras. Our net continuity assumption is given next.

Net Continuity Condition. *For any $W \in \mathcal{W}$ and any continuous collection $\{W_\epsilon\}_{\epsilon>0} \subset \mathcal{W}$ converging to W, the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ satisfies $\mathcal{R}(W) = (\bigcup_{\delta>0} \mathcal{R}(I_\delta))'' = \bigcap_{\delta>0} \mathcal{R}(A_\delta)$. Moreover, there exists a $\delta_0 > 0$ such that Ω is cyclic for the algebras $\mathcal{R}(I_\delta)$, with $0 < \delta < \delta_0$.*

We proceed with the following result, which establishes that the mentioned net continuity condition implies a certain continuity in nets of associated modular objects in the context of the CGMA. Condition (ii) of the CGMA entails that

Ω is cyclic for $\mathcal{R}(A_\delta)$, $0 < \delta < \delta_1$. And the Haag duality proven in Prop. 4.3.1 yields

$$\mathcal{R}(A_\delta)' = (\underset{0\leq\epsilon<\delta}{\cup}\mathcal{R}(W_\epsilon))' = \underset{0\leq\epsilon<\delta}{\cap}\mathcal{R}(W_\epsilon)' = \underset{0\leq\epsilon<\delta}{\cap}\mathcal{R}(W_\epsilon') \quad ,$$

for which Ω is cyclic whenever δ is sufficiently small, by hypothesis. Hence, there exists a $\delta_1 > 0$ such that Ω is cyclic and separating for $\mathcal{R}(A_\delta)$, $0 < \delta < \delta_1$. Moreover, since $W' \subset I_\delta'$, for all $\delta > 0$, Ω is also separating for $\mathcal{R}(I_\delta)$. Hence, the CGMA and the net continuity condition imply that the modular objects $J_{I_\delta}, \Delta_{I_\delta}$, resp. $J_{A_\delta}, \Delta_{A_\delta}$, corresponding to the pair $(\mathcal{R}(I_\delta), \Omega)$, resp. $(\mathcal{R}(A_\delta), \Omega)$, exist for all $0 < \delta < \min\{\delta_0, \delta_1\}$. Below we shall tacitly take $0 < \delta < \min\{\delta_0, \delta_1\}$ without further comment.

Proposition 4.3.2. *Assume the CGMA with the choices $\mathcal{M} = \mathbb{R}^4$ and $\mathcal{W}$ as described, as well as the mentioned net continuity condition. Let $\{W_\epsilon\}_{\epsilon>0}$ be a continuous net of wedges such that $W_\epsilon \to W$ as $\epsilon \to 0$. Then the net $\{J_{W_\epsilon}\}_{\epsilon>0}$ converges strongly to J_W as $\epsilon \to 0$. In addition, the net $\{\Delta_{W_\epsilon}^{it}\}_{\epsilon>0}$ converges strongly to Δ_W^{it} as $\epsilon \to 0$.*

Proof. By Corollary A.2 of [24], which is based upon a result of [25], it follows from the hypotheses that $\Delta_{I_\delta} \to \Delta_W$ and $\Delta_{A_\delta}^{-1} \to \Delta_W^{-1}$ in the strong resolvent sense, and $J_{I_\delta} \to J_W$ and $J_{A_\delta} \to J_W$ in the strong operator topology (note that $\mathcal{R}(W)' = (\underset{\delta>0}{\cap}\mathcal{R}(A_\delta))' = (\underset{\delta>0}{\cup}\mathcal{R}(A_\delta)')'')$. On the other hand, from equation (2.6) in [31], one has the inequality

$$(\mathbb{1} + \Delta_{A_\delta})^{-1} \leq (\mathbb{1} + \Delta_{W_\epsilon})^{-1} \leq (\mathbb{1} + \Delta_{I_\delta})^{-1} \quad , \tag{4.3.1}$$

for all $0 < \epsilon < \delta$. Employing this inequality, the polarization identity, and the stated strong resolvent convergence, it follows easily that $(\mathbb{1} + \Delta_{W_\epsilon})^{-1}$ converges weakly to $(\mathbb{1} + \Delta_W)^{-1}$. By the positivity of the operators in (4.3.1) and the operator monotonicity of the operation of taking square roots, (4.3.1) also entails

$$(\mathbb{1} + \Delta_{A_\delta})^{-1/2} \leq (\mathbb{1} + \Delta_{W_\epsilon})^{-1/2} \leq (\mathbb{1} + \Delta_{I_\delta})^{-1/2} \quad ,$$

for all $0 < \epsilon < \delta$, so that by the same argument, also $(\mathbb{1} + \Delta_{W_\epsilon})^{-1/2}$ converges weakly to $(\mathbb{1} + \Delta_W)^{-1/2}$. In order to make the following computations somewhat more transparent, let $R_W \equiv (\mathbb{1} + \Delta_W)^{-1}$ and $R_{W_\epsilon} \equiv (\mathbb{1} + \Delta_{W_\epsilon})^{-1}$. One observes then that for any vector $\Phi \in \mathcal{H}$ the expression

$$\|(R_{W_\epsilon}^{1/2} - R_W^{1/2})\Phi\|^2 = \langle \Phi, (R_{W_\epsilon} - R_{W_\epsilon}^{1/2} R_W^{1/2} - R_W^{1/2} R_{W_\epsilon}^{1/2} + R_W)\Phi \rangle$$

must converge to zero as $\epsilon \to 0$. Since $R_{W_\epsilon}^{1/2}$ is uniformly bounded, R_{W_ϵ} converges also strongly to R_W. Standard arguments then yield the strong convergence of $\{\Delta_{W_\epsilon}^{it}\}_{\epsilon>0}$ to Δ_W^{it} as $\epsilon \to 0$.

To proceed further, note that from the above it follows that $\Delta_W^{1/2}$ is the strong graph limit of the net $\{\Delta_{I_\delta}^{1/2}\}$. In particular, there exists a dense subset $\mathcal{K}$ of $\mathcal{H}$ such that for each $\Phi \in \mathcal{K}$ there exists a corresponding net $\{\Phi_\delta\}$ with $\Phi_\delta \in \mathcal{R}(I_\delta)\Omega$ satisfying $\Phi_\delta \to \Phi$ and

$$\Delta_{I_\delta}^{1/2}\Phi_\delta \to \Delta_W^{1/2}\Phi \quad .$$

Since the Tomita-Takesaki conjugations S_{I_δ} are restrictions of the corresponding conjugations S_{W_ϵ} to $\mathcal{R}(I_\delta)\Omega$ (for all $0 < \epsilon < \delta$), one sees that this implies

$$J_{W_\epsilon}\Delta_{W_\epsilon}^{1/2}\Phi_\delta = S_{W_\epsilon}\Phi_\delta = S_{I_\delta}\Phi_\delta = J_{I_\delta}\Delta_{I_\delta}^{1/2}\Phi_\delta \to J_W\Delta_W^{1/2}\Phi \quad ,$$

for all $0 < \epsilon < \delta$, since J_{I_δ} converges strongly to J_W. But this convergence of $J_{W_\epsilon}\Delta_{W_\epsilon}^{1/2}\Phi_\delta$ entails the convergence of $(\delta \to 0,\ 0 < \epsilon < \delta)$

$$\begin{aligned} \frac{1}{\mathbb{1} + \Delta_{W_\epsilon}^{1/2}} J_{W_\epsilon}\Phi_\delta &= \frac{1}{\mathbb{1} + \Delta_{W_\epsilon}^{-1/2}} J_{W_\epsilon}\Delta_{W_\epsilon}^{1/2}\Phi_\delta = \frac{1}{\mathbb{1} + \Delta_{W_\epsilon}^{-1/2}} J_{I_\delta}\Delta_{I_\delta}^{1/2}\Phi_\delta \\ &\to \frac{1}{\mathbb{1} + \Delta_W^{-1/2}} J_W\Delta_W^{1/2}\Phi = \frac{1}{\mathbb{1} + \Delta_W^{1/2}} J_W\Phi \quad . \end{aligned}$$

As the nets $\{\Phi_\delta\}_{\delta>0}$ and $\{\frac{1}{\mathbb{1}+\Delta_{W_\epsilon}^{1/2}}\}_{\epsilon>0}$ converge strongly, this proves the weak convergence

$$\frac{1}{\mathbb{1} + \Delta_W^{1/2}} J_{W_\epsilon}\Phi \to \frac{1}{\mathbb{1} + \Delta_W^{1/2}} J_W\Phi \quad .$$

Hence, J_{W_ϵ} converges weakly (and thus also strongly, since the operators are antiunitary) to J_W. □

The preceding proposition establishes that to every continuous net of wedges is associated a strongly continuous net of modular involutions. Using this fact and the explicit knowledge which Prop. 4.2.10 furnishes about the geometric action of the generators of the group $\mathcal{G}$, we shall show that there exists a choice of $J(\mathcal{P}_+^\uparrow)$ which is strongly continuous. In the following, $\mathcal{U}(\mathcal{H})$ denotes the group of unitary operators acting on the separable Hilbert space $\mathcal{H}$.

Proposition 4.3.3. *Assume the CGMA with the choices $\mathcal{M} = \mathbb{R}^4$ and $\mathcal{W}$ as described, along with the transitivity of the adjoint action of $\{J_W \mid W \in \mathcal{W}\}$ on the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$, and the net continuity condition stated at the beginning of this section. Then there exists a strongly continuous projective representation $V(\mathcal{P}_+^\uparrow) \subset \mathcal{J}$ of the group $\mathcal{P}_+^\uparrow$ which acts geometrically correctly upon the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$.*

As was already pointed out, *any* of the projective representations $J(\mathcal{P}_+^\uparrow)$ furnished by Corollary 2.3 acts geometrically correctly upon the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$. Here, we merely make a particular choice amongst these in order to explicitly assure that the projective representation is continuous. The unitarity of the representation is already guaranteed by Prop. 4.2.10.

We shall prove Prop. 4.3.3 in a series of steps. To begin, we shall define projective representations of certain subgroups of $\mathcal{P}_+^\uparrow$ and show that they actually yield continuous representations of their respective subgroups.

Consider a wedge $W^{(0)} \in \mathcal{W}_0$ containing the origin of $\mathbb{R}^4$ in its edge and denote by $x^{(0)}, y^{(0)}$, *etc.*, any translation in the two-dimensional subspace $\mathbb{R}^2_{W^{(0)}}$

generated by the two lightlike directions fixing the boundaries of $W^{(0)}$. Denote by $J_{z^{(0)}}$ the modular involution associated with $(\mathcal{R}(W^{(0)}+z^{(0)}),\Omega)$.

It follows from Props. 4.2.10 and 4.3.1 that

$$J_{z^{(0)}}\mathcal{R}(W)J_{z^{(0)}} = \mathcal{R}(\Lambda_{W^{(0)}}W + 2z^{(0)}) \quad ,$$

for all $W \in \mathcal{W}$, where $\Lambda_{W^{(0)}} \in \mathcal{L}_+$ is the reflection which is equal to -1 on $\mathrm{IR}^2_{W^{(0)}}$ and equal to 1 on the two-dimensional subspace of IR^4 which forms the edge of $W^{(0)}$. (This relation was Assumption (1) in [24].) One therefore sees that

$$J_{x^{(0)}}J_{y^{(0)}}\mathcal{R}(W)J_{y^{(0)}}J_{x^{(0)}} = \mathcal{R}(W + 2x^{(0)} - 2y^{(0)}) \quad ,$$

for any $W \in \mathcal{W}$, $W^{(0)} \in \mathcal{W}_0$, and $x^{(0)}, y^{(0)}$ as described above.

For $x^{(0)} \in \mathrm{IR}^2_{W^{(0)}} \subset \mathrm{IR}^4 \subset \mathcal{P}_+^{\uparrow}$, choose $V_{W^{(0)}}(2x^{(0)}) \equiv J_{x^{(0)}}J_{W^{(0)}}$. Then Prop. 4.3.2 entails immediately that $x^{(0)} \mapsto V_{W^{(0)}}(x^{(0)})$ is a strongly continuous family of unitary operators implementing the action of the subgroup $\mathrm{IR}^2_{W^{(0)}}$ of the translation group on the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$. This is true for any choice of $W^{(0)} \in \mathcal{W}_0$.

Next, consider the wedges $W_i^{(0)} = \{x \in \mathrm{IR}^4 \mid x_i > |x_0|\}$, $i = 1,2,3$, and the corresponding projective representations $V_i(\mathrm{IR}^2_{W_i^{(0)}})$, $i = 1,2,3$. These unitary operators will be used to build the desired representation of the translation group. We shall first show that they coincide on the subgroup of time translations. To this end we make use of the fact that the rotations in the time-zero plane are induced by unitary operators in $\mathcal{J}$, cf. Prop. 4.3.1. Hence, if R is a rotation by $\pi/2$ about the 1-axis, we obtain from Lemma 2.1 (2), using the abbreviation $x_0 = (x_0, 0, 0, 0)$, the equalities

$$J(R)V_1(x_0)J(R)^{-1} = J(R)J_{W_1^{(0)}+x_0}J_{W_1^{(0)}}J(R)^{-1} = J_{RW_1^{(0)}+x_0}J_{RW_1^{(0)}} = V_1(x_0) \quad ,$$

since $RW_1^{(0)} = W_1^{(0)}$. Here we have made use of the important fact, a consequence of Prop. 4.3.1 and the uniqueness of modular objects, that the modular conjugations associated with wedges transform covariantly under the adjoint action of the (anti)unitary operators in $\mathcal{J}$, *i.e.*

$$J(\lambda)J_W J(\lambda)^{-1} = J_{\lambda W} \quad , \tag{4.3.2}$$

for any choice of wedge $W \in \mathcal{W}$ and Poincaré transform $\lambda \in \mathcal{P}_+$. Secondly, we know from Corollary 2.3 that $V_1(x_0) = Z(x_0)V_2(x_0)$, where $Z(x_0)$ is an internal symmetry of the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ in the center of $\mathcal{J}$. In the light of the equalities

$$J(R)V_2(x_0)J(R)^{-1} = J(R)J_{W_2^{(0)}+x_0}J_{W_2^{(0)}}J(R)^{-1} = J_{R(W_2^{(0)}+x_0)}J_{RW_2^{(0)}} = V_3(x_0) \quad ,$$

using $R(W_2^{(0)}+x_0) = W_3^{(0)} + x_0$, we arrive at the relation

$$Z(x_0)V_2(x_0) = V_1(x_0) = J(R)V_1(x_0)J(R)^{-1} = Z(x_0)V_3(x_0) \quad .$$

Thus $V_2(x_0) = V_3(x_0)$, and in a similar way one proves $V_1(x_0) = V_3(x_0)$. We therefore write $V((x_0, 0, 0, 0))$ for $V_i(x_0)$. This technique of establishing the equality of unitary implementers will also be used in the subsequent arguments in order to solve the cohomological problems involved in the discussion of the projective representation.

Now, for any $x = (x_0, x_1, x_2, x_3) \in \mathrm{IR}^4$, we define

$$V(x) \equiv V((x_0, 0, 0, 0))V_1((0, x_1, 0, 0))V_2((0, 0, x_2, 0))V_3((0, 0, 0, x_3)) \quad .$$

As in the proof of Prop. 2.2 in [24], one verifies that $x \mapsto V(x)$ is a projective unitary representation of the translation subgroup of the Poincaré group acting geometrically correctly on the net $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$. In order to prove that it is actually a representation, we must show that the various factors in the definition of V commute. Let us consider, for example, the operator $V_1((0, x_1, 0, 0))$, which leaves Ω invariant and satisfies

$$V_1((0, x_1, 0, 0))\mathcal{R}(W_2^{(0)} + z^{(0)})V_1((0, x_1, 0, 0))^{-1} = \mathcal{R}(W_2^{(0)} + z^{(0)}) \quad ,$$

for every $z^{(0)} \in \mathrm{IR}^2_{W_2^{(0)}}$. So it must commute with the modular involutions of the coherent family $\{\mathcal{R}(W_2^{(0)} + z^{(0)}) \mid z^{(0)} \in \mathrm{IR}^2_{W_2^{(0)}}\}$ and therefore also with $V((x_0, 0, 0, 0)) = V_2((x_0, 0, 0, 0))$ and $V_2((0, 0, x_2, 0))$. Similarly, $V_1((0, x_1, 0, 0))$ commutes with $V_3((0, 0, 0, x_3))$, and by the same argument one can establish the commutativity of the remaining unitaries.

We next show that the unitaries in the definition of V define continuous representations of the respective one–dimensional subgroups.

Lemma 4.3.4. *Under the assumptions of Prop. 4.3.3, for any* $i = 1, 2, 3$ *and* $x^{(0)} \in \mathrm{IR}^2(W_i^{(0)})$, *the mapping* $\mathrm{IR} \ni t \mapsto V_i(tx^{(0)})$ *is a strongly continuous homomorphism.*

Proof. For convenience, set $J_t \equiv J_{W_i^{(0)} + tx^{(0)}}$ and $V(t) = J_{t/2}J_0$, $t \in \mathrm{IR}$. It follows from Prop. 4.3.2 that $J_{t/2}$ and hence $V(t)$ is strongly continuous in t. Since $V(t)J_0 = J_0V(t)^{-1}$ and similarly $V(t)J_{t/2} = J_{t/2}V(t)^{-1}$, one obtains with the help of relation (4.3.2), for $n \in \mathrm{IN}$,

$$V(t)^{2n}J_0 = V(t)^nJ_0V(t)^{-n} = J_{nt} \quad ,$$

and consequently one has

$$V(t)^{2n} = V(t)^{2n}J_0^2 = J_{nt}J_0 = V(2nt) \quad .$$

Similarly one finds

$$V(t)^{2n+1} = V(t)^{2n}J_{t/2}J_0 = V(t)^nJ_{t/2}V(t)^{-n}J_0 = J_{(n+1/2)t}J_0 = V((2n + 1)t) \quad .$$

From these relations one sees in particular that for $m_1, m_2 \in \mathrm{IN}$ and $0 \neq n \in \mathbb{Z}$,

$$V(m_1/n)V(m_2/n) = V(1/n)^{m_1}V(1/n)^{m_2} = V(1/n)^{m_1+m_2} = V((m_1 + m_2)/n) \quad .$$

Since m_1, m_2, n are arbitrary and $V(t)$ is continuous, the remaining portion of the assertion follows. □

Combining this lemma with the preceding results, we have thus established the fact that the unitary operators $V(x)$ introduced above define a continuous representation of the translations.

Lemma 4.3.5. *Under the assumptions of Prop. 4.3.3, there exists in $\mathcal{J}$ a strongly continuous unitary representation $V(\mathrm{IR}^4)$ of the translation subgroup which acts geometrically correctly upon the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$.*[7]

Let us now turn to the Lorentz transformations. As is well known, any Lorentz transformation $\Lambda \in \mathcal{L}_+^\uparrow$ can uniquely be decomposed in the chosen Lorentz system into a boost B and a rotation R, $\Lambda = BR$, where $B = \sqrt{\Lambda\Lambda^T}$ and $R = B^{-1}\Lambda$. It is apparent that the factors appearing in this decomposition are continuous in Λ. We first define unitary operators corresponding to the boosts and rotations individually.

Given a nontrivial boost B there exists a unique two-dimensional subspace IR^2_B in the time-zero plane $\{x \in \mathrm{IR}^4 \mid x_0 = 0\}$ of the chosen Lorentz system which is perpendicular to the boost direction and therefore pointwise invariant under the action of B. We pick an arbitrary unit vector $\vec{e} \in \mathrm{IR}^2_B$ and consider the corresponding wedge $W_{\vec{e}}^{(0)} = \{x \in \mathrm{IR}^4 \mid \vec{x}\cdot\vec{e} > |x_0|\}$. An elementary computation using Prop. 4.2.10 shows that the Poincaré transformations associated with the corresponding modular conjugations satisfy $g_{BW_{\vec{e}}^{(0)}} g_{W_{\vec{e}}^{(0)}} = B^2$. This leads us to define

$$V_{\vec{e}}(B) \equiv J_{B^{1/2}W_{\vec{e}}^{(0)}} J_{W_{\vec{e}}^{(0)}} \quad ,$$

where $B^{1/2}$ is the unique boost whose square is equal to B. If $B = 1$, we set $V_{\vec{e}}(1) = \mathbb{1}$. This definition is consistent since $J^2_{W_{\vec{e}}^{(0)}} = \mathbb{1}$ for any unit vector $\vec{e}$.

In a similar manner we construct implementers of the rotations. Given any proper rotation $R \neq 1$ there is a unique two-dimensional subspace IR^2_R which is perpendicular to the axis of revolution of R and therefore stable under the action of this rotation. As in the case of the boosts, we consider for $\vec{e} \in \mathrm{IR}^2_R$ the corresponding wedge $W_{\vec{e}}^{(0)}$ and find $g_{RW_{\vec{e}}^{(0)}} g_{W_{\vec{e}}^{(0)}} = R^2$. Correspondingly, we set

$$V_{\vec{e}}(R) \equiv J_{R^{1/2}W_{\vec{e}}^{(0)}} J_{W_{\vec{e}}^{(0)}} \quad ,$$

where $R^{1/2}$ is defined as the rotation with the same axis of revolution as R but with half the rotation angle.

This definition requires a consistency check because of the nonuniqueness of the square root of rotations. So let R_1, R_2 be two different square roots of R (differing by a rotation by π). Then $R_2 W_{\vec{e}}^{(0)} = -R_1 W_{\vec{e}}^{(0)}$ and, consequently, $J_{R_2 W_{\vec{e}}^{(0)}} J_{W_{\vec{e}}^{(0)}} = J_{-R_1 W_{\vec{e}}^{(0)}} J_{W_{\vec{e}}^{(0)}}$. Because of Haag duality, we have $\mathcal{R}(-W^{(0)}) = \mathcal{R}(W^{(0)\,\prime}) = \mathcal{R}(W^{(0)})^\prime$, for any wedge $W^{(0)}$, and consequently $J_{-W^{(0)}} = J_{W^{(0)}}$. Hence, we have the equality $J_{-R_1 W_{\vec{e}}^{(0)}} = J_{R_1 W_{\vec{e}}^{(0)}}$, proving the consistency of the definition of $V_{\vec{e}}(R)$. We shall show in the next lemma that the implementers of boosts and rotations defined above do not depend on the choice of the vector $\vec{e}$.

[7] It is possible to show the existence of a continuous representation of the translation group without the assumption of the net continuity condition. This argument will be presented in a subsequent publication.

Lemma 4.3.6. *Let $V_{\vec{e}}(B)$, $V_{\vec{e}}(R)$ be the unitary operators implementing the boost B and rotation R, respectively. These operators do not depend on the choice of the vector $\vec{e}$ within the above-stated limitations.*

Proof. Consider first the case of boosts. If $B = 1$, there is nothing to prove. So let $B \neq 1$, let IR^2_B be the corresponding two-dimensional invariant subspace and let B_1 be any other boost which leaves this subspace pointwise invariant. As in the case of the translations discussed in Lemma 4.3.4, it follows from relation (4.3.2) that for any $\vec{e} \in \mathrm{IR}^2_B$ one has $V_{\vec{e}}(B_1)^n = V_{\vec{e}}(B_1^n)$, for $n \in \mathrm{IN}$.

Now let R_ϕ be a rotation by ϕ about the axis established by the direction of the boost B and let $J(R_\phi)$ be a corresponding implementer. Then one obtains from relation (4.3.2)

$$J(R_\phi) V_{\vec{e}}(B_1) J(R_\phi)^{-1} = J_{R_\phi B_1^{1/2} W^{(0)}_{\vec{e}}} J_{R_\phi W^{(0)}_{\vec{e}}} = J_{B_1^{1/2} W^{(0)}_{R_\phi \vec{e}}} J_{W^{(0)}_{R_\phi \vec{e}}} = V_{R_\phi \vec{e}}(B_1) \quad ,$$

since R_ϕ and $B_1^{1/2}$ commute. On the other hand, according to Corollary 2.3, there exists some element Z_ϕ in the subgroup of internal symmetries $\mathcal{Z}$ of $\mathcal{J}$ such that

$$V_{R_\phi \vec{e}}(B_1) = Z_\phi V_{\vec{e}}(B_1) \quad .$$

Setting $\phi = 2m\pi/n$, for $m, n \in \mathrm{IN}$, one sees from the preceding two relations that

$$V_{\vec{e}}(B_1) = V_{R^n_{2m\pi/n}\vec{e}}(B_1) = J(R_{2m\pi/n})^n V_{\vec{e}}(B_1) J(R_{2m\pi/n})^{-n} = Z^n_{2m\pi/n} V_{\vec{e}}(B_1) \quad ,$$

and consequently $Z^n_{2m\pi/n} = \mathbb{1}$. Hence,

$$V_{R_{2m\pi/n}\vec{e}}(B_1^n) = V_{R_{2m\pi/n}\vec{e}}(B_1)^n = Z^n_{2m\pi/n} V_{\vec{e}}(B_1)^n = V_{\vec{e}}(B_1^n) \quad ,$$

and setting $B_1 = B^{1/n}$ one obtains

$$V_{R_{2m\pi/n}\vec{e}}(B) = V_{\vec{e}}(B) \quad .$$

According to Prop. 4.3.2, the operator $J_{R_\phi W^{(0)}}$ depends continuously on ϕ for any wedge $W^{(0)}$, and the same is thus also true of $V_{R_\phi \vec{e}}(B)$. It therefore follows from the preceding relation that $V_{R_\phi \vec{e}}(B) = V_{\vec{e}}(B)$ for any rotation R_ϕ, proving the assertion for the case of the boosts.

For the rotations R, one proceeds in exactly the same way as above. The role of R_ϕ is here played by the rotations about the axis of revolution fixed by R. □

In view of this result we may omit in the following the index $\vec{e}$ and set

$$V(B) \equiv V_{\vec{e}}(B), \quad V(R) \equiv V_{\vec{e}}(R) \quad .$$

We next discuss the continuity properties of these operators with respect to the boosts and rotations.

Lemma 4.3.7. *The unitary operators $V(B)$ and $V(R)$ depend (strongly) continuously on the boosts B and rotations R, respectively.*

Proof. Let B_n be a sequence of boosts which converges to B. If $B \neq 1$ it is clear that the distance between the unit disks in the corresponding invariant subspaces $\mathbb{R}^2_{B_n}$ and $\mathbb{R}^2_B$ converges to 0. In particular, there exists a sequence of unit vectors $\vec{e}_n \in \mathbb{R}^2_{B_n}$ which converges to some $\vec{e} \in \mathbb{R}^2_B$ and consequently the sequence of wedges $B_n^{1/2} W^{(0)}_{\vec{e}_n}$ converges to $B^{1/2} W^{(0)}_{\vec{e}}$. Because of the continuity of the modular operators J_W with respect to W, established in Prop. 4.3.2, one concludes that

$$V(B_n) = J_{B_n^{1/2} W^{(0)}_{\vec{e}_n}} J_{W^{(0)}_{\vec{e}_n}} \longrightarrow J_{B^{1/2} W^{(0)}_{\vec{e}}} J_{W^{(0)}_{\vec{e}}} = V(B) \quad .$$

If the sequence B_n converges to 1, the corresponding unit disks in $\mathbb{R}^2_{B_n}$ need not converge. But, because of the compactness of the unit ball in $\mathbb{R}^3$, for any sequence of unit vectors $\vec{e}_n \in \mathbb{R}^2_{B_n}$, there exists a subsequence $\vec{e}_{\sigma(n)}$ which converges to some unit vector $\vec{e}_\sigma$. Since $B^{1/2}_{\sigma(n)} \to 1$, the corresponding sequences of wedges $B^{1/2}_{\sigma(n)} W^{(0)}_{\vec{e}_{\sigma(n)}}$ and $W^{(0)}_{\vec{e}_{\sigma(n)}}$ converge to $W^{(0)}_{\vec{e}_\sigma}$, and consequently one has

$$V(B_{\sigma(n)}) = J_{B^{1/2}_{\sigma(n)} W^{(0)}_{\vec{e}_{\sigma(n)}}} J_{W^{(0)}_{\vec{e}_{\sigma(n)}}} \to J_{W^{(0)}_{\vec{e}_\sigma}} J_{W^{(0)}_{\vec{e}\sigma}} = \mathbb{1} \quad .$$

Since the choice of the sequence $\vec{e}_n \in \mathbb{R}^2_{B_n}$ was arbitrary, the proof of the continuity of the boost operators is complete. The argument for the rotations is analogous; the only difference being that the boost direction must be replaced by the axis of revolution. □

We are now in the position to prove Proposition 4.3.3. Given an element $(\Lambda, x) \in \mathcal{P}^\uparrow_+$, we proceed to the unique and continuous decomposition $(\Lambda, x) = (1, x)(B, 0)(R, 0)$ and set

$$V((\Lambda, x)) \equiv V(x)V(B)V(R) \quad ,$$

where the unitary operators corresponding to the translations, boosts and rotations have been defined above. Since these operators depend continuously on their arguments, the assertion of Prop. 4.3.3 follows. As a matter of fact, we shall see that the unitary operators $V((\Lambda, x))$ actually define a true representation of $\mathcal{P}^\uparrow_+$. A first step in this direction is the following lemma.

Lemma 4.3.8. *Let $V(\cdot)$ be the continuous unitary projective representation of $\mathcal{P}^\uparrow_+$ introduced above. One has*

(1) $V(R)V(B)V(R)^{-1} = V(RBR^{-1})$ and $V(R)V(R_0)V(R)^{-1} = V(RR_0R^{-1})$, for all boosts B and rotations R, R_0.

(2) $V(\cdot)$ defines a true representation of every continuous one-parameter subgroup of boosts or rotations.

Proof. The first statement in (1) follows from relation (4.3.2) and Lemma 4.3.6, which imply

$$\begin{aligned} V(R)V(B)V(R)^{-1} &= V(R) J_{B^{1/2} W^{(0)}_{\vec{e}}} J_{W^{(0)}_{\vec{e}}} V(R)^{-1} \\ &= J_{RB^{1/2}R^{-1} W^{(0)}_{R\vec{e}}} J_{W^{(0)}_{R\vec{e}}} = V(RBR^{-1}) \quad , \end{aligned}$$

where the last equality follows from the fact that RBR^{-1} is again a boost which leaves the subspace $R\,\mathbb{R}^2_B$ pointwise invariant. The argument for the rotations is analogous.

Now let $\{G(u) \mid u \in \mathbb{R}\}$, be a continuous one-parameter group of boosts or rotations. As in the proof of Lemma 4.3.4, one shows by an elementary computation on the basis of relation (4.3.2) that $V(G(u))^n = V(G(u)^n) = V(G(nu))$. Consequently, one finds that, for $m_1, m_2 \in \mathbb{N}$ and $0 \neq n \in \mathbb{Z}$,

$$\begin{aligned} V(G(m_1/n))V(G(m_2/n)) &= V(G(1/n))^{m_1} V(G(1/n))^{m_2} \\ &= V(G(1/n))^{m_1+m_2} = V(G((m_1+m_2)/n)) \quad . \end{aligned}$$

The stated assertion (2) thus follows once again from the continuity properties of $V(\cdot)$.

□

Instead of proving by explicit but tedious computations that $V(\cdot)$ defines a true representation of $\mathcal{P}_+^\uparrow$, we prefer to give a more abstract argument based on cohomology theory. In the appendix it is shown that the existence of a continuous unitary projective representation $V(\mathcal{P}_+^\uparrow)$ with values in $\mathcal{J}$ implies that there is a continuous unitary representation $U(\cdot)$ of the covering group $ISL(2,\mathbb{C})$ of $\mathcal{P}_+^\uparrow$. U takes values in the closure $\overline{\mathcal{J}}$ of $\mathcal{J}$ in the weak operator topology. Moreover, there exists a mapping $Z : ISL(2,\mathbb{C}) \mapsto \overline{\mathcal{Z}}$, the closure of the internal symmetry group $\mathcal{Z}$ in the center of $\overline{\mathcal{J}}$, such that $U(A) = Z(A)V(\mu(A))$, for all $A \in ISL(2,\mathbb{C})$, where $\mu : ISL(2,\mathbb{C}) \mapsto \mathcal{P}_+^\uparrow$ is the canonical covering homomorphism whose kernel is a subgroup of order 2, the center of $ISL(2,\mathbb{C})$.

The preceding results enable us to show that $U(\cdot)$ acts trivially on the center of $ISL(2,\mathbb{C})$ and therefore defines a representation of $\mathcal{P}_+^\uparrow$. For let $A_1, A_2 \in ISL(2,\mathbb{C})$ be two elements corresponding to rotations by π about two orthogonal axes, *i.e.* $\mu(A_i) = R_i(\pi)$, $i = 1,2$. It then follows that $A_1 A_2 A_1^{-1} A_2^{-1} = C$, where C is the nontrivial element in the center of $ISL(2,\mathbb{C})$. Consequently, we have

(4.3.3)
$$\begin{aligned} U(C) &= U(A_1)U(A_2)U(A_1)^{-1}U(A_2)^{-1} \\ &= Z(A_1)V(R_1(\pi))Z(A_2)V(R_2(\pi))V(R_1(\pi))^{-1}Z(A_1)^{-1}V(R_2(\pi))^{-1}Z(A_2)^{-1} \\ &= V(R_1(\pi))V(R_2(\pi))V(R_1(\pi))^{-1}V(R_2(\pi))^{-1} \quad , \end{aligned}$$

where we made use of the fact that the operators $Z(A_i), i = 1,2$, are elements of $\overline{\mathcal{Z}}$ and therefore commute through the product and cancel. Since $R_1(\pi)R_2(\pi)R_1(\pi)^{-1} = R_2(\pi)$, we see from Lemma 4.3.8 (1) that

$$V(R_1(\pi))V(R_2(\pi))V(R_1(\pi))^{-1} = V(R_1(\pi)R_2(\pi)R_1(\pi)^{-1}) = V(R_2(\pi)) \quad .$$

So we conclude that $U(C) = \mathbb{1}$, as claimed. We can therefore set

$$U(\mu(A)) \equiv U(A), \quad A \in ISL(2,\mathbb{C}) \quad .$$

In the final step of our argument we make use of the fact that the Poincaré group is perfect. (Recall that a group is perfect if it is equal to its commutator subgroup – see the appendix.) Given $\lambda_i \in \mathcal{P}_+^\uparrow$, $i = 1, 2$, one can show in the same way as in relation (4.3.3) that

$$U(\lambda_1\lambda_2\lambda_1^{-1}\lambda_2^{-1}) = V(\lambda_1)V(\lambda_2)V(\lambda_1)^{-1}V(\lambda_2)^{-1} \quad .$$

Since the elements on the right hand side of this equation are contained in $\mathcal{J}$, we conclude that the representation $U(\cdot)$ also has values in $\mathcal{J}$ (so one does not need to proceed to the closure $\overline{\mathcal{J}}$). It then follows from Prop. 4.3.1 that the unitary operators $U(\lambda), \lambda \in \mathcal{P}_+^\uparrow$, act geometrically correctly on the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$.

Now, given a wedge W and the corresponding modular conjugation J_W and reflection $g_W \in \mathcal{P}_+$ – see Prop. 4.2.10 – it follows from relation (4.3.2) that $J_W V(\lambda) J_W = Z V(g_W \lambda g_W^{-1})$, where $Z \in \mathcal{Z}$ is some internal symmetry. As these central elements drop out in group theoretic commutators of the operators $V(\lambda)$, we can compute the adjoint action of the modular conjugations J_W on $U(\mathcal{P}_+^\uparrow)$ by making use of the relation

$$\begin{aligned} J_W U(\lambda_1\lambda_2\lambda_1^{-1}\lambda_2^{-1}) J_W &= J_W V(\lambda_1)V(\lambda_2)V(\lambda_1)^{-1}V(\lambda_2)^{-1} J_W \\ &= V(g_W\lambda_1 g_W^{-1})V(g_W\lambda_2 g_W^{-1})V(g_W\lambda_1 g_W^{-1})^{-1}V(g_W\lambda_2 g_W^{-1})^{-1} \\ &= U(g_W\lambda_1\lambda_2\lambda_1^{-1}\lambda_2^{-1} g_W^{-1}) \quad . \end{aligned}$$

Since $\mathcal{P}_+^\uparrow$ is perfect, this shows that

$$J_W U(\lambda) J_W = U(g_W \lambda g_W^{-1}) , \quad \text{for} \quad \lambda \in \mathcal{P}_+^\uparrow \quad . \tag{4.3.4}$$

Hence the involution J_W induces the outer automorphism corresponding to g_W on $U(\mathcal{P}_+^\uparrow)$, so we may take $U(g_W) \equiv J_W$. The fact that $U(\lambda) \in \mathcal{J}$ acts geometrically correctly on the net implies, according to relation (4.3.2),

$$U(\lambda) J_W U(\lambda)^{-1} = J_{\lambda W} , \quad \text{for} \quad \lambda \in \mathcal{P}_+^\uparrow \quad . \tag{4.3.5}$$

Let $W_1, W_2 \in \mathcal{W}$ be arbitrary. There exists an element $\lambda \in \mathcal{P}_+^\uparrow$ such that $W_2 = \lambda W_1$. Hence the Poincaré covariance of $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ and condition (i) of the CGMA entail the relation $g_{W_2} = \lambda g_{W_1} \lambda^{-1}$. Using (4.3.4) and (4.3.5), we therefore have the equalities

$$\begin{aligned} U(g_{W_1})U(g_{W_2}) &= J_{W_1} J_{W_2} = J_{W_1} J_{\lambda W_1} \\ &= J_{W_1} U(\lambda) J_{W_1} U(\lambda)^{-1} \\ &= U(g_{W_1}\lambda g_{W_1}^{-1}) U(\lambda)^{-1} \\ &= U(g_{W_1}\lambda g_{W_1}^{-1}\lambda^{-1}) \\ &= U(g_{W_1} g_{W_2}) \quad , \end{aligned} \tag{4.3.6}$$

since $g_{W_1}^{-1} = g_{W_1}$. Hence, $U(\cdot)$ provides a representation for all of $\mathcal{G} = \mathcal{P}_+$.

Since $\mathcal{J}$ is generated by the conjugations $J_{\lambda W}, \lambda \in \mathcal{P}_+^\uparrow$, we conclude that $\mathcal{J} = U(\mathcal{P}_+^\uparrow) \cup J_{W^{(0)}} U(\mathcal{P}_+^\uparrow)$, for any fixed wedge $W^{(0)} \in \mathcal{W}_0$. Moreover, $\mathcal{J}^+ =$

$U(\mathcal{P}_+^\uparrow)$, where $\mathcal{J}^+$ is the subgroup of unitary operators in $\mathcal{J}$ which is generated by products of an even number of modular conjugations. As U is a faithful representation of $\mathcal{P}_+^\uparrow$ – cf. the standing assumptions in Chapter II – and $\mathcal{P}_+^\uparrow$ has trivial center, the center of $\mathcal{J}$ consists only of $\mathbb{1}$. Hence the representation $U(\cdot)$ must coincide with $V(\cdot)$. This shows finally that $V(\cdot)$ defines a representation of $\mathcal{P}_+^\uparrow$, as claimed. We summarize these findings in the following theorem.

Theorem 4.3.9. *Assume the CGMA with the choices $\mathcal{M} = \mathbb{R}^4$ and $\mathcal{W}$ the described set of wedges. If $\mathcal{J}$ acts transitively upon the set $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ and the net continuity condition mentioned at the beginning of Section 4.3 holds, then there exists a strongly continuous (anti)unitary representation $U(\mathcal{P}_+)$ of the proper Poincaré group which acts geometrically correctly upon the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ and which satisfies $U(g_W) = J_W$, for every $W \in \mathcal{W}$. Moreover, $U(\mathcal{P}_+^\uparrow)$ equals the subgroup of $\mathcal{J}$ consisting of all products of even numbers of J_W's and $\mathcal{J} = U(\mathcal{P}_+^\uparrow) \cup J_{W_R}U(\mathcal{P}_+^\uparrow)$. Furthermore, $U(\cdot)$ coincides with the representation $V(\cdot)$, which has been explicitly constructed above.*

V. Geometric Action of Modular Groups and the Spectrum Condition

A physically important property of a representation of the translation group on $\mathbb{R}^4$ is the spectrum condition, in other words, the condition that the generators of the given representation $U(\mathbb{R}^4)$ have their joint spectrum $\mathrm{sp}(U)$ in the closed forward light cone $\overline{V_+}$ (for the positive spectrum condition) or in the closed backward light cone $\overline{V_-}$ (for the negative spectrum condition). In Section 5.1 we examine how to incorporate the spectrum condition into our setting, using only the modular objects. We shall show that the (positive or negative) spectrum condition holds whenever the group $\mathcal{J}$ generated by the initial modular involutions contains also the initial modular groups. Some further consequences of the spectrum condition in our setting, such as the PCT and Spin & Statistics Theorems, will also be discussed.

We then turn our attention to the possible geometric action of the modular *unitaries*. In Section 5.2 we shall reconsider the condition of *modular covariance*, which has been extensively discussed in the literature [22][36][35][26]. If $W_0 \in \mathcal{W}$ is a wedge, $\{\Delta_{W_0}^{it}\}_{t\in\mathbb{R}}$ is the modular group corresponding to $(\mathcal{R}(W_0), \Omega)$, and $\{\lambda(t)\}_{t\in\mathbb{R}}$ is the one-parameter subgroup of (suitably Poincaré-transformed) boosts leaving W_0 invariant, then modular covariance is said to hold if

$$\Delta_{W_0}^{it}\mathcal{R}(W)\Delta_{W_0}^{-it} = \mathcal{R}(\lambda(t)W) \quad , \quad \text{for all} \quad t \in \mathbb{R} \quad , \quad W \in \mathcal{W} \quad ,$$

in other words, if the modular group associated to the algebra for the wedge W_0 implements the mentioned boost subgroup. In fact, the subgroup $\{\lambda(t)\}_{t\in\mathbb{R}}$ is usually more precisely specified: if $\ell_\pm$ are two positive lightlike translations such that $W_0 \pm \ell_\pm \subset W_0$, one has in $\mathcal{P}_+^\uparrow$ the relation

$$\lambda(t)(1, \ell_\pm)\lambda(t)^{-1} = (1, e^{\mp\alpha t}\ell_\pm) \quad ,$$

with $\alpha = \pm 2\pi$. The sign is a matter of convention fixing the direction of time.

Bisognano and Wichmann [9][10] (see also [29]) have shown that modular covariance holds for nets associated to Wightman fields in a Poincaré covariant vacuum representation. We shall show that if the adjoint action of the modular groups corresponding to the wedge algebras leaves the set $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ invariant, then modular covariance *follows* and either the positive or the negative spectrum condition holds. Moreover, under the same assumptions plus the locality of the net, the modular conjugations $\{J_W\}_{W\in\mathcal{W}}$ will be seen to act geometrically as reflections about spacelike lines, *i.e.* as in Prop. 4.2.10. In Section 5.3 we shall present some examples of nets satisfying all assumptions made in our program through Chapter IV, but violating the condition of modular covariance. In one of these examples the spectrum condition is violated, in the other the positive spectrum condition obtains. We then contrast the approaches to geometric modular action through the modular conjugations or through the modular groups in the light of the results of Section 5.2 and the mentioned examples.

5.1. The Modular Spectrum Condition

Let $V(\mathbb{R}^4)$ be any representation of the translation group acting covariantly on the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ and satisfying the relativistic spectrum condition with Ω as the ground state. Borchers [12] has isolated a condition on the modular group $\Delta_{W_0}^{it}$ associated with the pair $(\mathcal{R}(W_0),\Omega)$ which is intimately connected to the spectrum condition.

Borchers' Relation. *For every future-directed lightlike vector ℓ such that $W_0+\ell\subset W_0$, there holds the relation*

$$\Delta_{W_0}^{it} V(\ell) \Delta_{W_0}^{-it} = V(e^{-2\pi t}\ell) \quad , \quad \textit{for all} \quad t\in\mathbb{R} \quad . \tag{5.1.1}$$

Note that this is precisely the relationship which would result if $\Delta_{W_0}^{it}$ implemented the subgroup of boosts leaving the wedge W_0 invariant. It has turned out that this condition is *equivalent* to the representation $V(\mathbb{R}^4)$ satisfying the spectrum condition.[8] We cite the result as proven in [24]; the appearance of our theorem was preceded by that of an analogous result proven under slightly more restrictive conditions by Wiesbrock [68]. The proof of the deep result that the spectrum condition implies (5.1.1) is due to Borchers [12]. For a recent, considerably simplified proof of Borchers' theorem, we recommend [30] to the reader's attention.

Proposition 5.1.1. *Let $V(\mathbb{R}^4)$ be a strongly continuous unitary representation of the translation group on $\mathbb{R}^4$ which acts geometrically correctly upon the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ and leaves Ω invariant. Then $V(\mathbb{R}^4)$ satisfies the (positive) relativistic spectrum condition, i.e. $sp(V)\subset\overline{V}_+$, if and only if relation (5.1.1) holds for all wedges W_0, as described.*

We intend to utilize this proposition in a discussion of the spectral properties of the representation $U(\mathbb{R}^4)$ of the translation group obtained in the

[8] This connection has also been shown to be useful in applications to quantum fields defined on certain curved space-times associated with black holes [61].

previous chapter. Note that because we have a representation of $\mathcal{P}_+^\uparrow$ which acts geometrically correctly upon the net and which leaves the state invariant, if (5.1.1) holds (for $U(\cdot)$) for one such wedge W_0, it must hold for all such wedges.

In our approach, employing the modular involutions to derive symmetry groups and their representations, the *only* role played by the modular groups Δ_W^{it} is to characterize algebraically the spectrum condition as above. We next show that in our framework, the Borchers relation (5.1.1) (with $\pm 2\pi$ in the exponent on the right-hand side instead of -2π) already follows from the following assumption:

Modular Stability Condition. *The modular unitaries are contained in the group generated by the modular involutions, i.e.* $\Delta_W^{it} \in \mathcal{J}$, *for all* $t \in \mathrm{I\!R}$ *and* $W \in \mathcal{W}$.

In the situation described by this condition, the group generated by the modular unitaries and the modular conjugations associated to the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ by the vector Ω is minimal in a certain sense. The name of this condition is motivated by the *use* we envisage for it. We shall prove in Theorem 5.1.2 that the CGMA and the modular stability condition imply the spectrum condition, *i.e.* physical stability, in the special case of Minkowski space. Since both conditions are well-defined for nets based on arbitrary space-times, the modular stability condition, in the context of the CGMA, could perhaps serve as a substitute for the spectrum condition on space-times with no timelike Killing vector. In fact, as discussed below in Section 6.2, recent results [19][17] in de Sitter space support this picture.

We remark that the Poincaré covariance we have established entails that $\Delta_W^{it} \in \mathcal{J}$, for all $t \in \mathrm{I\!R}$ and *some* $W \in \mathcal{W}$, implies the modular stability condition. This follows from the transitive action of $\mathcal{P}_+^\uparrow$ upon the set $\mathcal{W}$ and the well-known fact that if Δ^{it} is the modular unitary for the pair $(\mathcal{M}, \Omega)$ and if the unitary U leaves Ω invariant, then $U\Delta^{it}U^*$ is the modular unitary for the pair $(U\mathcal{M}U^*, \Omega)$. We can now show that, within our framework, the condition that the modular unitaries are contained in the group $\mathcal{J}$ implies the spectrum condition, up to a sign. We shall see in the examples in Section 5.3 that each of the possible outcomes stated in this theorem can occur.

Theorem 5.1.2. *Assume the CGMA with the choices* $\mathcal{M} = \mathrm{I\!R}^4$ *and* $\mathcal{W}$ *the collection of wedgelike regions in* $\mathrm{I\!R}^4$, *the transitivity of the adjoint action of* $\mathcal{J}$ *on the net* $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$, *and the net continuity condition mentioned at the beginning of Section 4.3. Let* $U(\mathrm{I\!R}^4)$ *be the representation of the translation group obtained in Section 4.3. If* $\Delta_W^{it} \in \mathcal{J}$, *for all* $t \in \mathrm{I\!R}$ *and some* $W \in \mathcal{W}$, *i.e. if the modular stability condition obtains, then* $sp(U) \subset \overline{V_+}$ *or* $sp(U) \subset \overline{V_-}$. *Moreover, for every future-directed lightlike vector* ℓ *such that* $W + \ell \subset W$, *there holds the relation*

$$\Delta_W^{it} U(\ell) \Delta_W^{-it} = U(e^{-\alpha t}\ell) \quad , \quad \textit{for all} \quad t \in \mathrm{I\!R} \quad ,$$

where $\alpha = \pm 2\pi$.

Proof. Recall that $\mathcal{J}^+$ is the subgroup of $\mathcal{J}$ consisting of all products of even numbers of elements of $\{J_W \mid W \in \mathcal{W}\}$. Note that the relation $\Delta_W^{it/2}\Delta_W^{it/2} = \Delta_W^{it}$ and the assumption $\Delta_W^{it} \in \mathcal{J}$, for all $t \in \mathbb{R}$, imply that $\Delta_W^{it} \in \mathcal{J}^+ = U(\mathcal{P}_+^\uparrow)$ (using Theorem 4.3.9). Hence, for a fixed $W \in \mathcal{W}$ and each $t \in \mathbb{R}$, there exists an element $(\Lambda_t, a_t) \in \mathcal{P}_+^\uparrow$ such that

$$\mathcal{R}(W) = \Delta_W^{it}\mathcal{R}(W)\Delta_W^{-it} = U(\Lambda_t, a_t)\mathcal{R}(W)U(\Lambda_t, a_t)^{-1} = \mathcal{R}(\Lambda_t W + a_t) \quad .$$

Therefore, one must have $(\Lambda_t, a_t) \in \mathrm{InvP}_+^\uparrow(W)$, $t \in \mathbb{R}$, and the group $\mathcal{G}_W = \{(\Lambda_t, a_t) \mid t \in \mathbb{R}\}$ constituted by these transformations must be a one-parameter subgroup of $\mathrm{InvP}_+^\uparrow(W)$ which is abelian, since the unitaries Δ_W^{it}, $t \in \mathbb{R}$, mutually commute.

Let $W_0 \in \mathcal{W}$ be any wedge. One observes that if (ℓ_+, ℓ_-) is a pair of lightlike vectors such that $W_0 \pm \ell_\pm \subset W_0$, the adjoint action of any element of $\mathrm{InvL}_+^\uparrow(W)$ transforms the Poincaré group element $(1, \ell_\pm)$ to $(1, c\ell_\pm)$, with $c > 0$. In particular, for each $t \in \mathbb{R}$ there must exist an element $c_t^\pm > 0$ such that

$$\Delta_{W_0}^{it} U(u\ell_\pm)\Delta_{W_0}^{-it} = U(c_t^\pm u\ell_\pm) \quad , \tag{5.1.2}$$

for all $u \in \mathbb{R}$. Thus one has

$$\begin{aligned} U(c_{t+s}^\pm \ell_\pm) &= \Delta_{W_0}^{i(t+s)} U(\ell_\pm)\Delta_{W_0}^{-i(t+s)} = \Delta_{W_0}^{is}\Delta_{W_0}^{it} U(\ell_\pm)\Delta_{W_0}^{-it}\Delta_{W_0}^{-is} \\ &= U(c_s^\pm c_t^\pm \ell_\pm) \quad , \end{aligned}$$

which implies that $c_s^\pm c_t^\pm = c_{t+s}^\pm$, since $U(\mathbb{R}^4)$ acts geometrically correctly upon the net $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$ and since there exist wedges W_1 such that $W_1 + s\ell_\pm \neq W_1 + t\ell_\pm$ for $s \neq t$. From the left side of relation (5.1.2) one also sees that the map $(t, u) \mapsto U(c_t^\pm u\ell_\pm)$ is strongly continuous, uniformly on compact subsets of $\mathbb{R}^2$. As shall be shown, this implies that $c_t^\pm$ is continuous in t.

Assume that $c_t^\pm$ is discontinuous at $t = 0$. It then follows from the equation $c_s^\pm c_t^\pm = c_{t+s}^\pm$ that $c_t^\pm$ is unbounded in any neighborhood of $t = 0$. Thus, for any $r \neq 0$, there exist sequences $t_n \to 0$, $u_n \to 0$ such that $u_n c_{t_n} \to r$. Therefore, equation (5.1.2) and the mentioned strong continuity entail the equality $\mathbb{1} = U(r\ell_\pm)$, which is a contradiction. Thus, the function $t \mapsto c_t^\pm$ must be continuous at 0. The relation $c_s^\pm c_t^\pm = c_{t+s}^\pm$ then implies that there exist constants $\alpha_\pm \in \mathbb{R}$ such that $c_t^\pm = e^{\alpha_\pm t}$.

It is important to notice that $\alpha_\pm \neq 0$. If, for example, one had $c_t^+ = 1$ for all $t \in \mathbb{R}$, then one would have $[\Delta_{W_0}^{it}, U(\ell_+)] = 0$ and thus $\Delta_{W_0+\ell_+}^{it} = U(\ell_+)\Delta_{W_0}^{it}U(\ell_+)^{-1} = \Delta_{W_0}^{it}$, for all $t \in \mathbb{R}$. But since $\mathcal{R}(W_0 + \ell_+) \subsetneq \mathcal{R}(W_0)$, by the standing assumptions, this is in conflict with standard results in modular theory. For both algebras have Ω as a cyclic vector and the stability of the smaller algebra under the action of the modular group of the larger one would thus imply that these algebras must be equal (see [18]).

One can now apply the arguments of Prop. 2.3 in [24]. There it was shown that relation (5.1.2) implies that for any two vectors $\Phi, \Psi \in \mathcal{H}$ there exists a function $f(z)$ which is continuous and bounded on the strip $0 \leq \mathrm{Im}(z) \leq 1/2$, analytic in the interior, satisfies the bound $|f(z)| \leq \|\Phi\|\|\Psi\|$, and on the real axis has the boundary value

$$f(t) = \langle \Phi, U(-e^{\alpha_+ t}\ell_+)\Psi \rangle \quad .$$

Since Φ and Ψ are arbitrary, one may conclude that the operator function $z \mapsto U(-e^{\alpha_+ z}\ell_+)$ is weakly continuous on the strip $0 \leq \mathrm{Im}(z) \leq 1/2$, analytic in the interior, and bounded in norm by 1. In particular, one has

$$\|U(-i \sin(u\alpha_+)\ell_+)\| \leq 1 \quad ,$$

for $0 \leq u \leq 1/2$. Hence, it follows that either $P \cdot \ell_+ \geq 0$, where P is the generator of the strongly continuous abelian unitary group $U(\mathrm{I\!R}^4)$, or $P \cdot \ell_+ \leq 0$. By Lorentz covariance, these relations hold for arbitrary lightlike vector ℓ_+, hence the spectrum of P must be contained either in the closed forward light cone or the closed backward light cone. The final assertion of the theorem then follows from Borchers' theorem [12]. □

This observation reinforces our belief that the modular involutions are of primary interest in this context.

Theorem 5.1.2 seems to leave open the possibility that the modular group associated to the wedge algebra $\mathcal{R}(W)$ could conceivably act geometrically as some other subgroup of the invariance group $\mathrm{InvP}_+^\uparrow(W)$ of W besides the boost subgroup. However, this is not the case, as we shall prove in the next section – cf. Prop. 5.2.4 and Theorem 5.2.7.

We wish to emphasize the point that Borchers' relation (5.1.1) is truly an *additional* assumption in our framework, as is the modular stability condition. In Section 5.3 we present a simple example of a net satisfying our CGMA and all of the other assumptions made in this paper except the modular stability condition and (5.1.1). In this example the spectrum condition is therefore violated, and the action of the modular groups associated to wedge algebras does not coincide with the Lorentz boosts.

Next, we wish to make a few comments about the uniqueness of the representation of $\mathcal{P}_+^\uparrow$ which has been obtained above. There are uniqueness results for representations of the translation subgroup satisfying the spectrum condition in local quantum field theory - see [24] and references cited there. For the case of nets $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$ based on wedges, the assertion can be derived easily from Borchers' theorem. We state and prove this fact for completeness.

Proposition 5.1.3. *Let $V(\mathrm{I\!R}^4)$ be a continuous unitary representation of the translations on $\mathcal{H}$ which acts geometrically correctly on the net $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$, leaves Ω invariant, and satisfies the spectrum condition. Then there is no other representation on $\mathcal{H}$ with these properties.*

Proof. Let W be any wedge and let ℓ be any positive lightlike vector such that $W + \ell \subset W$. Since $V(\cdot)$ acts geometrically correctly on the net, one has $V(\ell)\Delta_W^{it}V(\ell)^{-1} = \Delta_{W+\ell}^{it}$, and because of the hypothesized spectral properties of $V(\cdot)$, Borchers' relation holds: $\Delta_W^{it}V(\ell)\Delta_W^{-it} = V(e^{-2\pi t}\ell)$. Combining these two relations yields $V(\ell - e^{-2\pi t}\ell) = \Delta_{W+\ell}^{it}\Delta_W^{-it}$, and the operators appearing on the right-hand side of this equation are fixed by the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ and the vector Ω. Hence, $V(\cdot)$ is uniquely determined by these data for all lightlike vectors; the group property then yields the desired conclusion. □

For the representation of the entire Poincaré group, the best result seems to be that of [22], which asserts that if the distal split property holds, then the representation of $\mathcal{P}_+^\uparrow$ is also unique. (See also the results in the recent article by Borchers [16].) In Section 5.3 we shall present an example of a well-behaved net covariant under two distinct representations of the Poincaré group, only one of which is selected by the CGMA.

With the additional condition (5.1.1) yielding the spectrum condition, algebraic PCT and Spin & Statistics theorems can be proven. A series of papers [36][35][45] (see also [26]) have demonstrated a purely algebraic version of the important relationship between spin and statistics, which was first pointed out by Fierz and Pauli and then proven rigorously in the context of Wightman quantum field theory by Burgoyne and Lüders and Zumino (see [60] for references). In the work [36][35][26] the assumption of modular covariance was made, which, as we shall see in Section 5.3, does not necessarily hold in our more general setting. But if the conditions of Theorem 5.1.2 are satisfied, then the results established above *do* imply the hypotheses made in the approach by Kuckert [45] in order to derive the PCT and Spin & Statistics theorems. We shall not take further space to formulate the obvious theorem and refer the reader to [45] for details.

5.2. Geometric Action of Modular Groups

To obtain a deeper insight into the nature of the property of modular covariance on the one hand and the relation between the geometric action of modular involutions and that of the modular groups on the other, we shall assume in this section that the modular groups have a geometric action similar to that which we have heretofore assumed for the modular involutions. In particular, we shall assume that the adjoint action of the modular groups of the wedge algebras leaves the set $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ invariant. Throughout this section we shall assume that $\mathcal{M} = \mathrm{IR}^4$ and $\mathcal{W}$ is the set of wedges, as previously described.

Condition of Geometric Action for the Modular Groups. *The Condition of Geometric Action for the modular groups is fulfilled if the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ and vector Ω satisfy the first three conditions of the CGMA stated in Chapter III and the fourth condition is replaced by the following requirement: For each $W_0 \in \mathcal{W}$, the adjoint action of $\{\Delta_{W_0}^{it}\}_{t\in\mathrm{IR}}$ leaves the set $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ invariant, i.e. for any $W \in \mathcal{W}$ and any $t \in \mathrm{IR}$ there exists a wedge $W_t \in \mathcal{W}$ such that*

$$\Delta_{W_0}^{it}\mathcal{R}(W)\Delta_{W_0}^{-it} = \mathcal{R}(W_t) \quad .$$

This *condition for the modular groups* will be called CMG for short. We shall show that the analysis carried out in the preceding chapters in the case of theories satisfying the CGMA can likewise be performed when one takes the CMG as the starting point.

We denote by $\mathcal{K}$ the unitary group generated by the set $\{\Delta_W^{it} \mid t \in \mathbb{R}, W \in \mathcal{W}\}$. As in Chapter II one sees that the CMG entails that each adΔ_W^{it} induces a bijection $\upsilon_W(t)$ on the set $\mathcal{W}$ of wedges. The group generated by these bijections will be denoted by $\mathcal{U}$. We state the following counterpart to Lemma 2.1.

Lemma 5.2.1. *The group $\mathcal{U}$ defined above has the following properties.*

(1) For every $\upsilon \in \mathcal{U}$ and $W \in \mathcal{W}$, one has $\upsilon\upsilon_W(t)\upsilon^{-1} = \upsilon_{\upsilon(W)}(t)$, $t \in \mathbb{R}$.

(2) If $\upsilon(W) = W$ for some $\upsilon \in \mathcal{U}$ and $W \in \mathcal{W}$, then $\upsilon\upsilon_W(t) = \upsilon_W(t)\upsilon$, $t \in \mathbb{R}$.

(3) One has $\upsilon_W(t)(W) = W$, for all $W \in \mathcal{W}$ and $t \in \mathbb{R}$.

(4) If $W_1 \in \mathcal{W}$ and $\upsilon_W(t)(W_1) \subset W$, for all $t \in \mathbb{R}$, then $W_1 = W$.

Proof. The first two statements can be established in the same way as part (2) and (3) of Lemma 2.1. The third statement follows from the fact that each algebra $\mathcal{R}(W)$ is stable under the adjoint action of the modular group $\{\Delta_W^{it} \mid t \in \mathbb{R}\}$. Finally, the fourth assertion is a consequence of the basic result from Tomita-Takesaki theory that the only weakly closed subalgebra of a von Neumann algebra $\mathcal{R}$ which has Ω as a cyclic vector and is stable under the action of the modular group of $(\mathcal{R}, \Omega)$ is $\mathcal{R}$ itself. □

Proposition 2.2 and Corollary 2.3, where $\mathcal{J}$ is replaced by $\mathcal{K}$ and $\mathcal{T}$ by $\mathcal{U}$, also hold in the setting of the CMG, and it is still true that $\mathcal{R}(W)$ is nonabelian for each $W \in \mathcal{W}$. Moreover, an analogue of Proposition 3.1 obtains. We omit the straightforward proofs of these statements. For the set of wedgelike regions $\mathcal{W}$ in $\mathbb{R}^4$, which we consider here, the elements $\upsilon_W(t)$ of the transformation group $\mathcal{U}$ satisfy the conditions (A) and (B) in Section 4.1. We can thus apply Theorem 4.1.15 to conclude the following result.

Lemma 5.2.2. *Let the CMG hold as described. If $\{\Delta_{W_0}^{it}\}_{t\in\mathbb{R}}$ is the modular group corresponding to an arbitrary wedge algebra $\mathcal{R}(W_0)$ and the vector Ω, then for each $t \in \mathbb{R}$ there exists an element $L_{W_0}(t)$ of the extended (by the dilatations $\mathbb{R}_+$) Poincaré group $\mathcal{DP}$ such that*

$$ad\Delta_{W_0}^{it}(\mathcal{R}(W)) = \mathcal{R}(L_{W_0}(t)W) \quad , \quad \textit{for all} \quad W \in \mathcal{W} \quad .$$

Because of the group law $\Delta_{W_0}^{is}\Delta_{W_0}^{it} = \Delta_{W_0}^{i(s+t)}$ and the standing assumption that the relation between wedges and wedge algebras is a bijection, one has

$$L_{W_0}(s)L_{W_0}(t) = L_{W_0}(s+t), \quad s,t \in \mathbb{R} \quad , \tag{5.2.1}$$

for the corresponding transformations. In particular, $L_{W_0}(t) = L_{W_0}(t/2)^2$, so each $L_{W_0}(t)$ lies in the identity component $\mathcal{DP}_+^\uparrow$ of the extended Poincaré group. We denote by $\mathcal{G}$ the subgroup of $\mathcal{DP}_+^\uparrow$ generated by the set $\{L_W(t) \mid t \in \mathbb{R}, W \in \mathcal{W}\}$. In the next step of our analysis we shall determine this group.

In order to abbreviate the argument, we shall make the additional simplifying assumption that $\mathcal{G}$ acts transitively on the set $\mathcal{W}$ of wedges (which follows from the assumption that the adjoint action of $\mathcal{K}$ upon $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ is transitive). However, this additional assumption can, in fact, be *derived* from the CMG as it stands; we shall present the proof in a subsequent publication.

Lemma 5.2.3. *If the CMG holds and $\mathcal{K}$ acts transitively upon $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$, then the group $\mathcal{G}$ of transformations coincides with the proper orthochronous Poincaré group: $\mathcal{G} = \mathcal{P}_+^\uparrow$.*

Proof. Note, to begin, that the elements of the commutator subgroup of $\mathcal{G}$ do not contain any nontrivial dilatations and therefore are contained in $\mathcal{P}_+^\uparrow$. Moreover, they act transitively on $\mathcal{W}$, as can be seen as follows: Let W_1 be any wedge and let $L_W(t) \in \mathcal{G}$ be any transformation associated with some wedge W. As $\mathcal{G}$ is assumed to act transitively on $\mathcal{W}$, there exists, according to part (1) of Lemma 5.2.1, a transformation $L \in \mathcal{G}$ such that $LL_{W_1}(t)L^{-1} = L_W(t)$. On the other hand, according to part (3) of that lemma, one has the relation $L_{W_1}(s)W_1 = W_1$, for all $s \in \mathrm{I\!R}$, and consequently

$$LL_{W_1}(t)L^{-1}L_{W_1}(t)^{-1}\, W_1 = L_W(t)\, W_1 \quad .$$

Since the wedge W_1 and the transformation $L_W(t)$ were arbitrary, the transitive action of the commutator subgroup follows. The first part of Prop. 4.2.9 then implies that this subgroup of $\mathcal{G}$ coincides with $\mathcal{P}_+^\uparrow$.

Now let W be any given wedge, let $L_W(t) = (\gamma_W(t), \Lambda_W(t), a_W(t)) \in \mathcal{G}$ be the corresponding transformation on Minkowski space, where $\gamma_W(t) > 0$ is a dilatation, $\Lambda_W(t)$ a Lorentz transformation and $a_W(t)$ a translation, and let $(1,1,a) \in \mathcal{G}, a \in \mathrm{I\!R}^4$, be any other nontrivial translation which leaves W invariant. Part (2) of Lemma 5.2.1 then implies that $L_W(t)(1,1,a)L_W(t)^{-1} = (1,1,a)$, for all $t \in \mathrm{I\!R}$. On the other hand, one obtains by explicit computation $L_W(t)(1,1,a)L_W(t)^{-1} = (1,1,\gamma_W(t)\Lambda_W(t)a)$. Hence a is an eigenvector of $\Lambda_W(t)$ and thus would have to be lightlike if $\gamma_W(t) \neq 1$, in conflict with its choice. Therefore, one has $\gamma_W(t) = 1$ and $\mathcal{G} = \mathcal{P}_+^\uparrow$, as claimed. □

In the next step we want to determine the geometric action of the transformations $L_W(t)$ associated with the modular groups. The preceding results suffice to show that these transformations are Lorentz boosts. More detailed information will be obtained by making use of the continuity and analyticity properties of the modular groups.

Proposition 5.2.4. *Given the CMG and the transitive action of $\mathcal{K}$ upon $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$, the transformations $L_{W_R}(t) \in \mathcal{G}$ associated with the standard wedge W_R are, for all $t \in \mathrm{I\!R}$, the boosts*

$$L_{W_R}(t) = \begin{pmatrix} B(t) & 0 \\ 0 & 1 \end{pmatrix} \quad \textit{with} \quad B(t) = \begin{pmatrix} \cosh \alpha t & \sinh \alpha t \\ \sinh \alpha t & \cosh \alpha t \end{pmatrix} \tag{5.2.2}$$

and $\alpha \in \{\pm 2\pi\}$. The form of $L_W(t)$ for arbitrary wedges W is obtained from $L_{W_R}(t)$ by Poincaré transformations – see the first part of Lemma 5.2.1.

Proof. According to the second part of Lemma 5.2.1 and Lemma 5.2.3, $L_{W_R}(t)$ commutes with all elements of the stability group of W_R in $\mathcal{P}_+^\uparrow$. It thus must be a boost which leaves W_R invariant and consequently has the block form given in (5.2.2). Moreover, because of relation (5.2.1), the matrix $B(t)$ has the form given in (5.2.2), where the argument αt of the hyperbolic functions could, however, be *a priori* any additive function (homomorphism) $\beta(t)$ on the reals. For the proof that $\beta(t)$ has the asserted form, it suffices to show that $\beta(t)$ is continuous - one then may apply standard results about continuous one-parameter subgroups of $GL(n,\mathbb{C})$ (see, *e.g.* Theorem 2.6 and Corollary 1.5 in [38]).

To this end one exploits the continuity properties of the group $\{\Delta_{W_R}^{it} \mid t \in \mathbb{R}\}$. According to the information about the action of $L_{W_R}(t)$ accumulated up to this point, if ℓ is any positive lightlike vector such that $W_R + \ell \subset W_R$, one has

$$\Delta_{W_R}^{it}\mathcal{R}(W_R + \ell)\Delta_{W_R}^{-it} = \mathcal{R}(W_R + e^{\beta(t)}\ell).$$

If $\beta(t)$ is discontinuous at $t = 0$, one may assume without restriction (since $\beta(\cdot)$ is additive) that there exists a $\beta_0 > 0$ and a sequence $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ such that $t_n \to 0$ and $\beta_{t_n} \geq \beta_0 > 0$. By isotony and the preceding equality of algebras, one thus obtains $\Delta_{W_R}^{it_n}\mathcal{R}(W_R + \ell)\Delta_{W_R}^{-it_n} \subset \mathcal{R}(W_R + e^{\beta_0}\ell)$. As $\Delta_{W_R}^{it}$ is continuous in the strong operator topology and $\mathcal{R}(W_R + e^{\beta_0}\ell)$ is weakly closed, one can proceed on the left-hand side of this inclusion to the limit, yielding $\mathcal{R}(W_R + \ell) \subset \mathcal{R}(W_R + e^{\beta_0}\ell)$. Since also $\mathcal{R}(W_R + e^{\beta_0}\ell) \subset \mathcal{R}(W_R + \ell)$, by isotony, one concludes that these two algebras are equal, in conflict with the CMG. So $\beta(\cdot)$ is continuous at 0, and since it is a homomorphism it must be continuous everywhere. This shows that for some constant α, $\beta(t) = \alpha t$, for all $t \in \mathbb{R}$.

In order to determine the value of this constant α, one can rely on results of Wiesbrock [71][72], cf. also [14]. If ℓ is a lightlike vector as above, the specific form of the action of $L_{W_R}(t)$, $t \in \mathbb{R}$, on $\mathcal{R}(W_R + \ell)$ implies that $\big(\mathcal{R}(W_R + \ell) \subset \mathcal{R}(W_R), \Omega\big)$ is a $\pm$-half-sided modular inclusion (where the $\pm$ depends on the sign of α). The claim $\alpha \in \{\pm 2\pi\}$ then follows from the results in the quoted references. □

We have therefore derived modular covariance from our *prima facie* less restrictive Condition of Geometric Action for the modular groups. We next show that we have a strongly continuous unitary representation of $\mathcal{P}_+^\uparrow$ satisfying the spectrum condition with either negative or positive energy.

Theorem 5.2.5. *Assume that the CMG is satisfied and that the adjoint action of $\mathcal{K}$ upon $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ is transitive. Then there is a strongly continuous unitary representation $U(\cdot)$ of the covering group $ISL(2,\mathbb{C})$ of $\mathcal{P}_+^\uparrow$ which generates $\mathcal{K}$ and acts geometrically correctly on the net. If, in addition, the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ satisfies locality, i.e. $\mathcal{R}(W) \subset \mathcal{R}(W')'$ for all $W \in \mathcal{W}$, then $U(\cdot)$ yields a strongly continuous unitary representation $U(\cdot)$ of $\mathcal{P}_+^\uparrow$ satisfying either the positive or negative spectrum condition, depending on the sign of α in Prop. 5.2.4.*

Proof. This may be proven analogously to the arguments of Section 4.3, but since Prop. 5.2.4 has already established that modular covariance holds, it suffices here simply to appeal to the results of [22][36] - particularly Lemma 2.6 and Corollary 1.8 in [22] and Prop. 2.8 in [36]. In fact, the mentioned results of [22] imply that $\mathcal{K}$ provides a strongly continuous unitary representation of the covering group $ISL(2,\mathbb{C})$.[9] For then the pair $(\mathcal{K},\xi)$, where $\xi : \mathcal{K} \mapsto \mathcal{U} \simeq \mathcal{P}_+^\uparrow$ is the canonical homomorphism, is what those authors call a central weak Lie extension of the group $\mathcal{P}_+^\uparrow$. With the additional assumption of locality, the results of [36] imply that the projective representation obtained above is actually a strongly continuous representation of $\mathcal{P}_+^\uparrow$. The sign of α in Prop. 5.2.4 determines whether the inclusions $(\mathcal{R}(W_R+\ell) \subset \mathcal{R}(W_R),\Omega)$, with W_R,ℓ as in the proof of Prop. 5.2.4, are all $+$-half-sided modular inclusions or $-$-half-sided modular inclusions. That, together with Poincaré covariance, then entails the spectrum condition with either positive or negative energy (see the argument of the proof of Theorem 5.1.2). □

It is of particular interest to note that the weak geometric action of the modular groups we have been studying in this section also entails the corresponding geometric action of the modular involutions, if and only if the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ is local.

Theorem 5.2.6. *If the CMG is satisfied and the group $\mathcal{K}$ generated by the modular unitaries of all wedge algebras acts transitively upon the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$, then $\mathcal{K}$ is equal to the group $\mathcal{J}^+$ consisting of all products of even numbers of modular conjugations $\{J_W \mid W\in\mathcal{W}\}$. The adjoint action of the modular conjugations in $\{J_W \mid W\in\mathcal{W}\}$ leaves the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ invariant (and so our CGMA holds) if and only if the net fulfills locality, i.e. $\mathcal{R}(W')\subset\mathcal{R}(W)'$, for all $W\in\mathcal{W}$.*

In that case, the modular conjugations $\{J_W \mid W\in\mathcal{W}\}$ have the same geometric action upon the net as was found in Prop. 4.2.10 under different hypotheses. Furthermore, the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ satisfies wedge duality and the modular conjugations yield a representation of the proper Poincaré group $\mathcal{P}_+$ which acts geometrically correctly upon the net.

(A simple and well-known example of a net which complies with the CMG but where locality and hence also the CGMA fails is the net generated by a Fermi field [10]. It satisfies a twisted form of locality, however.)

Proof. By the results of [22] appealed to in the proof of Theorem 5.2.5, $\mathcal{K}$ is isomorphic to either $\mathcal{P}_+^\uparrow$ itself or to its covering group, $ISL(2,\mathbb{C})$, and by Theorem 5.2.5 one knows that $\mathcal{K}=U(ISL(2,\mathbb{C}))$. Under the stated hypothesis, the conclusion of Corollary 2.7 in [36] still holds[10], *i.e.* one has also here the relation for the modular conjugations and groups associated with the wedges $W_k^{(0)}, k=1,2,3,$ based on the time-zero plane,

$$J_{W_R}\Delta^{it}_{W_k^{(0)}}J_{W_R}=\Delta^{-it}_{W_k^{(0)}} \quad k=2,3 \quad ,$$

[9] Note that, given our hypotheses, the assumptions of locality and additivity in [22] are not required for the cited results.

[10] In the proof of Prop. 2.6 in [36], which is appealed to in the argument for Corollary 2.7, one should replace $\mathcal{F}(W_1\cap\Lambda_2(-t)W_1)$ by $\mathcal{R}(W_R)\cap\mathcal{R}(\Lambda_2(-t)W_R)$. Since $W_R\cap\Lambda_2(-t)W_R$ is not empty, assumption (ii) in our CGMA entails that Ω is cyclic and separating for $\mathcal{R}(W_R)\cap\mathcal{R}(\Lambda_2(-t)W_R)$. The rest of the argument proceeds as before.

where $W_R = W_1^{(0)}$ is the standard wedge and J_{W_R} the corresponding modular involution. The corresponding relation for $k = 1$ is a basic result of Tomita-Takesaki theory. Furthermore, J_{W_R} commutes with those elements of $\mathcal{K}$ which act upon the net $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$ as translations in the direction of the 2- or 3-axes, since their adjoint action leaves $\mathcal{R}(W_R)$ invariant and they leave Ω fixed. The adjoint action of J_{W_R} on those elements of $\mathcal{K}$ which act upon $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$ as translations in the lightlike directions of $\ell_\pm$ fixed by W_R inverts these elements, by [71]. Let θ_1 denote the element $\mathrm{diag}(-1,-1,1,1) \in \mathcal{P}_+$. The above remarks imply the relations

$$J_{W_R} U(\mu^{-1}(\lambda)) J_{W_R} = U(\mu^{-1}(\theta_1 \lambda \theta_1)) \quad , \tag{5.2.3}$$

for any λ which is one of the translations or boosts just discussed, where μ is the canonical covering homomorphism from $ISL(2,\mathbb{C})$ onto $\mathcal{P}_+^\uparrow$. But since these boosts and these translations generate $\mathcal{P}_+^\uparrow$, it follows that (5.2.3) holds for any $\lambda \in \mathcal{P}_+^\uparrow$. Indeed, one has (5.2.3) for any wedge W, with θ_1 replaced by the corresponding involution, and it follows that $J_W \mathcal{K} J_W = \mathcal{K}$, for any $W \in \mathcal{W}$.

Since the Poincaré group acts transitively on $\mathcal{W}$, for any pair of wedges W_a, W_b there exists some Poincaré transformation $\lambda \in \mathcal{P}_+^\uparrow$ such that $\lambda W_a = W_b$. Consequently, one has $J_{W_b} = U(A(\lambda)) J_{W_a} U(A(\lambda))^{-1}$ for any $A(\lambda) \in ISL(2,\mathbb{C})$ with $\mu(A) = \lambda$, since $U(\cdot)$ acts geometrically correctly on the net and leaves Ω invariant. Hence, one has

$$J_{W_a} J_{W_b} = \big(J_{W_a} U(A(\lambda)) J_{W_a}\big) U(A(\lambda))^{-1} \in \mathcal{K} \quad ,$$

according to the preceding results, which shows that $\mathcal{J}^+ \subset \mathcal{K}$. On the other hand, it follows from relation (5.2.3) that for $\lambda \in \mathcal{P}_+^\uparrow$

$$J_{W_R} J_{\lambda^2 W_R} = U(A(\theta_1 \lambda \theta_1)^2 A(\lambda)^{-2}) \quad .$$

Hence the unitaries corresponding to the boosts in the 2- and 3-direction as well as to the lightlike translations in the direction of $\ell_{1\pm}$ are contained in $\mathcal{J}^+$. Similarly, one can reproduce these arguments with $W_R = W_1^{(0)}$ replaced by $W_2^{(0)}$ and $W_3^{(0)}$ to show that the unitaries corresponding to the boosts in the 1-direction as well as the lightlike translations in the direction of $\ell_{2\pm}$ and $\ell_{3\pm}$ are contained in $\mathcal{J}^+$. Since these unitaries together generate $U(ISL(2,\mathbb{C}))$, one concludes that $\mathcal{K} \subset \mathcal{J}^+$, and therefore the two groups are equal.

From the invariance of $\mathcal{R}(W_R)$ under the adjoint action of the unitaries implementing the stability group of W_R, it follows that also the algebra $\mathcal{R}(W_R)' = J_{W_R} \mathcal{R}(W_R) J_{W_R}$ is invariant under this action. Hence, if $\mathcal{R}(W_R)'$ is a wedge algebra, then it must be equal to $\mathcal{R}(W_R')$ – it cannot coincide with $\mathcal{R}(W_R)$, since otherwise it would be abelian. Therefore, if the adjoint action of the elements of $\{J_W \mid W \in \mathcal{W}\}$ leaves $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$ invariant, the net must satisfy wedge duality and hence locality. Conversely, if the net satisfies locality, then $\mathcal{R}(W') \subset \mathcal{R}(W)'$ is stable under the adjoint action of the modular group $\{\Delta_W^{-it} \mid t \in \mathbb{R}\}$, of $(\mathcal{R}(W)', \Omega)$ according to Prop. 5.2.4. Since Ω is cyclic and separating for both algebras, Tomita-Takesaki theory then entails the equality $\mathcal{R}(W') = \mathcal{R}(W)' = J_W \mathcal{R}(W) J_W$. But this implies that, for any $W_a, W_b \in \mathcal{W}$ with corresponding modular involutions J_{W_a}, J_{W_b}, one has

$$J_{W_a} \mathcal{R}(W_b) J_{W_a} = J_{W_a} J_{W_b} \mathcal{R}(W_b') J_{W_b} J_{W_a} \in \{\mathcal{R}(W)\}_{W \in \mathcal{W}} \quad ,$$

since $J_{W_a} J_{W_b} \in \mathcal{J}^+ = \mathcal{K}$ and $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$ is invariant under the adjoint action of $\mathcal{K}$. The remaining assertions are therefore immediate consequences of the results of Chapter IV. □

To close the circle of implications relating the geometric action of the modular involutions to that of the modular groups, we conclude this section with the following result.

Theorem 5.2.7. *Assume the CGMA, with the choices $\mathcal{M} = \mathrm{I\!R}^4$ and $\mathcal{W}$ the collection of wedgelike regions in $\mathrm{I\!R}^4$, and the transitivity of the adjoint action of $\mathcal{J}$ on the net $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$. If $\Delta_W^{it} \in \mathcal{J}$, for all $t \in \mathrm{I\!R}$ and some $W \in \mathcal{W}$, i.e. if the modular stability condition obtains, and the adjoint action of $\mathcal{K}$ upon $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$ is transitive, then modular covariance is satisfied.*

Proof. Since, by hypothesis, the adjoint action of any element of $\mathcal{J}$ leaves the set $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$ invariant and since their transitive action on $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$ implies $\Delta_W^{it} \in \mathcal{J}$, for all $t \in \mathrm{I\!R}$ and $W \in \mathcal{W}$, it is clear that the CMG is satisfied. Prop. 5.2.4 completes the proof. □

Hence, we *derive* modular covariance from our CGMA, whenever the modular stability condition also holds. We remark once again that in a later publication we shall show that the additional assumption of the transitive action of $\mathcal{K}$ is superfluous.

5.3. Modular Involutions Versus Modular Groups

As explained in the introduction, there have been two distinctly different approaches to the study of the geometric action of modular objects and its consequences. In the one, initiated in [24], geometric action of the modular involutions was assumed, whereas in the other, initiated in [12], the starting point was the geometric action of the modular groups. However, even within each of these approaches, differing forms of concrete action have been studied. In most of the papers concerned with the consequences of geometric action of the modular groups, the action was assumed in the form of modular covariance (see [22][36], among others). There are some variations of this condition in the literature [35][26], but they all have in common that from the outset one is given an action of the Lorentz group on the space-time.

Certain exceptions are the papers by Kuckert [46] and Trebels [65], where the geometric action was assumed in the guise of requiring the adjoint action of the modular groups (or the modular involutions) to leave the set of local algebras in Minkowski space invariant. However, in both approaches the starting point is a vacuum representation of a net on Minkowski space which is covariant with respect to the translation group satisfying the spectrum condition.

All of these approaches have in common that some *a priori* information about the geometric action of the modular groups or the spacetime symmetry group is required. But, as we have shown in the above analysis, this detailed information is *derived* if one starts from our CGMA. We also wish to emphasize that the condition of modular covariance and Borchers' relation (5.1.1) are not implied in our framework. To illustrate these assertions, we present a simple example of a net satisfying our CGMA and all of the other assumptions made in this paper, except the modular stability condition. This example thus violates the spectrum condition and the modular groups associated to wedge algebras do not coincide with the representation of the Lorentz boosts, *i.e.* modular covariance *fails* in this example, though it is Poincaré covariant. Subsequently, we give another example violating modular covariance but satisfying the spectrum condition and all of our assumptions. It is therefore clear that the assumption of modular covariance is more restrictive than the CGMA, even when the spectrum condition is posited.

Turning to our first example, let $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{C}}$ be the standard net of von Neumann algebras generated by a (hermitian, scalar, massive) free field on the Fock space $\mathcal{H}$. It is based on the set $\mathcal{C}$ of double cones in IR^4 and covariant under the standard action α_λ, $\lambda \in \mathcal{P}_+^\uparrow$, of the Poincaré group. Let Θ be the PCT-operator on $\mathcal{H}$ and θ be the corresponding reflection in Minkowski space. For each double cone $\mathcal{O}$ define $\mathcal{B}(\mathcal{O}) = \mathcal{A}(\theta\mathcal{O}) = \Theta\mathcal{A}(\mathcal{O})\Theta$. Let $\hat{\mathcal{A}}(\mathcal{O}) \equiv \mathcal{A}(\mathcal{O})\otimes\mathcal{B}(\mathcal{O})$ act on $\mathcal{H}\otimes\mathcal{H}$. The net $\{\hat{\mathcal{A}}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{C}}$ is clearly local, since Θ is antiunitary and thus behaves properly under the taking of algebraic commutants. We observe that $\hat{\alpha}_\lambda \equiv \alpha_\lambda \otimes \beta_\lambda$, with $\beta_\lambda \equiv \alpha_{\theta\lambda\theta}$, $\lambda \in \mathcal{P}_+^\uparrow$, defines an automorphic local action on $\{\hat{\mathcal{A}}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{C}}$, as can be seen as follows. With $\lambda \in \mathcal{P}_+^\uparrow$, one has

$$\begin{aligned}\hat{\alpha}_\lambda(\hat{\mathcal{A}}(\mathcal{O})) &= \alpha_\lambda(\mathcal{A}(\mathcal{O})) \otimes \beta_\lambda(\mathcal{B}(\mathcal{O})) = \mathcal{A}(\lambda\mathcal{O}) \otimes (\mathcal{A}((\theta\lambda\theta)\theta\mathcal{O})) \\ &= \mathcal{A}(\lambda\mathcal{O}) \otimes \mathcal{A}(\theta\lambda\mathcal{O}) = \hat{\mathcal{A}}(\lambda\mathcal{O}) \quad .\end{aligned}$$

With $U(\lambda)$ the unitary implementation of α_λ on $\mathcal{H}$, one easily checks that $V(\lambda) \equiv \Theta U(\lambda)\Theta$ implements the action of β_λ. Setting $U(x) = e^{ixP}$, where P is the generator of the translations satisfying the positive spectrum condition, one has $V(x) = \Theta e^{ixP}\Theta = e^{-ixP}$. Hence $V(\lambda)$ satisfies the negative spectrum condition, but $\hat{U}(\lambda) \equiv U(\lambda)\otimes V(\lambda)$ violates both the positive and the negative spectrum conditions.

By the results of Bisognano and Wichmann [9], applicable to the free field, one knows that for the standard wedge W_R the modular structure for the (weakly closed) wedge algebra $\mathcal{A}(W_R)$ and Ω is given by $J_{W_R} = \Theta_R = \Theta U_\pi$, where U_π implements the rotation by π about the 1-axis, and $\Delta_{W_R}^{it} = U(\lambda_R(t))$, $t \in \mathrm{IR}$, where the $\lambda_R(t)$ are the Lorentz boosts in the 1-direction. The corresponding modular objects for $(\mathcal{B}(W_R),\Omega) = (\Theta\mathcal{A}(W_R)\Theta,\Omega) = (\mathcal{A}(W_R)',\Omega)$ are given by ${}_{\mathcal{B}}J_{W_R} = \Theta_R$ and ${}_{\mathcal{B}}\Delta_{W_R}^{it} = U(\lambda_R(t))^{-1} = U(\lambda_R(-t))$. It follows that the modular objects for $(\hat{\mathcal{A}}(W_R) = \mathcal{A}(W_R)\otimes\mathcal{B}(W_R), \Omega\otimes\Omega)$ are given by

$$\hat{J}_{W_R} = \Theta_R \otimes \Theta_R \quad , \quad \hat{\Delta}_{W_R}^{it} = U(\lambda_R(t)) \otimes U(\lambda_R(-t)) \quad .$$

So, one has (with θ_R the transformation on Minkowski space corresponding to Θ_R)

$$\begin{aligned}\hat{J}_{W_R}\hat{\mathcal{A}}(\mathcal{O})\hat{J}_{W_R} &= \Theta_R\mathcal{A}(\mathcal{O})\Theta_R \otimes \Theta_R\mathcal{B}(\mathcal{O})\Theta_R = \mathcal{A}(\theta_R\mathcal{O}) \otimes \mathcal{A}(\theta_R\theta\mathcal{O}) \\ &= \mathcal{A}(\theta_R\mathcal{O}) \otimes \mathcal{A}(\theta\theta_R\mathcal{O}) = \hat{\mathcal{A}}(\theta_R\mathcal{O}) \quad ,\end{aligned}$$

and the modular conjugation $\hat{J}_{W_R}$ acts geometrically correctly on the net $\{\hat{\mathcal{A}}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{C}}$. By Poincaré covariance of the net, the same holds true for the modular involution $\hat{J}_W$, for any wedge W.

Turning to the modular groups, one sees

$$\begin{aligned}\hat{\Delta}^{it}_{W_R}\hat{\mathcal{A}}(\mathcal{O})\hat{\Delta}^{-it}_{W_R} &= U(\lambda_R(t))\mathcal{A}(\mathcal{O})U(\lambda_R(t))^{-1} \otimes U(\lambda_R(-t))\mathcal{B}(\mathcal{O})U(\lambda_R(-t))^{-1} \\ &= \mathcal{A}(\lambda_R(t)\mathcal{O}) \otimes \mathcal{A}(\lambda_R(-t)\theta\mathcal{O}) = \mathcal{A}(\lambda_R(t)\mathcal{O}) \otimes \mathcal{A}(\theta\lambda_R(-t)\mathcal{O}) \\ &= \mathcal{A}(\lambda_R(t)\mathcal{O}) \otimes \mathcal{B}(\lambda_R(-t)\mathcal{O}) \neq \hat{\mathcal{A}}(\lambda_R(t)\mathcal{O}) \quad .\end{aligned}$$

Hence, $\hat{\Delta}^{it}_{W_R}$ does not satisfy modular covariance. Note also that the modular groups are *not* contained in $\mathcal{J} = \hat{U}(\mathcal{P}_+)$, so that the modular stability condition is violated, in accord with Theorem 5.2.7.

We mention as an aside that in [36] Guido and Longo propose the split property, which yields the uniqueness of the representation of the Poincaré group, as a natural candidate for the hypothesis needed in order to conclude that the modular group of a wedge algebra satisfies modular covariance. However, in the preceding example, the split property holds, though modular covariance does not.

In our next example, we see that it is possible for all of our assumptions to hold, as well as the positive spectrum condition, but for modular covariance to be violated. For each $W \in \mathcal{W}$, the set of wedgelike regions in four-dimensional Minkowski space, we denote by $N(W)$ the unique wedge in the coherent family of wedges determined by W which contains the origin in its edge. Once again taking $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{C}}$ to be the usual net for the free field on four-dimensional Minkowski space, we consider the net $\{\mathcal{A}(W)\}_{W\in\mathcal{W}}$ indexed by the wedgelike regions and define for this example $\hat{\mathcal{A}}(W) \equiv \mathcal{A}(W) \otimes \mathcal{A}(-N(W))$. (Note that $-N(W) = N(W)'$.) This net is local, since $W_1 \subset W_2'$ entails $N(W_1) \subset N(W_2)'$ and since $\mathcal{A}(W)' = \mathcal{A}(W')$ (Haag duality). Moreover, for each $\mathcal{P}_+^\uparrow \ni \lambda = (\Lambda, a)$, we set $\hat{\alpha}_\lambda \equiv \alpha_\lambda \otimes \alpha_{(\Lambda,0)}$. Hence, the translation subgroup acts trivially upon the second factor of each local algebra. In this example, the unitary implementers of the action $\hat{\alpha}_\lambda$ are given by $\hat{U}(\Lambda, a) = U(\Lambda, a) \otimes U(\Lambda, 0)$, and the translation subgroup is implemented by $\hat{U}(a) = U(a) \otimes \mathbb{1}$. Thus, the positive spectrum condition holds (though the vacuum is infinitely degenerate), whereas modular covariance is violated. In fact, $\hat{\Delta}^{it}_{W_R} = U(\lambda_R(t)) \otimes U(\lambda_R(-t))$, since in the second factor of $\hat{\mathcal{A}}(W)$ there appears the algebra $\mathcal{A}(-N(W)) = \mathcal{A}(N(W)') = \mathcal{A}(N(W))'$. On the other hand, the modular conjugations corresponding to $(\hat{\mathcal{A}}(W), \Omega \otimes \Omega)$ are given by $J_W \otimes J_{N(W)}$ and hence satisfy the CGMA and the assumption of transitive action on $\mathcal{W}$.

It is of interest to note that this example also violates the condition of modular stability, $\hat{\Delta}_W^{it} \in \mathcal{J}$, in spite of the validity of the spectrum condition. Furthermore, the local algebras associated with double cones $\mathcal{O}$,

$$\hat{\mathcal{A}}(\mathcal{O}) \equiv \bigcap_{\mathcal{O}\subset W \in \mathcal{W}} \hat{\mathcal{A}}(W)$$

do not generate the wedge algebras. This resembles the situation which one expects to meet for the bosonic part of the field algebra in theories with topological or gauge charges.

We sketch a final illustrative example, which makes a number of points about the interrelationship of the CGMA, uniqueness of representation of the Poincaré group, and some further properties of interest. Consider an infinite component free hermitian Bose field with momentum space annihilation and creation operators satisfying the following canonical commutation relations [47]:

$$[a(\vec{p}\,', q'), a^*(\vec{p}, q)] = 2\omega_{\vec{p}}\, \delta^{(3)}(\vec{p} - \vec{p}\,')\delta^{(4)}(q - q') \quad ,$$

where $\vec{p}, \vec{p}\,' \in \mathrm{IR}^3$, $\omega_{\vec{p}} = \sqrt{\vec{p}^{\,2} + m^2}$, $m > 0$, and the variables $q, q' \in \mathrm{IR}^4$ label the internal degrees of freedom. One unitary representation of the Poincaré group on the corresponding Fock space of this field is determined by

$$U(\Lambda, x) a(\vec{p}, q) U(\Lambda, x)^{-1} \equiv e^{i\Lambda p\cdot x} a(\vec{\Lambda p}, q) \quad ,$$

where $p = (\omega_{\vec{p}}, \vec{p})$, while a second one is determined by

$$\tilde{U}(\Lambda, x) a(\vec{p}, q) \tilde{U}(\Lambda, x)^{-1} \equiv e^{i\Lambda p\cdot x} a(\vec{\Lambda p}, \Lambda q) \quad .$$

It is evident that both representations satisfy the spectrum condition.

Let $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{C}}$ be the net of von Neumann algebras generated by this free field. Clearly, this net transforms covariantly under both $U(\mathcal{P}_+^\uparrow)$ and $\tilde{U}(\mathcal{P}_+^\uparrow)$. The work of Bisognano and Wichmann [9] shows that, using the representation $U(\mathcal{P}_+^\uparrow)$, the net satisfies the special condition of duality, and hence it satisfies Haag duality for the wedge algebras, the CGMA, modular covariance and the modular stability condition, $\mathcal{K} \subset \mathcal{J}$. The arguments of Bisognano and Wichmann break down for the representation $\tilde{U}(\mathcal{P}_+^\uparrow)$, because the extra action on the dummy variable would destroy the analytic continuation crucial to their arguments.

Applying the CGMA to the net $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{C}}$ in the Fock vacuum state would result in the construction of the representation $U(\mathcal{P}_+^\uparrow)$ and not the representation $\tilde{U}(\mathcal{P}_+^\uparrow)$. Note further that since both representations act geometrically correctly upon the net, we have

$$\tilde{U}(\lambda) = \tilde{Z}(\lambda) U(\lambda) \quad ,$$

for all $\lambda \in \mathcal{P}_+^\uparrow$, with coefficients $\tilde{Z}$ which induce internal symmetries of the net and commute with $U(\mathcal{P}_+^\uparrow)$. But they are not contained in $\mathcal{J}$ and, for this reason, this example escapes the uniqueness statement in Theorem 4.3.9. On the other hand, the net $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{C}}$ violates the distal split property and, for this reason, the example also escapes the uniqueness theorem of [22].

As we have shown, the CMG implies both modular covariance and the CGMA for the involutions (the latter in the presence of locality). The results in Section 5.2 therefore generalize the results of both [22] and [46]. We have also seen that there exist Poincaré covariant nets of local algebras on Minkowski space which do not satisfy the condition of modular covariance but which satisfy all of our assumptions, with or without the additional condition of positive spectrum.

Though the CGMA (in application to the special case of Minkowski space) is weaker than the condition of modular covariance, it nonetheless allows one to systematically establish the same results which were proven under the assumption of modular covariance in the literature. Moreover, since the modular involutions depend only upon the characteristic cones of the pairs $(\mathcal{A}(W), \Omega)$, it would seem that they are more likely to encode some intrinsic information about the representation, as opposed to the modular unitaries, which are strongly state-dependent.

VI. Geometric Modular Action and De Sitter Space

As a further example of application of the program outlined in Chapter III, we consider three-dimensional de Sitter space. The restriction on the dimension is made for simplicity, as it will allow us to apply some of the results obtained in the preceding analysis.

It is well-known that three-dimensional de Sitter space dS^3 can conveniently be embedded into the ambient four-dimensional Minkowski space $\mathrm{I\!R}^4$. Choosing proper coordinates, it is described by

$$dS^3 \equiv \{x \in \mathrm{I\!R}^4 \mid x_0^2 - x_1^2 - x_2^2 - x_3^2 = -1\} \quad ,$$

with the induced metric and causal structure from Minkowski space. Accordingly, the restriction of the Lorentz group $\mathcal{L}$ in the ambient space $\mathrm{I\!R}^4$ to dS^3 is the isometry group of this space, simply called here the de Sitter group and commonly denoted by $O(1,3)$. As the elements of $\mathcal{L}$ are uniquely fixed by their action on dS^3, we will identify the de Sitter group with $\mathcal{L}$ for later convenience. Similarly, the proper de Sitter group and its identity component are identified with $\mathcal{L}_+$ and $\mathcal{L}_+^\uparrow$, respectively.

In this chapter we shall assume the CGMA for a net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ on $\mathcal{M} = dS^3$. Applying the reasoning advanced in Chapter III, one is presented once again with a unique minimal admissible family, namely $\mathcal{W} \equiv \{\tilde{W} \cap dS^3 \mid \tilde{W} \in \tilde{\mathcal{W}}_0\}$, where $\tilde{\mathcal{W}}_0$ is the family of wedgelike regions in the ambient four-dimensional Minkowski space $\mathrm{I\!R}^4$ containing the origin in their edges. Hence, we shall proceed with this choice of index set. Though there are clearly affinities between this setting and the Minkowski-space situation, there are nevertheless some nontrivial points to be worked out which do not automatically follow from the work in the previous chapters.

To begin, we shall prove in Section 6.1 a general Alexandrov-like result in dS^3 along the lines of Theorem 4.1.15. In view of the different geometric structure of de Sitter space, the construction of the induced point transformations in dS^3 differs from the corresponding construction in Minkowski space. (An alternative construction made under stronger assumptions may be found in [28].) In Section 6.2 it will be shown that the CGMA, with an additional technical postulate, implies that the bijections on $\mathcal{W}$ induced by the adjoint action of the modular involutions $\{J_W \mid W \in \mathcal{W}\}$ upon the net $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$ are obtained by elements of the de Sitter group. Then, under the assumption that the group generated by $\{\mathrm{ad} J_W \mid W \in \mathcal{W}\}$ acts transitively upon the set $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$, it will be shown that the group thereby generated is $\mathcal{L}_+$, whenever one of the algebras $\mathcal{R}(W)$ is nonabelian. In contradistinction to the Minkowski space situation, all elements of the index set $\mathcal{W}$ are atoms; hence it is entirely possible for the wedge algebras to be abelian here. If they are abelian, then the group induced upon dS^3 is equal to the identity component $\mathcal{L}_+^\uparrow$ of the de Sitter group.

After this analysis, we shall proceed analogously to the development in Chapter IV to obtain a strongly continuous unitary representation of $\mathcal{L}_+$, resp. $\mathcal{L}_+^\uparrow$, which acts geometrically correctly upon the net $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$.

6.1. Wedge Transformations in de Sitter Space

In this section, we shall work with bijections $\tau : \mathcal{W} \mapsto \mathcal{W}$ satisfying the condition

$$W_1 \cap W_2 = W_3 \cap W_4 \quad \Leftrightarrow \quad \tau(W_1) \cap \tau(W_2) = \tau(W_3) \cap \tau(W_4) \quad , \tag{6.1.1}$$

for arbitrary pairs W_1, W_2 and W_3, W_4 in $\mathcal{W}$. In the next section, we shall provide assumptions on the net $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$ which entail condition (6.1.1).

We shall use constantly without further mention the elementary fact that $W \in \mathcal{W}$ determines uniquely a wedgelike region $\tilde{W}$ in $\mathbb{R}^4$ such that $W = \tilde{W} \cap dS^3$ and *vice versa*. Hence we shall, where convenient for us, identify W with $\tilde{W}$. It will be clear from the context whether W is regarded as a subset of dS^3 or of the ambient space $\mathbb{R}^4$. Adopting the notation of Chapter IV, we shall write $W[\ell_1, \ell_2] \equiv \tilde{W}[\ell_1, \ell_2, 0] \cap dS^3$, where $\tilde{W}[\ell_1, \ell_2, 0] \in \tilde{\mathcal{W}}_0$ is the wedge in the ambient space fixed by the two positive lightlike vectors ℓ_1, ℓ_2 and the translation 0. For the analysis of condition (6.1.1) we must make some elementary geometric points about pairs of wedges.

Definition. *Let $W[\ell_1, \ell_2], W[\ell_3, \ell_4] \in \mathcal{W}$ be wedges. If the positive lightlike vectors ℓ_1, ℓ_4, respectively ℓ_3, ℓ_2, are not parallel, then the pair of wedges $(W[\ell_1, \ell_4], W[\ell_3, \ell_2])$ will be called the pair of wedges dual to $(W[\ell_1, \ell_2], W[\ell_3, \ell_4])$ (or simply the dual pair).*

If (W_3, W_4) is the pair dual to (W_1, W_2), then $W_1 \cap W_2 = W_3 \cap W_4$. If this intersection is nonempty, then (W_1, W_2) and (W_3, W_4) are the only pairs in $\mathcal{W}$ with this intersection. Hence, $\emptyset \neq W_1 \cap W_2 = W_3 \cap W_4$ implies that the (unordered) pairs (W_1, W_2) and (W_3, W_4) are either the same or dual (for details, see [28]).

We immediately have the following counterpart to Lemma 4.1.7.

Lemma 6.1.1. *Let* $\ell_{1\pm} = (1, \pm 1, 0, 0)$*,* $\ell_{2\pm} = (1, 0, \pm 1, 0)$ *and* $\ell = (1, a, b, c)$ *with* $a, b, c \in \mathrm{IR}$*,* $a^2 + b^2 + c^2 = 1$*,* $b \neq 1$*. The wedges* $W_1 = W[\ell_{1+}, \ell_{1-}]$ *and* $W_2 = W[\ell_{2+}, \ell]$ *have empty intersection if and only if* $0 < a \leq 1$*,* $0 \leq b < 1$ *and* $c = 0$*. The statement is still true if* W_1 *is replaced by* W_1' *and the condition* $0 < a \leq 1$ *is replaced by* $-1 \leq a < 0$*, or also if* ℓ_{2+} *is replaced by* ℓ_{2-} *and* $0 \leq b < 1$ *by* $-1 < b \leq 0$*.*

This result will be used in the proof of the next lemma.

Lemma 6.1.2. *Let* $\tau : \mathcal{W} \mapsto \mathcal{W}$ *be a bijection satisfying (6.1.1) and let* ℓ_0 *be a fixed future-directed lightlike vector. Then* τ *maps collections of wedges* $\{W[\ell_0, \ell] \mid \ell \,\textit{lightlike}, \ell \cdot \ell_0 > 0\}$ *and* $\{W[\ell, \ell_0] \mid \ell \,\textit{lightlike}, \ell \cdot \ell_0 > 0\}$ *onto sets of the same form.*[11] *Furthermore,*

$$W_1 \cap W_2 = \emptyset \quad \Leftrightarrow \quad \tau(W_1) \cap \tau(W_2) = \emptyset \quad , \tag{6.1.2}$$

for any $W_1, W_2 \in \mathcal{W}$*, and*

$$\tau(W') = \tau(W)' \quad , \quad \textit{for any} \quad W \in \mathcal{W} \quad . \tag{6.1.3}$$

Therefore, if $W_1 \cap W_2 \neq \emptyset$ *and the pair* (W_3, W_4) *is dual to* (W_1, W_2)*, then* $(\tau(W_1), \tau(W_2))$ *is dual to* $(\tau(W_3), \tau(W_4))$*.*

Henceforth, we shall abbreviate $\{W[\ell_0, \ell] \mid \ell \text{ lightlike}, \ \ell \cdot \ell_0 > 0\}$ by $\{W[\ell_0, \ell] \mid \ell\}$, *etc.*

Proof. Let $W_1 \cap W_2 = \emptyset$, with $W_1, W_2 \in \mathcal{W}$. There clearly exist infinitely many distinct pairs of disjoint wedges in $\mathcal{W}$. Let (W_1, W_2), (W_3, W_4) and (W_5, W_6) be any three of them. Then (6.1.1) implies

$$\tau(W_1) \cap \tau(W_2) = \tau(W_3) \cap \tau(W_4) = \tau(W_5) \cap \tau(W_6) \quad .$$

If this intersection is nonempty, $(\tau(W_1),\tau(W_2))$, $(\tau(W_3),\tau(W_4))$ and $(\tau(W_5),\tau(W_6))$ are distinct (since τ is a bijection), mutually dual pairs of wedges, which is impossible. Hence, the assertion (6.1.2) is proven. The final assertion of the lemma follows at once.

Let $\ell_1, \ell_2, \ell_3, \ell_4$ be given. There exist corresponding lightlike vectors $\ell_1', \ell_2', \ell_3', \ell_4'$ such that

$$\tau(W[\ell_1, \ell_2]) = W[\ell_1', \ell_2'] \quad \text{and} \quad \tau(W[\ell_3, \ell_4]) = W[\ell_3', \ell_4'] \quad .$$

Now $(W[\ell_1, \ell_2], W[\ell_3, \ell_4])$ equals its dual pair if and only if ℓ_1 is parallel to ℓ_3 or ℓ_2 is parallel to ℓ_4, and since self-dual pairs of nondisjoint wedges are mapped by τ to self-dual pairs of nondisjoint wedges, this is equivalent to ℓ_1' is parallel to ℓ_3' or ℓ_2' is parallel to ℓ_4', respectively. Thus, all pairs of images of the wedges $W[\ell_0, \ell]$, ℓ_0 fixed but ℓ arbitrary, are self-dual. Therefore, one has

$$\{\tau(W[\ell_0, \ell]) \mid \ell\} \subset \{W[\ell_0', \ell] \mid \ell\}$$

[11] Note that these collections of wedges are such that any two elements form a self-dual pair of wedges with nonempty intersection.

or

$$\{\tau(W[\ell_0,\ell]) \mid \ell\} \subset \{W[\ell,\ell'_0] \mid \ell\}$$

for a suitable ℓ'_0. Since the same statement holds for τ^{-1}, the equality of these sets follows.

It remains to prove (6.1.3). To this end assume $W[\ell_3,\ell_4] = W[\ell_1,\ell_2]'$, *i.e.* ℓ_3 is parallel to ℓ_2 and ℓ_4 is parallel to ℓ_1. The collection $\{\tau(W[\ell_1,\ell]) \mid \ell\}$, which contains the wedge $W[\ell'_1,\ell'_2]$, coincides with either $\{W[\ell'_1,\ell] \mid \ell\}$ or $\{W[\ell,\ell'_2] \mid \ell\}$. But by relation (6.1.1) and a straightforward application of Lemma 6.1.1, each element of $\{\tau(W[\ell_1,\ell]) \mid \ell\}$ is disjoint from $\tau(W[\ell_1,\ell_2]') = W[\ell'_3,\ell'_4]$. So ℓ'_1 is a positive multiple of ℓ'_4 in the first case (otherwise, one would have $W[\ell'_1,\ell'_4] \in \{\tau(W[\ell_1,\ell]) \mid \ell\}$ and $W[\ell'_1,\ell'_4] \cap W[\ell'_3,\ell'_4] = \emptyset$, in contradiction to Lemma 6.1.1); in the second case one concludes that ℓ'_2 is a positive multiple of ℓ'_3. On the other hand, by considering the collection $\{\tau(W[\ell,\ell_2]) \mid \ell\}$ instead, one can see that ℓ'_2 is a positive multiple of ℓ'_3, resp. that ℓ'_4 is a positive multiple of ℓ'_1. □

We next show that τ induces a map on the set of characteristic planes in the ambient space $\mathbb{R}^4$, as in Section 4.1. We use notation established there and recall that we identify $\mathcal{W}$ with $\tilde{\mathcal{W}}_0$.

Corollary 6.1.3. *Let $\tau : \mathcal{W} \mapsto \mathcal{W}$ be a bijection satisfying (6.1.1). Then τ induces a bijection of characteristic planes, which we shall also denote by τ, such that $\tau(H_0[\ell_1])$ and $\tau(H_0[\ell_2])$ are the characteristic planes determined by $\tau(W[\ell_1,\ell_2])$ (with $H_0[\ell_1] \neq H_0[\ell_2]$).*

Proof. According to Lemma 6.1.2, one has for fixed ℓ_0 either

$$\{\tau(W[\ell_0,\ell]) \mid \ell\} = \{W[\ell'_0,\ell] \mid \ell\} \quad \text{or} \quad \{\tau(W[\ell_0,\ell]) \mid \ell\} = \{W[\ell,\ell'_0] \mid \ell\} \quad ,$$

for a suitable ℓ'_0. Set $\tau(H_0[\ell_0]) = H_0[\ell'_0]$; the claim then follows easily. □

By considering disjoint pairs of wedges instead of maximal pairs of wedges, one can follow the argument of Lemma 4.1.11 to prove the following.

Lemma 6.1.4. *Let $\tau : \mathcal{W} \mapsto \mathcal{W}$ be a bijection satisfying (6.1.1). If $\ell_1,\ell_2,\ell_3,\ell_4$ are linearly dependent future-directed lightlike vectors such that any two of them are linearly independent, then*

$$\bigcap_{i=1}^{4} \tau(H_0[\ell_i]) = \bigcap_{i\neq k} \tau(H_0[\ell_i]) \quad \textit{for} \quad k = 1,2,3,4 \quad .$$

This leads to an induced map on spacelike lines through the origin.

Lemma 6.1.5. *Let $\tau : \mathcal{W} \mapsto \mathcal{W}$ be a bijection satisfying (6.1.1), and let $x \in \mathbb{R}^4$ be spacelike. Then the intersection*

$$\bigcap_{\{\ell \mid x \in H_0[\ell]\}} \tau(H_0[\ell])$$

is one-dimensional and spacelike. Hence, τ induces a bijection

$$\mathbb{R}x \mapsto \bigcap_{\{\ell \mid x \in H_0[\ell]\}} \tau(H_0[\ell])$$

on the set of spacelike one-dimensional subspaces of $\mathrm{I\!R}^4$. *This map will again be denoted by* τ.

Proof. Let $\ell_1, \ell_2, \ell_3, \ell_4$ be pairwise linearly independent lightlike vectors such that $x \in H_0[\ell_i]$, for $i = 1, 2, 3, 4$. Then this quadruple of vectors is linearly dependent and consequently, by Lemma 6.1.4, one has

$$\bigcap_{\{\ell | x \in H_0[\ell]\}} \tau(H_0[\ell]) = \bigcap_{i=1}^{3} \tau(H_0[\ell_i]) \quad .$$

□

We shall need the following geometric result about wedges.

Lemma 6.1.6. *Let* $x \in \mathrm{I\!R}^4$ *be spacelike and let* $\ell_k = (1, a_k, b_k, c_k)$, *where* $a_k, b_k, c_k \in \mathrm{I\!R}$ *satisfy* $a_k^2 + b_k^2 + c_k^2 = 1$, $k = 1, 2$. *Set* $W_0 = W[\ell_1, \ell_2]$. *Then* $\mathrm{I\!R}x \cap W_0 \neq \emptyset$ *if and only if* $W_0 \cap W \neq \emptyset$, *for all* $W \in \mathcal{W}$ *whose edge contains* $\mathrm{I\!R}x$. *For* $x = (0, 0, 1, 0)$, *this is also equivalent to the statement that* $b_1 b_2 < 0$. *Moreover,* $(0, 0, 1, 0) \in W_0$ *implies* $b_1 > 0$ *and* $-(0, 0, 1, 0) \in W_0$ *implies* $b_1 < 0$ *(when* $b_1 b_2 < 0$*).*

Proof. One may assume without loss of generality that $x = (0, 0, 1, 0)$. Since W_0 is open, it is trivial that $\mathrm{I\!R}x \cap W_0 \neq \emptyset$ implies $W_0 \cap W \neq \emptyset$, for all $W \in \mathcal{W}$ whose edge contains x.

For the converse, it will first be shown that $W_0 \cap W \neq \emptyset$, for all $W \in \mathcal{W}$ whose edge contains x, implies $b_1 b_2 < 0$. The case $b_1 = b_2 = 0$ is excluded, since it would imply that W_0 is invariant under the translations $\mathrm{I\!R}x$ and consequently also W_0' would be so invariant. Hence, it would follow that $W_0 \cap W_0' \neq \emptyset$, which is a contradiction. By considering W_0' instead of W_0 if $b_1 = 0$, one may assume that $b_1 \neq 0$. By applying suitable Lorentz transformations leaving $(0, 0, 1, 0)$ invariant, one may further assume that $a_1 = c_1 = 0$ and, after applying a suitable rotation, $a_2 > 0$ and $c_2 = 0$. Lemma 6.1.1 entails that if $b_1 > 0$, $a_2 > 0$ and $b_2 \geq 0$, or $b_1 < 0$, $a_2 > 0$ and $b_2 \leq 0$, then one has $W_0 \cap W[\ell_{1+}, \ell_{1-}] = \emptyset$, where $\ell_{1\pm}$ are as in the lemma. Since the wedge $W[\ell_{1+}, \ell_{1-}]$ contains the line $\mathrm{I\!R}x$ in its edge, this is a contradiction. Hence, there holds $b_1 b_2 < 0$.

Proceeding further, it may still be assumed that $a_1 = c_1 = c_2 = 0$. The remaining assertion of the lemma follows for $b_1 > 0$, $b_2 < 0$ (and similarly for $b_1 < 0$, $b_2 > 0$), if one notices that the vector

$$\begin{aligned}&(0, 0, (1 - b_2)^2, 0) \\ &\quad = -b_2(1 - b_2)(1, 0, 1, 0) - (1 - b_2)(1, a_2, b_2, 0) + a_2(a_2, 1 - b_2, a_2, 0)\end{aligned}$$

is an element of W_0. □

This enables us to prove this final preparatory lemma.

Lemma 6.1.7. *Let* $\tau : \mathcal{W} \mapsto \mathcal{W}$ *be a bijection satisfying (6.1.1), and let* $x \in \mathrm{I\!R}^4$ *be spacelike. If* $x \in W_1 \cap W_2$ *for* $W_1, W_2 \in \mathcal{W}$, *then*

$$\emptyset \neq \tau(\mathrm{I\!R}x) \cap \tau(W_1) = \tau(\mathrm{I\!R}x) \cap \tau(W_2) \quad .$$

Proof. It has been seen that τ maps the set of wedges in $\mathcal{W}$ whose edges contain the line $\mathbb{R}x$ onto the set of wedges in $\mathcal{W}$ whose edges contain the line $\tau(\mathbb{R}x)$. Lemmas 6.1.5 and 6.1.6 entail that both $\tau(\mathbb{R}x) \cap \tau(W_1)$ and $\tau(\mathbb{R}x) \cap \tau(W_2)$ are nonempty. There exist lightlike vectors $\ell_1, \ell_2, \ell_3, \ell_4$ such that $\tau(W_1) = W[\ell_1, \ell_2]$ and $\tau(W_2) = W[\ell_3, \ell_4]$. It is not possible for both the vectors ℓ_1, ℓ_4 and the vectors ℓ_2, ℓ_3 to be parallel, for otherwise one would have $W[\ell_3, \ell_4] = W[\ell_1, \ell_2]'$, which implies that $W_1 \cap W_2 = \emptyset$. Assuming that ℓ_1 and ℓ_4 are not parallel, Lemmas 6.1.2 and 6.1.6 then yield

$$\emptyset \neq \tau(\mathbb{R}x) \cap W[\ell_1, \ell_2] = \tau(\mathbb{R}x) \cap W[\ell_1, \ell_4] = \tau(\mathbb{R}x) \cap W[\ell_3, \ell_4] \quad ,$$

and a similar argument can be applied if ℓ_2 and ℓ_3 are not parallel. □

We have seen above that every point x in the three-dimensional de Sitter space can be identified with a spacelike $x \in \mathbb{R}^4$ with $x \cdot x = -1$. By Lemma 6.1.7, this then determines the nonempty intersection

$$\tau(\mathbb{R}x) \cap \tau(W_0) = \bigcap_{\substack{W \in \mathcal{W} \\ x \in W}} (\tau(\mathbb{R}x) \cap \tau(W)) = \tau(\mathbb{R}x) \cap \Big(\bigcap_{\substack{W \in \mathcal{W} \\ x \in W}} \tau(W)\Big) \quad ,$$

where $W_0 \in \mathcal{W}$ contains x. Since there exists a point $y \neq 0$ in this intersection, and $\tau(W_0) \in \mathcal{W}$, while $\tau(\mathbb{R}x)$ is a spacelike line, the intersection $\tau(\mathbb{R}x) \cap \tau(W_0)$ must contain the ray $\mathbb{R}_+ y$. Hence, there exists a unique point, call it $\delta(x)$, such that $\delta(x) \in \tau(\mathbb{R}x) \cap \tau(W_0)$ and $\delta(x) \cdot \delta(x) = -1$. It thus represents a point in three-dimensional de Sitter space. We have therefore proven the following result.

Proposition 6.1.8. *Let $\tau : \mathcal{W} \mapsto \mathcal{W}$ be a bijection satisfying (6.1.1). Then there exists a bijection $\delta : dS^3 \mapsto dS^3$ such that*

$$\tau(W) = \{\delta(x) \mid x \in W\} \quad ,$$

for all $W \in \mathcal{W}$.

The following Alexandrov-like theorem has been established for the case of de Sitter space by Lester [48]:

Lemma 6.1.9. *If $\phi : dS^3 \mapsto dS^3$ is a bijection such that lightlike separated points are mapped to lightlike separated points, then there exists a Lorentz transformation Λ of the ambient Minkowski space $\mathbb{R}^4$ such that $\phi(x) = \Lambda x$, for all $x \in dS^3$.*

We may therefore proceed to obtain the following extension of Lester's theorem. Details may be found in Section 1.5.2 of [28].

Theorem 6.1.10. *Let $\tau : \mathcal{W} \mapsto \mathcal{W}$ be a bijection satisfying (6.1.1), and let $\delta : dS^3 \mapsto dS^3$ be the associated bijection. Then there exists a Lorentz transformation Λ of the ambient Minkowski space $\mathbb{R}^4$ such that $\delta(x) = \Lambda x$, for all $x \in dS^3$, and $\tau(W) = \Lambda W$, for all $W \in \mathcal{W}$.*

6.2. Geometric Modular Action in de Sitter Space and the de Sitter Group

We now turn to the discussion of nets on de Sitter space satisfying the Condition of Geometric Modular Action given in Chapter III with the choices $\mathcal{M} = dS^3$ and the collection of wedges $\mathcal{W}$ specified in the previous section. In order to simplify the discussion, we work with the following somewhat more restrictive version of the CGMA.

Strong CGMA. *A theory complies with the strong form of the CGMA if the net* $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ *satisfies*

(i) $W \mapsto \mathcal{R}(W)$ *is an order-preserving bijection,*

(ii) Ω *is cyclic and separating for* $\mathcal{R}(W_1)\cap\mathcal{R}(W_2)$ *if and only if* $W_1\cap W_2 \neq \emptyset$*, for* $W_1, W_2 \in \mathcal{W}$*,*

(iii) for any $W_0, W_1, W_2 \in \mathcal{W}$ *with* $W_1 \cap W_2 \neq \emptyset$*, there holds*

$$\mathcal{R}(W_1)\cap\mathcal{R}(W_2) \subset \mathcal{R}(W_0) \quad \textit{if and only if} \quad W_1 \cap W_2 \subset W_0 \quad , \tag{6.2.1}$$

and

(iv) for each $W \in \mathcal{W}$*, the adjoint action of* J_W *leaves the set* $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ *invariant.*

The first and fourth conditions are the same as in Chapter III and entail the existence of an involution $\tau_W : \mathcal{W} \mapsto \mathcal{W}$ for each $W \in \mathcal{W}$ satisfying (3.1) and (3.2). The second condition is a strengthened version of the previous conditions (ii) and (iii). It directly implies relation (3.3).

The third condition is an additional natural assumption which has no counterpart in the original CGMA. We note that the restriction to intersecting pairs of wedges is motivated by a curious fact pointed out to us by E.H. Wichmann. Already for the standard net of von Neumann algebras of the free field, there exists a counterexample to relation (6.2.1) if W_1, W_2 are unrestricted wedges [67]. However, it has been shown that in a net satisfying the usual axioms as well as the condition of additivity of wedge algebras, the relation (6.2.1) holds for pairs satisfying $W_1' \cap W_2' \neq \emptyset$ [63][64]. But for wedges $W_1, W_2 \in \mathcal{W}_0$, it is easy to see that $W_1' \cap W_2' \neq \emptyset$ if and only if $W_1 \cap W_2 \neq \emptyset$.

Lemma 6.2.1. *Let the strong CGMA with the choices* $\mathcal{M} = dS^3$ *and the set of wedges* $\mathcal{W}$ *in* dS^3 *hold. Then for each* $W \in \mathcal{W}$ *the associated involution* $\tau_W : \mathcal{W} \mapsto \mathcal{W}$ *satisfies (6.1.1).*

Proof. As already pointed out, the strong CGMA entails relation (3.3). Therefore, in order to prove (6.1.1), it suffices to show that $W_1 \cap W_2 = W_3 \cap W_4 \neq \emptyset$ implies $\tau_W(W_1) \cap \tau_W(W_2) = \tau_W(W_3) \cap \tau_W(W_4)$. But $W_1 \cap W_2 = W_3 \cap W_4$ implies $\mathcal{R}(W_1) \cap \mathcal{R}(W_2) \subset \mathcal{R}(W_3)$, which itself entails $\mathcal{R}(\tau_W(W_1)) \cap \mathcal{R}(\tau_W(W_2)) \subset \mathcal{R}(\tau_W(W_3))$. In the light of (3.3), one concludes that also $\tau_W(W_1)\cap\tau_W(W_2) \neq \emptyset$, so by (6.2.1) one finds $\tau_W(W_1) \cap \tau_W(W_2) \subset \tau_W(W_3)$. By proving three similar inclusions, it follows that $\tau_W(W_1) \cap \tau_W(W_2) = \tau_W(W_3) \cap \tau_W(W_4)$. □

Given the hypotheses of Lemma 6.2.1, we conclude from Theorem 6.1.10 that $\mathcal{T}$ is isomorphic to a subgroup $\mathcal{G}$ of the de Sitter group $\mathcal{L}$. Since one has for $W_1, W_2 \in \mathcal{W}$ the fact that the inclusion $W_1 \subset W_2$ entails the equality $W_1 = W_2$, the index set $\mathcal{W}$ considered in this chapter consists exclusively of atoms, *i.e.* we cannot conclude from the argument of Chapter II that the algebras $\mathcal{R}(W)$ are nonabelian. Indeed, we shall see that this is quite possible. Note that none of the arguments in Section 4.2 relied upon the nonabelianness of the algebras $\mathcal{R}(W)$. Hence, with the additional assumption that the adjoint action of $\mathcal{J}$ upon the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ is transitive, we conclude that the entire identity component of $\mathcal{L}_+^\uparrow$ of $\mathcal{L}$ is contained in $\mathcal{G}$ (Prop. 4.2.2).

Lemma 6.2.2. *Let the strong CGMA with the choices $\mathcal{M} = dS^3$ and the set of wedges $\mathcal{W}$ in dS^3 hold. Moreover, let the adjoint action of $\mathcal{J}$ upon the set $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ be transitive. Then either the algebra $\mathcal{R}(W)$ is nonabelian for every $W \in \mathcal{W}$ and the geometric action of the modular involutions is precisely that found in Prop. 4.2.10, or all these wedge algebras are abelian and the geometric action of the modular involutions is that found in Prop. 4.2.10 times the reflection about the origin, θ.*

Proof. Because of the transitive action of $\mathcal{G}$ upon $\mathcal{W}$, it suffices to make the argument for the standard wedge W_R and the corresponding involution $g_{W_R} \in \mathcal{G}$. Since $\mathcal{L}_+^\uparrow \subset \mathcal{G}$, one sees from Lemma 2.1 that g_{W_R} commutes with the elements of the subgroup $\mathrm{InvL}_+^\uparrow(W_R)$ of $\mathcal{L}_+^\uparrow$ leaving W_R invariant. But W_R and W_R' are the only wedges which are stable under the action of $\mathrm{InvL}_+^\uparrow(W_R)$, so it follows that either $g_{W_R} W_R = W_R'$ or $g_{W_R} W_R = W_R$.

In both cases one can proceed in a manner similar to the proof of Prop. 4.2.10. Making use of the fact that g_{W_R} is an involution which commutes with $\mathrm{InvL}_+^\uparrow(W_R)$, it is not hard to show that g_{W_R} has the block form

$$g_{W_R} = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \quad ,$$

where $X, Y = \pm 1$. In the first case, $g_{W_R} W_R = W_R'$, one clearly has $X = -1$. If also $Y = -1$, then g_{W_R} commutes with all elements of $\mathcal{L}_+^\uparrow \subset \mathcal{G}$, which would be in conflict with the transitive action of $\mathcal{J}$ upon $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$. Hence $Y = 1$ and g_{W_R} has the form given in Prop. 4.2.10. The second case, $g_{W_R} W_R = W_R$, can be treated in the same manner.

Finally, the relation $\mathcal{R}(W_R)' = J_{W_R}\mathcal{R}(W_R)J_{W_R} = \mathcal{R}(g_{W_R} W_R)$ shows that if $g_{W_R} W_R = W_R$ then $\mathcal{R}(W_R)$ is abelian. Conversely, if $\mathcal{R}(W_R)$ is abelian (and hence maximally abelian by the cyclicity of Ω), then one has $\mathcal{R}(W_R) = \mathcal{R}(W_R)'$, and the above relation together with the first part of the CGMA implies $g_{W_R} W_R = W_R$. $\square$

It is now clear how to modify the arguments of Section 4.2 to obtain the following result.

Corollary 6.2.3. *Let the conditions of Lemma 6.2.2 be satisfied. If $\mathcal{R}(W)$ is nonabelian, for some $W \in \mathcal{W}$, then $\mathcal{G}$ coincides with $\mathcal{L}_+$. On the other hand, if $\mathcal{R}(W)$ is abelian, for some $W \in \mathcal{W}$, then $\mathcal{G}$ coincides with $\mathcal{L}_+^\uparrow$.*

The assumptions of Lemma 6.2.2 also directly yield an obvious counterpart to Proposition 4.3.1. The net continuity condition introduced in Section 4.3

and the arguments presented there again entail that there exists a strongly continuous projective representation of $\mathcal{L}_+$ in the nonabelian case (which is of primary interest here). Moreover, the reasoning in Section 4.3 implies that this gives a true representation $U(\mathcal{L}_+)$ of the proper de Sitter group. We summarize in the following theorem.

Theorem 6.2.4. *Let the strong CGMA with the choices $\mathcal{M} = dS^3$ and wedges $\mathcal{W}$ in dS^3 hold, and let the adjoint action of $\mathcal{J}$ upon the set $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ be transitive. If $\mathcal{R}(W)$ is nonabelian, for some $W \in \mathcal{W}$, then there exists a strongly continuous unitary representation of the proper de Sitter group $\mathcal{L}_+$ which acts geometrically correctly upon the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$. Moreover, the net satisfies Haag duality and is local.*

In light of the fact that the restricted Lorentz group $\mathcal{L}_+^\uparrow$ is also isomorphic to the group of motions of Lobaschewskian space, which can be modelled on a surface of transitivity of $\mathcal{L}_+^\uparrow$ in IR^4 (see, *e.g.*, [34]), it is likely that the preceding arguments can be employed to handle that space-time, as well.

To demonstrate that this theorem is not vacuous, we recall an example due to Fredenhagen [33]. Consider once again the net from Section 5.3 associated with the free scalar field on IR^4. We define for each region $W \in \mathcal{W}$ a corresponding algebra $\mathcal{R}(W) \equiv \mathcal{A}(\tilde{W})$, where $\tilde{W}$ is the wedge fixed by W in the ambient space IR^4 and $\mathcal{A}(\tilde{W})$ the corresponding algebra generated by the free field. The results of Bisognano and Wichmann [9] and Thomas and Wichmann [64] entail that this net is covariant under the de Sitter group, and the assumptions in Theorem 6.2.4 are satisfied by this net in the vacuum state.

Moreover, recent results in [17] and [19] concerning quantum field theory on de Sitter space-time are fully consistent with our findings, even though the starting point is quite different. These authors assume the existence of a preferred (vacuum-like) state vector Ω which is invariant under the de Sitter group $\mathcal{L}_+^\uparrow$ and satisfies a stability condition which can be expressed in terms of certain analyticity properties of the corresponding correlation functions. With this input they are then able to prove a Bisognano-Wichmann type theorem. In fact, they establish the Reeh-Schlieder property of Ω for wedge algebras (so the modular objects exist in their setting), and they also show that the modular conjugations associated with these algebras and Ω induce the geometric action upon the net found in the analysis presented here. Moreover, the modular groups comply with our proposal for a modular stability condition. These facts support our view of the relevance of our selection criterion for vacuum-like states in theories on curved space-times.

VII. Summary and Further Remarks

As this paper is lengthy and involves many steps, it is perhaps not amiss to provide a final summary here. First of all, we showed that our Condition of Geometric Modular Action, CGMA, in the abstract form of the Standing Assumptions, yielded special Coxeter groups $\mathcal{T}$ of automorphisms on the index set $(I, \leq)$ of the net $\{\mathcal{A}_i\}_{i\in I}$ and provided them with projective representations having coefficients in an abelian group $\mathcal{Z}$ of internal net symmetries. Some general properties of these groups, following from the modular theory, and a discussion of the finite case were given.

In Chapter III it was explained how, starting from a smooth manifold $\mathcal{M}$ and with a target space-time $(\mathcal{M}, g)$ in mind, one would go about identifying the index set $\mathcal{W}$ before testing states on the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ for the CGMA. The resultant program using the CGMA for the determination of much of the geometrical structure of the space-time was then described.

This program was then exemplified in application to the four-dimensional Minkowski space as target space. This involved a series of results of quite distinct natures. To begin, we showed that bijective inclusion-preserving mappings on the set of wedges which satisfy one additional condition are implemented by elements of the extended Poincaré group, thus extending the Alexandrov-type theorems for Minkowski space. Then, it was shown that subgroups of the Poincaré group which act transitively upon the set of wedges must contain the identity component $\mathcal{P}_+^\uparrow$ of the Poincaré group. These results enabled us to show that the CGMA, applied to nets indexed by wedges in $\mathbb{R}^4$ and supplemented by the transitivity condition, implied that the induced isometry group $\mathcal{G}$ is equal to the proper Poincaré group, and that the implementers for the generating involutions have exactly the geometric action found by Bisognano and Wichmann in their setting.

This explicit knowledge of the geometric nature of the adjoint action of the modular involutions J_W upon the net, along with the additional structure accompanying the modular theory, was used to construct a continuous projective representation of $\mathcal{P}_+^\uparrow$, under the assumption of the net continuity condition. Using Moore's Borel measurable cohomology theory, we showed that this projective representation of $\mathcal{P}_+^\uparrow$ lifted to a true representation of its universal covering group. The explicit geometric properties of the modular involutions already alluded to were then employed to prove that this representation of the covering group restricted to a strongly continuous unitary representation of $\mathcal{P}_+^\uparrow$ and actually coincided with the constructed projective representation. In other words, the projective representation constructed in Section 4.3 is actually a true representation.

In Section 5.1, we showed that if the modular unitaries are all contained in the group $\mathcal{J}$ generated by the modular involutions, *i.e.* if the modular stability condition holds, then the spectrum condition must hold. This is a purely algebraic stability condition which can be sensibly stated on any space-time. We next investigated the geometric action of the modular unitaries in detail. It was proven that, if the Condition of Geometric Action for modular groups is satisfied, then both modular covariance and the modular stability condition, $\mathcal{K} \subset \mathcal{J}$, hold and, if the net is local, the group $\mathcal{K}$ yields a strongly continuous unitary representation of $\mathcal{P}_+^\uparrow$ satisfying the spectrum condition. Moreover, under the same assumptions, the CGMA holds if and only if the net is local. Furthermore, if the CGMA and the modular stability condition are satisfied, then again modular covariance follows. In Section 5.3 a number of examples were given which make clear that modular covariance is, in fact, strictly stronger than the CGMA.

Finally, in Chapter VI we discussed the case of de Sitter space. In spite of its different geometric structure, results similar to the case of Minkowski space were recovered.

Among other space-times, we expect our approach to function with little change in such examples as the (static) Robertson-Walker space-times. It is an interesting problem whether also in these cases the maps induced upon the index sets of the corresponding nets of algebras are implemented by point transformations. In this regard, it is relevant to note that Alexandrov-type theorems are available for many of the classical Lorentzian space-times (see [8]). But even if not every element of the group $\mathcal{T}$ of transformations is implemented by a point transformation on the space-time (and we have already presented such an example in Section 4.1), we still anticipate that the CGMA could usefully select physically interesting states. Whatever the group of transformations which results, we would propose it as the symmetry group of the theory.

We complete our comments in this final chapter by returning briefly to the conceptually interesting question of whether one can derive the space-time itself from our initial algebraic data. In this paper we began with a particular smooth manifold $\mathcal{M}$ and saw how the CGMA, for a certain choice of index set which was determined by the target space-time $(\mathcal{M}, g)$, enabled us to derive a metric-characterizing isometry subgroup. But is it possible to do without these initial data?

We shall sketch here our program for meeting this question. We have shown that the abstract version of the CGMA in the form of the Standing Assumptions leads to a certain Coxeter group $\mathcal{T}$ of automorphisms on the index set $(I, \leq)$ of the net $\{\mathcal{R}_i\}_{i\in I}$. There exists in the mathematical literature a branch of geometry known as *absolute geometry*, whose point of departure is precisely an abstract group $\mathcal{T}$ generated by involutions and whose aim is to investigate which algebraic relations in the group $\mathcal{T}$ entail the existence of a space-time $(\mathcal{M}, g)$ such that the group $\mathcal{T}$ can be realized as a metric-characterizing subgroup of the isometry group of $(\mathcal{M}, g)$. This has been carried out for all planar geometries [5][75] and for three-dimensional Euclidean space [1].

So a first step in an attempt to characterize Minkowski space entirely in terms of the data $(\{\mathcal{R}_i\}_{i\in I}, \Omega)$ would be to find the algebraic relations in the group $\mathcal{T}$ which would enable one to derive in this manner four-dimensional Minkowski space. This has been accomplished in one form [44], but the particular geometric significance of the initial algebraic data in our setting entails that a different set of algebraic axioms be determined [66]. The second step would be to determine which additional structure on I, or equivalently, which relations among the algebras in the net $\{\mathcal{R}_i\}_{i\in I}$, imply *via* modular theory the requisite relations among the generating involutions J_i (equivalently, τ_i) found in the first step. In application to Minkowski space, this would give an intrinsic characterization of “wedge algebras” (equivalently “wedges”).

The results in this paper demonstrate that the CGMA is sufficiently strong to select physically interesting states and to actually determine metric-characterizing isometry groups in the examples of Minkowski and de Sitter space-times. We hope that the suggestive results and interesting perspectives of the present analysis will draw attention to the various mathematical problems opened up by our program.

Appendix: Cohomology and the Poincaré Group

In this appendix we shall prove the technical cohomological result used in the main text to the effect that the continuous projective representation $V(\mathcal{P}_+^\uparrow)$ constructed in Section 4.3 can be lifted to a true representation of the covering group. We include this appendix since we have not found in the literature the results in the form we need. Assume that $G \ni g \mapsto V(g) \in \mathcal{U}(\mathcal{H})$ is a continuous projective representation of a semisimple Lie group by unitary operators on a separable Hilbert space $\mathcal{H}$, which has coefficients in a closed[12] subgroup $\mathcal{Z} \subset \mathcal{U}(\mathcal{H})$ left pointwise fixed by the adjoint action of the elements of $\{V(g) \mid g \in G\}$. Note that the group $\mathcal{U}(\mathcal{H})$ of unitary operators acting on the separable Hilbert space $\mathcal{H}$ is, when provided with the strong (or weak) operator topology, a complete, metrizable, second countable topological group (cf. p. 33 in [27] and references cited there). It therefore follows that also $\mathcal{Z}$ is a complete, metrizable, second countable topological group; hence, it is a polonais (polish) group. In particular, $\mathcal{Z}$ is a trivial G-module. The first main theorem we want to prove is the following. (A related theorem with different assumptions and proof may be found in [22].)

Theorem A.1. *Let G, $V(G)$ and $\mathcal{Z}$ be as described above.*[13] *Then there exists a strongly continuous unitary representation of the covering group E of the group G.*

The proof of this theorem will proceed in several steps, which we present in separate lemmata for the sake of clarity. For the reader's convenience, we shall present some background information about the two-dimensional cohomology of groups, which can be found in textbooks on the subject (see, *e.g.* [20]). Since we are interested in the continuity of the representations, we shall need to work in the category of topological groups but find ourselves obliged to use the Borel cohomology on locally compact groups initiated by Mackey [49] and fully defined and extended by Moore [51]-[55], since the computational situation for continuous cohomologies seems to be exceedingly complicated. Fortunately, it can be shown that this will be sufficient for our purposes. For an overview of the various cohomologies for topological groups, see the review by Stasheff [59].

Let G be a group and $G' \equiv [G,G]$ denote its derived subgroup, *i.e.* the group generated by the set $\{ghg^{-1}h^{-1} \mid g,h \in G\}$ of commutators in G. If $G' = G$, the group G is said to be perfect, and any connected semisimple Lie group has this property.[14] In particular, the group of interest to us in this paper, the proper orthochronous Poincaré group $\mathcal{P}_+^\uparrow$, is a perfect group.

[12] It is no loss of generality to take $\mathcal{Z}$ closed. Though the subgroup $\mathcal{Z} \subset \mathcal{U}(\mathcal{H})$ in the main text is not *a priori* closed, closing it in the weak operator topology still yields a trivial $\mathcal{P}_+^\uparrow$-module, as used in this appendix. However, the restriction that $\mathcal{Z}$ be closed offers a technical problem in the main text which is dealt with there.

[13] In fact, the arguments presented below are valid for a larger class of groups G than semisimple Lie groups, but we shall not tax the reader's patience here with this generalization.

[14] Indeed, Moore [53] suggests the property $G = G'$ as the algebraic analogue of connectedness.

Let G be a group and A be an abelian group. A central extension of G by A is a triple $(\tilde{G}, \phi, \iota)$ with $\tilde{G}$ a group, ι an injective homomorphism from A to $\tilde{G}$ satisfying $\iota(A) \subset \mathrm{center}(\tilde{G})$ and ϕ a homomorphism from $\tilde{G}$ onto G satisfying $\mathrm{kernel}(\phi) = \iota(A)$. In other words, the sequence

$$\{1\} \longrightarrow A \xrightarrow{\iota} \tilde{G} \xrightarrow{\phi} G \longrightarrow \{1\} \tag{A.1}$$

is exact, with $\{1\}$ denoting the trivial group. Such a central extension is said to be equivalent to the central extension

$$\{1\} \longrightarrow A \xrightarrow{\iota'} \tilde{G}' \xrightarrow{\phi'} G \longrightarrow \{1\}$$

if there exists an isomorphism $\rho : \tilde{G} \mapsto \tilde{G}'$ such that the diagram

$$\begin{array}{ccccccccc}
\{1\} & \longrightarrow & A & \xrightarrow{\iota} & \tilde{G} & \xrightarrow{\phi} & G & \longrightarrow & \{1\} \\
 & & \Big\downarrow{\scriptstyle \mathrm{id.}} & & \Big\downarrow{\scriptstyle \rho} & & \Big\downarrow{\scriptstyle \mathrm{id.}} & & \\
\{1\} & \longrightarrow & A & \xrightarrow{\iota'} & \tilde{G}' & \xrightarrow{\phi'} & G & \longrightarrow & \{1\}
\end{array}$$

is commutative. The direct product $G \times A$ is an example of a central extension with the inclusion $a \mapsto (1, a)$ and the projection $(g, a) \mapsto g$, where $g \in G$ and $a \in A$. If the groups involved are topological groups and one wishes to keep track of continuity, as we do in this paper, then in the above the homomorphism ι is required also to be a homeomorphism onto a closed subgroup of $\tilde{G}$, ϕ must also be continuous and open (so that $\tilde{G}/\iota(A) \simeq \tilde{G}/\mathrm{kernel}(\phi) \simeq G$), and ρ must be an isomorphism in the category of topological groups.

If E is a topological group such that $[E, E]$ is dense in E and $p : E \mapsto G$ is a surjective continuous homomorphism, following Moore, we shall say that the pair (E, p) is a cover of G if the kernel of p is contained in the center of E. Then E is an extension of G by the trivial G-module kernel(p) (and, of course, $[G, G]$ is necessarily dense in G). Moore showed that if G is locally compact and separable, then G has at most one simply connected covering group (in this sense) up to isomorphism of topological group extensions (see Lemma 2.2 in [53]). Moreover, if G is perfect, then there does exist such a (unique) simply connected covering group (called the universal covering group) E, which turns out to be perfect and a Lie group itself (Theorem 2.2 in [53] and Theorem 10 in [55]). What will be important for our arguments below is that if G is a semisimple Lie group, then this universal covering group *coincides* with the standard, topologically defined, universal covering group (cf. p. 49 in [55]). A central extension $(U, \nu, \jmath)$ is called universal if for every central extension $(\tilde{G}, \phi, \iota)$ of G by A, there exists a (continuous, open) homomorphism h from U to $\tilde{G}$ such that $\phi \circ h = \nu$. If such a universal central extension exists, then it is unique up to isomorphism over G. And it is known (cf. Theorem 5.7 in [50]) that a group G admits a universal central extension if and only if G is perfect. From the remarks above, it is now clear that for semisimple Lie groups, the (standard) universal covering group coincides with the universal covering group in the sense of Moore, which coincides with the universal central extension.

Given a central extension (A.1) of G by A,[15] assume that $\sigma : G \mapsto \tilde{G}$ is a section with $\sigma(1) = 1$, in other words it is a (Borel measurable) set map such that $\phi(\sigma(g)) = g$ for all $g \in G$. The function $\gamma(\sigma) = \gamma : G \times G \mapsto \tilde{G}$ defined by $\gamma(g,h) \equiv \sigma(g)\sigma(h)\sigma(gh)^{-1}$ is a measure of the amount σ diverges from a homomorphism, and, of course, the associativity in $\tilde{G}$ implies that γ is a 2-cocycle. Note that because $\phi(\gamma(g,h)) = 1$, γ actually takes values in the subgroup A. Let $Z^2(G,A)$ denote the set of all such (Borel measurable) 2-cocycles (which turns out to be an abelian group). Let $B^2(G,A)$ denote the A-valued coboundaries, *i.e.* the subgroup of $Z^2(G,A)$ consisting of functions $\gamma : G \times G \mapsto A$ for which there exists a (Borel measurable) $\beta : G \mapsto A$ such that $\gamma(g,h) = \beta(g)\beta(h)\beta(gh)^{-1}$ for all $g, h \in G$. The quotient group $Z^2(G,A)/B^2(G,A)$ is precisely the second cohomology group $H^2(G,A)$. One therefore sees that if $H^2(G,A) = \{1\}$, then every (A-valued) projective representation σ of G in $\tilde{G}$ determines a 2-cocycle γ which is actually a 2-coboundary. Thus, by defining $\tilde{\sigma} \equiv \beta(g)^{-1}\sigma(g)$, a straightforward calculation shows that $\tilde{\sigma} : G \mapsto \tilde{G}$ is a (Borel measurable) homomorphism[16], *i.e.* a representation, as desired. And if $H^2(G,A)$ is nontrivial, then it is possible to start with a section σ for which there exists no β for which $\beta^{-1}\sigma$ yields a homomorphism. In this case, the question would have to be settled for a given section individually.

In the setting of relevance to this paper, $E \ni e \mapsto V(p(e))$ is a continuous projective representation of E with coefficients in $\mathcal{Z}$. We prove the relevant cohomological result for the covering group E.

Lemma A.2. *Let G be a connected semisimple Lie group and E be its universal covering group. Then the second cohomology group $H^2(E,\mathcal{Z})$ is trivial.*

Proof. In the proof of Theorem 9 in [55], it is shown that for a perfect, almost connected group G, the second cohomology group $H^2(E,S^1)$ in Moore's Borel measurable cohomology theory is trivial, where S^1 is the circle group. This result is thus applicable to the situation described by the hypothesis. Moreover, since G, and hence E, is perfect, it follows that also the first cohomology group $H^1(E,S^1)$ is trivial (see p. 48 in [55]). Thus Prop. 4 in [55] may be applied, yielding $H^2(E,A)$ is trivial for *any* unitary trivial G-module A, and, in particular, for $A = \mathcal{Z}$. □

Hence, there exists a function $Z : E \mapsto \mathcal{Z}$ such that $U(e) \equiv Z(e)V(p(e))$, $e \in E$, is a true representation of E. One does indeed obtain a (unitary) representation of the group E. But in Moore's cohomology, the cochains are only Borel measurable on the group; in other words, although the original section σ is continuous, the function β may only be Borel measurable, so that $\tilde{\sigma} \equiv \beta^{-1}\sigma$, *i.e.* U, may be only Borel measurable. However, the following result, attributed to Mackey in [78], closes this gap.

Lemma A.3. *If H_1 is a locally compact second countable group, H_2 is any second countable topological group, and $h : H_1 \mapsto H_2$ is a Borel measurable*

[15]For the purposes of his cohomology theory, in [51][52] Moore took A to be an abelian, locally compact and second countable topological group. However, in [54] he extended his results to include second countable, Hausdorff polonais groups A. We may, therefore, take $A = \mathcal{Z}$ below.

[16]The passage from Borel measurable to continuous will be addressed separately below.

homomorphism, then h is continuous.

Proof. This is Theorem B.3 in [78]. □

Hence, by taking $H_1 = E$ and $H_2 = \mathcal{U}(\mathcal{H})$, it follows that $E \ni e \mapsto U(e)$ is, in fact, a strongly continuous unitary representation of E, completing the proof of Theorem A.1.

In the more structured setting of the main text of this paper, G is the Poincaré group $\mathcal{P}_+^\uparrow$. There we get by an application of the preceding results:

Corollary A.4. *Let $V(\cdot)$ be the continuous unitary projective representation of $\mathcal{P}_+^\uparrow$ with values in $\mathcal{J}$ which has been constructed in Section IV.3, let $\overline{\mathcal{J}}$ be the closure of $\mathcal{J}$ in the weak operator topology and let $\overline{\mathcal{Z}}$ be the center of $\overline{\mathcal{J}}$. There exists a strongly continuous unitary representation $U(\cdot)$ of the covering group $ISL(2,\mathbb{C})$ of the Poincaré group $\mathcal{P}_+^\uparrow$ with values in $\overline{\mathcal{J}}$ and a mapping $Z : ISL(2,\mathbb{C}) \mapsto \overline{\mathcal{Z}}$ with $U(A) = Z(A)V(\mu(A)), A \in ISL(2,\mathbb{C})$. Here, $\mu : ISL(2,\mathbb{C}) \mapsto \mathcal{P}_+^\uparrow$ is the canonical covering homomorphism whose kernel is a subgroup of order 2, the center of $ISL(2,\mathbb{C})$.*

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(Buchholz) Institut für Theoretische Physik, Universität Göttingen, Bunsenstr. 9, D-37073 Göttingen, Germany , (Dreyer) Department of Physics, Pennsylvania State University, University Park, PA 16802, USA and (Florig and Summers) Department of Mathematics, University of Florida, Gainesville, FL 32611, USA