Universal Lower Bounds for Quantum Diffusion

J.M. Barbaroux[*] [†]

Lehrstuhl für Mathematik, Universität Regensburg

D-93040 Regensburg, Germany.

S. Tcheremchantsev[‡]

Département de Mathématiques, Université d'Orléans

F-45067 Orléans Cedex, France.

May 1998

Abstract

We study the connections between dynamical properties of Schrödinger operators on separable Hilbert space $\mathcal{H}$ and the properties of corresponding spectral measures. Our main result establish a direct relation between the Fourier transform of spectral measure and the moments of order p. This allows us to extend earlier results of Last [10] and Barbaroux, Combes, Montcho [2] based on the Strichartz Theorem. In particular, we obtain $\langle\langle |X|^p\rangle_{\psi(t)}\rangle(T) \geq h(T)$, where $h(T)$ is some function that has not necessarily the form T^α, and we derive lower bounds for $\langle\langle |X|^p\rangle_{\psi(t)}\rangle(T_k)$, for subsequences of time $T_k \nearrow +\infty$. We prove also a necessary condition for the dynamical localization in the presence of pure point spectrum. The results are applied to different concrete models of Schrödinger operators.

Key-Words: Schrödinger operators, spectral measure, double-space method, correlation dimensions, moment of order p, localization.

[*]Financially supported by the European Union through TMR network FMRX-CT 96-0001.

[†]E-mail address: jean-marie.barbaroux@mathematik.uni-regensburg.de

[‡]E-mail address: serguei.tcherem@labomath.univ-orleans.fr

1 Introduction

The study of dynamical properties of quantum systems through fractal dimensions of spectral measures has attracted much attention in recent years. Consider a separable Hilbert space $\mathcal{H}$, equal either to $\ell^2(\mathbf{Z}^d)$ or $L^2(\mathbf{R}^d)$, a vector ψ in $\mathcal{H}$ and a self-adjoint operator $H : \mathcal{H} \to \mathcal{H}$. We are interested in the asymptotic behaviour of the Cesaro mean at time T of the moments of order p for ψ, i.e.,

$$\langle\langle |X|^p \rangle_{\psi(t)}\rangle(T) \equiv T^{-1} \int_0^T (\psi(t),\, |X|^p \psi(t))\, dt \ , \tag{1}$$

where $\psi(t) = \mathrm{e}^{-itH}\psi = \int_{\lambda\in\mathbf{R}} \mathrm{e}^{-it\lambda} dP_H((-\infty,\, \lambda])\psi$ is the solution at time t of the Schrödinger equation for H with initial condition ψ. The projected valued measure P_H is the spectral family of H. One can also consider the case of discrete time evolution, where $\psi(t) = U^t\psi$, $t \in \mathbf{Z}$, for some unitary operator U (see Appendix A).

The first results concerning behaviour of $\langle\langle |X|^p \rangle_{\psi(t)}\rangle(T)$ and the fractal properties of spectral measure are due to Guarneri [7] who provides, for models on $\ell^2(\mathbf{Z}^d)$, a lower bound for the moments of order p from the knowledge of local behaviour of the spectral measure $d\mu_\psi \equiv d(P_H(-\infty,\lambda]\psi,\, \psi)$. Namely, assume that μ_ψ is uniformly α-Hölder (abbreviated UαH), i.e., there exist α in $[0,1]$ and a uniform constant c such that, for all ε in $(0,1)$ and μ_φ-a.e. x, one has $\mu_\psi(x-\varepsilon, x+\varepsilon) \leq c\varepsilon^\alpha$. Then, it is proven in [7] that for all $T > 1$,

$$\langle\langle |X|^p \rangle_{\psi(t)}\rangle(T) \geq c_{\mu_\psi} T^{\frac{\alpha p}{d}} \ , \tag{2}$$

where c_{μ_ψ} is a constant depending only on μ_ψ. Later on, Combes [3] extended this result to models in the continuous case, i.e., for operators H acting on $L^2(\mathbf{R}^d)$. An improvement of the techniques lead to a larger exponent α in (2) (see [10], [2]). All the methods used are based on the same argument. Consider the case $\mathcal{H} = \ell^2(\mathbf{Z}^d)$, and denote by $\{\delta_n\}_{n\in\mathbf{Z}^d}$ its canonical basis. If there exists c, depending only on μ_ψ, such that for all n in $\mathbf{Z}^d$ and T larger than 1

$$\langle |\psi(t,n)|^2 \rangle(T) \equiv T^{-1} \int_0^T |(\delta_n, \psi(t))|^2 dt \leq cT^{-\alpha} \ , \tag{3}$$

then,

$$B_\psi(T,N) \equiv \sum_{n\in\mathbf{Z}^d,\, |n|\leq N} \langle |\psi(t,n)|^2 \rangle(T) \leq c(2N+1)^d T^{-\alpha} \ , \tag{4}$$

where $|\cdot|$ denotes the sup norm in $\mathbf{Z}^d$. For $N(T) = \frac{1}{4}(\frac{T^\alpha}{c})^{\frac{1}{d}}$, we thus have,

$$\begin{aligned}\langle\langle|X|^p\rangle_{\psi(t)}\rangle(T) &\geq N(T)^p \sum_{n\in\mathbf{Z}^d,\,|n|>N(T)} \langle|\psi(t,n)|^2\rangle(T) \\ &= N(T)^p(1-B_\psi(T,N(T))) = c'T^{\frac{\alpha p}{d}} \ . \end{aligned} \tag{5}$$

Clearly, the key estimate in this method is inequality (3). This property can be obtained if, for all complex measures $d\nu_n(\lambda) \equiv d(P_H(-\infty,\lambda]\varphi,\,\delta_n)$, we can found a uniform upper bound of the type:

$$\langle|\hat{\nu}_n(t)|^2\rangle(T) \equiv T^{-1}\int_0^T \left|\int_{\mathbf{R}} \mathrm{e}^{-it\lambda} d\nu_n(\lambda)\right|^2 dt \leq c_{\mu_\psi} T^{-\alpha} \ , \tag{6}$$

for some finite constant c_{μ_ψ} depending only on μ_ψ. Actually, Strichartz Theorem [15] (see Theorem D.1 in Appendix D) provides such an estimate under the assumption that μ_ψ is UαH. One can observe that using the upper bound (3) uniform in n is not the best way to estimate $B_\psi(T,N)$. In fact, the values $|\psi(t,n)|^2$ are not independent for different n and one can expect that $B_\psi(T,N)$ is significantly less than $(2N+1)^d \sup_n\langle|\psi(t,n)|^2\rangle(T)$. The main idea of this paper is to evaluate $B_\psi(T,N)$ directly, representing this quantity as a scalar product in the space $\mathcal{H}\otimes\mathcal{H}$, and thus taking into account the quantum interference effects. This method was used by Tcheremchantsev [19], [20], to study dynamical properties for Markovian Anderson model. In our case we obtain

$$B_\psi(T,N) \leq c_0 N^{\frac{d}{2}} \left(S_\psi(T^{-1})\right)^{\frac{1}{2}}, \tag{7}$$

where $S_\psi(z) = z\int_0^{+\infty} \exp(-tz)|\hat{\mu}_\psi(t)|^2 dt$, and for some constant c_0 depending on ψ, giving thus

$$\langle\langle|X|^p\rangle_{\psi(t)}\rangle(T) \geq c_1 \left(S_\psi(T^{-1})\right)^{-\frac{p}{d}}. \tag{8}$$

The function $S_\psi(T^{-1})$ is very close (up to logarithmical factors) to the autocorrelation function $C_\psi(T) = T^{-1}\int_0^T |(\psi, e^{-itH}\psi)|^2 dt$. So, the inequality (8) establishes a direct relation between the two most important dynamical quantities of the quantum system. The advantage of the estimates (7) and (8) is that we do not assume any continuity properties for the spectral measure μ_ψ, and the result is valid for any T,ψ with some constants depending only on

p and d. In particular, if one has information on $S_\psi(T_k^{-1})$ for a subsequence $T_k \nearrow +\infty$, we get the corresponding lower bound for $\langle\langle |X|^p\rangle_{\psi(t)}\rangle(T_k)$.

The standard method to improve the estimates (5) or (8) consists in using the bounds (4) or (7) for some φ such that $(\psi, \varphi) \neq 0$ (see [10], [2]). That gives us the following result (Theorem 2.5): For any $\varphi \neq 0$ and $T > 1$,

$$\langle\langle |X|^p\rangle_{\psi(t)}\rangle(T) \geq c_{p,\psi} \|\varphi\|^{-2} |(\psi, \varphi)|^{2+\frac{4p}{d}} \left(S_\varphi(T^{-1})\right)^{-\frac{p}{d}} . \tag{9}$$

In $L^2(\mathbf{R}^d)$, inequality (9) is valid for φ such that $\varphi = P_H([-R, R])\varphi$, for some $R > 0$.

What is important in this result, is the control of the constant depending on φ, which is impossible in the classic approach where φ is $U\alpha H$.

There are two possible ways to apply this result. In a traditional manner, one can choose some φ and then study the behaviour of $S_\varphi(z)$ as $z \to +0$. This behaviour is determined by correlations dimensions of measure μ_φ. With this approach we obtain (Sections 3 and 4) most of existing results about the power-law estimates

$$\langle\langle |X|^p\rangle_{\psi(t)}\rangle \geq cT^{\frac{p\alpha}{d}}$$

uniform in time. In fact, our method can be also applied to obtain more general bounds like

$$\langle\langle |X|^p\rangle_{\psi(t)}\rangle(T) \geq \ h(T),$$

where $h(T) \geq 0$, and $\limsup_{T\to+\infty} h(T) = +\infty$. In particular, one can obtain lower bounds for subsequences of time $T_k \nearrow +\infty$.

The bounds (9), with a fixed φ, are not necessary optimal as $T \to +\infty$. Let us fix ψ, p and consider for simplicity the case $\mathcal{H} = \ell^2(\mathbf{Z}^d)$. For any $T > 0$ one can define the function

$$L(T) = \sup_{\varphi\neq 0} \|\varphi\|^{-2} |(\psi, \varphi)|^{2+\frac{4p}{d}} \left(S_\varphi(T^{-1})\right)^{-\frac{p}{d}} .$$

Thus, from (9), $\langle\langle |X|^p\rangle_{\psi(t)}\rangle(T) \geq c_{p,\psi} L(T)$ for any $T > 1$. The function $L(T)$ gives us the "ideal" lower bound at any time T. To reach this optimal bound, one should choose the function φ in (9) depending on T. This is a completely new possibility which does not exist in the traditional approach using the Strichartz Theorem. The examples show that the optimal lower bound $L(T)$ may be significantly better than any of the bounds obtained with a fixed φ.

In particular, for some ψ such that μ_ψ is pure point, we have $L(T) \geq T^\gamma$ with $\gamma > 0$.

The paper is organized as follows: In Section 2, we give our central Theorem 2.1 which implies the estimates (7) and (8). We then give refinement of the method in order to get better estimates on moments (Theorem 2.5). We show also that to get the optimal lower bound for $\langle\langle |X|^p\rangle_{\psi(t)}\rangle$, it is sufficient to take the functions φ in (9) from the cyclic subspace of vector ψ. In Section 3 we give some direct applications of Theorem 2.5. In particular, we obtain a necessary condition for dynamical localization in the case of pure point spectrum. The next section is devoted to the translation of our results in terms of fractal dimensions of spectral measure. We emphasize in particular the role of the so-called correlation dimensions. We give the links between the dimensions introduced in this paper and those already used in earlier works [10],[2]. All the results of Sections 3 and 4 are obtained in a traditional way, that is, choosing φ independent of T. In Appendix A we adapt our results to the discrete time case. In Appendix B we discuss a new possibility to choose φ depending on T in the case of operators with pure point spectrum. In Appendix C we give some applications of our results to concrete models of Schrödinger operators with singular continuous spectrum. Appendix D is devoted to Strichartz Theorem and a proof of its converse.

2 Main results

Our central result in this section is Theorem 2.1, whose proof is based on the idea used by Tcheremchantsev in [19] and [20]. It is the key estimate needed to establish theorem 2.5 and 3.1 on the dynamical properties for quantum systems. In all this section, $\mathcal{H}$ denotes the Hilbert space $\ell^2(\mathbf{Z}^d)$ or $L^2(\mathbf{R}^d)$, equipped with the usual scalar product $(\ ,\ )$. The statement of Theorem 2.1 remains true if one consider a general Hilbert space, but it is sufficient for our purpose to restrict ourselves to these specific cases.

THEOREM 2.1 *(Double-space method).*

Let H be a self-adjoint operator on $\mathcal{H}$ and A a Hilbert-Schmidt operator on $\mathcal{H}$. Let $\psi \in \mathcal{H}$ be such that for any $t > 0$, $(\psi(t), A\psi(t)) \geq 0$, where $\psi(t) = e^{-itH}\psi$ is the solution at time t of the Schrödinger equation associated to H, with initial condition ψ. We denote by μ_ψ the spectral measure of ψ

associated to H. For $z > 0$ define the function

$$S_\psi(z) = z \int_0^\infty \mathrm{e}^{-tz} |\hat{\mu}_\psi(t)|^2 \, dt = z^2 \int_{\mathbf{R}} \int_{\mathbf{R}} \frac{d\mu_\psi(x) d\mu_\psi(y)}{(x-y)^2 + z^2} \leq \|\psi\|^4, \tag{10}$$

where $\hat{\mu}_\psi(t) = \int_{\mathbf{R}} \mathrm{e}^{-itx} d\mu_\psi(x)$ is the Fourier transform of the measure μ_ψ. The following estimate, uniform in A, T and ψ, holds:

$$0 \leq \langle\langle A \rangle_{\psi(t)} \rangle(T) \equiv T^{-1} \int_0^T (\psi(t), \, A\psi(t)) \, dt \leq \mathrm{e} \|A\|_2 \left(S_\psi(T^{-1}) \right)^{\frac{1}{2}} , \tag{11}$$

where $\|A\|_2$ is the Hilbert-Schmidt norm of A.

Remarks 2.2. Note that the operator A is not necessarily positive or self-adjoint.

For the proof we first need an intermediate result from the theory of tensor product of Hilbert spaces:

LEMMA 2.3 *Let H be a self-adjoint operator in $\mathcal{H}$ with domain $\mathbf{D}(H)$. Consider the following operator acting on $\mathbf{D}^2 \equiv \mathbf{D}(H) \otimes \mathbf{D}(H)$,*

$$M_0 = H \otimes \mathbf{1} - \mathbf{1} \otimes H \, .$$

This operator is essentially self-adjoint in $\mathcal{H} \otimes \mathcal{H}$ (see e.g. [13]). We shall denote by M it's closure. Let

$$\psi(t) = e^{-itH} \psi, \quad f(t) \equiv \psi(t) \otimes C\psi(t) \ ,$$

where C is the complex conjugation operator, i.e., $(C\psi)(t,x) = \overline{\psi(t,x)}$. Then we have

$$f(t) = \mathrm{e}^{-itM} f, \quad f = \psi \otimes C\psi. \tag{12}$$

Proof of Theorem 2.1. Since A is a Hilbert-Schmidt operator, there exist two orthonormal basis of $\mathcal{H}$, $\{\varphi_n\}_{n=1}^\infty$ and $\{\chi_n\}_{n=1}^\infty$, and a monotonely decreasing sequence $\{E_n\}_{n=1}^\infty$, $E_n \geq 0$ such that

$$A = \sum_{n=1}^\infty E_n (\varphi_n, \, .) \chi_n \, . \tag{13}$$

Thus, $\|A\|_2^2 = \sum_{n=1}^{\infty} E_n^2$, and we have the identity

$$(\psi(t), A\psi(t)) = \sum_{n=1}^{\infty} E_n(\varphi_n,\, \psi(t))\,(\psi(t),\, \chi_n)\ . \tag{14}$$

Now, with the notations of Lemma 2.3, we write

$$(\varphi_n,\, \psi(t))\,(\psi(t),\, \chi_n) = (\varphi_n,\, \psi(t))\,(C\chi_n,\, C\psi(t)) = (g_n,\, f(t))_{\mathcal{H}\otimes\mathcal{H}}\ ,$$

where $g_n = \varphi_n \otimes C\chi_n$. Equation (14) together with (12) implies

$$(\psi(t),\, A\psi(t)) = \sum_{n=1}^{\infty} E_n(g_n,\, \mathrm{e}^{-itM} f)_{\mathcal{H}\otimes\mathcal{H}} = (g,\, \mathrm{e}^{-itM} f)_{\mathcal{H}\otimes\mathcal{H}}\ , \tag{15}$$

where $g = \sum_{n=1}^{\infty} E_n g_n = \sum_{n=1}^{\infty} E_n \varphi_n \otimes C\chi_n$. Since $\{\varphi_n\}_{n=1}^{\infty}$ and $\{\chi_n\}_{n=1}^{\infty}$ are orthonormal basis,

$$\|g\|_{\mathcal{H}\otimes\mathcal{H}}^2 = \sum_{n=1}^{\infty} E_n^2 = \|A\|_2^2\ . \tag{16}$$

We shall use now the spectral theorem for the self-adjoint operator M in $\mathcal{H}\otimes\mathcal{H}$. Let μ_f^M be the spectral measure of f. We have

$$(g,\, \mathrm{e}^{-itM} f)_{\mathcal{H}\otimes\mathcal{H}} = \int_{\mathbf{R}} \mathrm{e}^{-itx} \tilde{g}(x) d\mu_f^M(x)\ , \tag{17}$$

where $\tilde{g} \in L^2(\mathbf{R}, d\mu_f^M)$ and

$$\|\tilde{g}\|_{L^2(\mathbf{R}, d\mu_f^M)} \le \|g\|_{\mathcal{H}\otimes\mathcal{H}}\ , \tag{18}$$

(see e.g. [1] or [7]). The Fourier transform of μ_f^M is related to that of μ_ψ in a very simple manner, namely

$$\begin{aligned}\hat{\mu}_f^M(t) &= (f,\, \mathrm{e}^{-itM} f)_{\mathcal{H}\otimes\mathcal{H}} = (\psi \otimes C\psi,\, \psi(t) \otimes C\psi(t))_{\mathcal{H}\otimes\mathcal{H}} \\ &= |(\psi,\, \psi(t))|^2 = |\hat{\mu}_\psi(t)|^2\ .\end{aligned} \tag{19}$$

Since for any positive t we have $\langle A\rangle_{\psi(t)} = (\psi(t),\, A\psi(t)) \ge 0$, we obtain

$$\langle\langle A\rangle_{\psi(t)}\rangle(T) = T^{-1}\int_0^T (\psi(t),\, A\psi(t))\, dt \le \frac{\mathrm{e}}{T}\int_0^{\infty} \mathrm{e}^{-\frac{t}{T}}(\psi(t),\, A\psi(t))\, dt\ . \tag{20}$$

We now estimate the function

$$L_\psi(z) \equiv z \int_0^\infty \mathrm{e}^{-tz} (\psi(t),\, A\psi(t)) dt$$

for z in $(0,1)$. It follows from (15) and (17) that

$$0 \le L_\psi(z) = z \int_0^\infty \left(\mathrm{e}^{-tz} \int_{\mathbf{R}} \mathrm{e}^{-itx} \tilde{g}(x) d\mu_f^M(x) \right) dt = \int_{\mathbf{R}} \frac{z\tilde{g}(x)}{z+ix} d\mu_f^M(x) \ . \quad (21)$$

The Cauchy-Schwartz inequality applied to (21) implies

$$0 \le L_\psi(z) \le \left(\int_{\mathbf{R}} \frac{z^2 d\mu_f^M(x)}{z^2+x^2} \right)^{\frac{1}{2}} \|\tilde{g}\|_{L^2(\mathbf{R}, d\mu_f^M)} \ . \quad (22)$$

Let us calculate the Laplace transform of $\hat{\mu}_f^M$ for $z \in (0,1)$:

$$\int_0^\infty \mathrm{e}^{-tz} \hat{\mu}_f^M(t) dt = \int_0^\infty \left(\mathrm{e}^{-tz} \int_{\mathbf{R}} \mathrm{e}^{-itx} d\mu_f^M(x) \right) dt = \int_{\mathbf{R}} \frac{d\mu_f^M(x)}{z+ix} \quad (23)$$

Since from (19) one has $\hat{\mu}_f^M(t) = |\hat{\mu}_\psi(t)|^2$ for any positive t, then the integral in (23) is real and we get

$$S_\psi(z) \equiv z \int_0^\infty \mathrm{e}^{-tz} |\hat{\mu}_\psi(t)|^2 dt = z\mathrm{Re} \left(\int_{\mathbf{R}} \frac{d\mu_f^M(x)}{z+ix} \right) = z^2 \int_{\mathbf{R}} \frac{d\mu_f^M(x)}{z^2+x^2} \ . \quad (24)$$

Comparing (22) and (24), one obtains

$$0 \le L_\psi(z) \le (S_\psi(z))^{\frac{1}{2}} \|\tilde{g}\|_{L^2(\mathbf{R}, d\mu_f^M)} \ . \quad (25)$$

Using (16) and (18) thus gives

$$0 \le L_\psi(z) \le \|A\|_2 \, (S_\psi(z))^{\frac{1}{2}} \ . \quad (26)$$

Finally, from (20) and (26) we get the expected result:

$$0 \le \langle\langle A \rangle_{\psi(t)} \rangle(T) \le \mathrm{e} \|A\|_2 \left(S_\psi(T^{-1}) \right)^{\frac{1}{2}} \ .$$

The proof is completed. ∎

The obtained result needs some comments. First, if the measure μ we consider is UαH, then from Theorem D.1 due to Strichartz, we have $\langle|\hat{\mu}_\psi(t)|^2\rangle(T) = \langle|(\psi,\, \psi(t))|^2\rangle(T) \leq c_\psi T^{-\alpha}$, and thus $S_\psi(z) \leq \mathrm{e} c_\psi z^\alpha$. In this case, (11) is equivalent to the statement of the theorem 3.2 of Last [10] with $p = 2$ (for positive operators A). On the other hand, if one only knows that $\langle|\hat{\mu}_\psi(t)|^2\rangle(T) \leq c_\psi T^{-\alpha}$, then, with $A = (\varphi,\, \cdot\,)\varphi$ for some $\varphi \in \mathcal{H}$, Theorem 2.1 gives $\langle|(\varphi,\, \psi(t))|^2\rangle(T) \leq c_\psi T^{-\frac{\alpha}{2}}\|\varphi\|^2$, which is consistent with the fact that, in this case, μ_ψ is at most U$(\alpha/2)$H (see Theorem 3.1 in [10]).

The result of Theorem 2.1 is, however, much more general than the result of Last for $p = 2$. We do not assume any continuity properties of spectral measure, and the LHS of (11) is directly connected to the Fourier transform of the spectral measure through $S_\psi(z)$.

COROLLARY 2.4 *Let H be a self-adjoint operator on $\mathcal{H}$. If $\mathcal{H} = L^2(\mathbf{R}^d)$, we suppose, as in Theorem 6.2 in [10], that $H = \Delta + Q(x)$ with $Q_+ \in K_d^{loc}$ and $Q_- \in K_d$ (see e.g. [17] for the definitions). We pick ψ in $\mathcal{H}$ and denote $\psi(t) = \mathrm{e}^{-itH}\psi$. Furthermore, if $\mathcal{H} = L^2(\mathbf{R}^d)$, we assume that there exists $R > 0$ such that $P_H([-R, R])\psi = \psi$, where P_H is the spectral family for H. Define*

$$\begin{aligned} B_\psi(T, N) &\equiv T^{-1}\int_0^T \left(\sum_{n,\,|n|\leq N} |\psi(t, n)|^2\right) dt\ , \quad \textit{if } \mathcal{H} = \ell^2(\mathbf{Z}^d)\ , \\ B_\psi(T, N) &\equiv T^{-1}\int_0^T \left(\int_{|x|\leq N} |\psi(t, x)|^2 dx\right) dt\ , \quad \textit{if } \mathcal{H} = L^2(\mathbf{R}^d)\ . \end{aligned}$$

For all N and T, the following estimate holds,

$$B_\psi(T, N) \leq cN^{\frac{d}{2}}\left(S_\psi(T^{-1})\right)^{\frac{1}{2}}\ , \tag{27}$$

where c is a finite constant independent of T, N and ψ, and depending only on R if $\mathcal{H} = L^2(\mathbf{R}^d)$.

Proof. If $H = \ell^2(\mathbf{Z}^d)$, then one has

$$B_\psi(T, N) = \langle\langle A\rangle_{\psi(t)}\rangle(T)$$

where $A = \sum_{n\in\mathbf{Z}^d,\,|n|\leq N}(\delta_n,\,\cdot\,)\,\delta_n$ and $\delta_n(k) = \delta_{n,k}$. As

$$\|A\|_2 = (\mathrm{Card}\{n \in \mathbf{Z}^d\ :\ |n| \leq N\})^{1/2} \leq (2N+1)^{d/2},$$

the result of Theorem 2.1 gives immediately (27). If $\mathcal{H} = L^2(\mathbf{R}^d)$, we first define the operator K_N as follows: Let $\psi \in \mathcal{H}$, then

$$(K_N\psi)(x) = F(|x| \leq N)\psi(x),$$

where F is the characteristic function of the cube $\{x \in \mathbf{R}^d \,:\, |x| \leq N\}$. Since $\psi = P_H([-R, R])\psi$, we can write

$$\int_{|x|\leq N} |\psi(t,x)|^2 dx = (K_N P_H([-R,R])\psi(t),\, \psi(t)) \geq 0.$$

It follows from the result of Simon [17] that the operator

$$A = K_N P_H([-R,R]))$$

is Hilbert-Schmidt in $L^2(\mathbf{R}^d)$ and $\|A\|_2 \leq C(R)N^{d/2}$, which, together with Theorem 2.1, imply (27). ∎

THEOREM 2.5 *Let H be a self-adjoint operator which satisfies conditions of Corollary 2.4. We pick a vector ψ in $\mathcal{H}$.*

Let $\varphi \neq 0$ be any vector in $\mathcal{H}$. In the case $\mathcal{H} = L^2(\mathbf{R}^d)$, we also assume that $P_H([-R,R])\varphi = \varphi$ for some $R > 0$. Then, we have the following estimate uniform in φ, d and $T \geq 1$:

$$\begin{aligned} \langle\langle |X|^p\rangle_{\psi(t)}\rangle(T) &\equiv T^{-1}\int_0^T (\psi(t),\, |X|^p\psi(t))dt \\ &\geq c_{p,\psi}(R)\|\varphi\|^{-2}|(\psi,\varphi)|^{2+\frac{4p}{d}}\left(S_\varphi(T^{-1})\right)^{-\frac{p}{d}}\,, \end{aligned} \tag{28}$$

where $c_{p,\psi}(R)$ is a constant depending on p, ψ, and on R if $\mathcal{H} = L^2(\mathbf{R}^d)$.

Remarks 2.6. We do not assume that ψ or $\psi(t)$ belongs to the domain of $|X|^{\frac{p}{2}}$. This means that the quantity to the left hand side in (28) may be infinite. In this case the inequality is trivially true.

Proof. We shall give it in the case $\mathcal{H} = \ell^2(\mathbf{Z}^d)$, because in $L^2(\mathbf{R}^d)$ it is essentially the same. First suppose that $\psi = \varphi + \chi$, where $(\varphi,\chi) = 0$, $\varphi \neq 0$. Let us estimate the function $B_\psi(T,N)$. It is easy to check, (see e.g. [10] or [2]) that

$$B_\psi(T,N) \leq \|\chi\|^2 + B_\varphi(T,N) + 2\|\chi\|\left(B_\varphi(T,N)\right)^{\frac{1}{2}}\,. \tag{29}$$

As $B_\varphi(T,N) \le \|\varphi\|^2 \le \|\psi\|^2$, and $\|\psi\|^2 = \|\varphi\|^2 + \|\chi\|^2$, (29) implies

$$B_\psi(T,N) \le \|\psi\|^2 - \|\varphi\|^2 + c_\psi \left(B_\varphi(T,N)\right)^{\frac{1}{2}}. \tag{30}$$

Let us choose $N(T) = a\|\varphi\|^{\frac{8}{d}}(S_\varphi(T^{-1}))^{-\frac{1}{d}}$. With the result of Corollary 2.4 for the vector φ and (30) we obtain

$$B_\psi(T, N(T)) \le \|\psi\|^2 - \frac{\|\varphi\|^2}{2}$$

if $a \le a_0(\psi)$, where a_0 is a strictly positive constant depending only on ψ (and on R in the continuous case). We shall take $a = a_0(\psi)$ in the definition of $N(T)$. The moments of the position operator can be now estimated in a standard manner. For instance, if $\mathcal{H} = \ell^2(\mathbf{Z}^d)$, we write

$$\begin{aligned}
\langle\langle |X|^p\rangle_{\psi(t)}\rangle(T) &= T^{-1}\int_0^T \left(\sum_{n\in\mathbf{Z}^d} |n|^p|\psi(t,n)|^2\right) dt \\
&\ge T^{-1}\int_0^T \left(\sum_{|n|>N(T)} |n|^p|\psi(t,n)|^2\right) dt \\
&\ge N^p(T)\left(\|\psi\|^2 - B_\psi(T,N(T))\right) \\
&\ge \frac{a_0^p(\psi)}{2}\|\varphi\|^{2+\frac{8p}{d}}\left(S_\varphi(T^{-1})\right)^{-\frac{p}{d}}.
\end{aligned} \tag{31}$$

As $\psi = \varphi + \chi$, where $(\varphi,\chi) = 0$, one has $(\psi,\varphi) = \|\varphi\|^2$, therefore (31) implies (28).

Consider now the general case. Let $\varphi \in \mathcal{H}$ be such that $\varphi \ne 0$. If $(\psi,\varphi) = 0$, the estimate (28) is trivially true, so we can assume that $(\psi,\varphi) \ne 0$. One can write the identity

$$\psi = \frac{(\psi,\varphi)}{\|\varphi\|^2}\varphi + \chi \equiv \tilde{\varphi} + \chi,$$

where $(\tilde{\varphi},\chi) = 0$, $\tilde{\varphi} \ne 0$. It is easy to see that $\|\tilde{\varphi}\| = \|\varphi\|^{-1}|(\psi,\varphi)|$ and

$$S_{\tilde{\varphi}}(z) = \|\varphi\|^{-8}|(\psi,\varphi)|^4 S_\varphi(z).$$

The estimate (28) follows directly from the bound (31) for $\tilde{\varphi}$.

The proof in the continuous case is similar. ∎

COROLLARY 2.7 *Suppose that the conditions of Theorem 2.5 are satisfied. Let us fix $\psi \neq 0,\ p > 0$. For any $\varphi \neq 0$ define the function*

$$L(\varphi, T) = \|\varphi\|^{-2} |(\psi, \varphi)|^{2+\frac{4p}{d}} \left(S_\varphi(T^{-1}) \right)^{-\frac{p}{d}}.$$

1. Suppose that $\mathcal{H} = \ell^2(\mathbf{Z}^d)$. Let $L(T) = \sup_{\varphi : \varphi \neq 0} L(\varphi, T)$. Then, uniformly in $T \geq 1$, we have

$$\langle \langle |X|^p \rangle_{\psi(t)} \rangle(T) \geq c_{p,\psi} L(T).$$

2. Suppose that $\mathcal{H} = L^2(\mathbf{R}^d)$. For fixed $0 < R < \infty$, define the set

$$W_R = \{\varphi : \varphi \in \mathcal{H}, \varphi \neq 0,\ P_H([-R, R])\varphi = \varphi\}.$$

Let $L_R(T) = \sup_{\varphi : \varphi \in W_R} L(\varphi, T)$. The uniform estimate holds:

$$\langle \langle |X|^p \rangle_{\psi(t)} \rangle(T) \geq c_{p,\psi}(R) L_R(T).$$

The results follow directly from Theorem 2.5.

COROLLARY 2.8 *Let $\mathcal{H} = \ell^2(\mathbf{Z}^d)$. For any $\psi \neq 0$, $p > 0$ and d, define two exponents:*

$$\gamma^+(\psi, p/d) = \limsup_{T \to +\infty} \frac{\log L(T)}{\log T},$$

$$\gamma^-(\psi, p/d) = \liminf_{T \to +\infty} \frac{\log L(T)}{\log T},$$

In the case $\mathcal{H} = L^2(\mathbf{R}^d)$ we set

$$\gamma^+(\psi, p/d) = \sup_{R : R > 0} \limsup_{T \to +\infty} \frac{\log L_R(T)}{\log T},$$

$$\gamma^-(\psi, p/d)) = \sup_{R : R > 0} \liminf_{T \to +\infty} \frac{\log L_R(T)}{\log T}.$$

Define also

$$\alpha^+(\psi, p) = \limsup_{T \to +\infty} \frac{\log \langle \langle |X|^p \rangle_{\psi(t)} \rangle(T)}{\log T},$$

$$\alpha^-(\psi, p) = \liminf_{T \to +\infty} \frac{\log \langle \langle |X|^p \rangle_{\psi(t)} \rangle(T)}{\log T}.$$

In both discrete and continuous cases the following estimates hold:

$$\alpha^+(\psi, p) \geq \gamma^+(\psi, p/d), \quad \alpha^-(\psi, p) \geq \gamma^-(\psi, p/d).$$

The results is a direct consequence of Corollary 2.7.

One should make some observations about the functions $L(T)$, $L_R(T)$. First, it is easy to see from definition of $S_\varphi(z)$ that $L(a\varphi, T) = L(\varphi, T)$ for any complex number a. Therefore,

$$L(T) = \sup_{\varphi:\varphi\neq 0} L(\varphi, T) = \sup_{\varphi:\|\varphi\|=1} L(\varphi, T),$$

and the same is true for $L_R(T)$.

Second, it is sufficient to take φ from the cyclic subspace spanned by ψ.

LEMMA 2.9 *Denote by P_ψ the orthogonal projection on $\mathcal{H}_\psi$, the cyclic subspace spanned by ψ and H. For any $\varphi \neq 0$ and $T > 1$, we have $L(\varphi, T) \leq L(P_\psi\varphi, T)$. Therefore*

$$L(T) = \sup_{\varphi:\varphi\neq 0, \varphi\in\mathcal{H}_\psi} L(\varphi, T) = \sup_{\varphi:\|\varphi\|=1,\ \varphi\in\mathcal{H}_\psi} L(\varphi, T)\ , \tag{32}$$

$$L_R(T) = \sup_{\varphi:\varphi\in W_R\cap\mathcal{H}_\psi} L(\varphi, T) = \sup_{\varphi:\|\varphi\|=1,\ \varphi\in W_R\cap\mathcal{H}_\psi} L(\varphi, T)\ . \tag{33}$$

Proof. Let $\varphi \in \mathcal{H}$ be such that $\varphi \neq 0$ and $\varphi = \varphi_1 + \varphi_2$, where $\varphi_1 = P_\psi\varphi$. We have

$$(\psi, \varphi) = (\psi, \varphi_1), \quad \|\varphi\|^2 = \|\varphi_1\|^2 + \|\varphi_2\|^2. \tag{34}$$

Furthermore, for any Borel set S we can write

$$\begin{aligned} \mu_\varphi(S) &= (\varphi, P_H(S)\varphi) \\ &= (\varphi_1, P_H(S)\varphi_1) + (\varphi_2, P_H(S)\varphi_2) + 2\mathrm{Re}(\varphi_1, P_H(S)\varphi_2)\ . \end{aligned} \tag{35}$$

As $\varphi_1 = P_\psi\varphi$ and $\varphi_2 \perp \mathcal{H}_\psi$, we have

$$(\varphi_1, P_H(S)\varphi_2) = (\varphi, P_\psi P_H(S)\varphi_2) = (\varphi, P_H(S)P_\psi\varphi_2) = 0.$$

We obtain therefore from (35) that $\mu_\varphi = \mu_{\varphi_1} + \mu_{\varphi_2}$. As

$$S_\varphi(z) = z^2 \int\int \frac{d\mu_\varphi(x)d\mu_\varphi(y)}{z^2 + (x-y)^2},$$

it is clear that $S_\varphi(z) \geq S_{\varphi_1}(z)$. With (34) we estimate

$$\begin{aligned} L(\varphi, T) &= (\|\varphi_1\|^2 + \|\varphi_2\|^2)^{-1}|(\psi, \varphi_1)|^{2+\frac{4p}{d}} \left(S_\varphi(T^{-1})\right)^{-\frac{p}{d}} \\ &\leq \|\varphi_1\|^{-2}|(\psi, \varphi_1)|^{2+\frac{4p}{d}} \left(S_{\varphi_1}(T^{-1})\right)^{-\frac{p}{d}} = L(\varphi_1, T)\ , \end{aligned}$$

from which (32) follows. Now, (33) is an immediate consequence of the fact that, if $\varphi = P_H([-R,R])\varphi$, we have $\varphi_1 = P_\psi\varphi = P_\psi P_H([-R,R])\varphi = P_H([-R,R])P_\psi\varphi = P_H([-R,R])\varphi_1$. ∎

LEMMA 2.10 *Let $\psi \in \mathcal{H}$ and μ_ψ be its spectral measure. Consider the following sets in $L^2(\mathbf{R}, d\mu_\psi)$:*

$$G = \{g : g(x) \geq 0, \int_{\mathbf{R}} g^2(x) d\mu_\psi(x) = 1\},$$

$$G_R = \{g : g \in G, \quad \text{supp } g \subset [-R,R]\}.$$

For any $g \in L^2(\mathbf{R}, d\mu_\psi)$ define

$$K(g,z) = \left| \int g(x) d\mu_\psi(x) \right|^{2+\frac{4p}{d}} \left(z^2 \int\int \frac{|g(x)|^2 |g(y)|^2 d\mu_\psi(x) d\mu_\psi(y)}{z^2 + (x-y)^2} \right)^{-\frac{p}{d}}.$$

The following equalities hold:

$$L(T) = \sup_{g:g\in G} K(g, T^{-1}),$$

$$L_R(T) = \sup_{g:g\in G_R} K(g, T^{-1}).$$

Proof. It is well known (see e.g. [13] or [1]), that for any $\varphi \in \mathcal{H}_\psi$ there exists $g \in L^2(\mathbf{R}, d\mu_\psi)$ such that $\|\varphi\|_{\mathcal{H}} = \|g\|_{L^2(\mathbf{R},\, d\mu_\psi)}$ and

$$(\psi, \varphi) = \int g(x) d\mu_\psi(x), \quad d\mu_\varphi(x) = |g(x)|^2 d\mu_\psi(x).$$

The converse is also true: for any $g \in L^2(\mathbf{R}, d\mu_\psi)$ there exists corresponding $\varphi \in \mathcal{H}_\psi$. If $\|\varphi\| = 1$, one can easily verify that $L(\varphi, T) = K(g, T^{-1})$. One also has $\varphi \in W_R$ if and only if supp $g \subset [-R,R]$. Finally, one can observe that $|\int g(x) d\mu_\psi(x)| \leq \int |g(x)| d\mu_\psi(x)$. Therefore it is sufficient to take nonnegative $g(x)$. The result of the Lemma now follows directly from Lemma 2.9. ∎

The result of Lemma 2.10 shows that, for any $\psi \in \mathcal{H}$, the two exponents $\gamma^{\pm}(\psi, p/d)$ defined in the Corollary 2.8 are entirely determined by the properties of spectral measure μ_ψ (and depend also on the positive parameter $\beta = p/d$). Therefore one can call them "dynamical exponents" of the measure μ_ψ.

3 Some direct applications

The first result we shall prove gives lower bounds for the moments if the orthogonal projections of ψ on some closed subspaces are not zero. This is in spirit of [10], but the result is more general and the estimates are not necessarily power-law. Before proving it, let us first discuss the results of this kind which can be obtained in a traditional manner. For simplicity we shall discuss only discrete case.

If φ is such that μ_φ is $U\alpha H$, then the Strichartz Theorem implies $B_\varphi(T,N) \leq cN^dT^{-\alpha}$ (see inequality (4)) . With this result one can prove the result similar to that of Theorem 2.5: if $\psi \in \mathcal{H}$, $(\psi,\varphi) \neq 0$ and μ_φ is $U\alpha H$, then

$$\langle\langle |X|^p\rangle_{\psi(t)}\rangle(T) \geq c_{\psi,\varphi}T^{p\frac{\alpha}{d}}. \tag{36}$$

Consider the set $\mathcal{H}_{uh}(\alpha) = \{\varphi : \ \mu_\varphi \text{ is } U\alpha H\}$ and the closure of its finite linear combinations $\mathcal{H}_\alpha = \text{Span } \mathcal{H}_{uh}(\alpha)$. We easily see that the orthogonal projection $P_{\mathcal{H}_\alpha}\psi$ of ψ on $\mathcal{H}_\alpha$ is different from zero if and only if $(\psi,\varphi) \neq 0$ for some $\varphi \in \mathcal{H}_{uh}(\alpha)$. This gives us the following result :

Theorem $(\star)$ *Suppose that ψ is such that for some $\alpha > 0$, $P_{\mathcal{H}_\alpha}\psi \neq 0$, then estimate (36) holds.*

It is clear that this result is the best one can obtain with Strichartz Theorem and the power law in the estimate is the only possible one. What can be done further is the study of the subspaces $\mathcal{H}_\alpha$. In particular, one can obtain some sufficient conditions for $P_{\mathcal{H}_\alpha}\psi \neq 0$. Last showed [10] that $\mathcal{H}_{uh}(\alpha)$ is a linear set and that

$$\mathcal{H}_\alpha = \overline{\mathcal{H}_{uh}(\alpha)} = \mathcal{H}_{\alpha c} = \{\psi \,:\, \mu_\psi \text{ is } \alpha\text{−continuous}\}.$$

The result of [2] can be also considered from this point of view. In the proof of Theorem 3.1 of Barbaroux, Combes, Montcho, it was shown the following result: Let $\psi \in \mathcal{H}$ and $\gamma \equiv \mu_\psi - \text{ess.sup}\gamma^-(x)$, where $\gamma^-(x) = \liminf_{\varepsilon\to 0}\log(\mu_\psi(x-\varepsilon,x+\varepsilon))/\log\varepsilon$. Then for any $\nu > 0$ there exists φ such that μ_φ is $U(\gamma-\nu)H$ and $(\psi,\varphi) \neq 0$. So, in fact, $P_{\mathcal{H}_{\gamma-\nu}}\psi \neq 0$ for any $\nu > 0$. The Theorem $(\star)$ gives the estimate of Theorem 3.1 of [2]

$$\langle\langle |X|^p\rangle_{\psi(t)}\rangle(T) \geq cT^{\frac{p}{d}(\gamma-\nu)}.$$

For a given ψ, to get the best lower bounds for the moments, one should consider $\sup\{\alpha \,:\, P_{\mathcal{H}_\alpha}\psi \neq 0\}$. We shall discuss it in the next section.

Now we shall prove the result which generalises that of Theorem ($\star$).

THEOREM 3.1 *Let* $l : (0,1) :\to \mathbf{R}_+ \cup \{+\infty\}$ *be some positive function. Consider the following set in* $\mathcal{H}$*:*

$$\mathcal{D}_l \equiv \{\varphi \in \mathcal{H} \, : \, \exists c_\varphi < \infty \text{ s.t. } \; S_\varphi(z) \le c_\varphi l(z) \text{ for any } z \in (0,1)\} \; .$$

Let $\mathcal{L}$ *be the closure of the linear combinations of vectors in* $\mathcal{D}_l$*. If* $\psi \in \mathcal{H}$ *is such that the orthogonal projection* $P_{\mathcal{L}}\psi$ *of* ψ *on* $\mathcal{L}$ *is non-zero, then for all* $T \ge 1$*,*

$$\langle\langle |X|^p\rangle_{\psi(t)}\rangle(T) \ge c\left(l(T^{-1})\right)^{-\frac{p}{d}} \quad , \tag{37}$$

with some strictly positive constant c *uniform in* T*.*

Proof. Since $P_{\mathcal{L}}\psi \neq 0$, there exists $\varphi \in \mathcal{D}_l$ such that $(\psi,\, \varphi) \neq 0$. If $\mathcal{H} = \ell^2(\mathbf{Z}^d)$, the estimate (37) follows directly from Theorem 2.5. If $\mathcal{H} = L^2(\mathbf{R}^d)$, then one considers $\varphi_R = P_H([-R,R])\varphi$. For R large enough, $(\psi,\, \varphi_R) \neq 0$. Furthermore,

$$\begin{aligned} S_{\varphi_R}(z) &= z^2 \int_{\mathbf{R}}\int_{\mathbf{R}} \frac{d\mu_{\varphi_R}(x) d\mu_{\varphi_R}(y)}{(x-y)^2+z^2} \\ &= z^2 \int_{[-R,R]}\int_{[-R,R]} \frac{d\mu_{\varphi}(x) d\mu_{\varphi}(y)}{(x-y)^2+z^2} \le S_\varphi(z) \; . \end{aligned}$$

As $\varphi \in \mathcal{D}_l$, one obtains

$$S_{\varphi_R}(z) \le S_\varphi(z) \le c_\varphi l(z) \; . \tag{38}$$

The result now follows from (38) and the statement of Theorem 2.5. ∎

Remarks 3.2. Note that in the definition of $\mathcal{D}_l$, we have no assumption on the regularity of $l(z)$, except that it has to be defined for all z in $(0,1)$. In particular, given a decreasing sequence $\{z_k\}_{k\in\mathbf{N}}$, $z_k \searrow 0$, and a positive sequence $\{l_k\}_{k\in\mathbf{N}}$, $l_k \to 0$, we can set $l(z) = l_k$ if $z = z_k$ for some $k \in \mathbf{N}$, and $+\infty$ elsewhere. In this case the Theorem 3.1 gives a lower bound for the moments, for the values of time $T_k = z_k^{-1}$.

In the Appendix C we shall give some examples of applications of Theorem 2.5 and Theorem 3.1 to the concrete models. Here we shall use the Theorem 3.1 to give a simple proof of Theorems 6.1, 6.2 of [10]. We shall need the following elementary result.

LEMMA 3.3 *For any α in $[0,1]$, consider the set*

$$\mathcal{D}_\alpha \equiv \{\varphi \in \mathcal{H} \ : \ \exists c_\varphi < \infty \text{ s.t. } \langle |\hat{\mu}_\varphi|^2\rangle(T) \leq c_\varphi T^{-\alpha} \text{ for any } T > 1\} \ , \quad (39)$$

where $\langle |\hat{\mu}_\varphi|^2\rangle(T) \equiv T^{-1}\int_0^T |\hat{\mu}_\varphi|^2(t)dt$. Then the following identity holds:

$$\mathcal{D}_\alpha = \{\varphi \in \mathcal{H} \ : \exists c_\varphi < \infty \text{ s.t. } S_\varphi(z) \leq c_\varphi z^\alpha \text{ for any } z \in (0,1)\} \ .$$

Proof. If $S_\varphi(z) \leq c_\varphi z^\alpha$, then

$$\begin{aligned} \langle |\hat{\mu}_\varphi|^2\rangle(T) &\leq \mathrm{e}T^{-1}\int_0^\infty \mathrm{e}^{-\frac{t}{T}}|\hat{\mu}_\varphi|^2(t)dt \\ &= \mathrm{e}S_\varphi(T^{-1}) \leq \mathrm{e}c_\varphi T^{-\alpha} \ . \end{aligned} \quad (40)$$

On the other hand, let $\langle |\hat{\mu}_\varphi|^2\rangle(T) \leq c_\varphi T^{-\alpha}$, then for $F(T) = \int_0^T |\hat{\mu}_\varphi|^2(t)dt$, one has $F(T) \leq c_\varphi T^{1-\alpha}$. Integrating by parts, one can write the following identity:

$$\begin{aligned} S_\varphi(z) &= z\int_0^\infty \mathrm{e}^{-tz}|\hat{\mu}_\varphi|^2 dt = z\int_0^\infty \mathrm{e}^{-tz}F'(t)dt \\ &= z^2\int_0^\infty \mathrm{e}^{-tz}F(t)dt \leq z^2\int_0^\infty \mathrm{e}^{-tz}c_\varphi t^{1-\alpha}dt \\ &\leq c_\varphi z^\alpha \ . \end{aligned} \quad (41)$$

The estimates (40) and (41) give the statement of the Lemma. ∎

As a consequence of Theorem 3.1 and Lemma 3.3, one can establish:

COROLLARY 3.4 *Let $\mathcal{V}_\alpha = \mathrm{Span}\mathcal{D}_\alpha$ be the closure of the linear combinations of vectors from the set $\mathcal{D}_\alpha$ defined in Lemma 3.3 above. If $\psi \in \mathcal{H}$ is such that its orthogonal projection $P_{\mathcal{V}_\alpha}\psi$ on $\mathcal{V}_\alpha$ is non-zero, then uniformly in $T > 1$, one has*

$$\langle\langle |X|^p\rangle_{\psi(t)}\rangle(T) \geq c_{p,\psi}T^{\frac{p\alpha}{d}} \ , \quad (42)$$

for some strictly positive constant $c_{p,\psi}$.

If $P_{\mathcal{H}_{ac}}\psi \neq 0$, where $P_{\mathcal{H}_{ac}}$ is the orthogonal projection on the absolutely continuous subspace for H, then

$$\langle\langle |X|^p\rangle_{\psi(t)}\rangle(T) \geq c_{p,\psi}T^{\frac{p}{d}} \ . \quad (43)$$

Note that the proof of (43) is a straightforward consequence of the fact that (see e.g. [9])

$$\begin{aligned}\mathcal{V}_1 &= \overline{\{\varphi \in \mathcal{H} \ : \ \exists c_\varphi < \infty \text{ s.t. } \langle |\hat{\mu}_\varphi|^2 \rangle (T) \leq c_\varphi T^{-1} \text{ for any } T > 1\}} \\ &= \overline{\{\varphi \in \mathcal{H} \ : \ \hat{\mu}_\varphi \in L^2(\mathbf{R})\}} = \mathcal{H}_{ac} \ .\end{aligned}$$

Remarks 3.5. Given a self-adjoint operator H, we denote by $\mathcal{H}_{\alpha c}$ the set of φ in $\mathcal{H}$ such that the spectral measure μ_φ is absolutely continuous with respect to the Hausdorff measure h^α, i.e., for any $S \subset \mathbf{R}$, $h^\alpha(S) = 0$ implies $\mu_\varphi(S) = 0$. Last showed ([10], Theorem 5.3) that $\mathcal{H}_\alpha = \mathcal{H}_{\alpha c} \subset \mathcal{V}_\alpha$ for any α. This means that (42) implies the result of Theorems 6.1 and 6.2 in [10]. In the case $\alpha = 1$, it is known that $\mathcal{H}_{1c} = \mathcal{V}_1 = \mathcal{H}_{ac}$. If for some $\alpha \in (0, 1)$

$$\mathcal{V}_\alpha \neq \mathcal{H}_{\alpha c}, \tag{44}$$

then our result extends the result of Last. However, we do not know whether (44) occurs for some α.

In the last part of this section we shall apply our results to the problem of dynamical localization. Let H be a self-adjoint operator and ψ such that the spectral measure μ_ψ is pure point: $\mu_\psi = \sum_{m=1}^\infty \gamma_m \delta_{a_m}$, where $a_m \in \mathbf{R}$, $a_m \neq a_k$ for $m \neq k$, $\gamma_m > 0$ and $\sum_m \gamma_m = \|\psi\|^2$. Any function g in $L^2(\mathbf{R}, d\mu_\psi)$ is thus given by its values $g_m \equiv g(a_m)$. Let G, G_R be the sets defined in Lemma 2.10. It is easy to see that $g \in G$ if and only if $g_m \geq 0$ for any m and $\sum g_m^2 \gamma_m = 1$. One also has $g \in G_R$ if and only if $g \in G$ and $\sup_{m: g_m > 0} |a_m| \leq R$. One can also calculate, for g in G,

$$K(g, z) = \left(\sum_{m=1}^\infty g_m \gamma_m \right)^{2 + \frac{4p}{d}} \left(z^2 \sum_{m,k=1}^\infty \frac{g_m^2 g_k^2 \gamma_m \gamma_k}{z^2 + (a_m - a_k)^2} \right)^{-\frac{p}{d}} . \tag{45}$$

THEOREM 3.6 *For any $\beta > 0$, $g \in L^2(\mathbf{R}, d\mu_\psi)$ define*

$$K_\beta(g) = \left(\sum_{m=1}^\infty g_m \gamma_m \right)^{2+4\beta} \left(\sum_{m=1}^\infty g_m^4 \gamma_m^2 \right)^{-\beta} .$$

In the case $\mathcal{H} = \ell^2(\mathbf{Z}^d)$ we set

$$K_\beta = \sup_{g: g \in G} K_\beta(g)$$

and in the case $\mathcal{H} = L^2(\mathbf{R}^d)$ for any $R > 0$

$$K_{\beta,R} = \sup_{g: g \in G_R} K_\beta(g).$$

Suppose that $K_{p/d} = +\infty$ in the first case and there exists $R > 0$ such that $K_{p/d,R} = +\infty$ in the second case. Then

$$\lim_{T \to +\infty} \langle \langle |X|^p \rangle_{\psi(t)} \rangle (T) = +\infty.$$

Proof. It is obvious, by using dominated convergence theorem, that (45) implies for any g

$$\lim_{T \to +\infty} K(g, T^{-1}) = K_{p/d}(g).$$

The results of Corollary 2.7 and Lemma 2.10 yield

$$\liminf_{T \to +\infty} \langle \langle |X|^p \rangle_{\psi(t)} \rangle (T) \geq c K_{\frac{p}{d}}(g) \tag{46}$$

with a constant c uniform in $g \in G$ in the discrete case and uniform in $g \in G_R$ (but depending on R) in the continuous case. Taking the supremum on g in (46), one obtains the statement of the Theorem. ∎

One should stress that we do not control the upper bounds for the moments. Even if ψ is in the domain of $|X|^{\frac{p}{2}}$, it is possible that $\langle |X|^p \rangle_{\psi(t)} = +\infty$ for any $t > 0$. (See, however, [12], where it is proved in the case $H = -\Delta + Q$, for some class of potentials Q, that for ψ decreasing sufficiently fast at infinity, the moments always remain finite. See also [16] in the case of operator H with pure point spectrum). As we shall see in the Appendix B, the phenomenon of delocalization (i.e. $K_\beta = +\infty$) occurs if the weights γ_m decay slowly as $m \to \infty$. If $\psi = \delta_0$ and (e_m) is the orthonormal set of eigenfunctions of the operator H in $\ell^2(\mathbf{Z}^d)$ with a simple spectrum, we have $\gamma_m = |e_m(0)|^2$. This confirms that sufficiently fast decay of $|e_m(0)|$ is an essential condition for dynamical localisation [4], [6].

4 Bounds on the moments in terms of fractal exponents

LEMMA 4.1 *Let $H = H^*$, and $\psi \in \mathcal{H}$. We denote $C_\psi(T) \equiv \langle |\hat{\mu}_\psi|^2 \rangle(T) = T^{-1}\int_0^T |(\psi, \mathrm{e}^{-itH}\psi)|^2 dt$. Then there exists a finite constant c depending on ψ such that for any $T > 1$,*

$$\begin{aligned} C_\psi(T) &\leq \mathrm{e}\, S_\psi(T^{-1}) && (47) \\ S_\psi(T^{-1}) &\leq c \log T\, C_\psi(T \log T)\ . && (48) \end{aligned}$$

In particular, this implies

$$-\liminf_{z\to+0} \frac{\log S_\psi(z)}{\log z} = \limsup_{T\to\infty} \frac{\log C_\psi(T)}{\log T} = -D_2^-(\mu_\psi), \tag{49}$$

$$-\limsup_{z\to+0} \frac{\log S_\psi(z)}{\log z} = \liminf_{T\to\infty} \frac{\log C_\psi(T)}{\log T} = -D_2^+(\mu_\psi)\ . \tag{50}$$

The quantities $D_2^\pm(\mu_\psi)$ are the correlation dimensions of the measure μ_ψ (see e.g. [2]).

Proof. Inequality (47) is obvious. Deriving (48) is easily done by writing

$$\begin{aligned} S_\psi(z) &= z\left(\int_0^{\frac{|\log z|}{z}} \mathrm{e}^{-tz}|\hat{\mu}_\psi|^2(t)\, dt + \int_{\frac{|\log z|}{z}}^\infty \mathrm{e}^{-tz}|\hat{\mu}_\psi|^2(t)\, dt\right) \\ &\leq |\log z| C_\psi(\frac{|\log z|}{z}) + z\|\psi\|^4. \end{aligned}$$

The first two equalities in (49) and (50) follow directly from (47) and (48), and the last two equalities were proven in [8] and [2]. ∎

LEMMA 4.2 *For any $p > 0$, ψ define, as in Corollary 2.8, the two growth exponents*

$$\alpha^+(\psi, p) = \limsup_{T\to+\infty} \frac{\log\langle\langle |X|^p\rangle_{\psi(t)}\rangle(T)}{\log T},$$

$$\alpha^-(\psi, p) = \liminf_{T\to+\infty} \frac{\log\langle\langle |X|^p\rangle_{\psi(t)}\rangle(T)}{\log T}.$$

Let φ be such that $(\psi, \varphi) \neq 0$, and $\varphi = P_H([-R, R])\varphi$ for some $R > 0$ in the case $\mathcal{H} = L^2(\mathbf{R}^d)$. The following inequalities hold:

$$\alpha^+(p, \psi) \geq \frac{p}{d} D_2^+(\mu_\varphi), \quad \alpha^-(p, \psi) \geq \frac{p}{d} D_2^-(\mu_\varphi).$$

The result follows directly from Theorem 2.5 and (49), (50). This result is known for α^- (see e.g. [10], [2]), but is new for α^+.

To get the best lower bound for the moments, one should choose μ_φ with maximal possible values of D_2^- and D_2^+. For any ψ in $\mathcal{H}$ and any self-adjoint operator H, one can define

$$\begin{aligned} l(\psi) &= \sup_{\varphi:\ (\psi,\varphi)\neq 0} D_2^-(\mu_\varphi)\ , \\ u(\psi) &= \sup_{\varphi:\ (\psi,\varphi)\neq 0} D_2^+(\mu_\varphi)\ . \end{aligned}$$

One always has $0 \leq l(\psi) \leq u(\psi) \leq 1$.

LEMMA 4.3 *The numbers $l(\psi)$ and $u(\psi)$ remain unchanged if, in the definition, one takes the supremum over the φ such that $\varphi = P_H(K)\psi$ for some compact set K.*

Proof. Let φ be such that $(\psi, \varphi) \neq 0$. As in the proof of Lemma 2.9, one can represent φ as $\varphi = \varphi_1 + \varphi_2$, where $\varphi_1 = P_\psi \varphi$, P_ψ beeing the orthogonal projection on $\mathcal{H}_\psi$, the cyclic subspace generated by ψ. We have $(\psi, \varphi_1) = (\psi, \varphi) \neq 0$ and $S_{\varphi_1}(z) \leq S_\varphi(z)$. Equalities (49) and (50) imply $D_2^-(\mu_{\varphi_1}) \geq D_2^-(\mu_\varphi)$ and $D_2^+(\mu_{\varphi_1}) \geq D_2^+(\mu_\varphi)$. Therefore, it is sufficient to take $\varphi \in \mathcal{H}_\psi$ in the definition of $l(\psi)$, $u(\psi)$. Now let $\varphi \in \mathcal{H}_\psi$, $(\psi, \varphi) \neq 0$. We have for some $g \in L^2(\mathbf{R}, d\mu_\psi)$:

$$d\mu_\varphi(x) = |g(x)|^2 d\mu_\psi(x), \quad (\psi, \varphi) = \int_{\mathbf{R}} g(x) d\mu_{\psi(x)}\ .$$

Consider the sets $A_\delta = \{x \,|\, |g(x)| \geq \delta\}$. As $(\psi, \varphi) \neq 0$, there exists $\delta > 0$ small enough such that $\mu_\psi(A_\delta) > 0$. Since A_δ is a Borel set and μ_ψ is a positive Borel measure on $\mathbf{R}$, there exists a compact subset $K \subset A_\delta$ such that $\mu_\psi(K) > 0$. Define the vector $\tilde{\varphi} = P_H(K)\psi$. One can easily see that $(\psi, \tilde{\varphi}) = \|\tilde{\varphi}\|^2 = \mu_\psi(K) > 0$, and for any Borel set S

$$\mu_{\tilde{\varphi}}(S) \leq \delta^{-2} \int_S |g(x)|^2 d\mu_\psi(x) = \delta^{-2} \mu_\varphi(S).$$

Therefore, for any $z > 0$, we have $S_{\tilde{\varphi}}(z) \leq \delta^{-4} S_\varphi(z)$ and $D_2^\pm(\mu_{\tilde{\varphi}}) \geq D_2^\pm(\mu_\varphi)$, which gives the result of Lemma. ∎

THEOREM 4.4 *Let $\psi \in \mathcal{H}$, $\psi \neq 0$. The following estimates hold:*

$$\alpha^-(\psi,p) \geq \frac{p}{d} l(\psi), \;\; \alpha^+(\psi,p) \geq \frac{p}{d} u(\psi). \tag{51}$$

Therefore, for any $\varepsilon > 0$, there exists a positive constant $c_{\psi,\varepsilon}$ such that for all $T > 1$,

$$\langle\langle |X|^p\rangle_{\psi(t)}\rangle(T) \geq c_{\psi,\varepsilon} T^{\frac{p}{d}(l(\psi)-\varepsilon)} \;, \tag{52}$$

and there exists an increasing sequence $\{T_n\}_{n=1}^{\infty}$, $T_n \to \infty$ such that

$$\langle\langle |X|^p\rangle_{\psi(t)}\rangle(T_n) \geq c_{\psi,\varepsilon} T_n^{\frac{p}{d}(u(\psi)-\varepsilon)} \;. \tag{53}$$

Proof. First, Lemma 4.2 yields for any φ such that $(\psi,\varphi) \neq 0$ and $P_H([-R,R])\varphi = \varphi$ for some $0 < R < \infty$,

$$\alpha^-(\psi,p) \geq \frac{p}{d} D_2^-(\mu_\varphi), \;\; \alpha^+(\psi,p) \geq \frac{p}{d} D_2^+(\mu_\varphi).$$

Taking the supremum over φ, and with the result of Lemma 4.3, we obtain (51). ∎

Remarks 4.5.

i) One can easily see from definition of $L(T)$, $L_R(T)$, $u(\psi)$ and $l(\psi)$ that

$$\gamma^-(\mu_\psi, p/d) \geq \frac{p}{d} l(\psi), \;\; \gamma^+(\mu_\psi, p/d) \geq \frac{p}{d} u(\psi),$$

where the exponents $\gamma^\pm$ were defined in Corollary 2.8. Therefore, the result of Corollary 2.8 implies that of Theorem 4.4 (in fact, as it follows from the example of Appendix B, it may give significantly stronger estimates in some situations). However, we have given this result in terms of $u(\psi)$, $l(\psi)$ for the sake of completeness in order to establish connections with results of [2] and [10].

ii) There exists positive measures μ for which $D_2^+(\mu) > D_2^-(\mu)$ (see [11], Chapter 3, Example 8.1). Therefore, one can expect that for some measures μ_ψ, $u(\psi) > l(\psi)$. However, we cannot give examples of such measures.

LEMMA 4.6 *For a given self-adjoint operator H and α in $[0,1]$, we define the following closed subspaces of $\mathcal{H}$,*

$$\mathcal{L}_\alpha = \mathrm{Span}\{\varphi \in \mathcal{H} \ : \ D_2^-(\mu_\varphi) \geq \alpha\} \ , \ \ \mathcal{U}_\alpha = \mathrm{Span}\{\varphi \in \mathcal{H} \ : \ D_2^+(\mu_\varphi) \geq \alpha\} \ ,$$

where Span *denotes the closure of the set of finite linear combinations.*

Clearly one has $\mathcal{L}_\alpha \subset \mathcal{U}_\alpha$. Then, if $P_{\mathcal{L}_\alpha}$ and $P_{\mathcal{U}_\alpha}$ denote respectively the orthogonal projection on $\mathcal{L}_\alpha$ and $\mathcal{U}_\alpha$, we have the following identities:

$$l(\psi) = \sup\{\alpha \in [0,1] \text{ s.t. } P_{\mathcal{L}_\alpha}\psi \neq 0\}, \ u(\psi) = \sup\{\alpha \in [0,1] \text{ s.t. } P_{\mathcal{U}_\alpha}\psi \neq 0\} \ .$$

Proof. The results follow from the definition of $l(\psi)$, $u(\psi)$ and the obvious fact that for any set A, the orthogonal projection of ψ on Span$A$ is non zero if and only if there exists a vector φ in A such that $(\psi, \varphi) \neq 0$. ∎

For α in $[0,1]$, consider now the subspaces $\mathcal{H}_\alpha = \overline{\mathcal{H}_{uh}(\alpha)} = \mathcal{H}_{\alpha c}$ introduced in the previous section (see [10] for the definition). For any $\psi \in \mathcal{H}$, we already know that [10], [2]

$$\mu_\psi\text{−ess. sup } \gamma^-(x) = \dim_{\mathrm{H}}(\mu_\psi) = \sup\{\alpha \ : \ P_{\mathcal{H}_{\alpha c}}\psi \neq 0\} \tag{54}$$

where $\dim_{\mathrm{H}}(\mu_\psi)$ is the Hausdorff dimension of the measure μ_ψ, and $\gamma^-(x) = \liminf_{\varepsilon\to 0} \log(\mu_\psi(x-\varepsilon, x+\varepsilon))/\log\varepsilon$ is the lower local exponent of μ_ψ at point x. Define the number

$$\beta(\psi) = \sup\{\alpha \ : \ P_{\mathcal{V}_\alpha} \neq 0\}$$

where $\mathcal{V}_\alpha = $ Span $\{\varphi \in \mathcal{H} \ : \ \sup_{T>1} T^\alpha \langle |\hat{\mu}_\varphi|^2\rangle(T) < \infty\}$ was defined in Corollary 3.4.

LEMMA 4.7 *The following equality holds:*

$$\beta(\psi) = \dim_{\mathrm{H}}(\mu_\psi) \ .$$

Proof. In [10], Last has shown that for any α in $(0,1)$ and $\varepsilon > 0$ sufficiently small,

$$\mathcal{V}_{\alpha+\varepsilon} \subset \mathcal{H}_{\alpha c} \subset \mathcal{V}_\alpha. \tag{55}$$

On the other hand, if $\mathcal{H}_1$ and $\mathcal{H}_2$ are two closed subspaces such that $\mathcal{H}_1 \subset \mathcal{H}_2$, for any $\psi \in \mathcal{H}$, if $P_{\mathcal{H}_1}\psi \neq 0$ then $P_{\mathcal{H}_2}\psi \neq 0$. Definition of $\beta(\psi)$, (54) and (55) easily imply $\beta(\psi) = \dim_{\mathrm{H}}(\mu_\psi)$. ∎

LEMMA 4.8 *We have the following equality*

$$l(\psi) = \beta(\psi) \ .$$

Proof. As

$$D_2^-(\mu_\varphi) = -\limsup(\log C_\varphi(T)/\log T),$$

where $C_\varphi(T) = \langle|\hat{\mu}_\varphi|^2\rangle(T)$, it is easy to see that for any $\varepsilon > 0$,

$$\{\varphi : \langle|\hat{\mu}_\varphi|^2\rangle(T) \leq c_\varphi T^{-\alpha}\} \subset \{\varphi : D_2^-(\mu_\varphi) \geq \alpha\} \subset$$

$$\{\varphi : \langle|\hat{\mu}_\varphi|^2\rangle(T) \leq c'_\varphi T^{-\alpha+\varepsilon}\} \ ,$$

therefore

$$\mathcal{V}_\alpha \subset \mathcal{L}_\alpha \subset \mathcal{V}_{\alpha-\varepsilon} \ . \tag{56}$$

Then, as in the proof of Lemma 4.7, (56) implies

$$\beta(\psi) = \sup\{\alpha \ : \ P_{\mathcal{V}_\alpha}\psi \neq 0\} = \sup\{\alpha \ : \ P_{\mathcal{L}_\alpha}\psi \neq 0\} = l(\psi) \ .$$

∎

COROLLARY 4.9

$$l(\psi) = \dim_{\mathrm{H}}(\mu_\psi) \ .$$

Therefore, the result (52) of Theorem 4.4 is identical to Theorem 3.1 in [2].

Remarks 4.10. If ψ is such that $P_{\mathcal{V}_\alpha}\psi \neq 0$ for $\alpha = l(\psi)$, then Corollary 3.4 implies

$$\langle\langle|X|^p\rangle_{\psi(t)}\rangle(T) \geq c_{p,\psi} T^{\frac{pl(\psi)}{d}} \ ,$$

i.e., we do not have the term ε in the exponent.

A Discrete Unitary Groups

We present here an adaptation of our results in the case of a time evolution described by a discrete unitary group. This occurs, for example, when one considers Schrödinger operators with periodic time-dependent external forces, or more generally, periodic operators. We consider a one-parameter family of unitary operators $\{U_t\}_{t\in\mathbf{Z}}$ acting on a separable Hilbert space $\mathcal{H}$, such that $U(0) = I$, and for all s, t in $\mathbf{Z}$, $U(s+t) = U(s)U(t)$. We have thus $U(t) = U^t$ for any t, where $U = U(1)$ is some unitary operator in $\mathcal{H}$. It is well known (see e.g. [1]) that for all $\psi \in \mathcal{H}$ there exists a probability measure σ_ψ on $[0, 2\pi]$ such that for all $k \in \mathbf{Z}$

$$(\psi,\, U(k)\psi) = \int_0^{2\pi} \mathrm{e}^{-ikx} d\sigma_\psi(x) \ .$$

The analog of Theorem 2.1, for discrete time evolution, can be stated as follows:

THEOREM A.1 *Let A a be a Hilbert-Schmidt operator on $\mathcal{H}$. Let $\psi \in \mathcal{H}$ be such that for any $k \in \mathbf{N}$, $(\psi(k),\, A\psi(k)) \geq 0$, where $\psi(k) = U(k)\psi$. For $z \in (0,1)$, we define the function $S_\psi^{(d)}(z)$ as follows (the superscript (d) standing for "discrete")*

$$S_\psi^{(d)}(z) = z\sum_{k=0}^{\infty}\mathrm{e}^{-kz}|\hat{\sigma}_\psi(k)|^2 = \int_0^{2\pi}\int_0^{2\pi}\frac{1-\cos(x-y)\mathrm{e}^{-z}}{1-2\cos(x-y)\mathrm{e}^{-z}+\mathrm{e}^{-2z}}d\sigma_\psi(x)d\sigma_\psi(y) \ ,$$

where $\hat{\sigma}_\psi(k) = \int_0^{2\pi} \mathrm{e}^{-ikx} d\sigma_\psi(x)$. Then, for all $K \geq 1$

$$0 \leq \langle\langle A\rangle_{\psi(k)}\rangle(K) \equiv K^{-1}\sum_{k=0}^{K}(\psi(k),\, A\psi(k)) \leq \mathrm{e}\|A\|_2\left(S_\psi^{(d)}(K^{-1})\right)^{\frac{1}{2}} \ , \quad (57)$$

where $\|A\|_2$ is the Hilbert-Schmidt norm of A.

The proof of the theorem is very similar to the one of Theorem 2.1 and uses the spectral theorem for unitary operators.

Theorem 2.5 on the moment of order p can be rewritten, in the time-dependent case, as follows:

THEOREM A.2 *We assume that $\mathcal{H} = \ell^2(\mathbf{Z}^d)$. If ψ is a vector in $\ell^2(\mathbf{Z}^d)$, then, for any $\varphi \neq 0$, and all $K \in \mathbf{N} \setminus \{0\}$, we have*

$$\begin{aligned} \langle\langle |X|^p \rangle_{\psi(k)} \rangle (K) &\equiv K^{-1} \sum_{k=0}^{K} (\psi(k), |X|^p \psi(k)) \\ &\geq C_{p,\psi} \|\varphi\|^{-2} |(\psi, \varphi)|^{2+\frac{4p}{d}} \left(S_\varphi^{(d)}(T^{-1}) \right)^{-\frac{p}{d}} , \end{aligned} \tag{58}$$

where $C_{p,\psi}$ is a constant depending only on p and ψ.

Remarks A.3.

i) Equipped with these two theorems, one can easily deduce similar bounds as those provided for the time-independent case (Section 2, 3 and 4). In particular, the equivalent of Theorem 3.1 also holds. Thus, if $\gamma = \dim_{\mathrm{H}}(\sigma_\psi)$ is the Hausdorff dimension of the measure σ_ψ, then for all $\nu > 0$ there exists $c > 0$ such that for all $K \in \mathbf{N} \setminus \{0\}$,

$$\langle\langle |X|^p \rangle_{\psi(k)} \rangle (K) \geq c K^{\frac{p}{d}(\gamma - \nu)} .$$

Furthermore, for any increasing subsequence $\{K_n\}_{n \in \mathbf{N}}$ in $\mathbf{N}$, if $S_\varphi^{(d)}(K_n^{-1}) \leq c_\varphi K_n^{-\alpha}$ for some nonnegative constant c_φ, some $\alpha \in [0, 1]$ and some φ for which $(\psi, \varphi) \neq 0$, then, there exists a strictly positive constant c such that for all $n \in \mathbf{N}$,

$$\langle\langle |X|^p \rangle_{\psi(k)} \rangle (K_n) \geq c K_n^{\frac{p\alpha}{d}} .$$

ii) In the case $\mathcal{H}$ is any separable Hilbert space (not necessarily $\ell^2(\mathbf{Z}^d)$), the result of Theorem A.2 remains valid if, for a given orthonormal system $(e_n)_{n \in \mathbf{N}}$ of $\mathcal{H}$, we define the moment of order p at time K for a vector $\psi \in \mathcal{H}$ as

$$\langle\langle |X|^p \rangle_{\psi(k)} \rangle (K) = K^{-1} \sum_{k=0}^{K} \sum_{n=0}^{\infty} n^p |(e_n, U(k)\psi)|^2 .$$

This amounts to consider in Theorem A.1 the operator

$$A_N = \sum_{n=0}^{N} (e_n, \cdot) \, e_n .$$

Estimates (58) then holds in this case. Note that in the time independent case, these arguments are also correct, and the bound (28), with

$\langle\langle |X|^p\rangle_{\psi(t)}\rangle(T) = T^{-1}\int_0^T \left(\sum_{n=0}^{\infty} n^p|(e_n,\,\psi(t))|^2\right) dt$, is also true, and the projection assumption $P_H[-R,R]\varphi = \varphi$ is not any more necessary.

The connections with the autocorrelation function (as in Lemma 4.2 and Theorem 4.4) can be done through the following lemma:

LEMMA A.4 *Let* $\{U(k)\}_{k\in\mathbf{Z}}$ *be a unitary discrete group in* $\mathcal{H}$. *We define the autocorrelation function at time* K *for a state* $\psi \in \mathcal{H}$ *as*

$$C_\psi^{(d)}(K) \equiv K^{-1}\sum_{k=0}^{K} |(\psi,\, U(k)\psi)|^2 = K^{-1}\sum_{k=0}^{K} |\hat{\sigma}_\psi(k)|^2 \ .$$

For any α *in* $[0,1]$, *consider the set*

$$\mathcal{D}_\alpha^{(d)} \equiv \left\{\varphi \in \mathcal{H} \,:\, \exists c_\varphi < \infty \text{ s.t. } \ C_\varphi^{(d)}(K) \le c_\varphi K^{-\alpha} \text{ for any } K \in \mathbf{N}\setminus\{0\}\right\} \ ,$$

then we have

$$\mathcal{D}_\alpha^{(d)} = \left\{\varphi \in \mathcal{H} \,:\, \exists c_\varphi < \infty \text{ s.t. } \ S_\varphi^{(d)}(K^{-1}) \le c_\varphi K^{-\alpha} \text{ for any } K \in \mathbf{N}\setminus\{0\}\right\} \ .$$

Furthermore, there exists a finite constant c *depending only on* ψ *such that, for all* $K \in \mathbf{N}\setminus\{0\}$

$$\mathrm{e}^{-1}C_\psi^{(d)}(K) \le S_\psi^{(d)}(K) \le c\log K C_\psi^{(d)}(K\log K) \ . \tag{59}$$

Proof. If $S_\varphi^{(d)}(K^{-1}) \le c_\varphi K^{-\alpha}$, for some $c_\varphi < \infty$, then we have

$$C_\varphi^{(d)}(K) \le K^{-1}\sum_{k=0}^{\infty} \mathrm{e}^{1-\frac{k}{K}}|\hat{\sigma}_\varphi(k)|^2 \le \mathrm{e}S_\varphi^{(d)} \le \mathrm{e}c_\varphi K^{-\alpha}$$

Now suppose that there exists a finite constant c_φ such that for all $K \in \mathbf{N}\setminus\{0\}$, $C_\varphi^{(d)}(K) \le c_\varphi K^{-\alpha}$. We have:

$$S_\varphi^{(d)}(K^{-1}) = K^{-1}\sum_{k=0}^{\infty} \mathrm{e}^{-k/K}|\hat{\sigma}_\varphi(k)|^2 =$$

$$K^{-1}\left(1 + \mathrm{e}^{-\frac{1}{K}}|\hat{\sigma}_\varphi(1)|^2 + \sum_{k=2}^{\infty} \mathrm{e}^{-k/K}\left(kC_\varphi^{(d)}(k) - (k-1)C_\varphi^{(d)}(k-1)\right)\right) =$$

$$K^{-1}(1+\mathrm{e}^{-\frac{1}{K}}|\hat{\sigma}_\varphi(1)|^2)+$$
$$K^{-1}\left(\sum_{k=1}^{\infty}\left(\mathrm{e}^{-k/K}-\mathrm{e}^{-(k+1)/K}\right)kC_\varphi^{(d)}(k)-\mathrm{e}^{-1/K}C_\varphi^{(d)}(1)\right)$$
$$\leq c_\varphi K^{-2}\sum_{k=1}^{\infty}\mathrm{e}^{-k/K}k^{1-\alpha}+c_\varphi K^{-1}\leq \tilde{c}_\varphi K^{-\alpha}\ .$$

The proof of (59) is straightforward.

B Operators with pure point spectrum

In this section we shall consider some examples of applications of Theorem3.6 and Corollary 2.7. Let ψ be such that the spectral measure μ_ψ is pure point: $\mu_\psi=\sum_{m=1}^{\infty}\gamma_m\delta_{a_m}$, where $a_m\neq a_k$ for $m\neq k$, $\gamma_m>0$ and $\sum_m\gamma_m=\|\psi\|^2$. Suppose for simplicity that $\mathcal{H}=\ell^2(\mathbf{Z}^d)$. Let $\gamma_m=\frac{1}{m^\lambda}$ with some $\lambda>1$. One can show the following result.

LEMMA B.1 *If $\nu(\beta)=\beta(3-2\lambda)+1-\lambda>0$, then $K_\beta=+\infty$, where K_β was defined in Theorem 3.7.*

Proof. For any $M>0$ consider the function $f^{(M)}$ in $L^2(\mathbf{R},d\mu_\psi)$:

$$f_m^{(M)}\equiv f^{(M)}(a_m)=\chi_{(m\geq M)}.$$

One sees that $\|f^{(M)}\|^2=\sum_{m\geq M}\gamma_m\sim CM^{1-\lambda},\quad M\to+\infty$. Let $g^{(M)}=\|f^{(M)}\|^{-1}f^{(M)}$. The calculations give

$$K_\beta(g^{(M)})\sim CM^{\nu(\beta)},\quad M\to+\infty,$$

where $\nu(\beta)$ was defined above. If $\nu(\beta)>0$, clearly, $K_\beta=\sup_{g\in G}K_\beta(g)=+\infty$. ∎

One sees from definition of $\nu(\beta)$ that $\nu(\beta)>0$ if and only if $\lambda\in(1,\frac{3}{2})$, $\quad\beta>\frac{\lambda-1}{3-2\lambda}$. Now suppose that λ,β are such that $\nu(\beta)>0$. To get the lower bound for the function $L(T)$ in Corollary 2.7, we shall use Lemma 2.10. Consider the function $g^{(M)}$ defined in the proof of Lemma B.1 and calculate the function $K(g^{(M)},z)$. One sees easily that

$$K(g^{(M)},z)\geq CM^{1-\lambda+4\beta(1-\lambda)}\left(S^{(M)}(z)\right)^{-\beta}, \tag{60}$$

where

$$S^{(M)}(z) = \sum_{m\geq M} \gamma_m^2 + z^2 \sum_{k,m\geq M,k\neq m} \frac{\gamma_m\gamma_k}{z^2+(a_m-a_k)^2} \equiv S_1^{(M)} + S_2^{(M)}(z).$$

As to $S_1^{(M)}$, we have

$$S_1^{(M)} \sim CM^{1-2\lambda}, \quad M\to+\infty. \tag{61}$$

To get the optimal lower bound for $L(T)$, we shall take M depending on $z=T^{-1}$ so that M be large but at the same time $S_2^{(M)}(z)\leq S_1^{(M)}$. Denote by $M(T)$ the corresponding choice of M. Then one obtain from (60), (61) (in fact, the calculations here are the same as in the proof of Lemma B.1):

$$K(g^{(M(T))},T^{-1}) \geq CM(T)^{\nu(\beta)}.$$

Therefore, $L(T)\geq CM(T)^{\nu(\beta)}$.

The function $M(T)$ depends not only on the values γ_m but also on the distribution of eigenvalues a_m. One can consider two examples.

i) Let $a_m=m$. Then clearly

$$S_2^{(M)}(z) \leq Cz^2\left(\sum_{m\geq M}\gamma_m\right)^2 \sim Cz^2M^{2(1-\lambda)}.$$

One should take $M=T^2$ which gives the following lower bounds for the moments:

$$\langle\langle |X|^p\rangle_\psi\rangle(T) \geq CL(T) \geq CT^{2\nu(\beta)}, \quad \beta=p/d.$$

If λ is close to 1, the bounds looks like $CT^{p/d(2-\varepsilon)}$. This example is, however, rather theoretical. More realistic is the following.

ii) Suppose that $a_m=m^{-\delta}$, $\delta>0$. One should estimate the sum

$$S_2^{(M)}(z) = z^2\sum_{m,k\geq M,m\neq k}\frac{m^{-\lambda}k^{-\lambda}}{z^2+(m^{-\delta}-k^{-\delta})^2}.$$

Comparing the sum with the double integral, one can show that if $\eta=\lambda-1-\delta>0$, then $S_2^{(M)}(z)\leq CzM^{-2\eta-\delta}$. Therefore, one takes $M(T)=T^{\frac{1}{1+\delta}}$ which gives $L(T)\geq CT^{\frac{\nu(\beta)}{1+\delta}}$. For the moments of order p we have thus the lower bounds like $CT^{\frac{p}{d}\rho}$, where ρ is always less than 1, which is more realistic than the bounds of the first example.

The same kind of considerations can be made in the case of dense pure point spectrum. The main problem is the estimation of the function $S_2^{(M)}(z)$.

The spectral measure μ_ψ being pure point, one can easily see that $u(\psi) = l(\psi) = \dim_H(\mu_\psi) = 0$, where dimensions $u(\psi), l(\psi)$ were defined in section 4. In the considered examples, however, we have $\gamma^{\pm}(\psi, \beta) > 0$ for some β. This shows that the result of Corollary 2.7 is significantly stronger than that of Theorem 4.4.

One should stress that the considered examples are purely theoretical. To get this kind of estimates for "concrete" Schrödinger operators like $H = -\Delta + Q(x)$, one should study the rate of decay of the coefficients γ_m. In the case $\mathcal{H} = \ell^2(\mathbf{Z}^d)$, $\psi = \delta_0$, one has $\gamma_m = |e_m(0)|^2$, where e_m are normalised eigenfunctions of H. If $|e_m(0)|$ decay slowly as $m \to \infty$, the phenomenon of delocalisation described above takes place.

C Operators with singular continuous spectrum

Example 1: Schrödinger operator with high barriers potential.

We consider here the operator studied by Simon in [18]. i.e,

$$H = -\frac{d^2}{dx^2} + \sum_{n=1}^{\infty} \left(V_n(x - L_n) + V_n(-x - L_n)\right)$$

where V_n is a sequence of non-negative compactly supported "trapping" potentials (see [18]). He proved that for a dense set $\mathcal{D} \subset L^2(\mathbf{R})$, $\forall \varphi, \psi \in \mathcal{D}$, and for $t \geq 2$, $|(\varphi, \mathrm{e}^{-itH}\psi)| \leq C_{\varphi,\psi}(\log t)/t^{1/2}$. Therefore, for any $\varphi \in \mathcal{D}$, $|\hat{\mu}_\varphi|(t) \leq C_\varphi \log t/t^{\frac{1}{2}}$. As at the same time $|\hat{\mu}_\varphi|(t) \leq \|\varphi\|^2$ for any t, it is easy to estimate for $z \in (0, 1/2)$:

$$S_\varphi(z) = z \int_0^{+\infty} \exp(-tz)|\hat{\mu}_\varphi|^2(t)dt \leq C_\varphi z|\log z|^3.$$

The result of Theorem 3.1 applied to $l(z) = z|\log z|^3$ gives us that for any $\psi \in L^2(\mathbf{R})$, $\psi \neq 0$, there exists a positive constant c_ψ such that for all $T > 0$,

$$\langle\langle |X|^p \rangle_{\psi(t)}\rangle(T) \geq c_\psi \left(\frac{T}{(\log T)^3}\right)^p .$$

Remarks C.1 *The result obtained here provides a ballistic lower bound, up to a logarithmic term instead of a polynomial one as in [18]. Moreover, it holds for all states in* $L^2(\mathbf{R})$.

Example 2: Rank one perturbation of operators with SULE.

We consider

$$H_\lambda = H + \lambda(\delta_0, .)\,\delta_0\ ,$$

where H is a self-adjoint operator on $\ell^2(\mathbf{Z}^d)$ with SULE (see [4]), λ is a real parameter. Then, in [4], for $p \geq 2$, it is proven for $\delta(t) = \mathrm{e}^{itH_\lambda}\delta_0$ that for all $t \geq 2$,

$$(\delta(t), |X|^p \delta(t)) \leq c(\log t)^p\ . \tag{62}$$

On the other hand, from Theorem 2.5, we get

$$\langle\langle |X|^p\rangle_{\delta(t)}\rangle(T) \geq C_\psi \left(S_\psi(T^{-1})\right)^{-\frac{p}{d}}\ , \tag{63}$$

for any ψ such that $(\psi, \delta_0) = \psi(0) \neq 0$. Then, it follows from (62) and (63) that, for any ψ, $\psi(0) \neq 0$,

$$S_\psi(z) \geq \frac{C_\psi}{|\log z|^d}\ .$$

Now, since from Lemma 4.1 we have $S_\psi(T^{-1}) \leq D\log T \cdot \langle|\hat{\mu}_\psi|^2\rangle(T\log T)$, we obtain

$$\langle|\hat{\mu}_\psi|^2\rangle(T) \geq \frac{\tilde{C}_\psi}{(\log T)^{d+1}}\ ,$$

for some nonnegative constant $\tilde{C}_\psi$.

This result proves that the correlation functions are, like the moment of order p, stable under rank one perturbation for such operators.

D Strichartz Theorem

The aim of this section is to recall Strichartz Theorem and to establish its converse. Although we do not explicitly use this Theorem for our main result, it is useful to state it in order to compare with the different approachs proposed in [7], [3], [10], [2].

THEOREM D.1 *Strichartz [15]*

We assume that μ is a uniformly α-Hölder measure (UαH), i.e., there exists a finite constant c such that for μ-a.e. x in $\mathbf{R}$ *and all ε in* $(0,1)$,

$$\mu(x-\varepsilon, x+\varepsilon) \leq c\varepsilon^{\alpha} \ .$$

Let f be in $L^2(\mathbf{R},\, d\mu)$, then there exists a finite constant c_μ depending only on μ such that:

$$\sup_{T\geq 1} T^{\alpha-1} \int_0^T \ |(f\mu)\hat{}(t)|^2 dt \leq c_\mu \, \|f\|_2{}^2 \tag{64}$$

where $\hat{}$ *denotes the Fourier transform.*

The converse of this theorem is also true,

THEOREM D.2 *Let μ be a bounded positive measure on* $\mathbf{R}$*, and suppose that there exists a constant c_μ such that for all f in $L^2(\mathbf{R},\, d\mu)$ the inequality (64) holds. Then μ is a uniformly α-Hölder measure.*

Proof. Pick any x_0 in $\mathbf{R}$, and ε in $(0,1)$. Define $f(y) = \chi_{(x_0-\varepsilon,\, x_0+\varepsilon)}$ to be the characteristic function of the interval $I_\varepsilon = (x_0-\varepsilon,\, x_0+\varepsilon)$. Note that $\|f\|^2_{L^2(\mathbf{R},d\mu)} = \mu(x_0-\varepsilon,\, x_0+\varepsilon)$. Then, from Lemma 3.2 of Last [10], for $T = \frac{\pi}{2}\varepsilon^{-1}$ and $d\mu_f \equiv f d\mu$, we get

$$T^{-1}\int_0^T \ |(f\mu)\check{}(t)|^2 dt == T^{-1}\int_0^T \ |\hat{\mu}_f|^2(t) dt \geq$$

$$\frac{1}{2}\int_{I_\varepsilon}\int_{I_\varepsilon} \frac{\sin^2\frac{(x-y)T}{2}}{(\frac{(x-y)T}{2})^2} d\mu(x) d\mu(y) \geq \frac{4}{\pi^2}\mu^2(x_0-\varepsilon,\, x_0+\varepsilon). \tag{65}$$

Thus, (64) and (65) imply

$$\mu^2(x_0-\varepsilon,\, x_0+\varepsilon) \leq c_\mu \left(\frac{\pi}{2}\right)^{2-\alpha} \varepsilon^\alpha \mu(x_0-\varepsilon,\, x_0+\varepsilon) \ .$$

Therefore, μ is UαH.

Acknowledgments

The authors want to thank M. Holschneider for stimulating discussions about various aspects of this paper, and S. Vaienti for the example cited in Remark 4.5 ii).

References

[1] I. N. Akhiezer, I.M. Glazman: "Theory of Linear Operators in Hilbert Spaces", Vol II, (F. Ungar Publishing Co.), New-York 1963.

[2] J.M. Barbaroux, J.M. Combes, R. Montcho: Remarks on the relation between quantum dynamics and fractal spectra, *J. Math. Anal. Appl.* **213**, 698-722 (1997).

[3] J.M. Combes: Connections between quantum dynamics and spectral properties of time-evolution operators, in "Differential Equations with Appplications to Mathematical Physics" (W.F.Ames, E.M.Harrell, and J.V.Herod, Eds.), Academic Press, Boston, 1993.

[4] R. Del Rio, S. Jitomirskaya, Y. Last and B. Simon: Operators with singular continuous spectrum, IV. Hausdorff dimensions, rank one perturbations and localization, *J. Anal. Math*, **69**, 153-200 (1996).

[5] K.J. Falconer: "The Geometry of Fractal Sets", Cambridge Tracts in Mathematics, Cambridge Univ. Press, Cambridge, UK, 1985.

[6] F. Germinet, S. De Bièvre: Dynamical Localization for Random Schrödinger Operators, *Commun. Math. Phys*, to appear.

[7] I. Guarneri: Spectral properties of quantum diffusion on discrete lattices , *Europhysics Letters*, **10**(2), 95-100 (1989).

[8] C.A. Guerin and M. Holschneider: Three equivalent definitions of the correlation dimension of a probability measure *J. Statist. Phys* **86**, no3-4, 707-720 (1997).

[9] T. Kato: "Perturbation Theory for Linear Operators". 2nd ed., Berlin: Springer 1980.

[10] Y. Last: Quantum dynamics and decompositions of singular continuous spectra, *J. Funct. Anal.* , **142**, 402-445 (1996).

[11] Y. B. Pesin: "Dimension Theory in Dynamical Systems", Chicago Univ. Press, 1997.

[12] C. Radin, B. Simon: Invariant domains for the time-dependent Schrödinger equation, *J. Diff. Eq.*, **29** , 289-296 (1978).

[13] M. Reed and B. Simon: Methods of Modern Mathematical Physics, Vol 1, Academic Press. New-York 1978.

[14] C.A. Rogers: "Hausdorff Measures", London: Cambridge University Press 1970

[15] R.S. Strichartz: Fourier asymptotics of fractal measures, *J. Funct. Anal.* , **89**, 154-187 (1990).

[16] B. Simon: Absence of ballistic motion, *Commun. Math. Phys* **134**, 209-212 (1990).

[17] B. Simon: Schrödinger semigroups, *Bull. Am. Math. Soc.* **7**, 447-526 (1982).

[18] B. Simon: Operators with singular continuous spectrum, VII: Examples with borderline time decay, *Commun. Math. Phys.* **176**, 713-722 (1996).

[19] S. Tcheremchantsev: Markovian Anderson Model: Bounds for the rate of propagation, *Commun. Math. Phys.* **187**, 441-469 (1997).

[20] S. Tcheremchantsev: Transport properties of Markovian Anderson model, *Commun. Math. Phys.* , to appear.