SOME QUESTIONS OF INTEGRAL GEOMETRY ON ANOSOV MANIFOLDS

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1. Posing the problem and formulating the results

The author is grateful to B. Kleiner for the following remark that has been used as a start point of the present work: there is an analogy between the modified horizontal derivative constructed in author's works [Sh1, Sh3] and asymptotic Jacobi fields first introduced by E. Hopf [H] and then developed by many authors [G, Es, Eb]. Indeed, absence of conjugate points plays the crucial role in both the cases, and the same Riccati equation is used. The author is also grateful to N. S. Dairbekov for fruitful discussions of the subject.

Let (M, g) be a closed (= compact without boundary) Riemannian manifold. We are interesting in questions like the following one: to what extent is a smooth function $f \in C^\infty(M)$ determined by its integrals over all closed geodesics?

Of course, the question is sensible only in the case when (M, g) has sufficiently many closed geodesics. Therefore the question was first investigated for symmetric Riemannian manifolds. The simplest of such manifolds is the sphere with the standard metric. P. Funk [F] proved that the even part of a function on the two-dimensional sphere is determined by integrals over great circles. This work is traditionally considered as the start point of integral geometry. Later the problem was investigated for some other symmetric spaces [Hl].

Anosov manifolds constitute another wide class of manifolds with sufficiently many closed geodesics. Recall that a closed Riemannian manifold is called the *Anosov manifold* if its geodesic flow is of Anosov type. For such a manifold, the set of closed geodesics is dense in the set of all geodesics. The class of Anosov manifolds is opposite in some sense to the class of homogeneous Riemannian manifolds because the isometry group of an Anosov manifold is discrete [Es].

Theorem 1.1. *Let (M, g) be an Anosov manifold. If a function $f \in C^\infty(M)$ integrates to zero over every closed geodesic then f must itself be zero.*

A closed Riemannian manifold is said to have a simple length spectrum if there do not exist two different closed geodesics such that the ratio of their lengths is a rational number. This is a generic condition.

Corollary 1.2. *Let (M, g) be an Anosov manifold with simple length spectrum, and $\Delta : C^\infty(M) \to C^\infty(M)$ be the corresponding Laplace — Beltrami operator. If*

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real functions $q_1, q_2 \in C^\infty(M)$ are such that the operators $\Delta + q_1$ and $\Delta + q_2$ have coincident spectra, then $q_1 \equiv q_2$.

Theorem 1.1 implies Corollary 1.2 by the same arguments as in [GK].
The similar result is valid for 1-forms.

Theorem 1.3. *Let (M,g) be an Anosov manifold, and f be a smooth 1-form on M. If f integrates to zero around every closed geodesic, then f is an exact form.*

We now consider the corresponding question for symmetric tensor fields of arbitrary degree. Let τ'_M be the cotangent bundle of a closed Riemannian manifold M, and $C^\infty(S^m\tau'_M)$ be the space of smooth symmetric covariant tensor fields of degree m. For such a field $f \in C^\infty(S^m\tau'_M)$ and a closed geodesic $\gamma : [a,b] \to M$, we may consider the integral

$$If(\gamma) = \oint\limits_\gamma \langle f, \dot\gamma^m\rangle dt = \int\limits_a^b f_{i_1\dots i_m}(\gamma(t))\dot\gamma^{i_1}(t)\dots\dot\gamma^{i_m}(t)\,dt. \tag{1.1}$$

The integrand on (1.1) is written with use made by local coordinates. Nevertheless, it is evidently invariant, i.e., independent of the choice of coordinates. Let $Z(S^m\tau'_M)$ denote the subspace of $C^\infty(S^m\tau'_M)$ consisting of all fields such that $If(\gamma) = 0$ for every closed geodesic γ. For $m > 0$ this subspace is not zero as is seen from the following argument. Let us introduce the first order differential operator

$$d = \sigma\nabla : C^\infty(S^{m-1}\tau'_M) \to C^\infty(S^m\tau'_M), \tag{1.2}$$

where ∇ is the covariant derivative and σ is the symmetrization. This operator is called the *inner differentiation*. A tensor field f is called the *potential field* if it can be represented in the form $f = dv$ for some $v \in C^\infty(S^{m-1}\tau'_M)$. Let $P(S^m\tau'_M)$ denote the space of all potential fields. If $f = dv$, then the integrand on (1.1) equals to $d(v_{i_1\dots i_{m-1}}(\gamma(t))\dot\gamma^{i_1}(t)\dots\dot\gamma^{i_{m-1}}(t))/dt$. Therefore there is the inclusion

$$P(S^m\tau'_M) \subset Z(S^m\tau'_M). \tag{1.3}$$

The principal question of integral geometry of tensor fields is formulated as follows: for what classes of closed Riemannian manifolds and for what values of m is inclusion (1.3) in fact the equality? For instance, Theorems 1.1 and 1.3 say that (1.3) is the equality for an Anosov manifold in the cases $m = 0$ and $m = 1$. The equality is also proved in the case of arbitrary m for some symmetric spaces [M1, M2].

The following result is proved in [CSh].

Theorem 1.4. *For an Anosov manifold of nonpositive sectional curvature, inclusion (1.3) is the equality for all m.*

In [CSh] this theorem is formulated for negatively curved manifolds. Nevertheless, only nonpositivity of the curvature and Anosov type of the geodesic flow are used in the proof.

Let us recall some standard definitions. For an open $U \subset \mathbf{R}^n$ and $p \geq 1$, the *Sobolev space* $W_p^1(U)$ is the completion of $C^1(U)$ with respect to the norm

$$\|f\|_{W_p^1} = \|f\|_{L_p} + \|\nabla f\|_{L_p} = \left(\int\limits_U |f(x)|^p \, dx \right)^{1/p} + \left(\int\limits_U |\nabla f(x)|^p \, dx \right)^{1/p}.$$

A k-dimensional *distribution* on a manifold N is a section of the Grassmanian bundle $G_k(N)$ whose fiber at a point $x \in N$ is the manifold $G_k(T_xN)$ of all k-dimensional subspaces of T_xN. A distribution is said to belong to the class W_p^1 if in local coordinates it is represented by functions of the class $W_{p,loc}^1$.

Recall that the *stable* and *unstable distributions* participate in the definition of an Anosov flow [An]. In the case of a smooth Anosov flow, these distributions are absolutely continuous [An] and Hölder-continuous [AS] but in general they are not smooth.

In the case of an n-dimensional Anosov manifold M, the stable and unstable distributions are defined on the manifold ΩM of unit tangent vectors. Note that $\dim(\Omega M) = 2n - 1$.

The main result of the present article is the following

Theorem 1.5. *Let (M, g) be an n-dimensional Anosov manifold whose stable and unstable distributions belong to the class W_p^1 with some $p > 2n - 1$. Then inclusion (1.3) has a finite codimension for every m.*

In the case of an Anosov surface (= two-dimensional Anosov manifold) the stable nd unstable distributions are C^1-smooth [KH]. We thus have

Corollary 1.6. *For an Anosov surface, inclusion (1.2) has a finite codimension for every m.*

In view of the theorem, we pose the following

Conjecture 1.7. *The stable and unstable distributions of an n-dimensional Anosov manifold belong to the class W_p^1 with some $p > 2n - 1$.*

The conjecture agrees with all known results on regularity of the stable and unstable distributions of Anosov flows.

In the case of $m = 2$, the question under consideration relates to the spectral rigidity problem. Let us recall the corresponding definitions. A smooth one-parameter family g^τ $(-\varepsilon < \tau < \varepsilon)$ of metrics on a manifold M is called the *deformation* of a metric g if $g^0 = g$. Such a family is called the *isospectral deformation* if the spectrum of the Laplace — Beltrami operator Δ^τ of the metric g^τ is independent of τ. A deformation g^τ is called the *trivial deformation* if there exists a family φ^τ of diffeomorphisms of M such that $g^\tau = (\varphi^\tau)^* g$. A manifold (M, g) is called *spectrally rigid* if it does not admit a nontrivial isospectral deformation.

The following result is proved in [GK].

Theorem 1.8. *An Anosov manifold is spectrally rigid if and only if inclusion (1.3) is the equality for $m = 2$.*

The theorem is formulated in [GK] for negatively curved manifolds. Nevertheless, the same proof is valid for Anosov manifolds.

Our method of proving Theorem 1.5 is a combination of methods used in [Sh2] and [CSh]. The principal idea of [Sh2] is constructing a semibasic tensor field $a = (a^{ij})$ such that the curvature tensor of the modified horizontal derivative $\overset{a}{\nabla}$ meets the equation $\overset{a}{R}_{ijkl}\xi^i\xi^k = 0$. In this case the Pestov identity does not contain the term dependent on curvature. The stable and unstable distributions give us the possibility to construct a similar modifying tensor field $a = (a^{ij})$ in the case of an Anosov manifold. Unfortunately, this tensor field is not smooth. Therefore we have to approximate it by a smooth field. In the process of approximation, we should control boundedness of first order derivatives of the smoothing field. For such the control we need the hypothesis that the stable and unstable distributions belong to the class W^1_p.

Theorems 1.1 and 1.3 are proved in the same way. But in this case we need no estimate of first order derivatives of the modifying tensor field.

We conclude the introduction by posing the following question.

Problem 1.9. *Does there exist an Anosov manifold such that inclusion (1.3) is not equality for some values of m?*

2. Reduction of Theorem 1.5 to an estimate for the kinetic equation

Given a Riemannian manifold (M, g), by TM we denote its tangent space. Points of TM are denoted by pairs (x, ξ), where $x \in M$ and $\xi \in T_xM$. Let ΩM be the manifold of unit tangent vectors. By $G^t : TM \to TM$ we denote the geodesic flow, and by H, the corresponding vector field on TM. At points of ΩM H is tangent to ΩM. Therefore H can be considered as a first order differential operator $H : C^\infty(\Omega M) \to C^\infty(\Omega M)$.

The operator

$$-\delta : C^\infty(S^m\tau'_M) \to C^\infty(S^{m-1}\tau'_M)$$

is defined as the formal dual of operator (1.2). We call δ the *divergence operator*.

Here we will show that Theorem 1.5 is a corollary of the following

Lemma 2.1. *Let (M, g) be an n-dimensional Anosov manifold whose stable and unstable distributions belong to the class W^1_p with some $p > 2n - 1$. If a function $u \in C^\infty(\Omega M)$ and tensor field $f \in C^\infty(S^m\tau'_M)$ are connected by the kinetic equation*

$$Hu(x, \xi) = f_{i_1 \dots i_m}(x)\xi^{i_1} \dots \xi^{i_m}, \tag{2.1}$$

then the estimate

$$\|u\|^2_{H^1(\Omega M)} \le C\left(\|u\|^2_{L_q(\Omega M)} + \|\delta f\|_{L_2(S^{m-1}\tau'_M)} \cdot \|u\|_{L_2(\Omega M)}\right) \tag{2.2}$$

holds with $q = 2p/(p - 2)$ and a constant C independent of u and f.

Proof of Theorem 1.5. Let $N = 2n - 1 = \dim(\Omega M)$. The inequality $p > N$ implies that $q < 2N/(N - 2)$, and there is the compact imbedding $H^1(\Omega M) \subset L_q(\Omega M)$.

If the theorem is not true, there exists an infinite sequence of tensor fields $z_k \in Z(S^m\tau'_M)$ $(k = 1, 2, \dots)$ which is linearly independent mod$\,(P(S^m\tau'_M))$. Applying

Theorem 2.2 of [CSh], we decompose every field z_k into potential and solenoidal parts

$$z_k = y_k + dv_k, \qquad \delta y_k = 0. \tag{2.3}$$

Then the sequence $y_k \in Z(S^m\tau'_M)$ is linearly independent. Applying the smooth version of the Livčic theorem [LMM], we find functions $w_k \in C^\infty(\Omega M)$ satisfying the kinetic equation

$$Hw_k = \langle y_k(x), \xi^m\rangle. \tag{2.4}$$

The sequence w_k is linearly independent since in the other case (2.4) would imply linear dependence of the sequence y_k. Applying the Riesz theorem on an almost perpendicular [Y, p. 124], we construct a new sequence of functions $u_k \in C^\infty(\Omega M)$ such that

$$\|u_k\|_{L_q(\Omega M)} = 1, \qquad \|u_k - u_l\|_{L_q(\Omega M)} > 1/2 \quad \text{for} \quad k \neq l, \tag{2.5}$$

and every u_k is a linear combination of $w_1, \dots, w_k$. Equation (2.4) implies that

$$Hu_k = \langle f_k(x), \xi^m\rangle,$$

where $f_k \in C^\infty(S^m\tau'_M)$ is a linear combination of $y_1, \dots, y_k$. Therefore (2.3) implies that

$$\delta f_k = 0. \tag{2.6}$$

By equalities (2.5) and (2.6), estimate (2.2) has the following form for the pair u_k, f_k:

$$\|u_k\|_{H^1(\Omega M)} \leq C\|u_k\|_{L_q(\Omega M)} = C.$$

In other words, the sequence u_k is bounded in $H^1(\Omega M)$. Since the imbedding $H^1(\Omega M) \subset L_q(\Omega M)$ is compact, the sequence u_k contains a subsequence converging in $L_q(\Omega M)$. But this contradicts to inequality (2.5).

3. The modifying tensor fields $\overset{s}{a}$ and $\overset{u}{a}$

Given a manifold M and open set $U \subset TM$, by $C(\beta^r_s M; U)$ we denote the space of continuous semibasic tensor fields of degree (r, s) over U. The definition of a semibasic tensor field is given in Chapter 3 of [Sh1], see also [PSh].

Lemma 3.1. *Let (M, g) be an n-dimensional Anosov manifold. There exist continuous semibasic tensor fields $\overset{s}{a} = (\overset{s}{a}_{ij}(x,\xi)) \in C(\beta^0_2 M; T^0M)$ and $\overset{u}{a} = (\overset{u}{a}_{ij}(x,\xi)) \in C(\beta^0_2 M; T^0M)$ defined on $T^0M = \{(x,\xi) \in TM \mid \xi \neq 0\}$ such that*

(1) the fields are symmetric

$$\overset{s}{a}_{ij} = \overset{s}{a}_{ji}, \quad \overset{u}{a}_{ij} = \overset{u}{a}_{ji}$$

and orthogonal to the vector ξ

$$\xi^i \overset{s}{a}_{ij}(x,\xi) = 0, \quad \xi^i \overset{u}{a}_{ij}(x,\xi) = 0;$$

(2) they are positively homogeneous of degree 1 in ξ

$$\overset{s}{a}(x,t\xi)=t\overset{s}{a}(x,\xi),\quad \overset{u}{a}(x,t\xi)=t\overset{u}{a}(x,\xi)\quad for\quad t>0;$$

(3) the rank of the matrix $(\overset{s}{a}_{ij}-\overset{u}{a}_{ij})$ *equals* $n-1$ *at every point* $(x,\xi)\in T^0M$*;*

(4) along every geodesic $\gamma:\mathbf{R}\to M$*, the fields* $\overset{s}{a}{}^i_j(t)=(g^{ik}\overset{s}{a}_{kj})(\gamma(t),\dot\gamma(t))$ *and* $\overset{u}{a}{}^i_j(t)=(g^{ik}\overset{u}{a}_{kj})(\gamma(t),\dot\gamma(t))$ *are smooth and satisfy the Riccati equation*

$$a'+a^2+R=0, \tag{3.1}$$

where the prime denotes the covariant derivative, and $R=R(t)$ *is the curvature operator,* $R^i_j=R^i_{kjl}\dot\gamma^k\dot\gamma^l$*;*

(5) if the stable and unstable distributions belong to the class W^1_p*, then the fields* $\overset{s}{a}$ *and* $\overset{u}{a}$ *belong also to* W^1_p*.*

Before proving the lemma we recall some notions concerning the geodesic flow and Jacobi fields.

Let $\tau_M=(TM,\pi,M)$ be the tangent bundle of a Riemannian manifold (M,g). Given a point $(x,\xi)\in TM$, the canonical isomorphism

$$T_{(x,\xi)}(TM)\cong T_xM\oplus T_xM,\quad v\mapsto(d\pi(v),Kv)$$

is defined, where $K:TTM\to TM$ is the *connection mapping*. The subspaces of $T_{(x,\xi)}(TM)$ corresponding to the summands of the right-hand side are called the *horizontal* and *vertical spaces* respectively. This isomorphism defines the *Sasaki metric* on TM

$$\langle v,w\rangle=\langle d\pi(v),d\pi(w)\rangle+\langle Kv,Kw\rangle.$$

The metric g determines the isomorphism of the tangent and cotangent bundles. The standard symplectic structure of the cotangent bundle, being shifted to TM with the help of the isomorphism, is defined by the 2-form

$$\omega(v,w)=\langle d\pi(v),Kw\rangle-\langle d\pi(w),Kv\rangle.$$

The tangent space of the manifold ΩM of unit tangent vectors is defined at a point $(x,\xi)\in\Omega M$ by the equality

$$T_{(x,\xi)}(\Omega M)=\{v\in T_{(x,\xi)}(TM)\mid\langle Kv,\xi\rangle=0\}. \tag{3.2}$$

Let H be the geodesic vector field on TM generating the geodesic flow G^t : $TM\to TM$. Note that H is horizontal, $KH=0$. Given a vector $v\in T_{(x,\xi)}(TM)$, the vector field $Y_v(t)=d\pi\circ dG^t(v)$ is a Jacobi vector field along the geodesic $\gamma(t)=\exp_x t\xi$ with the covariant derivative $Y_v'(t)=K\circ dG^t(v)$.

Let now (M,g) be an Anosov manifold. Given $(x,\xi)\in\Omega M$, two $(n-1)$-dimensional subspaces $X_s(x,\xi)$ and $X_u(x,\xi)$ of $T_{(x,\xi)}(\Omega M)$ are defined which are called the *stable* and *unstable spaces* respectively. We will use the following properties of these spaces which follow from Proposition 1.7 and Theorem 3.2 of [Eb].

(i) The distributions $(x,\xi) \mapsto X_s(x,\xi)$ and $(x,\xi) \mapsto X_u(x,\xi)$ are continuous and invariant with respect to the geodesic flow, i.e.,

$$dG^t(X_s(x,\xi)) = X_s(G^t(x,\xi)), \quad dG^t(X_u(x,\xi)) = X_u(G^t(x,\xi)).$$

(ii) Each of the spaces $X_s(x,\xi)$ and $X_u(x,\xi)$ is orthogonal to the vector $H(x,\xi)$. The space $T_{(x,\xi)}(\Omega M)$ splits to the (not orthogonal) direct sum of tree subspaces

$$T_{(x,\xi)}(\Omega M) = X_s(x,\xi) \oplus X_u(x,\xi) \oplus \{H(x,\xi)\}.$$

(iii) The restriction of the mapping $d\pi$ to each of the spaces $X_s(x,\xi)$ and $X_u(x,\xi)$ is an isomorphism onto $N_{(x,\xi)} = \{\eta \in T_xM \mid \langle \xi, \eta \rangle = 0\}$.

(iv) $X_s(x,\xi)$ and $X_u(x,\xi)$ are Lagrangian spaces, i.e., $\omega(v,w) = 0$ for $v,w \in X_s(x,\xi)$ $(X_u(x,\xi))$.

Proof of Lemma 3.1. Given $(x,\xi) \in \Omega M$, we define two endomorfisms $b_s(x,\xi)$ and $b_u(x,\xi)$ of the space $N_{(x,\xi)}$ as compositions of the following mappings:

$$b_s(x,\xi) : N_{(x,\xi)} \xrightarrow{(d\pi)^{-1}} X_s(x,\xi) \xrightarrow{K} N_{(x,\xi)},$$

$$b_u(x,\xi) : N_{(x,\xi)} \xrightarrow{(d\pi)^{-1}} X_u(x,\xi) \xrightarrow{K} N_{(x,\xi)}.$$

Note that $Kv \in N_{(x,\xi)}$ for every $v \in T_{(x,\xi)}(\Omega M)$ because of equality (3.2). This definition is equivalent to the following rule that is more comfortable for using: two vectors $\eta, \zeta \in N_{(x,\xi)}$ are connected by the equality $b_s(x,\xi)\eta = \zeta$ if and only if there exists $v \in X_s(x,\xi)$ such that $d\pi(v) = \eta$ and $Kv = \zeta$. The similar rule is valid for the operator $b_u(x,\xi)$.

We establish the following properties of the operators $b_s(x,\xi)$ and $b_u(x,\xi)$.

1. The operators $b_s(x,\xi)$ and $b_u(x,\xi)$ continuously depend on $(x,\xi) \in \Omega M$. If the stable and unstable distributions belong to the class W^1_p, then the functions $(x,\xi) \to b_s(x,\xi)$ and $(x,\xi) \to b_u(x,\xi)$ belong also to W^1_p.

2. The operators b_s and b_u are selfdual. Indeed, let $\eta_i \in N_{(x,\xi)}$ and $\zeta_i = b_s(x,\xi)\eta_i$ $(i = 1,2)$. Then there are $v_i \in X_s(x,\xi)$ such that $d\pi(v_i) = \eta_i$ and $Kv_i = \zeta_i$. Therefore

$$\langle b_s\eta_1, \eta_2 \rangle - \langle \eta_1, b_s\eta_2 \rangle = \langle \zeta_1, \eta_2 \rangle - \langle \eta_1, \zeta_2 \rangle$$

$$= \langle Kv_1, d\pi(v_2) \rangle - \langle d\pi(v_1), Kv_2 \rangle = \omega(v_1, v_2) = 0$$

since $X_s(x,\xi)$ is a Lagrangian space.

3. The operator $b_s - b_u$ is nondegenerate. Indeed, let $b_s\eta = b_u\eta$ for some vector $\eta \in N_{(x,\xi)}$. There exist vectors $v \in X_s$ and $w \in X_u$ such that

$$d\pi(v) = \eta = d\pi(w), \quad Kv = b_s\eta = b_u\eta = Kw.$$

These relations imply that $v = w \in X_s \cap X_u = 0$. Consequently, $v = w = 0$ and $\eta = 0$.

4. Along every unit speed geodesic $\gamma : \mathbf{R} \to M$, each of the operator functions

$$b_s(t) = b_s(\gamma(t), \dot\gamma(t)) : N_{(\gamma(t),\dot\gamma(t))} \to N_{(\gamma(t),\dot\gamma(t))},$$

$$b_u(t) = b_u(\gamma(t), \dot\gamma(t)) : N_{(\gamma(t),\dot\gamma(t))} \to N_{(\gamma(t),\dot\gamma(t))}$$

satisfies the Riccati equation

$$b' + b^2 + R = 0. \tag{3.3}$$

Indeed, fix a geodesic γ and denote $N_t = N_{(\gamma(t),\dot\gamma(t))}$. Define the operator function $D(t) : N_t \to N_t$ as the solution to the Jacobi equation

$$D'' + RD = 0 \tag{3.4}$$

satisfying the initial conditions

$$D(0) = E \ (= \text{ identity}), \quad D'(0) = b_s(0).$$

Establish validity of the equality

$$D'(t) = b_s(t)D(t). \tag{3.5}$$

To this end fix a vector $\eta \in N_0$ and denote by $\eta(t) \in N_t$ the result of parallel translating the vector η along γ. Then the vector function $Y(t) = D(t)\eta(t)$ is the Jacobi vector field along γ satisfying the initial conditions

$$Y(0) = \eta, \quad Y'(0) = b_s(0)\eta.$$

On the other hand, if a vector $v \in X_s(\gamma(0), \dot\gamma(0))$ is such that $d\pi(v) = \eta$, then

$$D(t)\eta(t) = Y(t) = d\pi \circ dG^t(v), \quad D'(t)\eta(t) = Y'(t) = K \circ dG^t(v).$$

The vector $v_t = dG^t(v)$ belongs to $X_s(\gamma(t), \dot\gamma(t))$ because the distribution X_s is invariant with respect to the geodesic flow. The proceeding equalities can be rewritten as follows:

$$D(t)\eta(t) = d\pi(v_t), \quad D'(t)\eta(t) = Kv_t, \quad v_t \in X_s(\gamma(t), \dot\gamma(t)).$$

These relations imply that

$$D'(t)\eta(t) = b_s(t)D(t)\eta(t).$$

The latter equality is equivalent to (3.5) because η is an arbitrary vector.

The operator $D(t)$ is nondegenerate for sufficiently small $|t|$, and equality (3.5) can be rewritten as follows: $b_s(t) = D'(t)D^{-1}(t)$. ¿From this and Jacobi equation (3.4), we see that $b_s(t)$ satisfies Riccati equation (3.3) at least for sufficiently small $|t|$. Since the Riccati equation is invariant with respect to the shift $t \mapsto t + t_0$, it is satisfied for all t.

Given $(x,\xi)\in\Omega M$, we define the operators

$$\overset{s}{a}(x,\xi):T_xM\to T_xM,\quad \overset{u}{a}(x,\xi):T_xM\to T_xM$$

by the equalities

$$\overset{s}{a}(x,\xi)|_{N_{(x,\xi)}}=b_s(x,\xi),\quad \overset{s}{a}(x,\xi)\xi=0;$$

$$\overset{u}{a}(x,\xi)|_{N_{(x,\xi)}}=b_u(x,\xi),\quad \overset{u}{a}(x,\xi)\xi=0.$$

We then extend the functions $\overset{s}{a}$ and $\overset{u}{a}$ to T^0M in such the way that they are positively homogeneous in ξ

$$\overset{s}{a}(x,t\xi)=t\overset{s}{a}(x,\xi),\quad \overset{u}{a}(x,t\xi)=t\overset{u}{a}(x,\xi)\quad \text{for}\quad t>0.$$

We thus have constructed the semibasic tensor fields $\overset{s}{a}=(\overset{s}{a}{}^i_j(x,\xi))\in C(\beta^1_1M;T^0M)$ and $\overset{u}{a}=(\overset{u}{a}{}^i_j(x,\xi))\in C(\beta^1_1M;T^0M)$. The above proved properties of b_s and b_u imply that the semibasic tensor fields $\overset{s}{a}_{ij}=g_{ik}\overset{s}{a}{}^k_j$ and $\overset{u}{a}_{ij}=g_{ik}\overset{u}{a}{}^k_j$ satisfy all statements of Lemma 3.1.

Remark. Observe that, proving Lemma 3.1, we did not use the main property of the distributions X_s and X_u participating in the definition of an Anosov flow: X_s is contracted and X_u is expanded by dG^t. Nevertheless, this property is implicitly used in the proof of Theorem 1.5 via the above reference to the Livčic theorem.

4. Smoothing the tensor fields $\overset{s}{a}$ and $\overset{u}{a}$

We would like to use two modified horizontal derivatives defined by the scheme of Section 8.2 of [Sh1] (see also [Sh3]) with the modifying tensors $\overset{s}{a}$ and $\overset{u}{a}$ constructed in the previous section. Unfortunately, the fields $\overset{s}{a}$ and $\overset{u}{a}$ are only continuous but are not smooth. For constructing a modified horizontal derivative, we need at least C^2-smoothness of the modifying tensor because the definition of the curvature tensor assumes existence of second order derivatives. Therefore we have to smooth the tensor fields $\overset{s}{a}$ and $\overset{u}{a}$. We will choose the smoothing tensors in such a way that they would satisfy all statements of Lemma 3.1 with the following exception: Riccati equation (3.1) will be satisfied approximately.

First of all we will discuss some questions concerning smoothing sections of a vector bundle.

Let $\pi:E\to N$ be a smooth m-dimensional vector bundle over a compact manifold N. Choose a finite atlas $\{U_a,\varphi_a\}_{a=1}^A$ of the manifold N, a partition of unity $\{\mu_a\}_{a=1}^A$ subordinated to the atlas, and local trivializations $(e^a_1,\dots,e^a_m)$ of the bundle over U_a (this means that $e^a_\alpha\in C^\infty(E;U_a)$, and the vectors $e^a_1(x),\dots,e^a_m(x)$ constitute a basis of the fiber E_x for every point $x\in U_a$). Every section $f\in C^\infty(E)$ can be uniquely represented in the form

$$f(x)=\sum_{\alpha=1}^m f^\alpha_a(x)e^a_\alpha(x),\quad x\in U_a. \tag{4.1}$$

For $0 \le k < \infty$, let $C^k(E)$ be the space of sections f such that $(\mu_a f_a^\alpha) \circ \varphi_a^{-1} \in C^k(\mathbf{R}^n)$. The norm on the space is defined by the equality

$$\|f\|_{C^k(E)} = \sum_{a=1}^{A} \sum_{\alpha=1}^{m} \|(\mu_a f_a^\alpha) \circ \varphi_a^{-1}\|_{C^k(\mathbf{R}^n)}.$$

The W_p^1-norm on $C^1(E)$ is defined in the similar way

$$\|f\|_{W_p^1(E)} = \sum_{a=1}^{A} \sum_{\alpha=1}^{m} \|(\mu_a f_a^\alpha) \circ \varphi_a^{-1}\|_{W_p^1(\mathbf{R}^n)}.$$

Up to equivalence, the norms are independent of the choice of the atlas, partition of unity, and trivializations.

Let $H \in C^\infty(\tau_N)$ be a smooth vector field on N. Choosing a connection on E, we can define the derivative Hf of a section f with respect to H. A section $f \in C(E)$ is said *differentiable along* H if the derivative Hf exists and belongs to $C(E)$. In such the case the norm $\|Hf\|_{C(E)}$ is defined and independent, up to equivalence, of the choice of the connection.

Lemma 4.1. *Let $H \in C^\infty(\tau_N)$ be a smooth vector field on a compact manifold N which does not vanish at any point, and E be a smooth vector bundle over N. Fix a C-norm on $C(E)$ and a connection on E. For a differentiable along H section $f \in C(E)$ and for $\varepsilon > 0$, there exists a smooth section $\widetilde{f} \in C^\infty(E)$ such that*

$$\|f - \widetilde{f}\|_{C(E)} < \varepsilon, \quad \|H(f - \widetilde{f})\|_{C(E)} < \varepsilon.$$

Moreover, if the section f belongs to the class W_p^1, then for a fixed W_p^1-norm the estimate

$$\|\widetilde{f}\|_{W_p^1(E)} \le C$$

holds with some constant C independent of ε.

Proof. A chart (U, φ) of the manifold N is called *straightening the vector field* H if $\varphi_* H$ coincides with the coordinate vector field $\partial/\partial x^1$ on the range $\varphi(U) \subset \mathbf{R}^n$. There exists a finite atlas $\{U_a, \varphi_a\}_{a=1}^A$ consisting of straightening charts [Ar]. Choose a partition of unity subordinated to the atlas and trivializations of E over U_a. Given a differentiable along H section $f \in C(E)$, representation (4.1) holds where, for every a and α, $\widetilde{f}_a^\alpha = (\mu_a f^\alpha) \circ \varphi_a^{-1}$ is a continuous compactly supported function on $\mathbf{R}^n$ with the continuous derivative $\partial \widetilde{f}_a^\alpha / \partial x^1$. If f belongs to the class W_p^1, then every $\widetilde{f}_a^\alpha$ belongs to $W_p^1(\mathbf{R}^n)$.

Fix a nonnegative function $\lambda \in C_0^\infty(\mathbf{R}^n)$ such that $\int_{\mathbf{R}^n} \lambda\, dx = 1$, and put $\lambda_\delta(x) = \lambda(x/\delta)/\delta^n$ for $\delta > 0$. For every indices a and α, the function $\overset{\delta}{f}{}_a^\alpha = f_a^\alpha * \lambda_\delta$ is C^∞-smooth on $\mathbf{R}^n$, and $\operatorname{supp} \overset{\delta}{f}{}_a^\alpha \subset \varphi_a(U_a)$ for sufficiently small δ. The differences $\overset{\delta}{f}{}_a^\alpha - \widetilde{f}_a^\alpha$ and $\partial(\overset{\delta}{f}{}_a^\alpha - \widetilde{f}_a^\alpha)/\partial x^1$ tend to zero uniformly on $\mathbf{R}^n$ as $\delta \to 0$. If f belongs to the class W^1_p, then the difference $\overset{\delta}{f}{}^\alpha_a - \widetilde{f}^\alpha_a$ tends to zero in the space $W^1_p(\mathbf{R}^n)$. Therefore the section

$$\widetilde{f} = \sum_{a=1}^{A} \sum_{\alpha=1}^{m} \mu_a (\overset{\delta}{f}{}^\alpha_a \circ \varphi_a) e^a_\alpha$$

possesses all the desired properties for sufficiently small $\delta > 0$. The lemma is proved.

Given an Anosov manifold (M, g), let $\beta^0_2 M|_{\Omega M}$ be the restriction of the bundle $\beta^0_2 M$ to the compact manifold $N = \Omega M$, and E be its subbundle consisting of semibasic tensors $f = (f_{ij})$ satisfying the conditions $f_{ij} = f_{ji}$ and $\xi^i f_{ij} = 0$. Let $\overset{s}{a}, \overset{u}{a} \in C(E)$ be the sections constructed in Lemma 3.1, and H be the geodesic vector field on ΩM. Applying Lemma 4.1 to $\overset{s}{a}$ and $\overset{u}{a}$ and extending the so-obtained smooth fields to $T^0 M$ by homogeneity, we obtain the following

Lemma 4.2. *Let (M, g) be an n-dimensional Anosov manifold. For every $\varepsilon > 0$, there exist smooth semibasic tensor fields $\overset{s}{a} = (\overset{s}{a}_{ij}(x, \xi)) \in C^\infty(\beta^0_2 M; T^0 M)$ and $\overset{u}{a} = (\overset{u}{a}_{ij}(x, \xi)) \in C^\infty(\beta^0_2 M; T^0 M)$ satisfying statements (1)–(3) of Lemma 3.1 and the following ones:*

(4′) along every geodesic $\gamma : \mathbf{R} \to M$, the fields $\overset{s}{a}{}^i_j(t) = (g^{ik} \overset{s}{a}_{kj})(\gamma(t), \dot\gamma(t))$ and $\overset{u}{a}{}^i_j(t) = (g^{ik} \overset{u}{a}_{kj})(\gamma(t), \dot\gamma(t))$ satisfy the inequality

$$|a' + a^2 + R| < \varepsilon$$

for all $t \in \mathbf{R}$;

(4″) the estimates

$$\|\overset{s}{a}\|_{C(\beta^0_2 M|_{\Omega M})} \le C, \quad \|\overset{u}{a}\|_{C(\beta^0_2 M|_{\Omega M})} \le C$$

hold with some constant C independent of ε;

(5′) if the stable and unstable distributions belong to the class W^1_p, then the estimates

$$\|\overset{s}{a}\|_{W^1_p(\beta^0_2 M|_{\Omega M})} \le C, \quad \|\overset{u}{a}\|_{W^1_p(\beta^0_2 M|_{\Omega M})} \le C$$

hold with some constant C independent of ε.

5. The modified horizontal derivatives $\overset{s}{\nabla}$ and $\overset{u}{\nabla}$

Here we will use the vertical and horizontal derivatives $\overset{v}{\nabla}, \overset{h}{\nabla} : C^\infty(\beta^r_s M) \to C^\infty(\beta^r_{s+1} M)$ defined for a Riemannian manifold (M, g). The definition of the derivatives is presented in Chapter 3 of [Sh1] (see also [PSh]). There are canonical isomorphisms $\beta^r_s M \cong \beta^{r+s}_0 M \cong \beta^0_{r+s} M$ defined by the metric g, and we will sometimes identify these bundles.

Let us now recall the construction of a modified horizontal derivative exposed in Chapter 8 of [Sh1] and in [Sh3]. A *modifying tensor* is a semibasic tensor field $a = (a^{ij}(x, \xi)) \in C^\infty(\beta^2_0 M)$ which is symmetric, $a^{ij} = a^{ji}$; orthogonal to ξ, $\xi_i a^{ij}(x, \xi) =$

0; and positively homogeneous of degree one in ξ. Given such a field, the *modified horizontal derivative* is the first order differential operator

$$\overset{a}{\nabla} : C^\infty(\beta^r_s M) \to C^\infty(\beta^{r+1}_s M)$$

which is defined in local coordinates as follows. For $u = (u^{i_1\dots i_r}_{j_1\dots j_s}) \in C^\infty(\beta^r_s M)$,

$$\overset{a}{\nabla}{}^k u^{i_1\dots i_r}_{j_1\dots j_s} = \overset{h}{\nabla}{}^k u^{i_1\dots i_r}_{j_1\dots j_s} + a^{kp}\overset{v}{\nabla}_p u^{i_1\dots i_r}_{j_1\dots j_s}$$
$$-\sum_{\alpha=1}^{r}\overset{v}{\nabla}_p a^{ki_\alpha}\cdot u^{i_1\dots i_{\alpha-1}pi_{\alpha+1}\dots i_r}_{j_1\dots j_s} + \sum_{\alpha=1}^{s}\overset{v}{\nabla}_{j_\alpha} a^{kp}\cdot u^{i_1\dots i_r}_{j_1\dots j_{\alpha-1}pj_{\alpha+1}\dots j_s}. \tag{5.1}$$

In contrast to the ordinary horizontal derivative, the metric tensor g is not parallel with respect to $\overset{a}{\nabla}$. This is just the reason why k is an upper index in definition (5.1). Nevertheless, we will also use the operator $\overset{a}{\nabla}_k = g_{kl}\overset{a}{\nabla}{}^l$.

Since the modifying tensor is orthogonal to ξ, the geodesic vector field considered as the differential operator $H : C^\infty(TM) \to C^\infty(TM)$ can be expressed in terms of $\overset{a}{\nabla}$

$$Hu = \xi_k \overset{a}{\nabla}{}^k u \qquad (u \in C^\infty(TM)). \tag{5.2}$$

The curvature tensor $\overset{a}{R} = (\overset{a}{R}_{ijkl}(x,\xi)) \in C^\infty(\beta^0_4 M)$ corresponding to $\overset{a}{\nabla}$ is defined by the equality

$$\overset{a}{R}_{ijkl} = R_{ijkl} + \overset{h}{\nabla}_l\overset{v}{\nabla}_j a_{ik} - \overset{h}{\nabla}_k\overset{v}{\nabla}_j a_{il} +$$
$$+a_{lp}\overset{v}{\nabla}{}^p\overset{v}{\nabla}_j a_{ik} - a_{kp}\overset{v}{\nabla}{}^p\overset{v}{\nabla}_j a_{il} + \overset{v}{\nabla}{}^p a_{ik}\cdot\overset{v}{\nabla}_j a_{lp} - \overset{v}{\nabla}{}^p a_{il}\cdot\overset{v}{\nabla}_j a_{kp},$$

where (R_{ijkl}) is the Riemannian curvature tensor. This formula implies that

$$\overset{a}{R}_{ijkl}\xi^j\xi^l = (Ha)_{ik} + a^l_i a_{kl} + R_{ijkl}\xi^j\xi^l. \tag{5.3}$$

Let now (M,g) be an Anosov manifold, and $\overset{s}{a}$ ($\overset{u}{a}$) be the semibasic tensor field constructed in Lemma 4.2. Setting $a = \overset{s}{a}$ ($\overset{u}{a}$) in (5.1), we define the modified horizontal derivative on $C^\infty(\beta^r_s M; T^0M)$ which will be denoted by $\overset{s}{\nabla}$ ($\overset{u}{\nabla}$). The corresponding curvature tensor will be denoted by $\overset{s}{R}$ ($\overset{u}{R}$).

Comparing (5.3) with statement ($4'$) of Lemma 4.2, we arrive at the following important conclusion: for every semibasic vector fields $v, w \in C^\infty(\beta^1_0 M)$, the estimates

$$|\overset{s}{R}_{ijkl}\xi^i\xi^k v^j w^l| < \varepsilon|v||w|, \quad |\overset{u}{R}_{ijkl}\xi^i\xi^k v^j w^l| < \varepsilon|v||w| \tag{5.4}$$

hold uniformly on ΩM.

By statement (3) of Lemma 4.2, the set

$$\overset{s}{\nabla}{}^i u, \quad \overset{u}{\nabla}{}^i u \ (i = 1,\dots,n), \quad \xi^k\overset{v}{\nabla}_k u$$

is a full family of derivatives of a function $u \in C^\infty(T^0M)$, i.e., every first order derivative of u is a linear combination of the set. This observation is specified by the following

Lemma 5.1. *For every function $u \in C^\infty(T^0M)$, the estimates*

$$|\overset{v}{\nabla} u - \langle \xi, \overset{v}{\nabla} u\rangle \xi| \le C(|\overset{s}{\nabla} u| + |\overset{u}{\nabla} u|), \tag{5.5}$$

$$|\overset{h}{\nabla} u| \le C(|\overset{s}{\nabla} u| + |\overset{u}{\nabla} u|) \tag{5.6}$$

hold on ΩM with some constant C independent of u.

Proof. It suffices to consider a function u whose support is contained in the domain $U \subset T^0M$ of a local coordinate system. The semibasic vector field $y = \overset{v}{\nabla} u - \langle \xi, \overset{v}{\nabla} u\rangle \xi$ is orthogonal to ξ on ΩM. By the definition of the modified derivatives

$$\overset{s}{\nabla}{}^i u = \overset{h}{\nabla}{}^i u + \overset{s}{a}{}^{ij} \overset{v}{\nabla}_j u, \quad \overset{u}{\nabla}{}^i u = \overset{h}{\nabla}{}^i u + \overset{u}{a}{}^{ij} \overset{v}{\nabla}_j u. \tag{5.7}$$

Substituting $\overset{v}{\nabla}_j u = y_j + \langle \xi, \overset{v}{\nabla} u\rangle \xi_j$ into these equalities and using orthogonality of $\overset{s}{a}$ and $\overset{u}{a}$ to ξ, we obtain

$$\overset{s}{\nabla}{}^i u = \overset{h}{\nabla}{}^i u + \overset{s}{a}{}^{ij} y_j, \quad \overset{u}{\nabla}{}^i u = \overset{h}{\nabla}{}^i u + \overset{u}{a}{}^{ij} y_j.$$

This implies that

$$(\overset{s}{a}{}^{ij} - \overset{u}{a}{}^{ij}) y_j = \overset{s}{\nabla}{}^i u - \overset{u}{\nabla}{}^i u. \tag{5.8}$$

By statements (1) and (3) of Lemma 4.2, the operator $\overset{s}{a} - \overset{u}{a}$ is an automorphism of the space $N_{(x,\xi)} = \{\eta \in T_xM \mid \langle \xi, \eta\rangle = 0\}$. The right-hand side of (5.8) belongs to this space because $\langle \xi, \overset{s}{\nabla} u - \overset{u}{\nabla} u\rangle = \xi_i \overset{s}{\nabla}{}^i u - \xi_i \overset{u}{\nabla}{}^i u = Hu - Hu = 0$. Therefore equation (5.8) has a unique solution

$$y_i = \alpha_{ij}(\overset{s}{\nabla}{}^j u - \overset{u}{\nabla}{}^j u)$$

with some coefficients α_{ij} that are smooth functions in U and are independent of u. ¿From this we obtain (5.5). The estimate (5.6) follows from (5.5) and (5.7). The lemma is proved.

For a semibasic tensor field $u \in C^\infty(\beta^r_s M; T^0M)$, we will use the notations

$$\|u\|^2_{L_2} = \int\limits_{\Omega M} |u(x,\xi)|^2 \, d\Sigma(x,\xi), \quad \|u\|^2_{H^1} = \|u\|^2_{L_2} + \|\overset{h}{\nabla} u\|^2_{L_2} + \|\overset{v}{\nabla} u\|^2_{L_2},$$

where $d\Sigma(x,\xi) = d\omega_x(\xi) \wedge dV^n(x)$ is the symplectic volume form on ΩM. Lemma 5.1 has the following

Corollary 5.2. *The two norms $\|u\|_{H^1}$ and $(\|\overset{s}{\nabla} u\|^2_{L_2} + \|\overset{u}{\nabla} u\|^2_{L_2} + \|u\|^2_{L_2})^{1/2}$ are equivalent on the space of positively homogeneous in ξ functions $u(x,\xi) \in C^\infty(T^0M)$ with the same degree of homogeneity.*

Indeed, $\langle \xi, \overset{v}{\nabla} u\rangle = \lambda u$ if u is homogeneous of degree λ.

Remark. The numbers ε and C participating in (5.4)–(5.6) are independent in the following sense: ε can be chosen arbitrary small with a fixed value of C. Indeed, C is determined by the continuous fields $\overset{s}{a}$ and $\overset{u}{a}$ constructed in Lemma 3.1, while ε depends on the degree of approximating these fields by smooth ones.

6. Proof of Lemma 2.1

Let a tensor field $f \in C^\infty(S^m\tau'_M)$ and a function $u \in C^\infty(\Omega M)$ satisfy equation (2.1). We extend the function u to T^0M in such the way that it is positively homogeneous of degree $m-1$ in ξ. Then equation (2.1) holds on T^0M. By (5.2), this equation can be rewritten in the form

$$Hu = \xi_k \overset{s}{\nabla}{}^k u = \xi_k \overset{u}{\nabla}{}^k u = \langle f(x), \xi^m\rangle. \tag{6.1}$$

For the modified horizontal derivative $\overset{s}{\nabla}$, the following *Pestov identity* is valid (see Section 8.2 of [Sh1] or [Sh3]):

$$2\langle \overset{s}{\nabla} u, H\overset{v}{\nabla} u\rangle = |\overset{s}{\nabla} u|^2 + \overset{s}{\nabla}{}^i v_i + \overset{v}{\nabla}_i w^i - \overset{s}{R}_{ijkl}\xi^i\xi^k \overset{v}{\nabla}{}^j u \cdot \overset{v}{\nabla}{}^l u, \tag{6.2}$$

where

$$v_i = \xi_i \overset{s}{\nabla}{}^j u \cdot \overset{v}{\nabla}_j u - \xi_j \overset{v}{\nabla}_i u \cdot \overset{s}{\nabla}{}^j u, \tag{6.3}$$

$$w^i = \xi_j \overset{s}{\nabla}{}^i u \cdot \overset{s}{\nabla}{}^j u. \tag{6.4}$$

We transform the left-hand side of identity (6.2). From (6.1) we obtain

$$\overset{v}{\nabla}_k(Hu) = \overset{v}{\nabla}_k(\langle f(x), \xi^m\rangle) = m f_{ki_2\dots i_m}\xi^{i_2}\dots\xi^{i_m}.$$

With the help of the relation $\overset{s}{\nabla}{}^i \xi_j = 0$, the latter formula implies

$$2\langle \overset{s}{\nabla} u, \overset{v}{\nabla} Hu\rangle = 2m\overset{s}{\nabla}{}^i u \cdot f_{ii_2\dots i_m}\xi^{i_2}\dots\xi^{i_m} =$$

$$= \overset{s}{\nabla}{}^i(2mu f_{ii_2\dots i_m}\xi^{i_2}\dots\xi^{i_m}) - 2mu\overset{s}{\nabla}{}^i(f_{ii_2\dots i_m}\xi^{i_2}\dots\xi^{i_m}).$$

Introducing the notation

$$\widetilde{v}_i = 2mu f_{ii_2\dots i_m}\xi^{i_2}\dots\xi^{i_m},$$

we have

$$2\langle \overset{s}{\nabla} u, \overset{v}{\nabla} Hu\rangle = \overset{s}{\nabla}{}^i \widetilde{v}_i - 2mu\overset{s}{\nabla}{}^i(f_{ii_2\dots i_m}\xi^{i_2}\dots\xi^{i_m}). \tag{6.5}$$

By the definition of the modified derivative

$$\overset{s}{\nabla}{}^i(f_{ii_2\dots i_m}\xi^{i_2}\dots\xi^{i_m}) = \overset{h}{\nabla}{}^i(f_{ii_2\dots i_m}\xi^{i_2}\dots\xi^{i_m}) +$$

$$+ \overset{s}{a}{}^{ip}\overset{v}{\nabla}_p(f_{ii_2\dots i_m}\xi^{i_2}\dots\xi^{i_m}) + \overset{v}{\nabla}_i \overset{s}{a}{}^{ip}\cdot(f_{pi_2\dots i_m}\xi^{i_2}\dots\xi^{i_m}) =$$

$$= (\delta f)_{i_2\dots i_m}\xi^{i_2}\dots\xi^{i_m} + (m-1)\overset{s}{a}{}^{ip} f_{ipi_3\dots i_m}\xi^{i_3}\dots\xi^{i_m} + \overset{v}{\nabla}_i \overset{s}{a}{}^{ip}\cdot f_{pi_2\dots i_m}\xi^{i_2}\dots\xi^{i_m}.$$

Substituting the latter expression into (6.5), we obtain

$$2\langle \overset{s}{\nabla} u, \overset{v}{\nabla} Hu\rangle = \overset{s}{\nabla}{}^i \widetilde{v}_i - 2mu(\delta f)_{i_2\dots i_m}\xi^{i_2}\dots\xi^{i_m} -$$

$$-2m(m-1)\overset{s}{a}{}^{ip}uf_{ipi_3\dots i_m}\xi^{i_3}\dots\xi^{i_m}-2m\overset{v}{\nabla}_i\overset{s}{a}{}^{ip}\cdot uf_{pi_2\dots i_m}\xi^{i_2}\dots\xi^{i_m}.$$

With the help of the latter formula, we transform Pestov identity (6.2) to the form

$$|\overset{s}{\nabla}u|^2=\overset{s}{\nabla}{}^i(\widetilde{v}_i-v_i)-\overset{v}{\nabla}_iw^i-2mu(\delta f)_{i_2\dots i_m}\xi^{i_2}\dots\xi^{i_m}-2m(m-1)\overset{s}{a}{}^{ip}uf_{ipi_3\dots i_m}\xi^{i_3}\dots\xi^{i_m}$$

$$-2m\overset{v}{\nabla}_i\overset{s}{a}{}^{ip}\cdot uf_{pi_2\dots i_m}\xi^{i_2}\dots\xi^{i_m}+\overset{s}{R}_{ijkl}\xi^i\xi^k\overset{v}{\nabla}{}^ju\cdot\overset{v}{\nabla}{}^lu. \tag{6.6}$$

With the help of statement ($4''$) of Lemma 4.2 and inequality (5.4), the last tree terms on the right-hand side of (6.6) can be estimated at a point $(x,\xi)\in\Omega M$ as follows:

$$|\overset{s}{a}{}^{ip}uf_{ipi_3\dots i_m}\xi^{i_3}\dots\xi^{i_m}|\le C|u||f|,$$

$$|\overset{v}{\nabla}_i\overset{s}{a}{}^{ip}\cdot uf_{pi_2\dots i_m}\xi^{i_2}\dots\xi^{i_m}|\le|\overset{v}{\nabla}a||u||f|,$$

$$|\overset{s}{R}_{ijkl}\xi^i\xi^k\overset{v}{\nabla}{}^ju\cdot\overset{v}{\nabla}{}^lu|<\varepsilon|\overset{v}{\nabla}u|^2.$$

With the help of these estimates, (6.6) gives the inequality

$$|\overset{s}{\nabla}u|^2\le\overset{s}{\nabla}{}^i(\widetilde{v}_i-v_i)-\overset{v}{\nabla}_iw^i+C|u||f|+2m|u||\delta f|+\varepsilon|\overset{v}{\nabla}u|^2+2m|\overset{v}{\nabla}a||u||f| \tag{6.7}$$

that is valid on ΩM.

We integrate inequality (6.7) over ΩM and transform the integrals of divergent terms by the Gauss—Ostrogradskiĭ formulas (Theorem 3.6.3 and formula (8.2.30) of [Sh1]). The integral of $\overset{s}{\nabla}{}^i(\widetilde{v}_i-v_i)$ equals to zero because ΩM is a closed manifold. The integral of the second term is nonpositive; indeed

$$-\int\limits_{\Omega M}\overset{v}{\nabla}_iw^i\,d\Sigma=-(n+2m-2)\int\limits_{\Omega M}\langle w,\xi\rangle\,d\Sigma=-(n+2m-2)\int\limits_{\Omega M}|Hu|^2\,d\Sigma\le 0.$$

The latter equality is written because $\langle w,\xi\rangle=|Hu|^2$ as is seen from (6.4). Thus, after integration (6.7) gives us the inequality

$$\|\overset{s}{\nabla}u\|^2_{L_2}\le C(\|u\|_{L_2}\cdot\|f\|_{L_2}+\|u\|_{L_2}\cdot\|\delta f\|_{L_2})+\varepsilon\|u\|^2_{H^1}+\int\limits_{\Omega M}|\overset{v}{\nabla}a||u||f|\,d\Sigma. \tag{6.8}$$

Let $p>2n-1$ be such that the stable and unstable distributions belong to the class W^1_p, and let $q=2p/(p-2)$. Applying the Hölder inequality, we estimate the integral on (6.8)

$$\int\limits_{\Omega M}|\overset{v}{\nabla}a||u||f|\,d\Sigma\le\left(\int\limits_{\Omega M}|\overset{v}{\nabla}a|^p\,d\Sigma\right)^{1/p}\|u\|_{L_q}\cdot\|f\|_{L_2}.$$

By statement (5′) of Lemma 4.2, the integral on the right-hand side can be estimated by some constant C independent on u, and we obtain

$$\int\limits_{\Omega M} |\overset{v}{\nabla} a||u||f|\,d\Sigma \le C\|u\|_{L_q}\cdot\|f\|_{L_2}. \tag{6.9}$$

With the help of the latter estimate, (6.8) implies the inequality

$$\|\overset{s}{\nabla} u\|^2_{L_2} \le C(\|u\|_{L_q}\cdot\|f\|_{L_2} + \|u\|_{L_2}\cdot\|\delta f\|_{L_2}) + \varepsilon\|u\|^2_{H^1}. \tag{6.10}$$

We can repeat our arguments with $\overset{s}{\nabla}$ replaced with $\overset{u}{\nabla}$. In such the way we obtain the following analog of (6.10):

$$\|\overset{u}{\nabla} u\|^2_{L_2} \le C(\|u\|_{L_q}\cdot\|f\|_{L_2} + \|u\|_{L_2}\cdot\|\delta f\|_{L_2}) + \varepsilon\|u\|^2_{H^1}. \tag{6.11}$$

With the help of Corollary 5.2, inequalities (6.10) and (6.11) give

$$\|u\|^2_{H^1} \le CC'(\|u\|_{L_q}\cdot\|f\|_{L_2} + \|u\|_{L_2}\cdot\|\delta f\|_{L_2}) + C'\|u\|^2_{L_2} + C'\varepsilon\|u\|^2_{H^1}. \tag{6.12}$$

As we have emphasized, the number ε can be chosen arbitrary small with some fixed values of C and C'. In particular, we can assume that $C'\varepsilon < 1$, and inequality (6.12) can be rewritten in the form

$$\|u\|^2_{H^1} \le C(\|u\|_{L_q}\cdot\|f\|_{L_2} + \|u\|_{L_2}\cdot\|\delta f\|_{L_2} + \|u\|^2_{L_2}) \tag{6.13}$$

with some new constant C independent of u.

The kinetic equation $Hu = \xi^i\overset{h}{\nabla}_i u = \langle f, \xi^m\rangle$ implies the estimate $\|f\|_{L_2} \le C\|\overset{h}{\nabla} u\|_{L_2} \le C\|u\|_{H^1}$ that allows us to rewrite (6.13) in the form

$$\|u\|^2_{H^1} \le C(\|u\|_{L_q}\cdot\|u\|_{H^1} + \|u\|_{L_2}\cdot\|\delta f\|_{L_2} + \|u\|^2_{L_2}). \tag{6.14}$$

Considering (6.14) as a quadratic inequality in $x = \|u\|_{H^1}$, one can easily see that it implies (2.2) with another constant C. The lemma is proved.

Observe that the hypothesis on membership of the stable and unstable distributions in the class W^1_p was used in the proof only once, in estimate (6.9). As we will see, there is no need of this estimate in the proofs of Theorems 1.1 and 1.3.

Proof of Theorem 1.1. Let a function $f \in C^\infty(M)$ integrates to zero over every closed geodesic. Applying the Livčic theorem, we obtain the function $u \in C^\infty(\Omega M)$ satisfying the kinetic equation

$$Hu(x,\xi) = f(x) \tag{6.15}$$

on ΩM. Extending u to T^0M in such the way as $u(x, t\xi) = t^{-1}u(x,\xi)$ for $t > 0$, equation (6.15) is satisfied on T^0M. The left-hand side of the Pestov identity (6.2) is identical zero in our case. After integration over ΩM, the identity gives

$$\|\overset{s}{\nabla} u\|^2_{L_2} + (n-2)\|Hu\|^2_{L_2} = \int\limits_{\Omega M} \overset{s}{R}_{ijkl}\xi^i\xi^k\overset{v}{\nabla}{}^j u\cdot\overset{v}{\nabla}{}^l u\,d\Sigma. \tag{6.16}$$

Let $y^i = \overset{v}{\nabla}{}^i u - \langle \xi, \overset{v}{\nabla} u\rangle \xi^i$. Then, using the symmetries of the curvature tensor, we obtain

$$\overset{s}{R}_{ijkl}\xi^i\xi^k\overset{v}{\nabla}{}^j u \cdot \overset{v}{\nabla}{}^l u = \overset{s}{R}_{ijkl}\xi^i\xi^k y^j y^l.$$

With the help of (5.4) and (5.5), the latter equality implies the estimate

$$|\overset{s}{R}_{ijkl}\xi^i\xi^k\overset{v}{\nabla}{}^j u \cdot \overset{v}{\nabla}{}^l u| \le \varepsilon |y|^2 \le C\varepsilon(|\overset{s}{\nabla} u|^2 + |\overset{u}{\nabla} u|^2).$$

Combining this estimate with (6.16), we obtain

$$\|\overset{s}{\nabla} u\|^2_{L_2} \le C\varepsilon(\|\overset{s}{\nabla} u\|^2_{L_2} + \|\overset{u}{\nabla} u\|^2_{L_2}). \tag{6.17}$$

In the same way we obtain the similar estimate

$$\|\overset{u}{\nabla} u\|^2_{L_2} \le C\varepsilon(\|\overset{s}{\nabla} u\|^2_{L_2} + \|\overset{u}{\nabla} u\|^2_{L_2}). \tag{6.18}$$

If $C\varepsilon < 1/2$, inequalities (6.17) and (6.18) imply that $\overset{s}{\nabla} u = \overset{u}{\nabla} u = 0$. Consequently, $f = \xi_i \overset{s}{\nabla}{}^i u = 0$. The theorem is proved.

Proof of Theorem 1.3. In this case the kinetic equation looks as follows:

$$Hu(x,\xi) = f_i(x)\xi^i, \tag{6.19}$$

and $\overset{v}{\nabla} Hu = f$. Therefore the Pestov identity (6.2) has the form

$$2\langle \overset{s}{\nabla} u, f\rangle = |\overset{s}{\nabla} u|^2 + \overset{s}{\nabla}{}^i v_i + \overset{v}{\nabla}_i w^i - \overset{s}{R}_{ijkl}\xi^i\xi^k\overset{v}{\nabla}{}^j u \cdot \overset{v}{\nabla}{}^l u.$$

After integration over ΩM this gives

$$\|\overset{s}{\nabla} u\|^2_{L_2} - 2(\overset{s}{\nabla} u, f)_{L_2} + n\|Hu\|^2_{L_2} = \int\limits_{\Omega M} \overset{s}{R}_{ijkl}\xi^i\xi^k\overset{v}{\nabla}{}^j u \cdot \overset{v}{\nabla}{}^l u \, d\Sigma. \tag{6.20}$$

¿From (6.19), we obtain

$$\|Hu\|^2_{L_2} = \int\limits_{\Omega M} f_i(x) f_j(x)\xi^i\xi^j \, d\Sigma(x,\xi) =$$

$$= \int\limits_M f_i(x) f_j(x) \left[\int\limits_{\Omega_x M} \xi^i\xi^j \, d\omega_x(\xi)\right] dV^n(x) = \frac{1}{n}\|f\|^2_{L_2}.$$

With the help of the latter equality, (6.20) takes the form

$$\|\overset{s}{\nabla} u - f\|^2_{L_2} = \int\limits_{\Omega M} \overset{s}{R}_{ijkl}\xi^i\xi^k\overset{v}{\nabla}{}^j u \cdot \overset{v}{\nabla}{}^l u \, d\Sigma.$$

Estimating the right-hand side integral with the help of (5.4), we obtain

$$\|\overset{s}{\nabla}u - f\|^2_{L_2} \le \varepsilon C \|\overset{v}{\nabla}u\|^2_{L_2}. \tag{6.21}$$

Repeating our arguments with interchanged $\overset{s}{\nabla}$ and $\overset{u}{\nabla}$, we obtain the similar inequality

$$\|\overset{u}{\nabla}u - f\|^2_{L_2} \le \varepsilon C \|\overset{v}{\nabla}u\|^2_{L_2}. \tag{6.22}$$

Let now ε tend to zero in (6.21) and (6.22). The vector fields f and $\overset{v}{\nabla}u$ are independent of ε as well as the constant C. The fields $\overset{s}{\nabla}u$ and $\overset{u}{\nabla}u$ tend respectively to

$$\overset{s}{\nabla}_i u = \overset{h}{\nabla}_i u + \overset{s}{a}{}_i^p \overset{v}{\nabla}_p u, \quad \overset{u}{\nabla}_i u = \overset{h}{\nabla}_i u + \overset{u}{a}{}_i^p \overset{v}{\nabla}_p u \tag{6.23}$$

with the continuous tensors $\overset{s}{a}$ and $\overset{u}{a}$ constructed in Lemma 3.1. Thus, passing to limit in (6.21) and (6.22) as $\varepsilon \to 0$, we obtain

$$\overset{s}{\nabla}_i u = f_i = \overset{u}{\nabla}_i u. \tag{6.24}$$

Now equalities (6.23) give us

$$(\overset{s}{a}{}_i^p - \overset{u}{a}{}_i^p)\overset{v}{\nabla}_p u = 0. \tag{6.25}$$

In our case the function $u(x,\xi)$ is positively homogeneous of zero degree, and therefore

$$\xi^k \overset{v}{\nabla}_k u = 0. \tag{6.26}$$

With the help of statement (3) of Lemma 3.1, (6.25) and (6.26) imply that $\overset{v}{\nabla}u = 0$, i.e., the function u is independent of ξ; $u = u(x)$. Equalities (6.23) and (6.24) take now the form $f_i = \overset{h}{\nabla}_i u = \partial u/\partial x^i$. Therefore f is the exact form, $f = du$. The theorem is proved.

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