# Approximation of nonessential spectrum of transfer operators

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Abstract. We give sufficient conditions to approximate the "nonessential" spectrum of a bounded operator $\mathcal{L}$ acting on a Banach space $\mathcal{B}$ by part of the spectra of a sequence of compact (or finite rank) operators $\mathcal{L}_j = (\mathrm{Id} - \Pi_j)\mathcal{L}(\mathrm{Id} - \Pi_j)$, where $\mathrm{Id} - \Pi_j$ is a suitable family of uniformly bounded operators which approach the identity. (By nonessential spectrum we mean here all the spectrum outside of the disc of radius equal to the essential spectral radius.) For this, we combine the formulas

$$\rho_{\mathrm{ess}}(\mathcal{L}) = \lim_{m\to\infty} (\limsup_{j\to\infty}(\inf)\|\Pi_j \mathcal{L}^m\|)^{1/m} = \lim_{m\to\infty} (\limsup_{j\to\infty}(\inf)\|\mathcal{L}^m \Pi_j\|)^{1/m} ,$$

for the essential spectral radius with nonstandard perturbative results on the stability of the nonessential spectrum of quasicompact operators. We present concrete applications to transfer operators of smooth expanding maps using multiresolution analysis (large scale approximation projections).

## 1. Introduction

Matrices, and more generally linear operators on infinite-dimensional vector spaces, are ubiquitous tools which permeate pure and applied mathematics. A natural problems, which has kept mathematicians busy for centuries, is to determine, or at least approximate, their spectrum (in infinite-dimensional situations, sometimes only a discrete part of it). In this work, we are concerned with the infinite-dimensional (Banach space) situation, and we deal with bounded linear operators which are not necessarily compact. Our main result (Proposition 3 in Section 2) is a list of conditions guaranteeing that a subset of the eigenvalues of a sequence of compact or finite-rank operators $(\mathrm{Id} - \Pi_j)\mathcal{L}(\mathrm{Id} - \Pi_j)$ (together with the corresponding eigenspaces) converge to those eigenvalues of the original operator $\mathcal{L}$ which are outside of a disc containing the essential spectrum. The simple proof combines a convenient exact formula for the essential spectral radius (Theorem 1 from [H1]) with a non-standard – and somewhat unexpected – perturbative result (Theorem 2 from [BY]), which had originally been used to control the spectrum of randomly perturbed dynamical systems.

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Sequences of compact or finite rank operators $\mathrm{Id} - \Pi_j$ for which our results hold can sometimes be explicited in some cases (sometimes via multiresolution analysis, using wavelets). In Section 3, we explain a specific dynamical systems setting where our scheme works. The linear operator there is the Ruelle transfer operator associated to a differentiable uniformly expanding dynamical system on a torus. (Transfer operators, sometimes also called Perron-Frobenius operators, are very powerful tools to study the ergodic properties of dynamical systems. We refer e.g. to [R] and the references therein for the framework of Section 3.) The Banach spaces are Sobolev or Hölder spaces, and the finite-rank operators $\mathrm{Id} - \Pi_j$ are constructed using the Meyer [Me] orthonormal wavelet basis (it would be interesting to actually run the algorithm on a computer).

An important and famous finite-dimensional matrix scheme used in ergodic theory of dynamical systems (to approximate the physical, or "SRB" measure, together with its rate of mixing) is the Ulam method. Recent numerical and theoretical work has shown that not only the maximal eigenvalue, but also further spectral values of Ulam matrices approximate well part the spectrum associated to various types of chaotic dynamics (see in particular the sequence of papers and effective algorithms of Dellnitz and collaborators [DJ], and the rigorous results of Hunt [Hu], and Froyland [Fr]; we also mention the recent paper [MKY] — see also [K1] — for similar approximation results, together with quantified estimates on the speed of convergence, finally [BIS] contains results obtained using Theorem 2 below from [BY]). It seems to us, however, that since Ulam matrices are obtained by locally constant approximations, they cannot describe the action of the dynamics on observables smoother than Hölder or Lipschitz. Our scheme, on the other hand, is applicable to a wider range of smoothness classes.

We end this introduction by mentioning, in order of expected difficulty, three directions for future research. As soon as one proves that a mathematical object can be approximated by a sequence, one obvious question is the speed of convergence. For the case considered in Section 3, we believe that exponential speeds of convergence hold (by analogy to the results in [KMY], e.g.).

A second natural problem consists in extending our dynamical results from Section 3 to compact boundaryless manifolds more general than the $n$-torus $\mathbb{T}^n$. This should be possible by developing and/or applying the necessary multi-resolution analysis.

Last, but definitely not least, we have limited ourselves to uniformly expanding dynamical systems for which the transfer operator has nice spectral properties when acting on smooth function. When the dynamics is uniformly hyperbolic, the inverse maps improve smoothness along unstable manifolds but make functions less smooth along stable manifolds. Although recent progress has been made in our understanding of analytic [Rg], but also differentiable [Li, Ki], hyperbolic settings, one still does not have a good Banach space framework for the transfer operator. Perhaps our approximation scheme can be extended to the hyperbolic setting via the use of "directional" Banach spaces. (Further extensions to nonuniformly hyperbolic dynamics would also be desirable.)

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## 2. Approximation of the discrete spectrum: two abstract results

We first recall a few basic definitions and facts (see [K] and [DS] for more information). Let $(\mathcal{B}, \|\cdot\|)$ be a Banach space, that will always be assumed infinite-dimensional. (Since we mostly have function spaces in mind, we denote vectors in $\mathcal{B}$ by $\varphi$, $\psi$ etc.). Denote by $B(\mathcal{B})$ the set of bounded linear operators acting in $\mathcal{B}$ (noting $\|\mathcal{L}\|$ for the operator norm of $\mathcal{L} \in B(\mathcal{B})$), by $K(\mathcal{B}) \subset B(\mathcal{B})$ the ideal of compact operators, and by $F(\mathcal{B}) \subset K(\mathcal{B})$ the ideal of finite rank operators. For $\mathcal{L} \in B(\mathcal{B})$ the *resolvent set* of $\mathcal{L}$ is the set of complex numbers $z$ so that $\mathcal{L} - z\,\mathrm{Id} : \mathcal{B} \to \mathcal{B}$ is an invertible operator with a bounded inverse $(\mathcal{L} - z\,\mathrm{Id})^{-1} \in B(\mathcal{B})$. The *spectrum* $\sigma(\mathcal{L})$ of $\mathcal{L}$ is the set of $z \in \mathbb{C}$ which are not in the resolvent set of $\mathcal{L}$. The *spectral radius* $\rho(\mathcal{L})$ of $\mathcal{L}$ is

$$\rho(\mathcal{L}) = \sup\{|z| \text{ s.t. } z \in \sigma(\mathcal{L})\}\,.$$

As is well known, the spectral radius of $\mathcal{L}$ can be obtained as the following limit:

$$\rho(\mathcal{L}) = \lim_{m\to\infty} \|\mathcal{L}^m\|^{1/m}\,.$$

An element $z \in \sigma(\mathcal{L})$ is an *eigenvalue* of $\mathcal{L}$ if $\mathcal{L} - z\mathrm{Id}$ is not invertible. The *geometric multiplicity* of an eigenvalue $z$ is the dimension $1 \le m_1(z) \le \infty$ of its *eigenspace* $\{\varphi \in \mathcal{B} \text{ s.t. } (\mathcal{L} - z)\varphi = 0\}$, and its *(algebraic) multiplicity* is the dimension $m_1(z) \le m_2(z) \le \infty$ of the generalised eigenspace $\{\varphi \in \mathcal{B} \text{ s.t. } \exists m \ge 1\,, (\mathcal{L} - z)^m \varphi = 0\}$. The supremum $1 \le i(z) \le \infty$ of those $m$ which occur in the definition of the generalised eigenspace of an eigenvalue $z$ is called the *index* of $z$.

The *essential spectral radius* $\rho_{\mathrm{ess}}(\mathcal{L})$ of $\mathcal{L}$ is the largest number $\kappa \ge 0$ such that any $\lambda \in \sigma(\mathcal{L})$ with modulus $|\lambda| > \kappa$ is an isolated eigenvalue of finite (algebraic) multiplicity of $\mathcal{L}$. We sometimes use the informal terminology "nonessential spectrum" or "discrete spectrum" to denote the spectrum of $\mathcal{L}$ outside of the disc of radius $\rho_{\mathrm{ess}}(\mathcal{L})$.

There exist several definitions for the *essential spectrum* of a linear bounded operator. *Browder's* [Br, Section 6] *essential spectrum* is the set of those $z \in \mathbb{C}$ such that at least one of the three following possibilities holds: $z$ is a limit point of $\sigma(\mathcal{L})$, or $(\mathcal{L} - z\,\mathrm{Id})\mathcal{B}$ is not closed, or the generalised eigenspace $\{\varphi \in \mathcal{B} \text{ s.t. } \exists m \ge 1\,, (\mathcal{L} - z)^m\varphi = 0\}$ has infinite dimension. *Wolf's* [W] *essential spectrum* is the set of those $z \in \mathbb{C}$ such that $\mathcal{L} - z\,\mathrm{Id}$ is not Fredholm. In general the Wolf and Browder essential spectrum do not coincide (as noted in [N], the complement of the Browder essential spectrum is the set of those components of the complement of the Wolf essential spectrum which meet the resolvent set).

Our definition for the essential spectral radius is consistent both with Browder's and Wolf's definition of essential spectrum as we explain now. Firstly, $\rho_{\mathrm{ess}}(\mathcal{L})$ is the radius of the smallest disc containing the Browder essential spectrum (because of [Br, Lemma 17, p. 110]). Secondly, $\rho_{\mathrm{ess}}(\mathcal{L})$ is also the radius of the smallest disc containing the Wolf essential spectrum. This second property can be deduced from two facts: on the one hand $z$ is in the Wolf essential spectrum if and only if $\mathcal{L} - z\,\mathrm{Id}$ is invertible modulo $K(\mathcal{B})$ if and only if $\mathcal{L} - z\mathrm{Id}$ is invertible modulo $F(\mathcal{B})$, see [L, Chapter IX, Theorem

6]. On the other hand Nussbaum's formula [N] states that the radius $\rho_{\mathrm{ess}}(\mathcal{L})$ of the smallest disc containing the Browder essential spectrum coincides with

$$\rho_{\mathrm{ess}}(\mathcal{L}) = \lim_{m\to\infty} (\inf\{\|\mathcal{L}^m - \mathcal{K}\| \text{ s.t. } \mathcal{K} \in K(\mathcal{B})\})^{1/m} . \tag{2.1}$$

By the above equivalent formulations of the Wolf spectrum, Nussbaum's formula can be modified to

$$\rho_{\mathrm{ess}}(\mathcal{L}) = \lim_{m\to\infty} (\inf\{\|\mathcal{L}^m - \mathcal{F}\| \text{ s.t. } \mathcal{F} \in F(\mathcal{B})\})^{1/m} . \tag{2.2}$$

This is a non-trivial result since the set $F(\mathcal{B})$ of finite rank operators is not necessarily dense in the ideal $K(\mathcal{B})$ of compact operators. (The paper [F] of Fried was extremely useful in clarifying the above points.)

We denote by $\mathcal{B}^*$ the dual space of the Banach space $\mathcal{B}$, i.e., the space of bounded linear functionals $\nu : \mathcal{B} \to \mathbb{C}$. (Having in mind mainly complex measures and distributions we write $\nu$, $\mu$ for elements of $\mathcal{B}^*$.) For $\nu \in \mathcal{B}^*$, we use the notation $\nu(\varphi) = (\nu|\varphi)$. For $\mathcal{L} \in B(\mathcal{B})$, we write $\mathcal{L}^* \in B(\mathcal{B}^*)$, for the dual operator defined by $(\mathcal{L}^*\nu|\varphi) = (\nu|\mathcal{L}(\varphi))$.

Recall that operators of finite rank on $\mathcal{B}$ can be written as

$$\psi \mapsto \sum_{\alpha\in A} \nu_\alpha(\psi)\varphi_\alpha ,$$

where $A$ is a finite index set, $\nu_\alpha \in \mathcal{B}^*$, and $\varphi_\alpha \in \mathcal{B}$.

Our first abstract result is a list of exact formulas (probably well-known "in spirit") for the essential spectral radius (see [H1] for proofs). They hold for Banach spaces for which suitable families of bounded operators $\Pi_j$ converging to zero and for which $(\mathrm{Id} - \Pi_j)\mathcal{L}$ is compact exist. Such families of operators can be constructed via multiresolution analysis for many classical function spaces (most notably Sobolev, but more generally Triebel, and partly Besov-Hölder classes), of periodic functions on the torus $\mathbb{T}^n$, say (see e.g. [H1, H2], see also Section 3 below for applications to transfer operators, where the $\mathrm{Id} - \Pi_j$ are in fact projections).

**Definition (Compact approximation of the identity for $(\mathcal{L}, \mathcal{B})$).** A sequence of operators $\mathrm{Id} - \Pi_j \in B(\mathcal{B})$, for $j \in \mathbb{Z}^+$, is a compact uniformly bounded approximation of the identity for $(\mathcal{L}, \mathcal{B})$ if:

$$(i)\ \exists \quad \text{const} > 0 \text{ s.t. } \|\Pi_j\| \le \text{const} , \forall j \in \mathbb{Z}^+ ;$$
$$(ii)\ (\mathrm{Id} - \Pi_j)\mathcal{L} \text{ is compact}, \forall j \in \mathbb{Z}^+ ;$$

and

$$(iii)\ \mathcal{B}_0 = \mathcal{B}_0(\{\Pi_j\}) = \{\varphi \in \mathcal{B} \text{ s.t. } \lim_{j\to\infty} \Pi_j(\varphi) = 0\} \text{ is dense in } \mathcal{B} .$$

We shall also need a dual notion:

**Definition ($*$-compact approximation of the identity for $(\mathcal{L}, \mathcal{B})$).** A sequence of operators $\mathrm{Id} - \Pi_j \in B(\mathcal{B})$, for $j \in \mathbb{Z}^+$, is a $*$-compact uniformly bounded approximation of the identity for $(\mathcal{L}, \mathcal{B})$ if it satisfies $(i)$ together with:

$$(ii^*)\ \mathcal{L}(\mathrm{Id} - \Pi_j) \text{ is compact, } \forall j \in \mathbb{Z}^+ ;$$

and

$$(iii^*)\ \mathcal{B}_0^* = \mathcal{B}_0^*(\{\Pi_j\}) = \{\nu \in \mathcal{B}^* \text{ s.t. } \lim_{j\to\infty} \Pi_j^*(\nu) = 0\} \text{ is dense in } \mathcal{B}^* .$$

(Note that a sequence of operators $\mathrm{Id} - \Pi_j$ is a $*$-compact uniformly bounded approximation of the identity for $(\mathcal{L}, \mathcal{B})$ if and only if the $\mathrm{Id} - \Pi_j^*$ form a compact uniformly bounded approximation of the identity for $(\mathcal{L}^*, \mathcal{B}^*)$.)

Stronger results will hold for sequences of operators satisfying certain hierarchical constraints (which hold in particular in the setting of Section 3):

**Definition (Hierarchical compact approximation of the identity for $(\mathcal{L}, \mathcal{B})$).** A sequence of operators $\mathrm{Id} - \Pi_j \in B(\mathcal{B})$ is called a *hierarchical* compact (respectively $*$-compact) approximation of the identity for $(\mathcal{L}, \mathcal{B})$ if it satisfies $(i)$, $(ii)$ and $(iii)$ (respectively $(ii^*)$, $(iii^*)$) together with

$$(iv)\ \Pi_j \Pi_{j+1} = \Pi_{j+1} \Pi_j = \Pi_{j+1} , \quad \forall\, j \in \mathbb{Z}^+ . \tag{2.3}$$

The operator $\mathcal{L}$ is said to act *in scales*, respectively $*$-*scales*, for $k$ with respect to the hierarchical compact approximation of the identity, if there is $k \in \mathbb{Z}^+$ such that

$$\begin{gathered}(v)\ \lim_{j\to\infty} \|\Pi_{j+k} \mathcal{L}(\mathrm{Id} - \Pi_j)\| = 0 ; \\ \text{respectively} \\ (v^*)\ \lim_{j\to\infty} \|(\mathrm{Id} - \Pi_j)\mathcal{L}\Pi_{j+k}\| = 0 .\end{gathered} \tag{2.4}$$

If $(v)$ (respectively $(v^*)$) in (2.4) holds for $k = 0$ then $\mathcal{L}$ is said to act *exactly* (respectively **-exactly*) *in scales* on $\{\Pi_j\}$.

**Theorem 1 (Holschneider [H1], 1996).** *Let $\mathcal{L} \in B(\mathcal{B})$. Suppose there is a compact uniformly bounded approximation of the identity $\{Id - \Pi_j\}$ for $(\mathcal{L}, \mathcal{B})$. (i.e., satisfying $(i, ii, iii)$). Then $\mathcal{B}_0(\{\Pi_j\}) = \mathcal{B}$, and*

$$\rho_{\mathrm{ess}}(\mathcal{L}) = \lim_{m\to\infty} (\limsup_{j\to\infty} \|\Pi_j \mathcal{L}^m\|)^{1/m} = \lim_{m\to\infty} (\liminf_{j\to\infty} \|\Pi_j \mathcal{L}^m\|)^{1/m} . \tag{2.5}$$

*If there is $\{Id - \Pi_j\}$ a $*$-compact uniformly bounded approximation of the identity in $(\mathcal{L}, \mathcal{B})$ (i.e., satisfying $(i, ii^*, iii^*)$), then $\mathcal{B}_0^*(\{\Pi_j\}) = \mathcal{B}^*$ and*

$$\rho_{\mathrm{ess}}(\mathcal{L}) = \lim_{m\to\infty} (\limsup_{j\to\infty} \|\mathcal{L}^m \Pi_j\|)^{1/m} = \lim_{m\to\infty} (\liminf_{j\to\infty} \|\mathcal{L}^m \Pi_j\|)^{1/m} . \tag{2.6}$$

*If there is a hierarchical compact (or $*$-compact) approximation of the identity $\{Id - \Pi_j\}$, and if there is $k \in \mathbb{Z}^+$ such that $\mathcal{L}$ acts in scales for $k$ on $\{\Pi_j\}$ (i.e., $(i, iv)$ and either $(ii, iii, v)$ or $(ii^* - iii^*, v^*)$ hold), then any of the four limits in* (2.5–2.6) *coincide with* $\rho_{\mathrm{ess}}(\mathcal{L})$.

*Remark 1.* In the last assertion of the theorem, if $\Pi_j$ satisfies $\|\Pi_j\| \le 1$, then the interior limit actually exists. Indeed, for any bounded $\mathcal{A}$ we have by $(iv)$

$$\|\mathcal{A}\Pi_j\| = \|\mathcal{A}\Pi_{j-1}\Pi_j\| \le \|\mathcal{A}\Pi_{j-1}\| .$$

Therefore the sequence $\|\mathcal{A}\Pi_j\|$ is monotonically decreasing. The same argument applies to $\|\Pi_j\mathcal{A}\|$.

*Remark 2.* If the sequence $\Pi_j$ satisfies only $(i)$ and $(ii)$ (respectively $(ii^*)$) then the Nussbaum formula (2.1) clearly gives the upper bounds in (2.5), respectively (2.6) (see also the proof in the Appendix). In some applications neither $(iii)$ nor $(iii^*)$ can be assumed to hold, but an assumption similar to the one appearing in [K2], and that we state now, may be used. (Note that a key idea to obtain lower bounds for the essential spectral radius by constructing almost eigenvalues was contained in the beautiful short paper [Ma] of Mather.) Suppose that $\Pi_j$ and $\mathcal{L}$ are such that $(i, ii)$ hold and that there is a double sequence of infinite dimensional closed spaces $V_j^m \subset \Pi_j\mathcal{B}$, for $j, m \in \mathbb{Z}^+$ and a constant $C > 0$ so that for each fixed $m$ and all sequences $\varphi_j^m \in V_j^m$ with $\|\varphi_j\| = 1$ we have

$$\limsup_{j\to\infty} \|\mathcal{L}^m \varphi_j\| \ge C \limsup_{j\to\infty} \|\mathcal{L}^m \Pi_j\| . \tag{2.7}$$

Then

$$\rho_{\mathrm{ess}}(\mathcal{L}) = \lim_{m\to\infty} (\limsup_{j\to\infty} \|\mathcal{L}^m \Pi_j\|)^{1/m} .$$

(See [H2, Theorem 3.3, and also Section 9] for an analysis of transfer operators acting on homogeneous $(s+\alpha)$-Hölder spaces, with $0 < \alpha < 1$ and $s \in \mathbb{Z}^+$, using condition (2.7).)

The idea to use families of projection operators to exploit the essential spectrum is not new. For example, Persson [Pe] gives a formula for the lower bound of the essential spectrum of a self-adjoint operator in $L^2(\mathbb{R}^n)$ by using restrictions of the operator to complements of compact sets of $\mathbb{R}^n$. More recently, Keller [K3] developed a theory of quasi-nuclear operators and applied it to construct dynamical Fredholm determinants associated to transfer operators in an analytic setting (his Proposition 2.2 is related to our Theorem 1, but it requires additional assumptions on the approximating projections, in particular the nontrivial hypothesis [K3,(2.12)]).

Theorem 1 will be used in combination with the following result on spectral approximations (which was originally proved in view of understanding small stochastic perturbations):

**Theorem 2 (Baladi–Young [BY], 1993).** *For $\mathcal{L} \in B(\mathcal{B})$, let $\mathcal{L}_j \in B(\mathcal{B})$ be a sequence of operators with*

$$\|\mathcal{L}_j\| \le const\,, \forall\, j \in \mathbb{Z}^+ \;\; and \;\; \lim_{j\to\infty} \mathcal{L}_j(\varphi) = \mathcal{L}(\varphi)\,, \forall\, \varphi \in \mathcal{B}\,.$$

*Assume that for any $\kappa > \rho_{\mathrm{ess}}(\mathcal{L})$, there is $m_0 \in \mathbb{Z}^+$ such that for each $m \ge m_0$, there is $j_0(m)$ such that for all $j \ge j_0(m)$*

$$\|\mathcal{L}^m - \mathcal{L}_j^m\| \le \kappa^m\,. \tag{2.8}$$

*Then for any $\kappa > \rho_{\mathrm{ess}}$ there is $j_0(\kappa)$ such that the essential spectral radius $\rho_{\mathrm{ess}}(\mathcal{L}_j) < \kappa$ for each $j \ge j_0$.*

*Furthermore, writing $X$, respectively $X_j$ (with $j \ge j_0$), for the (finite-dimensional) direct sum of the generalised eigenspaces associated to the eigenvalues of $\mathcal{L}$, respectively $\mathcal{L}_j$, of modulus larger than $\kappa$, and denoting the corresponding spectral projections by $\mathbb{P}$, $\mathbb{P}_j$, we have*

(1) *The norms $\|\mathbb{P} - \mathbb{P}_j\|$ tend to zero as $j \to \infty$. More precisely, there is $\delta > 0$ such that*

$$\|\mathbb{P} - \mathbb{P}_j\| \le \exp^{-\delta m(j)}\,, \forall\, j \ge j_0\,, \tag{2.9}$$

*where*

$$m(j) = \max\{m \in \mathbb{Z}^+ \; s.t. \; j \ge j_0(m)\}\,.$$

(2) *The Hausdorff distance between the spectrum of $\mathcal{L}|_X$ and that of $\mathcal{L}_j|_{X_j}$ tends to zero as $j \to \infty$. More precisely, for all $j \ge j_0$*

$$HD(\sigma(\mathcal{L}|_X), \sigma(\mathcal{L}_j|_{X_j})) \le const\, (C_X^{(m(j))}(j) + C_X^{(1)}(j))^{1/d}\,, \tag{2.10}$$

*where $d \ge 1$ is the maximum of the indices of eigenvalues of $\mathcal{L}$ of modulus larger than $\kappa$ and*

$$C_X^{(m)}(j) = \max_{\substack{\varphi\in X\\ \|\varphi\|=1}} \|(\mathcal{L}_j^m - \mathcal{L}^m)\varphi\|\,. \tag{2.11}$$

*(Note that $C_X^{(m(j))}(j) \le const\, \kappa^{m(j)}$ for $j \ge j_0$.)*

(3) *If $\varphi \in X$ is an eigenvector for $\mathcal{L}$ and an eigenvalue $\lambda$ of algebraic multiplicity $d_a$ and index $d_i \ge 1$, then for each $j \ge j_0$ the operator $\mathcal{L}_j$ has $1 \le \ell \le d_a$ eigenvectors $\varphi_{i,j}$ for eigenvalues $\lambda_{i,j}$ ($i = 1, \dots, \ell$), with sum over $i = 1, \dots, \ell$ of the algebraic multiplicities of $\lambda_{i,j}$ equal to $d_a$, and*

$$\max_{1\le i\le \ell} \max(|\lambda - \lambda_{i,j}|, \|\varphi - \varphi_{i,j}\|) \le const\, (C_X^{(m(j))}(j) + C_X^{(1)}(j))^{1/d_i}\,. \tag{2.12}$$

Theorem 2 is obtained by combining Lemmas 1, 2 and 3 from Section 2 in [BY], noting that assumptions (A.1) and (A.3) there follow from the definition of the essential spectral radius, while assumption (A.2) in [BY] is just our hypothesis (2.8).

The main new result of this paper is obtained by putting together Theorem 1 and Theorem 2:

**Proposition 3.** *For $\mathcal{L} \in B(\mathcal{B})$, let $\{Id - \Pi_j\}$ be either both a compact and a $*$-compact uniformly bounded approximation of the identity, or a compact or $*$-compact uniformly bounded hierarchical approximation of the identity on which $\mathcal{L}$ acts in scales for some $k$, i.e., either (a) [$(i, ii, ii)$ and $(i, ii^*, iii^*)$], or (b) $(i, ii, iii, iv, v)$, or (c) $(i, ii^*, iii^*, iv, v^*)$ hold.*

*Assume further that for some large enough $M \in \mathbb{Z}^+$, $\mathcal{L}^M$ acts exactly in scales on the sequence $\Pi_j$ (More precisely: in case (a) we assume [$(v)$ and $(v^*)$] for $k = 0$, in case (b) we assume $(v)$ for $k = 0$, and in case (c) the convergence $(v^*)$ for $k = 0$.)*

*Then for any fixed $\kappa > \rho_{\text{ess}}(\mathcal{L})$, the spectrum and generalised eigenspaces of the compact operators*

$$\mathcal{L}_j = (1 - \Pi_j)\mathcal{L}(1 - \Pi_j)$$

*outside of the disc of radius $\kappa$ converge to those of $\mathcal{L}$ in the sense of* (1)-(3) *of Theorem 2 (including bounds* (2.9-2.12)*).*

Obviously, Theorems 1 and 2 as well as Proposition 3 are of interest only when $\rho_{\text{ess}}(\mathcal{L}) < \rho(\mathcal{L})$ (especially when there are "many" eigenvalues between $\rho_{\text{ess}}(\mathcal{L})$ and $\rho(\mathcal{L})$). Proposition 3 is especially interesting if $\mathcal{L}(1 - \Pi_j)$ or $(1 - \Pi_j)\mathcal{L}$ are finite rank operators. Section 3 contains specific examples where both conditions in this paragraph are satisfied.

*Proof of Proposition 3.* We start by replacing $\mathcal{L}$ by $\mathcal{M} = \mathcal{L}^M$ and $\mathcal{L}_j$ by $\mathcal{M}_j = (1 - \Pi_j)\mathcal{L}^M(1 - \Pi_j)$, at the end of the proof we shall see how to recover results about $\mathcal{L}$ itself. (Note that if $(ii)$ respectively $(ii^*)$ is satisfied for $\mathcal{L}$ then it also holds for $\mathcal{L}^M$. Iterating assumption $(iv)$ does not lead into difficulties.)

The only thing which requires checking is assumption (2.8) of Theorem 2. For this, we observe that the conditions for the strongest statement of Theorem 1 are satisfied. Thus

$$\rho_{\text{ess}}(\mathcal{L}^M) = \lim_{m\to\infty} (\limsup_{j\to\infty} \|\Pi_j \mathcal{M}^m\|)^{1/m} = \lim_{m\to\infty} (\limsup_{j\to\infty} \|\mathcal{M}^m \Pi_j\|)^{1/m} .$$

In other words, for any $\tilde{\kappa}$ with $\tilde{\kappa} > \rho_{\text{ess}}(\mathcal{L}^M)$, there is $m_0 \geq 1$ such that for all $m \geq m_0$ and each $\epsilon > 0$ there is $j_1(m)$ such that for all $j \geq j_1(m)$

$$\|\Pi_j \mathcal{M}^m\| \leq \tilde{\kappa}^m + \epsilon \text{ and } \|\mathcal{M}^m \Pi_j\| \leq \tilde{\kappa}^m + \epsilon . \tag{2.13}$$

(We may, and will, take $\epsilon = \tilde{\kappa}^m$.) For a constant $\widehat{C} > 0$ (independent of $j$ and $m$) to be defined below, and for each fixed $m \geq m_0(\tilde{\kappa})$, we set ,

$$\delta = \delta(m) = \frac{1}{(m-1)(\widehat{C}\|\mathcal{M}\|)^{m-1}} ,$$

and take $j_2 \geq \max(j_0(m), j_1(m))$ large enough so that

$$\|\Pi_j \mathcal{M}(1 - \Pi_j)\| \leq \delta\, \tilde{\kappa}^m \tag{2.14}$$

for all $j \geq j_2$ (recall that $\mathcal{M}$ acts exactly in scales). We then have the bounds

$$\begin{aligned}
\|\mathcal{M}^m - \mathcal{M}_j^m\| &= \|\mathcal{M}^m - [(1-\Pi_j)\mathcal{M}(1-\Pi_j)]^m\| \\
&= \|\mathcal{M}^m - (1-\Pi_j)\mathcal{M}^m(1-\Pi_j) \\
&\qquad - \sum_{k=1}^{m-1}(1-\Pi_j)\mathcal{M}^k(2-\Pi_j)\Pi_j\mathcal{M}[(1-\Pi_j)^2\mathcal{M}]^{m-k-1}(1-\Pi_j)\| \\
&\leq \|\Pi_j\mathcal{M}^m\Pi_j + (1-\Pi_j)\mathcal{M}^m\Pi_j + \Pi_j\mathcal{M}^m(1-\Pi_j)\| \\
&\qquad + (m-1)\delta\tilde{\kappa}^m(\widehat{C}\|\mathcal{M}\|)^{m-1} \\
&\leq (\sup_j \|\Pi_j\|)^2 2\tilde{\kappa}^m + 2(\sup_j \|\Pi_j\|)(1+\sup_j \|\Pi_j\|)2\tilde{\kappa}^m \\
&\qquad + \tilde{\kappa}^m \,,
\end{aligned}$$

(we used $\hat{C} = (1+\sup_j \|\Pi_j\|)(2+\sup_j \|\Pi_j\|)^2$) obtaining an estimate which is clearly equivalent to (2.8) for $\mathcal{M} = \mathcal{L}^M$ since all constant factors are uniform in $m$.

We now explain how to prove our statements for the original operator $\mathcal{L}$. First note that, since $\rho_{\mathrm{ess}}(\mathcal{L}^M) = (\rho_{\mathrm{ess}}(\mathcal{L}))^M$, if $\tilde{\kappa} > \rho_{\mathrm{ess}}(\mathcal{L})$ then $\kappa = \tilde{\kappa}^M > \rho_{\mathrm{ess}}(\mathcal{L}^M)$. Also, since the spectral projection associated to $\mathcal{L}$ and an eigenvalue $\lambda$ coincides with that of $\mathcal{L}^M$ and $\lambda^M$, the assertion regarding $\|\mathbb{P} - \mathbb{P}_j\|$ in (1) of Theorem 2 is clearly valid. Since $X$ is finite-dimensional, we may conclude by applying perturbation theory of finite dimensional matrices as in the proof of Lemma 3 in [BY, pp. 361-362]. (Note that algebraic multiplicity is preserved when taking powers of an operator, whereas the index, which is the size of the largest Jordan block, may only decrease.) $\square$

**Nonconvergence of the determinants**

In Section 3 we shall discuss situations where Proposition 3 furnishes us with a sequence of finite rank operators $\mathcal{L}_j$ whose eigenvalues of large enough modulus (together with the corresponding generalised eigenspaces) converge to the corresponding data for a given noncompact bounded operator $\mathcal{L}$. It is tempting to consider the associated sequence of "Fredholm determinants" $d_j(z) = \det(\mathrm{Id} - z\mathcal{L}_j)$. The functions $d_j(z)$ are polynomials, of degree increasing with $j$, and whose zeroes of small enough modulus converge as $j \to \infty$ to the inverse eigenvalues of $\mathcal{L}$. In many situations, however, the functions $d_j(z)$ do *not converge* as holomorphic functions in any disc. (The "small (essential) spectrum" of $\mathcal{L}$ seems to be the reason for this lack of convergence.) Such counter-examples can be obtained in the framework of smooth expanding maps of the circle, by considering a sequence $\Pi_j$ associated to finite smoothness.

## 3. Dynamical transfer operators and wavelet approximations: an application

A typical situation where Theorem 1 applies is when $\mathcal{L}$ is the transfer operator associated to a smooth expanding map of the $n$-dimensional torus $\mathbb{T}^n$, the Banach space is a space of smooth distributions, a Sobolev space, or more generally a Triebel space (see [H2]), and the $\Pi_j$ are obtained by orthogonal projections on a multi-resolution analysis (see [Me]). We now give precise definitions and statements, without striving for the fullest generality.

**Definition of the transfer operator**

Let $f : \mathbb{T}^n \to \mathbb{T}^n$ be a $C^\infty$ uniformly expanding map (i.e., there exists $\gamma > 1$ with $\|Dfv\| \geq \gamma \|v\|$ for each $x \in \mathbb{T}^n$ and each $v \in T_x\mathbb{T}^n$). Let $g : \mathbb{T}^n \to \mathbb{C}$ be a $C^\infty$ function. The Ruelle transfer operator $\mathcal{L}$ associated to the pair $(f, g)$ is defined (e.g. on $L^2(\mathbb{T}^n) = L^2(\mathbb{T}^n, d\mu)$, with $\mu$ Lebesgue measure on the torus) by

$$\mathcal{L}\varphi(x) = \sum_{f(y)=x} \frac{g(y)\varphi(y)}{|\det Df(y)|} . \tag{3.1}$$

(The choice $g \equiv 1$ leads to the usual Perron-Frobenius-type transfer operator, for which there exists a maximal eigenfunction which is the density of the unique absolutely continuous invariant probability measure for $f$.)

**Definition of the hierarchical approximation of the identity**

We now give an explicit example of operators $\Pi_j$ that will satisfy the assumptions of Proposition 3. We start with an orthonormal wavelet basis for $L^2(\mathbb{R}^n)$, e.g., the Meyer basis obtained from multiresolution analysis of $L^2(\mathbb{R}^n)$ (see [Me]). This is a set of $2^n - 1$ functions $\psi_k$ ($k = 1, \ldots , 2^n - 1$), in the Schwartz space $S(\mathbb{R}^n)$ of rapidly decreasing functions

$$S(\mathbb{R}^n) = \{\varphi \in L^2(\mathbb{R}^n) \text{ s.t. } \sup_{\mathbb{R}^n} |x^m \partial^p \varphi(x)| < \infty , \forall p, m \in (\mathbb{Z}^+)^n\}$$

such that all moments of each $\psi_k$ vanish (i.e., $\int_{\mathbb{R}^n} x^m \psi_k(x)\, d\mu = 0$), and such that

$$\{\psi_k^{j,\ell} = 2^{jn/2}\psi_k(2^j x - \ell) \text{ s.t. } k = 1, \ldots , 2^n - 1 , j \in \mathbb{Z}, \ell \in \mathbb{Z}^n\}$$

is an orthonormal basis of $L^2(\mathbb{R}^n)$. Moreover, the $\psi_k$ have compactly supported Fourier transforms. Let $\mathcal{P} : S(\mathbb{R}^n) \to L^2(\mathbb{T}^n)$ be the periodisation operator

$$\mathcal{P}\varphi(x) = \sum_{\ell \in \mathbb{Z}^n} \varphi(x + \ell) .$$

Then we get an orthonormal basis of $L^2(\mathbb{T}^n)$ by considering

$$\{P\psi_k^{j,\ell} , k = 1, \ldots , 2^n - 1 , j \in \mathbb{Z}^- , \ell \in \{0, \ldots 2^{-j} - 1\}^n\} \cup \{\psi_0 \equiv 1\} .$$

Finally, we define the sequence $\Pi_j : L^2(\mathbb{T}^n) \to L^2(\mathbb{T}^n)$, for $j \in \mathbb{Z}^+$, by setting $\mathrm{Id} - \Pi_j$ to be the (finite-rank) orthogonal projection on the finite dimensional space generated by the $\mathcal{P}\psi_k^{j',\ell}$ for $k = 1, \dots, 2^n - 1$, $-j \le j' \le 0$, and $\ell \in \{0, \dots, 2^{-j'} - 1\}^n$:

$$
\begin{aligned}
(\mathrm{Id} - \Pi_j)(\varphi) = &\int_{\mathbb{T}^n} \varphi\, \psi_0 \, d\mu \\
&+ \sum_{k=1}^{2^n-1} \sum_{j'=-j}^{0} \sum_{\ell \in \{0,\dots,2^{-j'}-1\}^n} \mathcal{P}\psi_k^{j',\ell} \cdot \int_{\mathbb{T}^n} \varphi\, \overline{\mathcal{P}\psi_k^{j',\ell}}\, d\mu
\end{aligned}
\tag{3.2}
$$

**Two (scales of) Banach spaces**

We shall consider the transfer operator $\mathcal{L}$ acting on two scales of Banach spaces (the sequence of projections just introduced will work for both). The first is the well-known scale of Sobolev spaces $H^s(\mathbb{T}^n)$, for fixed $s \in \mathbb{R}^+$, i.e.,

$$
H^s(\mathbb{T}^n) = \{\varphi \in L^2(\mathbb{T}^n) \text{ s.t. } \|\varphi\|_{H^s} := \sqrt{|\hat{\varphi}(0)|^2 + \sum_{m \in \mathbb{Z}^n} |m|^{2s} |\hat{\varphi}(m)|^2} < \infty\}\,. \tag{3.3}
$$

where we used the notation $\hat{\varphi}(m)$, for the Fourier coefficients of $\varphi$, and we write $|m| = \sqrt{\sum_i m_i^2}$.

The second is derived from the Hölder – Zygmund scale of spaces $\Lambda^\alpha(\mathbb{T}^n)$. This is the space of periodic functions $f \in C^{[s]}(\mathbb{T}^n)$, for $s = [s] + \{s\}$, $[s] \in \mathbb{N}$, $\{s\} \in [0, 1)$, for which for all multi-indices $\beta$, $|\beta| = [s]$ we have

$$
\|\varphi\|_\infty + \sup_{\beta, x, y} \frac{|\partial^\beta \varphi(x) - \partial^\beta \varphi(y)|}{|x - y|^{\{s\}}} < \infty \tag{3.4}
$$

The left-hand side defines the norm of $\Lambda^s(\mathbb{T}^n)$ Here we suppose $s \notin \mathbb{N}$, otherwise we have to use the Zygmund spaces (see, e.g., [Tr] for a definition). In Lemma 4 below we will use the closure of $C^\infty(\mathbb{T}^n)$ in $\Lambda^s(\mathbb{T}^n)$, denoted by $\lambda^s(\mathbb{T}^n)$. The point is that both Sobolev and Hölder scales of spaces (together with many others) are well characterised through wavelet coefficients. More precisely, a distribution $\varphi$ is in $H^s(\mathbb{T}^n)$ if and only if its *wavelet coefficients*

$$
\alpha_k^{j,\ell} := (\mathcal{P}\psi_k^{j,\ell} | \varphi)
$$

satisfy

$$
\sqrt{\sum_{j,k,\ell} 2^{-2js} |\alpha_k^{j,\ell}|^2} < \infty\,.
$$

The left-hand side defines a norm equivalent to the standard Sobolev norm. The scale $\Lambda^s(\mathbb{T}^n)$ is characterised in wavelet space via

$$
\sup_{j,k,\ell} 2^{-sj} |\alpha_k^{j,\ell}| < \infty
$$

Again we have equivalence of norms. The closed subspace $\lambda^s(\mathbb{T}^n)$ consists of precisely those functions for which, in addition,

$$\lim_{j\to-\infty} \max_{\ell,k} 2^{-sj}|\alpha_k^{j,\ell}| = 0\,.$$

The projections $\mathrm{Id} - \Pi_j$ from (3.2) are finite-rank on both the Sobolev and Hölder spaces. Since our transfer operator $\mathcal{L}$ is associated with a smooth map $f$ and weight $g$, it is also bounded on these spaces (see [H1]). As the following key technical lemma shows, we may apply Proposition 3:

**Lemma 4. (Exact scaling for suitable Banach spaces)** *Let $\mathcal{L}$ be a transfer operator associated to a smooth expanding map $f$ and a smooth weight $g$ as in* (3.1)*, acting on a Banach space $\mathcal{B} = H^s(\mathbb{T}^n)$ or $\mathcal{B} = \lambda^s(\mathbb{T}^n)$. Let $Id - \Pi_j$ be the sequence of finite-rank projections defined by* (3.2)*. Then $\Pi_j$ and $\mathcal{L}$ satisfy properties $(i, ii, iii, iv, v)$. Furthermore, there is $M \in \mathbb{Z}^+$ such that $\mathcal{L}^M$ acts exactly in scales on the hierarchical sequence $\Pi_j$.*

*Proof of Lemma 4.*

Assertions $(ii)$ and $(iv)$ are obvious consequences of the definitions. Properties $(i)$ and $(iii)$ follow from the above wavelet characterisation of the scales $H^s(\mathbb{T}^n)$ and $\lambda^s(\mathbb{T}^n)$ in wavelet space. (Note that property $(iii)$ does not hold for $\Lambda^s(\mathbb{T}^n)$.)

To prove the exact scaling property we recall a result proved in [H1, Sections 7-8]. There it was shown that the transfer operator may be written as

$$\mathcal{L} = \mathcal{L}^\partial + \mathcal{R}\,,$$

where $\mathcal{R}$ is smoothing: it is continuous from $H^r(\mathbb{T}^n)$ to $H^{r+1}(\mathbb{T}^n)$, and from $\lambda^r(\mathbb{T}^n)$ to $\lambda^{r+1}(\mathbb{T}^n)$ for all $r > -1$.

$\mathcal{L}^\partial$ is a linearised version of $\mathcal{L}$, its precise definition can be found in [H1, Lemma 7.3]. (Note that the operators analogous to $\mathcal{L}^\partial$ and $\mathcal{R}$ in Sections 7–8 of [H1] were defined in the wavelet coordinates, while here we work with their versions in the original space $\mathbb{T}^n$; also, [H1] considered the non periodic case of $\mathbb{R}^n$, this does not lead to difficulties.) It suffices to recall that $\mathcal{L}^\partial$ is completely determined by the derivatives of the dynamics $f$ at all points of $\mathbb{T}^n$. In particular, since, by hypothesis $\mathcal{L}$ is obtained by composing with uniform contractions, $\mathcal{L}^\partial$ maps functions supported by a disk in Fourier space to functions supported by a $\gamma$-times smaller disk, where $\gamma > 1$ is our bound for the expansion rate of the dynamics. On the other hand, the image of $(\mathrm{Id} - \Pi_j)$ consists of functions supported by a disk of radius $\le c\,2^j$ in Fourier space, whereas the image of $\Pi_j$ is supported outside a disk of radius $C\,2^j$ with some $C > c > 0$. Upon replacing $\mathcal{L}$ by $\mathcal{L}^m$ ($m$ depends on $C/c$) we see that

$$\Pi_j(\mathcal{L}^m)^\partial(\mathrm{Id} - \Pi_j) = 0\,.$$

It remains to analyse the effect of $\mathcal{R}$. Since $\mathcal{R}$ is acting in scales, it is continuous from $H^{s-1}(\mathbb{T}^n)$ to $H^s(\mathbb{T}^n)$ and from $\lambda^{s-1}(\mathbb{T}^n)$ to $\lambda^s(\mathbb{T}^n)$. It can thus be written as $\mathcal{R} = \Gamma\mathcal{R}'$, where $\Gamma$ is defined via $\Gamma : \mathcal{P}\psi_k^{j,\ell} \to 2^j \mathcal{P}\psi_k^{j,\ell}$. Thus $\mathcal{R}'$ is continuous from $H^s(\mathbb{T}^n)$ to $H^s(\mathbb{T}^n)$. Now, by definition

$$\|\Pi_j \Gamma\|_{H^s(\mathbb{T}^n)} = \|\Pi_j \Gamma\|_{\lambda^s(\mathbb{T}^n)} \le \text{const } \; 2^{-j} \to 0 \quad (j \to \infty)\,,$$

and the exact scaling property follows. $\square$

*Remark 3.* For $y \in \mathbb{T}^n$ we introduce the bounded linear operator $\mathcal{D}_{f,y}$ acting either on the Sobolev space $H^s(\mathbb{R}^n)$ or the derived from Hölder space $\lambda^s(\mathbb{T}^n)$ (defined analogously to (3.3–3.4)) by

$$\mathcal{D}_{f,y}\varphi = \frac{\varphi \circ D(f^{-1}_{f(y)})}{|\det Df(y)|}\,, \tag{3.5}$$

(where the inverse branch $f^{-1}_{f(y)}$ is unambiguously defined by $f^{-1}(f(y)) = y$). For each $m \ge 1$ an operator $\mathcal{D}_{f^m,y}$ can be defined similarly as in (3.5).

In fact, Holschneider [H1, Theorems 21.–2.2] applies Theorem 1 in wavelet coordinates to show a more explicit formula for the essential spectral radius. The following bounds are easy consequences of his formula:

$$\rho_{\text{ess}}(\mathcal{L}) \le \begin{cases} \lim_{m\to\infty} \left( \sup_{x\in\mathbb{T}^n} \sum_{f^m(y)=x} |g^{(m)}(y)| \, \|\mathcal{D}_{f^m,y}\|_{\lambda^s(\mathbb{R}^n)} \right)^{1/m} \\ \qquad \text{for } \mathcal{B} = \lambda^s(\mathbb{T}^n)\,, \\ \lim_{m\to\infty} \left( \sup_{x\in\mathbb{T}^n} \sum_{f^m(y)=x} |g^{(m)}(y)|^2 \, \|\mathcal{D}_{f^m,y}\|^2_{H^s(\mathbb{R}^n)} \right)^{1/(2m)} \\ \qquad \text{for } \mathcal{B} = H^s(\mathbb{T}^n)\,, \end{cases} \tag{3.6}$$

(for $m \ge 1$, we write $g^{(m)}(x) = \prod_{i=0}^{m-1} g(f^i(x)))$. Using that each inverse branch of $f^m$ is a contraction by $\gamma^{-m}$, it is not difficult to show that for any $y$

$$\|\mathcal{D}_{f^m,y}\|_{\lambda^s(\mathbb{R}^n)} \le \gamma^{-ms} \text{ and, for } s > \frac{n}{2}\,, \|\mathcal{D}_{f^m,y}\|_{H^s(\mathbb{R}^n)} \le \gamma^{-m(s-n/2)}\,. \tag{3.7}$$

For $n = 1$, the bound obtained from (3.6) and the refined version $\|\mathcal{D}_{f^m,y}\|_{\lambda^s(\mathbb{R}^n)} \le |(f^m)'(y)|^{-s}$ of (3.7) is similar to the one-dimensional exact formula for $\Lambda^s([0,1])$ with $0 < s < 1$ in [BJL]. (See also [CI] for $s \ge 1$, and, in a one-dimensional bounded variation setting, [K2], for earlier and different expressions.) Campbell–Latushkin [CL] and Gundlach–Latushkin [GL] have recently obtained, via a different (Oseledec theorem) approach, other exact expressions of the essential spectral radius in higher dimensional smooth expanding settings.

If $g$ is nonnegative, the spectral radius of $\mathcal{L}$ acting on $C^s$ functions (for any $s \ge 0$) is just (see [R])

$$\rho(\mathcal{L}) = \lim_{m\to\infty} \left( \sup_{x\in\mathbb{T}^n} \sum_{f^m(y)=x} \frac{g^{(m)}(y)}{|\det Df^m(y)|} \right)^{1/m}.$$

Therefore, since there is an eigenvalue equal to $\rho(\mathcal{L})$ with a nonegative eigenfunction in $C^\infty(\mathbb{T}^n) \subset H^s(\mathbb{T}^n) \cap \lambda^s(\mathbb{T}^n)$ (use that $f, g$ are $C^\infty$, see [R]), the essential spectral radius of $\mathcal{L}$ on $\lambda^s(\mathbb{T}^n)$ satisfies

$$\rho_{\text{ess}}(\mathcal{L}) \le \rho(\mathcal{L})/\gamma^s < \rho(\mathcal{L})$$

(this double inequality was proved by Ruelle in [R] for $\mathcal{L}$ acting on $C^s(\mathbb{T}^n)$). Similarly, for $s > n/2$, the essential spectral radius of $\mathcal{L}$ on $H^s(\mathbb{T}^n)$ satisfies

$$\rho_{\text{ess}}(\mathcal{L}) \le \rho(\mathcal{L})/\gamma^{s-n/2} < \rho(\mathcal{L})\,.$$

*Remark 4.* Ordinary Fourier analysis could be used in the Sobolev space framework, in particular, the analogue of Lemma 4 would be true. However, Fourier series would not be suitable for the Hölder space analysis, or for other Banach spaces in which multiresolution analysis and wavelets are applicable. Also, in situations where Fourier and wavelet analysis are both applicable, numerical algorithms based on wavelets usually converge much faster. (See, e.g., [Me].)

*Remark 5.* Because our $C^\infty$ assumptions on $(f, g)$, our transfer operator $\mathcal{L}$ preserves any Hölder or Sobolev space. We have chosen to work with an orthonormal wavelet basis which is $r$-regular for each $r > 0$, and which can thus be applied to approximate the "nonessential" spectrum of $\mathcal{L}$ on any $H^s(\mathbb{T}^n)$. Since the essential spectral radius of $\mathcal{L}$ on $H^s(\mathbb{T}^n)$ is a monotone function of $s$ which tends to zero as $s \to \infty$, in fact *all* the eigenvalues of $(\mathrm{Id} - \Pi_j)\mathcal{L}(\mathrm{Id} - \Pi_j)$ will converge to eigenvalues of $\mathcal{L}$ for eigenfunctions in $\cap_{s>0} H^s(\mathbb{T}^n)$. (In other words, we only see the "embedded smooth spectrum" of $\mathcal{L}$.)

If we had chosen a wavelet basis of a given regularity $r > 0$, and let $\mathcal{L}$ act on $H^s(\mathbb{T}^n)$ for $r \ge s > 0$, then the eigenvalues of modulus greater than the essential spectral radius of $\mathcal{L}$ acting on $H^r(\mathbb{T}^n)$ (and only them) obtained by the approximation scheme are guaranteed to exhaust the eigenvalues of $\mathcal{L}$ acting on $H^s(\mathbb{T}^n)$ in the corresponding annulus (their eigenfunctions will in fact lie in $H^r(\mathbb{T}^n) \subset H^s(\mathbb{T}^n)$).

Finally, one could extend the results of [H1], and therefore the results of the present paper, to the case when the dynamics and weights involved in the construction of the transfer operator have a given finite regularity (this would restrict the regularity of the Banach spaces which can be considered).

## Appendix: Proof of the abstract spectral radius theorem (Theorem 1)

We reproduce for the reader's convenience the proof of Theorem 1, adapted from [H1].

*Proof of Theorem 1.* We first suppose that the compactness condition ($ii^*$) and the density property ($iii^*$) hold and show (2.6) (the proof of (2.5) assuming ($ii$) and ($iii$) is completely analogous and is left to the reader). We consider only the "lim sup" and leave the "lim inf"-part to the reader. We decompose the argument into five steps:

*Step 1*: We have for all $m$,

$$\inf\{\|\mathcal{L}^m - \mathcal{K}\| \text{ s.t. } \mathcal{K} \text{ compact}\} \le \|\mathcal{L}^m \Pi_j\|\,.$$

Indeed, we may write

$$\mathcal{L}^m - \mathcal{L}^m\Pi_j = \mathcal{L}^m(\mathrm{Id} - \Pi_j)\,.$$

The right hand side is a compact operator, since it contains a compact factor by hypothesis $(ii)$, and since $\mathcal{L}$ is bounded.

*Step 2*: We have $\mathcal{B}_0^* = \mathcal{B}^*$. Indeed, suppose $\mathcal{B}_0^* \ni \psi_k \to \psi \in \mathcal{B}^*$. Then for every $\epsilon > 0$, we find $K$ such that $k \geq K$ implies $\|\psi_k - \psi\| \leq \epsilon$. It follows that

$$\|\Pi_j^t\psi\| \leq \|\Pi_j^*\psi_k\| + \|\Pi_j^*(\psi - \psi_k)\| \leq \|\Pi_j^*\psi_k\| + \mathrm{const}\,\epsilon \to \mathrm{const}\,\epsilon \quad (j \to \infty)\,.$$

Since $\epsilon$ was arbitrary, the statement follows.

*Step 3*: For all $\mathcal{K} \in F(\mathcal{B})$, we have $\lim_{j\to\infty} \|\mathcal{K}\Pi_j\| = 0$. Indeed, for all $s$ with $\|s\| = 1$ we find

$$\|\mathcal{K}\Pi_j\psi\| = \|\sum_\alpha(\nu_\alpha|\Pi_j\psi)\psi_\alpha\| \leq \sum_\alpha \|\Pi_j^*\nu_\alpha\|_{\mathcal{B}^*}\|\nu_\alpha\|_{\mathcal{B}}\,.$$

This tends to 0 as $j \to \infty$, since, by Step 2, $\mathcal{B}_0^* = \mathcal{B}^*$, and since the sum contains only finitely many terms.

*Step 4*: We have for all $\mathcal{K} \in F(\mathcal{B})$

$$\|\mathcal{L} - \mathcal{K}\| \geq \frac{1}{\mathrm{const}}\ \limsup_{j\to\infty}\ \|\mathcal{L}\Pi_j\|\,.$$

Here $\infty > \mathrm{const} \geq 1$ follows from condition $(i)$. Indeed, since $\|\Pi_j\| \leq \mathrm{const}$, we find

$$\|(\mathcal{L} - \mathcal{K})\Pi_j\| \leq \mathrm{const}\,\|\mathcal{L} - \mathcal{K}\|\,,$$

for each $j$. Now

$$\|(\mathcal{L} - \mathcal{K})\Pi_j\| \geq \|\mathcal{L}\Pi_j\| - \|\mathcal{K}\Pi_j\|\,.$$

We may take the limit superior $j \to \infty$ and obtain the stated estimate.

*Step 5*: We have for all $m$, upon replacing $\mathcal{L}$ by $\mathcal{L}^m$,

$$\mathrm{const}^{\,-1/m}\,(\limsup_{j\to\infty}\ \|\mathcal{L}^m\Pi_j\|)^{1/m} \leq (\inf\{\|\mathcal{L}^m - \mathcal{K}\| \text{ s.t. } \mathcal{K} \text{ finite rank}\})^{1/m}\,,$$

and

$$(\inf\{\|\mathcal{L}^m - \mathcal{K}\| \text{ s.t. } \mathcal{K} \text{ compact}\})^{1/m} \leq (\limsup_{j\to\infty}\ \|\mathcal{L}^m\Pi_j\|)^{1/m}\,.$$

We now may go to the limit $m \to \infty$, showing (2.5–2.6).

Assuming now that $\Pi_j$ is hierarchical and that $(iii^*), (v^*)$ hold, we prove the last assertion of Theorem 1 in three steps:

*Step 6*: Note that if $\mathcal{A}$ satisfies $(v^*)$ for some $k$, it also satisfies it for $k+1$ (and hence for all $k' > k$). Indeed, we may write as before

$$\|(\mathrm{Id} - \Pi_j)\mathcal{A}\Pi_{j+k+1}\| = \|(\mathrm{Id} - \Pi_j)\mathcal{A}\Pi_{j+k}\Pi_{j+k+1}\| \leq \mathrm{const}\,\|(\mathrm{Id} - \Pi_j)\mathcal{A}\Pi_{j+k}\|\,.$$

The last expression tends to 0 by hypothesis on $\mathcal{A}$. An analogous argument applies for the second limit.

*Step 7*: The set of bounded operators satisfying $(v^*)$ for a given $k$ forms an algebra. Indeed if both $\mathcal{M}$ and $\mathcal{L}$ satisfy condition $(v^*)$, then their sum and scalar multiples obviously satisfy it, and we are left to check the product. By Step 6 we may suppose that $\mathcal{L}$ and $\mathcal{M}$ satisfy $(v^*)$ for the same $k$. Then we have, thanks to $(iv)$,

$$\begin{aligned}&\|(\mathrm{Id} - \Pi_j)\mathcal{L}\mathcal{M}\Pi_{j+2k+1}\| \\ &= \|(\mathrm{Id} - \Pi_j)\mathcal{L}[\Pi_{j+k} + (\mathrm{Id} - \Pi_{j+k})(\mathrm{Id} - \Pi_{j+k+1})]\mathcal{M}\Pi_{j+2k+1}\| \, .\end{aligned}$$

Using the triangular inequality and $(i)$, the last expression may be bounded above by

$$\leq \mathrm{const}\, \|\mathcal{M}\| \|(\mathrm{Id} - \Pi_j)\mathcal{L}\Pi_{j+k}\| + (1 + \mathrm{const}\,)^2 \|\mathcal{L}\| \|(\mathrm{Id} - \Pi_{j+k+1})\mathcal{M}\Pi_{j+2k+1}\| \, .$$

By hypothesis, both expressions tend to 0 as $j \to \infty$.

*Step 8*: Conclusion. For any $\mathcal{M} \in B(\mathcal{B})$ satisfying $(v^*)$ and thus in particular for $\mathcal{M} = \mathcal{L}^m$, we have

$$\|\mathcal{M}\Pi_{j+k}\| \leq \|\Pi_j \mathcal{M}\Pi_{j+k}\| + \|(\mathrm{Id} - \Pi_j)\mathcal{M}\Pi_{j+k}\|$$

In the limit $j \to \infty$, we obtain

$$\limsup_{j\to\infty} \|\mathcal{M}\Pi_{j+k}\| \leq \limsup_{j\to\infty} \|\Pi_j \mathcal{M}\| \|\Pi_{j+k}\| \leq \mathrm{const}\, \limsup_{j\to\infty} \|\Pi_j \mathcal{M}\| \, .$$

In the same way way we obtain

$$\limsup_{j\to\infty} \|\Pi_{j+k}\mathcal{M}\| \leq \mathrm{const}\, \limsup_{j\to\infty} \|\mathcal{M}\Pi_j\| \, .$$

Since the constant appearing in the right-hand side does not depend on $\mathcal{M}$, the stated equality of all limits follows. $\square$

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