Trajectory and Attractor Convergence for a Nonlocal Kuramoto-Sivashinsky Equation

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Abstract

The nonlocal Kuramoto-Sivashinsky equation arises in the modeling of the flow of a thin film of viscous liquid falling down an inclined plane, subject to an applied electric field. In this paper, the authors show that, as the coefficient of the nonlocal integral term goes to zero, the solution trajectories and the maximal attractor of the nonlocal Kuramoto-Sivashinsky equation converge to those of the usual Kuramoto-Sivashinsky equation.

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1 Introduction

In this paper we investigate some dynamical issues of a Kuramoto-Sivashinsky equation with a nonlocal spatial integral term

$$u_t + u_{xxxx} + u_{xx} + uu_x + \alpha H(u_{xxx}) = 0, \tag{1.1}$$

where α is a non-negative constant coefficient, and

$$H(f) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(\xi)}{x-\xi} d\xi, \tag{1.2}$$

is the Hilbert transform and the integral is understood in the sense of the Cauchy Principle Value.

The equation under consideration arises in the modeling of the flow of a thin film of viscous liquid falling down an inclined plane, subject to an applied electric field [9]. The application of a uniform electric field at infinity, perpendicular to the inclined plane, is to destabilize the liquid films on the surface of the plane. In an industrial setting it is hoped that this destabilization will lead to an enhancement of heat transfer. In this engineering context, $\alpha W_e/\sqrt{W|\cot\beta - \frac{4}{5}R_e|}$, where W_e is the electrical Weber number, W the Weber number, R_e the Reynolds number and β the angle between the plane and the horizontal. When the electrical field is absent, i.e. $W_e = 0$, the Hilbert transform term is gone and we obtain the usual (local) Kuramoto-Sivashinsky equation

$$u_t + u_{xxxx} + u_{xx} + uu_x = 0. \tag{1.3}$$

Throughout this paper we restrict our attention to the case of $u(x,t)$ periodic in x with period $l > 0$. We denote the interval $I := (-l, l)$. Then, (1.2) is replaced by (see [1]) the "periodic" Hilbert transform

$$H(f) = -\frac{1}{2l}\int_I cot\frac{\pi(x-\xi)}{2l}\, f(\xi)d\xi.$$

Observe that $u = C$, a constant, satisfies (1.1). Moreover, for

$$u(x,0) \ = u_0(x) \ \text{ for } x \in I\,, \text{ and } \bar{u}_0 \ := \ \frac{1}{2l}\int_I u_0(x)\,dx\,, \tag{1.4}$$

integrating (1.1) over the interval I yields

$$\frac{d}{dt}\int_I u(x,t)\,dx \ = \ 0\,, \ \text{ i.e. } \frac{1}{2l}\int_I u(x,t)\,dx \ = \ \bar{u}_0\,.$$

Thus, we have that the mean of the solution is conserved with respect to time. This may be interpreted in the sense that the "dynamics" of u satisfying (1.1) are centered around the mean value of the initial data. Therefore, without loss of generality, we will work in a periodic functional space with zero mean. That is, we will assume the following condition

$$u(x,t) \text{ periodic on } I, \text{ with } u(x,0) = u_0(x)\,, \text{ and } \int_I u_0(x)\,dx \;=\; 0\,. \quad (1.5)$$

In this paper, we will show that individual solution trajectories and the maximal attractor of the nonlocal Kuramoto-Sivashinsky equation approach the solution trajectories and the maximal attractor of the usual local Kuramoto-Sivashinsky equation, as the coefficient for the nonlocal term goes to zero.

2 Preliminaries

We denote by $L^2_{per}(I)$, $H^k_{per}(I)$, $k = 1, 2, \cdots$, the usual Sobolev spaces of periodic functions on I. Let $\dot{L}^2_{per}(I)$, $\dot{H}^k_{per}(I)$ denote the spaces of functions g in $L^2_{per}(I)$, $H^k_{per}(I)$, respectively, with mean zero, i.e. $\bar{g} := \frac{1}{2l}\int_I g(x)dx \;=\; 0$. In the following, $||\cdot||$ denotes the usual $L^2_{per}(I)$ norm. Due to the Poincare inequality, $\|D^k u\|$ is an equivalent norm in $\dot{H}^k_{per}(I)$. All integrals are with respect to $x \in I$, unless specified otherwise. We recall the following two inequalities:

Poincaré inequality ([25], p.49):

$$\int_I g^2\,dx \le (2l)^2 \int_I g_x^2\,dx, \quad (2.1)$$

for $g \in \dot{H}^1_{per}(I)$.

Agmon inequality ([25], p.50):

$$||g||^2_\infty \le 2||g||\,||g_x||, \quad (2.2)$$

for $g \in \dot{H}^1_{per}(I)$. This inequality also follows from $g^2(x) = 2\int_{x_0}^x g g_x\,dx$, where $g(x_0) = 0$ (as g is continuous and has zero mean).

The Hilbert transform (1.2) is a linear, invertible, bounded operator from L^2 to L^2, and from Sobolev space H^k to H^k. Several noteworthy properties of the transform are (see [1], [2] or [24]):

$$\begin{aligned}
D_x H &= H(D_x)\,, \\
H^{-1} &= -H, \\
\int vH(u) &= -\int uH(v), \\
\int H(u)H(v) &= \int uv, \\
\int uH(u) &= 0, \\
\|H(u)\| &= \|u\|, \\
(H(u))_x &= H(u_x).
\end{aligned}$$

These properties hold for the Hilbert transformation on both the real line and periodic intervals ([1]). On the periodic interval $(-l,l)$, the Hilbert transformation has a simple representation

$$H(f)(x) = i\sum_{k\in Z} sgn(k) f_k e^{ik\pi x/l},$$

for $f(x)=\sum_{k\in Z} f_k e^{ik\pi x/l}$, with f_k's the Fourier coefficients of f.

Duan and Ervin [6] (or, for similar arguments, see [4], [5] or [25]), have shown that the nonlocal dynamical system (1.1) and (1.5) has global solutions and a maximal attractor in $\dot{L}^2_{per}(I)$. Moreover the following time-uniform estimates hold for the nonlocal Kuramoto-Sivashinsky equation:

$$\|u\|_{L^2_{per}} \le \rho_0, \tag{2.3}$$

$$\|u\|_{H^1_{per}} \le \rho_1, \tag{2.4}$$

$$\|u\|_{H^2_{per}} \le \rho_2, \tag{2.5}$$

where ρ_0, ρ_1, ρ_2 are absolute positive constants for $0 \le \alpha \le 1$. These estimates have also been proved for the usual Kuramoto-Sivashinsky equation ([3], [26] and references therein).

The usual Kuramoto-Sivashinsky equation (1.3) together with (1.5) also has global solutions and a maximal attractor in $\dot{L}^2_{per}(I)$ as well as other dynamical properties; see, for example, [20], [21], [16], [10], [17] and [23].

3 Localization Limit

In this section, we will show that individual solution trajectories and the maximal attractor for the nonlocal Kuramoto-Sivashinsky equation (1.1) converge to the solution trajectories and the maximal attractor of the usual Kuramoto-Sivashinsky equation (1.3), as the coefficient α of the nonlocal term goes to zero. We note that Duan et al. [5] have studied the this convergence for a Kuramoto-Sivashinsky equation with an extra dispersive term u_{xxx}. Ercolani et al. [8] have also studied the impact of u_{xxx} on the dynamics of the usual Kuramoto-Sivashinsky equation.

General convergence result for the maximal attractors is available in Hale et al. [12], Temam [25] and Hale-Lin-Raugel [13]. Similar applications can also be found in Hill and Sulin [14].

Let us denote by $T_\alpha(t)$ and $T(t)$ the solution operators for the nonlocal Kuramoto-Sivashinsky equation (1.1), and the usual Kuramoto-Sivashinsky equation (1.3), respectively. We need to verify a few conditions that will ensure the convergence of the solution trajectories and attractors. In particular, we should show that operators $T_\alpha(t)$ and $T(t)$ satisfy

$$\|T_\alpha(t)u_0 - T(t)u_0\| \le \eta(\alpha, t, u_0),$$

where $\eta(\alpha, t, u_0) \to 0$ as $\alpha \to 0$, and that the domain of attraction of $T_\alpha(t)$ is independent of α. For more details of these conditions see Theorem I.1.2 of Temam [25], or Theorem 2.4 of Hale-Lin-Raugel [13].

Assume that u, v are the solutions of the nonlocal and usual Kuramoto-Sivashinsky equations, respectively, with the same initial data $u_0(x)$, i.e., $u = T_\alpha(t)u_0$, $v = T(t)u_0$. We denote also $w = u - v$, and hence $\|w(0)\| \equiv 0$.

Using the equations (1.1) and (1.3), we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w\|^2 + \|w_{xx}\|^2 - \|w_x\|^2 &+ \int (uw_x + wv_x)w dx \\ &+ \alpha \int H(w_{xxx})w dx = 0. \qquad (3.1)\end{aligned}$$

After integration by parts, and using the Cauchy-Schwarz inequality and Agmon inequality (2.2), we get

$$\begin{aligned}\|w_x\|^2 &\le \|w\|^2 + \frac{1}{4}\|w_{xx}\|^2, \qquad (3.2)\\ \int (uw_x + wv_x)w dx &= \int (-\frac{1}{2}u_x + v_x)w^2 dx \\ &\le (\frac{1}{2}\|u_x\|^{\frac{1}{2}}\|u_{xx}\|^{\frac{1}{2}} + \|v_x\|^{\frac{1}{2}}\|v_{xx}\|^{\frac{1}{2}})\|w\|^2 \\ &\le 2\rho_1^{\frac{1}{2}}\rho_2^{\frac{1}{2}}\|w\|^2, \qquad (3.3)\\ \alpha \int H(w_{xxx})w dx = -\alpha \int H(w_{xx})w_x dx &\le \alpha^2\|w_x\|^2 + \frac{1}{4}\|w_{xx}\|^2 \\ &\le \alpha^2\rho_1^2 + \frac{1}{4}\|w_{xx}\|^2. \qquad (3.4)\end{aligned}$$

Here we have used the estimates in (2.4) and (2.5). Substituting (3.2), (3.3) and (3.4) into (3.1), we obtain,

$$\frac{d}{dt}\|w\|^2 + \|w_{xx}\|^2 \le 2\|w\|^2 + 4\rho_1^{\frac{1}{2}}\rho_2^{\frac{1}{2}}\|w\|^2 + 2\alpha^2\rho_1^2. \qquad (3.5)$$

Let

$$b_1 = 2 + 4\rho_1^{\frac{1}{2}}\rho_2^{\frac{1}{2}}, \text{ and } b_2 = 2\rho_1^2.$$

Thus we have

$$\frac{d}{dt}\|w\|^2 \le b_1\|w\|^2 + b_2\alpha^2, \qquad (3.6)$$

or

$$\|w(t)\|^2 \le \|w(0)\|^2 e^{b_1 t} + \frac{b_2}{b_1}(e^{b_1 t} - 1)\alpha^2, \qquad (3.7)$$

Since $\|w(0)\| \equiv 0$, we conclude that $\|w(t)\| \to 0$ as $\alpha \to 0$. This shows that the solution u of the nonlocal Kuramoto-Sivashinsky equation (1.1) approaches the solution v of the usual Kuramoto-Sivashinsky equation (1.3).

Moreover, solution operator $T_\alpha(t)$ has a uniform domain of attraction, i.e., $\dot{L}^2_{per}$, which is independent of α.

Therefore, by using Theorem I.1.2 of Temam [25] or Theorem 2.4 of Hale-Lin-Raugel [13], we have the following theorem.

Theorem 3.1 *When the coefficient α of the nonlocal integral term goes to zero,*

(i) A solution trajectory of the nonlocal Kuramoto-Sivashinsky equation (1.1) converges in L^2 to a solution trajectory of the usual Kuramoto-Sivashinsky equation (1.3), as long as both trajectories start at the same initial point.

(ii) The maximal attractor of the the nonlocal Kuramoto-Sivashinsky equation (1.1) converges in L^2 to the maximal attractor of the usual Kuramoto-Sivashinsky equation (1.3).

4 Discussions

Nonlocal spatial integral terms appear in mathematical models in the form of partial integro-differential equations for various physical phenomena, for example, in beam theory [15], in electric systems [22], chemical diffusion [11], heat conduction processes [18], temperature evolution in an atmospheric system [19], in ferromagnetic system [7], and in chemical frontal development [27].

In this paper we have studied the limiting behavior of a prototypical partial-integral differential equation arising in fluid dynamics, as the coefficient of the nonlocal integral term goes to zero. This work is a part of our effort on the investigation of impact of nonlocal spatial interactions on the dynamics of (local) infinite dimensional dynamical systems.

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