Spectral shift function of the Schrödinger operator in the large coupling constant limit

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Abstract

We consider the spectral shift function $\xi(\lambda; H_0 - \alpha V, H_0)$, where H_0 is a Schrödinger operator with a variable Riemannian metric and an electro-magnetic field and V is a perturbation by a multiplication operator. We prove the Weyl type asymptotic formula for $\xi(\lambda; H_0 - \alpha V; H_0)$ in the large coupling constant limit $\alpha \to \infty$.

0 Introduction

0.1 Let $H_0 = -\triangle$ in $L_2(\mathbf{R}^d)$, $d \geq 1$ and let $V = V(x)$, $x \in \mathbf{R}^d$, be a perturbation potential which decays rapidly enough as $|x| \to \infty$. For a coupling constant $\alpha > 0$ and a spectral parameter $\lambda < 0$ denote

$$N(\lambda, \alpha) = \#\{n \in \mathbf{N} \mid \lambda_n(H_0 - \alpha V) < \lambda\}, \tag{0.1}$$

where $\lambda_n(H_0 - \alpha V)$ are the eigenvalues of $H_0 - \alpha V$, numbered with the multiplicities taken into account. Under certain restrictions on V (which are different for $d = 1$, $d = 2$ and $d \geq 3$), the following Weyl type asymptotic formula is valid (see, e.g., [24, 4] and references therein):

$$\lim_{\alpha \to \infty} \alpha^{-d/2} N(\lambda, \alpha) = (2\pi)^{-d} \omega_d \int V_+^{d/2}(x) dx, \quad \lambda < 0, \tag{0.2}$$

where ω_d is the volume of a unit ball in $\mathbf{R}^d$ and $V_+(x) = \max\{0, V(x)\}$.

Next, let H_0 be a Schrödinger operator of a more general type:

$$H_0 = (-i\nabla - \mathbf{A}(x))^2 + U(x), \tag{0.3}$$

where $\mathbf{A}$ is a magnetic vector potential and U is an electric potential, which satisfy some regularity conditions. Then $N(\lambda, \alpha)$ again obeys (0.2) for $\lambda < \inf \sigma(H_0)$. This fact is well known and has been proved in many particular cases; to the author's knowledge, the most general situation was considered in [3] and [7] (see also the review [4] and references therein). The spectrum $\sigma(H_0)$ of the operator (0.3) may have gaps (apart from the semi-infinite gap $(-\infty, \inf \sigma(H_0))$). One can study the behaviour of the discrete spectrum of

$H_0 - \alpha V$ in these gaps for $\alpha \to \infty$. Let us discuss two generalizations of (0.2) known in this situation.

0.2 Let $V \geq 0$; for $\alpha > 0$ put[1]

$$N_\pm(\lambda; H_0, \sqrt{V}; \alpha) := \#\{t \in (0, \alpha) \mid \lambda_n(H_0 \mp tV) = \lambda\}, \quad \lambda \in \mathbf{R} \setminus \sigma(H_0). \tag{0.4}$$

In other words, $N_\pm(\lambda; H_0, \sqrt{V}; \alpha)$ is the number of eigenvalues (counting multiplicities) of $H_0 \mp tV$, which pass the point λ as t grows monotonically from 0 to α. Since $V \geq 0$, it follows that the eigenvalues of $H_0 \mp \alpha V$ are monotone functions of α. It is clear that

$$N(\lambda, \alpha) = N_+(\lambda; H_0, \sqrt{V}; \alpha), \quad \lambda < \inf \sigma(H_0). \tag{0.5}$$

For the function N_+, a relation similar to (0.2) is valid in the gaps of $\sigma(H_0)$ — see [12, 13, 1, 3, 7, 4] and references therein:

$$\lim_{\alpha \to \infty} \alpha^{-d/2} N_+(\lambda; H_0, \sqrt{V}; \alpha) = (2\pi)^{-d} \omega_d \int V^{d/2}(x) dx, \quad \lambda \in \mathbf{R} \setminus \sigma(H_0). \tag{0.6}$$

Allowing some naive speculation, one can consider (0.6) as an expression of some intuitively attractive "preservation law" for the eigenvalues of $H_0 - \alpha V$: as α grows, the eigenvalues which "disappear" at the left edge of a gap, eventually reappear at the right edge of the next gap (located to the left from the first one). Thus, we deal with a "flow of eigenvalues" leftwards, which sometimes "disappears under the spectrum".

0.3 Now suppose that the potential V is not necessarily of a definite sign: $V = V_+ - V_-$, $V_\pm \geq 0$. For $\alpha > 0$ put[2]

$$N(\lambda; H_0, \sqrt{V_+}, \sqrt{V_-}; \alpha) := \#\{t \in (0, \alpha) \mid \lambda_n(H_0 - tV) = \lambda, \frac{d}{ds}\lambda_n(H_0 - sV) \mid_{s=t} < 0\}$$

$$-\#\{t \in (0, \alpha) \mid \lambda_n(H_0 - tV) = \lambda, \frac{d}{ds}\lambda_n(H_0 - sV) \mid_{s=t} > 0\}, \quad \lambda \in \mathbf{R} \setminus \sigma(H_0). \tag{0.7}$$

In other words, $N(\lambda; H_0, \sqrt{V_+}, \sqrt{V_-}; \alpha)$ is a difference of the numbers of eigenvalues (counting multiplicities) of $H_0 - tV$ which cross λ leftwards and rightwards[3] as t grows from 0 to α. Obviously, for the perturbations of a definite sign, the counting function (0.7) coincides with $\pm N_\pm$:

$$N(\lambda; H_0, \sqrt{V_+}, 0; \alpha) = N_+(\lambda; H_0, \sqrt{V_+}; \alpha), \qquad \lambda \in \mathbf{R} \setminus \sigma(H_0), \tag{0.8}$$
$$N(\lambda; H_0, 0, \sqrt{V_-}; \alpha) = -N_-(\lambda; H_0, \sqrt{V_-}; \alpha), \qquad \lambda \in \mathbf{R} \setminus \sigma(H_0),$$

and for $\lambda < \inf \sigma(H_0)$ — with $N(\lambda, \alpha)$ of (0.1):

$$N(\lambda; H_0, \sqrt{V_+}, \sqrt{V_-}; \alpha) = N(\lambda, \alpha), \quad \lambda < \inf \sigma(H_0). \tag{0.9}$$

[1] We write $\sqrt{V}$ in (0.4) in order to make our notation coherent with that of [3, 27].
[2] Definition (0.7) is correct due to the analyticity of $\lambda_n(H_0 - \alpha V)$ in α — see [27] for the details.
[3] The eigenvalues which "turn" at the point λ, do not enter the expression (0.7) — see [27, 28].

Note also that the function (0.7) admits the representation (see [27])

$$N(\lambda; H_0, \sqrt{V_+}, \sqrt{V_-}; \alpha) = N_+(\lambda; H_0 + \alpha V_-, \sqrt{V_+}; \alpha) - N_-(\lambda; H_0, \sqrt{V_-}; \alpha). \qquad (0.10)$$

For the function (0.7), the following generalization of (0.2), (0.6) is valid (see [27]):

$$\lim_{\alpha\to\infty} \alpha^{-d/2} N(\lambda; H_0, \sqrt{V_+}, \sqrt{V_-}; \alpha) = (2\pi)^{-d}\omega_d \int V_+^{d/2}(x)dx, \quad \lambda \in \mathbf{R} \setminus \sigma(H_0). \qquad (0.11)$$

In this connection, see also [1, 28] and references therein.

0.4 The aim of this paper is to present a certain analogue of (0.2), (0.6), (0.11) for λ lying *on the spectrum* of H_0. In this case the generalization of the counting functions (0.1), (0.4), (0.7) is given by the I. M. Lifshits–M. G. Krein *spectral shift function* (SSF). Let, as above, H_0 be the Schrödinger operator (0.3) and $V = V(x)$ be a potential which decays sufficiently fast as $|x| \to \infty$. We prove the following asymptotic formula for the SSF:

$$\lim_{\alpha\to\infty} \alpha^{-d/2} \xi(\lambda; H_0 - \alpha V, H_0) = -(2\pi)^{-d}\omega_d \int V_+^{d/2}(x)dx, \quad \lambda \in \mathbf{R} \qquad (0.12)$$

(the precise statements are given in §1.4). Note that

$$\xi(\lambda; H_0 - \alpha V, H_0) = -N(\lambda; H_0, \sqrt{V_+}, \sqrt{V_-}; \alpha), \quad \lambda \in \mathbf{R} \setminus \sigma(H_0) \qquad (0.13)$$

(see Remark 6.4) and thus, by (0.8), (0.9),

$$\begin{aligned} \xi(\lambda; H_0 - \alpha V, H_0) &= -N_+(\lambda; H_0, \sqrt{V}; \alpha), \quad V \geq 0, \quad \lambda \in \mathbf{R} \setminus \sigma(H_0); \\ \xi(\lambda; H_0 - \alpha V, H_0) &= -N(\lambda, \alpha), \qquad\qquad \lambda < \inf \sigma(H_0). \end{aligned} \qquad (0.14)$$

We see that (0.12) can be considered as a natural generalization of (0.2), (0.6), (0.11). Note, however, that the fact of existence of the SSF imposes strong restrictions on the rate of decay of V at infinity. For this reason, (0.12) makes sense for a poorer class of V's, than (0.2), (0.6), (0.11).

Formula (0.12) adds a new feature to our naive picture of the "flow of eigenvalues" of $H_0 - \alpha V$ leftwards as $\alpha \to \infty$. It shows that this "flow" can be noticed and controlled not only in the gaps, but also on the spectrum of H_0, where it obeys the same asymptotic "preservation law".

The relation (0.12) means that the family of functions in the l.h.s., which depend on the parameter α, converges to a constant function in the r.h.s. as $\alpha \to \infty$. The type of convergence has to be specified. In fact, we prove two main results. Theorem 1.5 gives convergence in (0.12) for almost every $\lambda \in \mathbf{R}$ (see Remark 1.6). Theorem 1.7 gives convergence in the weighted space $L_1(\mathbf{R}; d\mu(\lambda))$ with some power type weight $d\mu(\lambda)$.

The hypothesis of Theorem 1.5 includes the requirement $V \leq 0$. On the other hand, Theorem 1.7 is valid only for $d \geq 3$. Thus, neither of these theorems is exhaustive, but together they provide a fairly complete picture. As a by-product of the proof of Theorems 1.5, 1.7, we obtain some information on the asymptotic behaviour of $\xi(\lambda; H_0 + \alpha V, H_0)$ as $\alpha \to \infty$ (for $V \geq 0$) — see Theorem 1.8.

0.5 The proof of Theorems 1.5 and 1.7 is based on some new integral representation for the SSF, which was found in [21]. Namely, let a nonnegative operator V be factorized as $V = G^*G$. The above mentioned representation for the SSF reads (see Propositions 3.3, 3.4 below):

$$\xi(\lambda; H_0 \pm \alpha V, H_0) = \pm \mathcal{N}_\mp(\lambda; H_0, G; \alpha). \tag{0.15}$$

Here $\mathcal{N}_\pm(\lambda; H_0, G; \alpha)$ (see (3.7)) is an integral of a counting function of eigenvalues of a family of compact operators, related to G and the resolvent of H_0. The functions $\mathcal{N}_\pm$ coincide with $N_\pm$ for $\lambda \in \mathbf{R} \setminus \sigma(H_0)$. The relation (0.15) is of an abstract nature. To a certain extent, it can be considered as a *Birman–Schwinger principle on the continuous spectrum.*

The proof of Theorem 1.5 results from a straightforward analysis of the functions $\mathcal{N}_+(\lambda; H_0, \sqrt{V}; \alpha)$ as $\alpha \to \infty$. The proof of Theorem 1.7 is based on the integral estimates on $\mathcal{N}_\pm$, which were obtained in [22]. These proofs are independent of each other.

0.6 The paper is organized as follows. In §1 we give some necessary definitions and present the main results. We consider Schrödinger operators H_0 of a more general type than (0.3); namely, we allow for a variable metric of the space (see §1.3). The asymptotic formula (0.2) in this case has to be "corrected": the metric enters the r.h.s. The proof of this formula is given in §2. In §3 we discuss the representation (0.15) and formulate some necessary results of [21, 22]. In §4, we prove some auxiliary estimates on the Schrödinger operator and discuss the existence of the SSF. Theorem 1.5 is proved in §5, and Theorem 1.7 — in §6.

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1 Main results

1.1 Notation

1) The standard inner product in $\mathbf{C}^d$ is denoted by $\langle \cdot, \cdot \rangle$; $\mathbf{1}$ is a unit $d \times d$-matrix. Integral without the domain of integration explicitly specified implies integration over $\mathbf{R}^d$. By $\operatorname{meas} \delta$ we denote the Lebesgue measure of a Borel set $\delta \subset \mathbf{R}$. We denote $\omega_d = \text{volume}\{x \in \mathbf{R}^d \mid |x| \leq 1\}$. Formulas and statements with double indices ($\pm$ and $\mp$) should be read as pairs of statements, in one of which all the indices take upper values and in another — the lower ones. A constant which appears in formula $(i.j)$ is denoted by $C_{i.j}$.

2) *Functions.* The spaces $L_p(\mathbf{R}^d)$ and $L_{p,\text{loc}}(\mathbf{R}^d)$ are defined in a usual way. The space $l_\tau(\mathbf{Z}^d; L_\sigma(\mathbf{Q}^d))$, $\tau > 0$, $\sigma \geq 1$, consists of the functions $u \in L_{\sigma,\text{loc}}(\mathbf{R}^d)$ such that the

following functional is finite:

$$\|u\|^{\tau}_{l_\tau(L_\sigma)} := \sum_{j\in\mathbf{Z}^d} \left(\int_{\mathbf{Q}^d+j} |u|^\sigma dx\right)^{\tau/\sigma}, \quad \mathbf{Q}^d = (0,1)^d \subset \mathbf{R}^d.$$

For a real-valued function F we put $F_\pm := (|F| \pm F)/2$. For an open set $\Omega \subset \mathbf{R}^d$, $H^1(\Omega)$ is a Sobolev space and $H^1_0(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$.

3) *Operators.* Below $\mathcal{H}$, $\mathcal{H}_1$, $\mathcal{H}_2$ are separable Hilbert spaces. For a linear operator A, the notations $\operatorname{Dom} A$, $\operatorname{Ran} A$, $\operatorname{Ker} A$, A^*, $\sigma(A)$, $\rho(A)$ are standard; $\overline{A}$ is the closure of A, I is the identity operator. For a selfadjoint operator A the symbol $E_A(\delta)$ denotes the spectral measure of a Borel set $\delta \subset \mathbf{R}$; $A_\pm := (|A| \pm A)/2$. Resolvent of a selfadjoint operator H_0 is denoted by $R_0(z) = (H_0 - zI)^{-1}$. By $\mathbf{B}(\mathcal{H}_1,\mathcal{H}_2)$ and $\mathbf{S}_\infty(\mathcal{H}_1,\mathcal{H}_2)$ we denote respectively the spaces of bounded and compact operators acting from $\mathcal{H}_1$ into $\mathcal{H}_2$; $\mathbf{B}(\mathcal{H}) := \mathbf{B}(\mathcal{H},\mathcal{H})$, $\mathbf{S}_\infty(\mathcal{H}) := \mathbf{S}_\infty(\mathcal{H},\mathcal{H})$. For $T = T^* \in \mathbf{S}_\infty(\mathcal{H})$ and $s > 0$ we denote $n_\pm(s,T) := \operatorname{rank} E_{T_\pm}((s,+\infty))$, and for $T \in \mathbf{S}_\infty(\mathcal{H}_1,\mathcal{H}_2)$ put $n(s,T) := n_+(s^2, T^*T)$.

4) *Classes of compact operators.* For $0 < p < \infty$ the Neumann-Schatten class $\mathbf{S}_p(\mathcal{H}_1,\mathcal{H}_2) \subset \mathbf{S}_\infty(\mathcal{H}_1,\mathcal{H}_2)$ is a set of operators V such that the following functional is finite:

$$\|T\|_{\mathbf{S}_p} := \left(p\int_0^\infty s^{p-1} n(s,T)ds\right)^{1/p}.$$

The functional $\|\cdot\|_{\mathbf{S}_p}$ is a norm for $p \geq 1$ and a quasinorm for $p < 1$. For $0 < p < \infty$ the class $\mathbf{\Sigma}_p(\mathcal{H}_1,\mathcal{H}_2) \subset \mathbf{S}_\infty(\mathcal{H}_1,\mathcal{H}_2)$ is a set of all compact operators T such that the following functional is finite:

$$\|T\|_{\mathbf{\Sigma}_p} := \left(\sup_{s>0} s^p n(s,T)\right)^{1/p}.$$

The functional $\|\cdot\|_{\mathbf{\Sigma}_p}$ is a quasinorm. The classes $\mathbf{\Sigma}_p(\mathcal{H}_1,\mathcal{H}_2)$ are not separable (if $\dim\mathcal{H}_1 = \dim\mathcal{H}_2 = \infty$); a separable subspace $\mathbf{\Sigma}_p^0 \subset \mathbf{\Sigma}_p$ is defined by

$$\mathbf{\Sigma}_p^0 := \{T \in \mathbf{\Sigma}_p \mid \lim_{s\to+0} s^p n(s,T) = 0\}.$$

Note that $\mathbf{S}_p \subset \mathbf{\Sigma}_p^0$. For $T = T^* \in \mathbf{S}_\infty$ the following functionals are introduced:

$$\Delta_p^{(\pm)}(T) := \limsup_{s\to\infty} s^p n_\pm(s,T), \tag{1.1}$$

$$\delta_p^{(\pm)}(T) := \liminf_{s\to\infty} s^p n_\pm(s,T), \tag{1.2}$$

so that $0 \leq \delta_p^{(\pm)}(T) \leq \Delta_p^{(\pm)}(T) \leq \infty$. The functionals $\Delta_p^{(\pm)}$, $\delta_p^{(\pm)}$ are continuous in $\mathbf{\Sigma}_p$ and do not change if their argument changes by an operator of the class $\mathbf{\Sigma}_p^0$ (the last statement is essentially due to H. Weyl).

1.2 Spectral shift function

Let H_0 and H be selfadjoint operators in a Hilbert space $\mathcal{H}$ and let their difference, V, be a trace class operator:

$$V := H - H_0 \in \mathbf{S}_1(\mathcal{H}). \tag{1.3}$$

Then the following *Lifshits—Krein trace formula* [17, 15] holds:

$$\operatorname{Tr}\left(\phi(H)-\phi(H_0)\right)=\int_{-\infty}^{\infty}\xi(\lambda;H,H_0)\phi'(\lambda)d\lambda. \tag{1.4}$$

Here ϕ is any function of some functional class and $\xi(\lambda;H,H_0)$ is the SSF for the pair H_0, H, which is given by the *Krein formula* [15]

$$\xi(\lambda;H,H_0)=\frac{1}{\pi}\lim_{\varepsilon\to+0}\arg\det(I+VR_0(\lambda+i\varepsilon)),\ \text{ a. e. } \lambda\in\mathbf{R}. \tag{1.5}$$

The branch of the argument in (1.5) is fixed by the condition $\arg\det(I+VR_0(z))\to 0$, $\operatorname{Im} z\to+\infty$. For an exposition of the SSF theory, see [10, 35]. The SSF is *monotonic* in V [15]:

$$\pm V\geq 0 \quad\Rightarrow\quad \pm\xi(\lambda;H,H_0)\geq 0. \tag{1.6}$$

If H_0 is semibounded from below, by choosing an appropriate ϕ in (1.4) one finds that

$$\xi(\lambda;H,H_0)=-\operatorname{rank}E_H((-\infty,\lambda)),\quad \lambda<\inf\sigma(H_0), \tag{1.7}$$

which coincides with (0.14). If the operators H_0, H_1, H_2 in $\mathcal{H}$ are such that H_2-H_1 and H_1-H_0 are trace class operators, then, clearly,

$$\xi(\lambda;H_2,H_0)=\xi(\lambda;H_2,H_1)+\xi(\lambda;H_1,H_0). \tag{1.8}$$

In applications, instead of (1.3), it is usually possible to check the inclusion

$$f(H)-f(H_0)\in\mathbf{S}_1(\mathcal{H}). \tag{1.9}$$

Here $f:(a,b)\to\mathbf{R}$ is some smooth enough, monotone function and $(a,b)\subseteq\mathbf{R}$ is an interval which contains $\sigma(H)\cup\sigma(H_0)$. Then, the SSF for the pair H_0, H is *defined* by a natural formula

$$\xi(\lambda;H,H_0):=\operatorname{sign} f'\cdot\xi(f(\lambda);f(H),f(H_0)). \tag{1.10}$$

In this case the trace formula (1.4) is, of course, still valid (only the class of admissible functions ϕ may have to be changed). Thus, (1.7) is also valid. The relation (1.6) in this case has been proved in [14] (see also [35, §8.10]) for $f(\lambda)=(\lambda-\lambda_0)^{-m}$ with any $m>0$ and $\lambda_0<\inf(\sigma(H)\cup\sigma(H_0))$. Finally, note that if the operators H_0, H_1, H_2 in $\mathcal{H}$ are such that $f(H_2)-f(H_1)$ and $f(H_1)-f(H_0)$ are trace class operators, then (1.8) is still valid.

1.3 Schrödinger operator

In $\mathbf{R}^d$, $d\geq 1$, we fix a real $d\times d$–matrix valued function $g=g(x)$ such that

$$g_-\mathbf{1}\leq g(x)\leq g_+\mathbf{1},\quad 0<g_-\leq g_+<\infty, \tag{1.11}$$

a magnetic real vector potential $\mathbf{A}=\mathbf{A}(x)$,

$$\mathbf{A}\in L_{2,\mathrm{loc}}(\mathbf{R}^d), \tag{1.12}$$

and a scalar electric potential $U = U(x)$,

$$U \in L_{1,\mathrm{loc}}(\mathbf{R}^d), \quad U \geq 0. \tag{1.13}$$

Define a quadratic form

$$h^{(0)}_{g,\mathbf{A},U}[u,u] = \sum_{k,j=1}^{d} \int g_{kj}(x) \left(-i\frac{\partial u}{\partial x_k} - \mathbf{A}_k u\right) \overline{\left(-i\frac{\partial u}{\partial x_j} - \mathbf{A}_j u\right)} dx + \int U|u|^2 dx \tag{1.14}$$

on the domain $d[h^{(0)}_{g,\mathbf{A},U}] = C_0^\infty(\mathbf{R}^d)$.

Proposition 1.1 *Under the conditions* (1.11)–(1.13) *the form* $h^{(0)}_{g,\mathbf{A},U}$ *is closable and its closure* $h_{g,\mathbf{A},U}$ *has the domain*

$$d[h_{g,\mathbf{A},U}] = \{u \in L_2(\mathbf{R}^d) \mid i\nabla u + \mathbf{A}u \in L_2(\mathbf{R}^d), \quad U^{1/2}u \in L_2(\mathbf{R}^2)\}. \tag{1.15}$$

Proposition 1.1 follows from the two-sided estimate

$$g_- h^{(0)}_{\mathbf{1},\mathbf{A},U}[u,u] \leq h^{(0)}_{g,\mathbf{A},U}[u,u] \leq g_+ h^{(0)}_{\mathbf{1},\mathbf{A},U}[u,u] \tag{1.16}$$

and the fact that $h^{(0)}_{\mathbf{1},\mathbf{A},U}$ is closable and the domain of its closure is given by the expression in the right hand side of (1.15) — see [29]. Denote by $H_0 = H_0(g,\mathbf{A},U)$ the selfadjoint operator corresponding to the form $h_{g,\mathbf{A},U}$. By (1.16),

$$g_- H_0(\mathbf{1},\mathbf{A},U) \leq H_0(g,\mathbf{A},U). \tag{1.17}$$

The relation (1.17), in particular, implies

Proposition 1.2 *Let* V *be a multiplication by the function* $V = V(x) \geq 0$, $x \in \mathbf{R}^d$. *If* V *is* $(-\triangle)$*–form compact, then* V *is* $H_0(g,\mathbf{A},U)$*–form compact.*

Proof By (1.17), the problem is reduced to that of a flat metric $g \equiv \mathbf{1}$, in which case one just has to follow the reasoning of Theorems 2.5, 2.6 of [2]. □

Let V be a multiplication by the function $V = V(x)$, $x \in \mathbf{R}^d$, such that $|V|$ is $(-\triangle)$–form compact. Define the operators

$$H_+(\alpha) = H_0 + \alpha V_-, \quad H_-(\alpha) = H_0 - \alpha V_+, \quad H(\alpha) = H_0 - \alpha V, \quad \alpha > 0 \tag{1.18}$$

as the form sums. This definition is correct by Proposition 1.2. Put

$$N(\lambda,\alpha) := \operatorname{rank} E_{H(\alpha)}((-\infty,\lambda)), \quad \lambda < \inf\sigma(H_0), \quad \alpha > 0. \tag{1.19}$$

Proposition 1.3 *Let the conditions* (1.11) *hold true and*

$$\left.\begin{array}{lll} |\mathbf{A}| \in L_{d,\mathrm{loc}}(\mathbf{R}^d) & \textit{for} & d \geq 3; \\ |\mathbf{A}|^2 \log(1+|\mathbf{A}|) \in L_{1,\mathrm{loc}}(\mathbf{R}^2) & \textit{for} & d = 2; \\ \mathbf{A} \equiv 0 & \textit{for} & d = 1; \end{array}\right\} \tag{1.20}$$

$$\left.\begin{array}{lll} U \in L_{d/2,\mathrm{loc}}(\mathbf{R}^d),\ U \geq 0 & \textit{for} & d \geq 3; \\ U\log(1+U) \in L_{1,\mathrm{loc}}(\mathbf{R}^2),\ U \geq 0 & \textit{for} & d = 2; \\ U \in L_{1,\mathrm{loc}}(\mathbf{R}),\ U \geq 0 & \textit{for} & d = 1; \end{array}\right\} \tag{1.21}$$

$$\left.\begin{array}{lll} V \in L_{d/2}(\mathbf{R}^d) & \textit{for} & d \geq 3; \\ V \in l_1(\mathbf{Z}^2; L_\sigma(\mathbf{Q}^2)),\ \sigma > 1 & \textit{for} & d = 2; \\ V \in l_{1/2}(\mathbf{Z}^1; L_1(\mathbf{Q}^1)) & \textit{for} & d = 1. \end{array}\right\} \tag{1.22}$$

Then, for every $\lambda < 0$:

$$\lim_{\alpha \to \infty} \alpha^{-d/2} N(\lambda, \alpha) = C_{1.23}, \tag{1.23}$$

where

$$C_{1.23} := (2\pi)^{-d} \omega_d \int V_+^{d/2}(x) (\det g(x))^{-1/2} dx.$$

The proof of Proposition 1.3 and references to the literature are given in §2. Note that the condition $\mathbf{A} \equiv 0$ for $d = 1$ is motivated by the fact that in the one-dimensional case the magnetic field can be always gauged away.

The assumptions of Proposition 1.3 on the negative part of V may be considerably relaxed. It is sufficient to require that

$$V_- \in L_{d/2,\mathrm{loc}}(\mathbf{R}^d) \text{ for } d \geq 3,$$
$$V_- \log(1 + V_-) \in L_{1,\mathrm{loc}}(\mathbf{R}^2) \text{ for } d = 2,$$
$$V_- \in L_{1,\mathrm{loc}}(\mathbf{R}) \text{ for } d = 1.$$

Moreover, if there exists an open set $\Omega \subset \mathbf{R}$ such that $V(x) \geq 0$ for $x \in \Omega$ and $V(x) \leq 0$ for $x \in \mathbf{R} \setminus \Omega$, then it is sufficient to assume that $V_- \in L_{1,\mathrm{loc}}(\mathbf{R}^d)$ for all $d \geq 1$ (see, e.g., [9, Appendix 6]). For $d \geq 3$ one can take $\lambda = 0$ in (1.23). If the potential U has a negative part which is $(-\triangle)$–form bounded with a relative bound zero, formula (1.23) is still valid for $\lambda < \inf \sigma(H_0)$ — see [3]. For $d \geq 3$, the condition $V \in L_{d/2}(\mathbf{R}^d)$ is not only sufficient but also necessary for the validity of the asymptotics (1.23).

1.4 SSF of the Schrödinger operator. Main results

Below $H_0 = H_0(g, \mathbf{A}, U)$ is the Schrödinger operator defined in the previous subsection and $H(\alpha)$, $H_\pm(\alpha)$ are the operators (1.18). First discuss the existence of the SSF.

Proposition 1.4 *Let* (1.11)– (1.13) *hold true and let the perturbation potential $V = V(x)$, $x \in \mathbf{R}^d$, be such that $|V|$ is $(-\triangle)$–form compact and*

$$V \in L_1(\mathbf{R}^d) \textit{ for } d \leq 3, \tag{1.24}$$
$$V \in l_1(\mathbf{Z}^d; L_2(\mathbf{Q}^d)) \textit{ for } d \geq 4. \tag{1.25}$$

Then, the inclusion

$$(H(\alpha) - \lambda_0 I)^{-k} - (H_0 - \lambda_0 I)^{-k} \in \mathbf{S}_1 \tag{1.26}$$

holds for all integer $k > (d-1)/2$ and all $\lambda_0 < 0$ with large enough $|\lambda_0|$ if $d \geq 4$ and for $k = 1$ and all $\lambda_0 \in \rho(H_0) \cap \rho(H(\alpha))$ if $d \leq 3$.

The proof (given in §4.2) follows the reasoning of [2, Theorem 2.11, Corollary 2.13] (where the case $g \equiv \mathbf{1}$ has been considered). The inclusion (1.26) allows us to define the SSF for the pair H_0, $H(\alpha)$ according to (1.10) for $f(\lambda) = (\lambda - \lambda_0)^{-k}$.

Theorem 1.5 *Let* (1.11), (1.20), (1.21) *hold true and suppose that the perturbation potential $V = V(x) \geq 0$, $x \in \mathbf{R}^d$, satisfies the conditions*

$$\left.\begin{array}{ll} V \in L_{d/2}(\mathbf{R}^d) \cap l_1(\mathbf{Z}^d; L_2(\mathbf{Q}^d)) & \textit{for } d \geq 4; \\ V \in L_{d/2}(\mathbf{R}^3) \cap L_1(\mathbf{R}^3) & \textit{for } d = 3; \end{array}\right\} \tag{1.27}$$

$$\left.\begin{array}{ll} V \in l_\tau(\mathbf{Z}^2; L_\sigma(\mathbf{Q}^2)),\ \tau < 1,\ \sigma > 1 & \textit{for } d = 2; \\ V \in l_{1/2}(\mathbf{Z}^1; L_1(\mathbf{Q}^1)) & \textit{for } d = 1. \end{array}\right\} \tag{1.28}$$

Then, for a.e. $\lambda \in \mathbf{R}$*:*

$$\lim_{\alpha\to\infty} \alpha^{-d/2}\xi(\lambda; H(\alpha), H_0) = -C_{1.23}. \tag{1.29}$$

Remark 1.6 For any α, the SSF $\xi(\lambda; H(\alpha), H_0)$ is defined for almost every $\lambda \in \mathbf{R}$ (see (1.5)). Thus, the meaning of (1.29) has to be clarified. We understand (1.29) in the following sense:

For any sequence $\alpha_n \to \infty$ *there exists such a set* $M \subset \mathbf{R}$, $\mathrm{meas}\,(\mathbf{R} \setminus M) = 0$, *that* $\xi(\lambda; H(\alpha_n), H_0)$ *is defined for all* $(\lambda, n) \in M \times \mathbf{N}$ *and*

$$\lim_{n\to\infty} \alpha_n^{-d/2}\xi(\lambda; H(\alpha_n), H_0) = -C_{1.23}, \quad \forall \lambda \in M. \tag{1.30}$$

In the following theorem the potential V is not supposed to be of a definite sign.

Theorem 1.7 *Let* $d \geq 3$*; suppose that the conditions* (1.11), (1.20), (1.21) *hold true and the perturbation potential* $V = V(x)$, $x \in \mathbf{R}^d$, *satisfies* (1.27)*. Then, for any* $E > 0$ *and* $p > d/2$*:*

$$\lim_{\alpha\to\infty} \int_{-E}^{\infty} |\alpha^{-d/2}\xi(\lambda; H(\alpha), H_0) + C_{1.23}|(\lambda + 2E)^{-p} d\lambda = 0. \tag{1.31}$$

Obviously, one can replace $(\lambda + 2E)^{-p}$ in (1.31) by $(\lambda + \tilde{E})^{-p}$ with any $\tilde{E} > E$. Note that the conditions (1.20), (1.21) for $d \geq 3$ mean simply that

$$|\mathbf{A}|^2 + U \in L_{d/2,\mathrm{loc}}(\mathbf{R}^d), \quad U \geq 0.$$

1.5 Comments

Theorem 1.5 raises a natural question on the behaviour of $\xi(\lambda; H_+(\alpha), H_0)$ as $\alpha \to \infty$. The approach of the present paper does not allow us to obtain any asymptotics for this quantity. However, some bounds for it can be proved. The integral estimates for $\xi(\lambda; H_+(\alpha), H_0)$ were obtained in [22] (see (3.24) below). As a by-product of the construction of the proof of Theorem 1.5, we get the following statement.

Theorem 1.8 *Let the conditions* (1.11)–(1.13) *hold true and suppose that the perturbation potential* $V = V(x) \geq 0$, $x \in \mathbf{R}^d$, *is* $(-\triangle)$*-form compact, satisfies* (1.24), (1.25) *and*

$$V \in l_\tau(\mathbf{Z}^d; L_1(\mathbf{Q}^d)), \quad 1/2 \leq \tau \leq 1. \tag{1.32}$$

Then, for a.e. $\lambda \in \mathbf{R}$,

$$\lim_{\alpha\to\infty} \alpha^{-\rho}\xi(\lambda; H_+(\alpha), H_0) = 0, \tag{1.33}$$

where $\rho = \tau$ *if* $\tau < 1$ *and* $\rho > 1$ — *any number if* $\tau = 1$.

Formula (1.33) is understood in the same sense as (1.29) — see Remark 1.6. The proof of Theorem 1.8 is given in §5.

Note that the spectrum of $H_0(g, \mathbf{A}, U)$ may have extremely "unregular" structure; for example, the absolutely continuous spectrum may be empty.. At the same time, our results are formulated in terms which are "not sensible" to the spectrum of H_0. On the other hand, one can consider H_0's with absolutely continuous spectrum and V's, which are H_0–smooth in some appropriate sense. In this framework, one can obtain estimates and asymptotics of the SSF which are locally uniform in λ. In this connection, see [33]. There exists a lot of literature devoted to the large energy and semiclassical asymptotics of the SSF; we do not touch this subject here.

Theorems 1.5, 1.7, 1.8 can be generalized for some differential operators H_0 of the order $l \neq 2$ — see Remarks 5.8, 6.5.

2 Proof of Proposition 1.3

Proposition 1.3 has been proved in many particular cases — see, e.g., [9, 7, 26] and references therein. Once the spectral bounds of a correct order in α (see (2.1)–(2.3)) are established, the proof of asymptotics (1.23) by means of the local compactness technique of [7] becomes a routine procedure. Nevertheless, for the reader's convenience, we give the complete proof.

2.1 The spectral bounds

Proposition 2.1 *Suppose that the conditions* (1.11)–(1.13) *and* (1.22) *hold. Then, the function* $N(\lambda,\alpha)$*, defined in* (1.19)*, obeys:*
(i) *for* $d \geq 3$*:*

$$N(\lambda,\alpha) \leq C_{2.1}(d)\alpha^{d/2} g_-^{-d/2} \int V_+^{d/2}(x)dx, \quad \lambda < 0; \tag{2.1}$$

(ii) *for* $d = 2$ *and any* $\sigma > 1$*:*

$$N(\lambda,\alpha) \leq C_{2.2}(\lambda,\sigma)\alpha g_-^{-1} \|V_+\|_{l_1(L_\sigma)}, \quad \lambda < 0; \tag{2.2}$$

(iii) *for* $d = 1$*:*

$$N(\lambda,\alpha) \leq C_{2.3}(\lambda)\alpha^{1/2} g_-^{-1/2} \|V_+\|^{1/2}_{l_{1/2}(L_1)}, \quad \lambda < 0. \tag{2.3}$$

The bound (2.1) follows from (1.17) and the magnetic variant of the Cwikel–Lieb–Rozenblum estimate — see [16, 30] or [18]. The bound (2.2) follows from (1.17) and the corresponding estimate for $H_0(\mathbf{1}, \mathbf{A}, U)$, which has been recently proved in [26]. For $d = 1$, by using (1.17) and gauging away the magnetic field, the problem is reduced to $H_0 = -\triangle$; in this case, the bound (2.3) is well-known (see, e.g., [8]).

Obviously, the bounds (2.1)–(2.3) do not, in fact, require any conditions on V_-, apart from the inclusion $V_- \in L_{1,\mathrm{loc}}(\mathbf{R}^d)$, which is assumed in order to ensure that $C_0^\infty(\mathbf{R}^d)$ is a form core for $H(\alpha)$.

Remark 2.2 Denote

$$Z_\pm(\lambda) := R_0^{1/2}(\lambda) V_\pm R_0^{1/2}(\lambda), \quad \lambda < 0. \tag{2.4}$$

The Birman–Schwinger principle (see also (3.10) below) reads:

$$N(\lambda,\alpha) = n_+(\alpha^{-1}; Z_+(\lambda) - Z_-(\lambda)), \quad \lambda < 0,\ \alpha > 0. \tag{2.5}$$

It follows that (2.1)–(2.3) are equivalent to the estimates:

$$\|Z_\pm(\lambda)\|^{d/2}_{\mathbf{\Sigma}_{d/2}} \le C_{2.1}(d) g_-^{-d/2} \int V_\pm^{d/2}(x)dx, \quad d \ge 3; \tag{2.6}$$

$$\|Z_\pm(\lambda)\|_{\mathbf{\Sigma}_1} \le C_{2.2}(\lambda,\sigma) g_-^{-1} \|V_\pm\|_{l_1(L_\sigma)}, \quad d = 2; \tag{2.7}$$

$$\|Z_\pm(\lambda)\|^{1/2}_{\mathbf{\Sigma}_{1/2}} \le C_{2.3}(\lambda) g_-^{-1/2} \|V_\pm\|^{1/2}_{l_{1/2}(L_1)}, \quad d = 1. \tag{2.8}$$

2.2 Preliminary statements

Proposition 2.3 *Let Ω be an open ball in $\mathbf{R}^d$, $d \ge 1$, and let*

$$\begin{array}{ll} F \in L_{d/2}(\Omega), & d \ge 3; \\ F \log(1+|F|) \in L_1(\Omega), & d = 2; \\ F \in L_1(\Omega), & d = 1. \end{array} \tag{2.9}$$

Then the quadratic form $\int_\Omega F|u|^2 dx$, $u \in H^1(\Omega)$, is compact in $H^1(\Omega)$.

For $d \ge 3$ and $d = 1$, this follows from Hölder inequality and the standard Sobolev theorems on compactness of the embeddings $H^1(\Omega) \subset L_{\frac{2d}{d-2}}(\Omega)$, $d \ge 3$ and $H^1(\Omega) \subset L_\infty(\Omega)$, $d = 1$. For $d = 2$, this is the result of [34].

Lemma 2.4 *Let Ω be an open ball in $\mathbf{R}^d$, $d \ge 1$; suppose that* (1.11), (1.20), (1.21) *hold. Then the form*

$$h_{g,\mathbf{A},U}[u,u] - h_{g,0,0}[u,u], \quad u \in H^1(\Omega), \tag{2.10}$$

is compact in $H^1(\Omega)$.

Proof Expand formula (2.10):

$$h_{g,\mathbf{A},U}[u,u] - h_{g,0,0}[u,u] = \int_\Omega \langle g\mathbf{A}u, \mathbf{A}u\rangle dx + 2\mathrm{Re} \int_\Omega \langle g\mathbf{A}u, i\nabla u\rangle dx + \int_\Omega U|u|^2 dx.$$

By Proposition 2.3 and the conditions (1.11), (1.20), (1.21), the first and the last summands in the right hand side are compact in $H^1(\Omega)$. The second summand is compact due to the estimate

$$\left|\int_\Omega \langle g\mathbf{A}u, i\nabla u\rangle\right| \le \|g\mathbf{A}u\|_{L_2(\Omega)} \|u\|_{H^1(\Omega)}. \quad \square$$

Let Ω be an open ball in $\mathbf{R}^d$, $d \ge 1$, $V \in C_0^\infty(\Omega)$ and let g obey (1.11). Consider the consecutive maxima of the quotient

$$\int_\Omega V|u|^2 dx / (h_{g,0,0}[u,u] - \lambda \|u\|^2_{L_2(\Omega)}), \quad u \in H^1(\Omega), \tag{2.11}$$

where $\lambda < 0$. Denote by $n_\pm(s, (2.11))$, $s > 0$, the distribution function of the maxima of (2.11). In other words, $n_\pm(s,(2.11)) = n_\pm(s, X)$, where X is a compact operator generated by the form in the numerator in the Hilbert space $H^1(\Omega)$ with the metric $h_{g,0,0}[u,u] - \lambda\|u\|^2_{L_2(\Omega)}$. In what follows, we use the notations of the form $\Delta_p^{(\pm)}((2.11))$, $\delta_p^{(\pm)}((2.11))$ instead of $\Delta_p^{(\pm)}(X)$, $\delta_p^{(\pm)}(X)$. Along with (2.11), consider the same quotient in $H_0^1(\Omega)$:

$$\int_\Omega V|u|^2 dx / (h_{g,0,0}[u,u] - \lambda \|u\|^2_{L_2(\Omega)}), \quad u \in H_0^1(\Omega). \tag{2.12}$$

Proposition 2.5 (see, e.g., [9]) *For any $\lambda < 0$,*

$$\Delta^{(+)}_{d/2}((2.11)) = \delta^{(+)}_{d/2}((2.11)) = \Delta^{(+)}_{d/2}((2.12)) = \delta^{(+)}_{d/2}((2.12)) = C_{1.23}.$$

Lemma 2.6 *Let Ω be an open ball in $\mathbf{R}^d$, $d \geq 1$, $V \in C_0^\infty(\Omega)$ and $\lambda < 0$; suppose that* (1.11), (1.20), (1.21) *hold. For the quotients*

$$\int_\Omega V|u|^2 dx/(h_{g,\mathbf{A},U}[u,u] - \lambda\|u\|^2_{L_2(\Omega)}), \quad u \in H^1(\Omega), \tag{2.13}$$

$$\int_\Omega V|u|^2 dx/(h_{g,\mathbf{A},U}[u,u] - \lambda\|u\|^2_{L_2(\Omega)}), \quad u \in H^1_0(\Omega), \tag{2.14}$$

one has

$$\Delta^{(+)}_{d/2}((2.13)) = \delta^{(+)}_{d/2}((2.13)) = \Delta^{(+)}_{d/2}((2.14)) = \delta^{(+)}_{d/2}((2.14)) = C_{1.23}.$$

Proof By Lemma 2.4, the form in the denominators of (2.13), (2.14) differs from that in the denominators of (2.11), (2.12) by a compact term. Relatively compact perturbations of the metric in the Hilbert space do not affect the leading term in asymptotics — see, e.g., [9, Lemma 1.16]. Thus, the statement follows from Proposition 2.5. □

2.3 Proof of Proposition 1.3

By (2.5), the desired formula (1.23) is equivalent to

$$\Delta^{(+)}_{d/2}(Z_+(\lambda) - Z_-(\lambda)) = \delta^{(+)}_{d/2}(Z_+(\lambda) - Z_-(\lambda)) = C_{1.23}. \tag{2.15}$$

By (2.6)–(2.8) and continuity of $\Delta^{(\pm)}_{d/2}$, $\delta^{(\pm)}_{d/2}$ in $\mathbf{\Sigma}_{d/2}$, it suffices to prove (2.15) for $V \in C_0^\infty(\mathbf{R}^d)$. Let $\Omega \subset \mathbf{R}^d$ be an open ball and $\operatorname{supp} V \subset \Omega$. Due to the variational principle (Dirichle-Neumann bracketing), the upper and lower bounds for the spectrum of $Z_+(\lambda) - Z_-(\lambda)$ are given by the spectra of (2.13), (2.14):

$$n_+(s, (2.14)) \leq n_+(s, Z_+(\lambda) - Z_-(\lambda)) \leq n_+(s, (2.13)), \quad s > 0.$$

From here and Lemma 2.6 follows (2.15). □

3 Representation for the SSF

In this section we collect the necessary statements from [21, 22] concerning a certain new representation for the SSF (see Propositions 3.3, 3.4 below).

3.1 Definition of $\mathcal{N}_\pm$

Let $\mathcal{H}$ be a "basic" and $\mathcal{K}$ — an "auxiliary" Hilbert space, H_0 be a selfadjoint semibounded from below operator in $\mathcal{H}$, G be a closed linear operator which acts from $\mathcal{H}$ into $\mathcal{K}$. For simplicity assume that

$$H_0 \geq 0; \tag{3.1}$$

besides, we suppose that
$$GR_0^{1/2}(-1) \in \mathbf{S}_\infty(\mathcal{H}, \mathcal{K}). \tag{3.2}$$
For $z \in \rho(H_0)$ define compact in $\mathcal{K}$ operators
$$T(z; H_0, G) := \overline{GR_0(z)G^*} = (GR_0^{1/2}(-1))(H_0 + I)R_0(z)(GR_0^{1/2}(-1))^*, \tag{3.3}$$
$$A(z; H_0, G) := \operatorname{Re} T(z; H_0, G), \quad K(z; H_0, G) := \operatorname{Im} T(z; H_0, G). \tag{3.4}$$

We shall write $T(z)$, $A(z)$, $K(z)$ instead of $T(z; H_0, G)$, etc., if the choice of H_0, G is clear from the context. Suppose that for some $\lambda \in \mathbf{R}$ a pair of operators H_0, G satisfies the following condition.

Condition 3.1 *The limit*
$$\lim_{\varepsilon \to +0} T(\lambda + i\varepsilon; H_0, G) =: T(\lambda + i0; H_0, G), \tag{3.5}$$
exists in the operator norm and
$$K(\lambda + i0; H_0, G) \in \mathbf{S}_1(\mathcal{K}). \tag{3.6}$$

Then define
$$\mathcal{N}_\pm(\lambda; H_0, G; \alpha) := \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{dt}{1+t^2} n_\pm(\alpha^{-1}; A(\lambda + i0) + tK(\lambda + i0)), \quad \alpha > 0. \tag{3.7}$$

It is easy to see that conditions (3.5), (3.6) ensure the convergence of the integral (3.7). The functions $\mathcal{N}_\pm(\lambda; H_0, G; \alpha)$ enjoy some natural properties — see [22].

Proposition 3.2 [22, Corollary 3.8] *Suppose that for some open interval $\delta \subset \mathbf{R}$ the following inclusion holds:*
$$GE_{H_0}(\delta) \in \mathbf{S}_2(\mathcal{H}, \mathcal{K}). \tag{3.8}$$
Then for a. e. $\lambda \in \delta$ the pair H_0, G satisfies Condition 3.1 and for any $\alpha > 0$
$$\mathcal{N}_\pm(\lambda; H_0, G; \alpha) \in L_{1,\mathrm{loc}}(\delta).$$
In particular, if (3.8) *is true for any bounded interval $\delta \subset \mathbf{R}$, then*
$$\mathcal{N}_\pm(\lambda; H_0, G; \alpha) \in L_{1,\mathrm{loc}}(\mathbf{R}).$$

Define the operators
$$H_\pm(\alpha) = H_0 \pm \alpha V, \quad V = \overline{G^*G}, \quad \alpha > 0 \tag{3.9}$$
as form sums and let the counting function $N_\pm(\lambda; H_0, G; \alpha)$ be as defined by (0.4). The following relation is known as the *Birman–Schwinger principle*:
$$N_\pm(\lambda; H_0, G; \alpha) = n_\pm(\alpha^{-1}, T(\lambda; H_0, G)), \quad \lambda \in \mathbf{R} \setminus \sigma(H_0). \tag{3.10}$$

Comparing (3.10) and (3.7) and noting that for $\lambda \in \rho(H_0) \cap \mathbf{R}$ one has $K(\lambda + i0) = 0$ and $A(\lambda + i0) = T(\lambda + i0) = T(\lambda)$, we arrive at the formula

$$\mathcal{N}_\pm(\lambda; H_0, G; \alpha) = N_\pm(\lambda; H_0, G; \alpha), \quad \lambda \in \mathbf{R} \setminus \sigma(H_0). \tag{3.11}$$

In particular, by (0.5),

$$\mathcal{N}_+(\lambda; H_0, G; \alpha) = N(\lambda, \alpha), \quad \lambda < 0. \tag{3.12}$$

3.2 Connection between $\mathcal{N}_\pm$ and SSF

1) Trace class perturbations. Let H_0 be a selfadjoint operator in $\mathcal{H}$ and $G \in \mathbf{S}_2(\mathcal{H}, \mathcal{K})$. It is well-known that under these conditions[4] for a.e. $\lambda \in \mathbf{R}$ the limits (3.5) exist in the Hilbert-Schmidt norm and the inclusion (3.6) holds — see [5] or [35]. More precise results can be found in [19, 20] — see Proposition 5.3 below.

Proposition 3.3 [21, Theorem 1.1] *Let H_0 be a selfadjoint operator in $\mathcal{H}$ and $G \in \mathbf{S}_2(\mathcal{H}, \mathcal{K})$; then for any $\alpha > 0$*

$$\xi(\lambda; H_0 \pm \alpha G^* G, H_0) = \pm \mathcal{N}_\mp(\lambda; H_0, G; \alpha), \quad a.e.\ \lambda \in \mathbf{R}. \tag{3.13}$$

Note that for $\lambda \in \mathbf{R} \setminus \sigma(H_0)$ the relation (3.13) turns into the formula

$$\xi(\lambda; H_0 \pm \alpha G^* G, H_0) = \pm n_\mp(\alpha^{-1}; T(\lambda; H_0, G)),$$

which was proved earlier in [33].

2) Relatively trace class perturbations. Let H_0, G satisfy (3.1), (3.2) and for some $m > 0$:

$$G R_0^m(-1) \in \mathbf{S}_2(\mathcal{H}, \mathcal{K}). \tag{3.14}$$

Then, by Proposition 3.2 (see also Lemma 5.7 below), for a.e. $\lambda \in \mathbf{R}$ the pair H_0, G satisfies Condition 3.1. Fix some $\alpha > 0$ and define the operators (3.9) as form sums. In order to define the SSF for the pair H_0, $H_\pm(\alpha)$, assume that for some $k > 0$ and $\lambda_0 < \inf(\sigma(H_0) \cup \sigma(H_\pm(\alpha)))$ the inclusion

$$(H_\pm(\alpha) - \lambda_0 I)^{-k} - (H_0 - \lambda_0 I)^{-k} \in \mathbf{S}_1(\mathcal{H}) \tag{3.15}$$

holds. The inclusion (3.15) enables one to define the SSF via (1.10) with $f(\lambda) = (\lambda - \lambda_0)^{-k}$.

Proposition 3.4 [21, Theorem 1.2] *Suppose that the conditions* (3.1), (3.2), (3.14), (3.15) *hold. Then,*

$$\xi(\lambda; H_\pm(\alpha), H_0) = \pm \mathcal{N}_\mp(\lambda; H_0, G; \alpha), \quad a.e.\ \lambda \in \mathbf{R}. \tag{3.16}$$

3.3 Pointwise estimates for $\mathcal{N}_\pm$

For the operators

$$A = A^* \in \mathbf{S}_\infty(\mathcal{K}), \quad K = K^* \in \mathbf{S}_q(\mathcal{K}), \quad q \leq 1, \tag{3.17}$$

[4] Condition (3.1) in this case can be dropped; H_0 should not be necessarily semibounded.

introduce the quantities, "simulating" the integral (3.7):

$$\eta_\pm(s;A,K)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{dt}{1+t^2}n_\pm(s,A+tK),\quad s>0. \tag{3.18}$$

One easily checks that the conditions (3.17) (for $q=1$) are sufficient for the convergence of the integral (3.18).

Proposition 3.5 [21, Lemma 2.1] *For the quantities* (3.18)*, the following estimates hold true:*

$$\eta_\pm(s;A,K)\le\inf_{0<\tau<1}(n_\pm(s(1-\tau),A)+(s\tau)^{-q}\mu_q(s\tau,K)\|K\|^q_{\mathbf{S}_q}), \tag{3.19}$$

$$\eta_\pm(s;A,K)\ge\sup_{\tau>0}(n_\pm(s(1+\tau),A)-(s\tau)^{-q}\mu_q(s\tau,K)\|K\|^q_{\mathbf{S}_q}), \tag{3.20}$$

where $\mu_q(\tau,K)\ge 0$ *obeys*

$$\mu_q(\tau,K)\to 0 \ \textit{as}\ \tau\to+0. \tag{3.21}$$

3.4 Integral estimates for $\mathcal{N}_\pm$

Proposition 3.6 [22] *Suppose that the operators H_0, G satisfy* (3.1), (3.2) *and* (3.14) *for some $m>1/2$. Then for any $E>0$ and $\alpha>0$ the following estimates hold:*

$$\int_{-E}^{\infty}(\mathcal{N}_+(\lambda;H_0,G;\alpha)-N_+(-E;H_0,G;\alpha(1-\theta)^{-2}))_+(\lambda+2E)^{-2m}d\lambda$$
$$\le\alpha\theta^{-1-m}\|GR_0^m(-E)\|^2_{\mathbf{S}_2},\quad\forall\theta\in(0,1); \tag{3.22}$$

$$\int_{-E}^{\infty}(\mathcal{N}_+(\lambda;H_0,G;\alpha)-N_+(-2E;H_0,G;\alpha(1+\theta)^{-1}))_-(\lambda+2E)^{-2m}d\lambda$$
$$\le\alpha\theta^{-1}(2/(2m-1)+4)\|GR_0^m(-2E)\|^2_{\mathbf{S}_2},\quad\forall\theta>0; \tag{3.23}$$

$$\int_{0}^{\infty}\mathcal{N}_-(\lambda;H_0,G;\alpha)(\lambda+2E)^{-2m}d\lambda\le\alpha\|GR_0^m(-2E)\|^2_{\mathbf{S}_2}. \tag{3.24}$$

Estimates (3.22)–(3.24) are immediate consequences of Lemmas 4.3, 4.4 and 3.3 of [22] respectively.

4 Proof of Proposition 1.4

In this section we present some auxiliary statements on the Schrödinger operator $H_0=H_0(g,\mathbf{A},U)$, defined in §1.3, and prove Proposition 1.4. Our reasoning follows almost literally that of [2, §2]. Below we suppose that (1.11)–(1.13) hold true.

4.1 Schrödinger operator: auxiliary statements

Proposition 4.1 *For some constants $M>0$ and $\beta>0$, which depend only on g, the following pointwise estimate holds:*

$$|e^{-tH_0}\psi|\le Me^{t\beta\triangle}|\psi|,\quad\forall t>0,\ \psi\in L_2(\mathbf{R}^d). \tag{4.1}$$

For a flat metric, Proposition 4.1 is a well-known diamagnetic inequality (see [29] and references therein); in this case $M = \beta = 1$. Essentially, in the above stated form Proposition 4.1 has been proved in [11]. An alternative scheme of proof was outlined in [22, §7].

Proposition 4.2 *Let F be a multiplication by a function $F \in l_{2q}(\mathbf{Z}^d; L_2(\mathbf{Q}^d))$, $1/2 \le q \le 1$. Then*

$$\|FR_0^m(-1)\|_{\mathbf{S}_{2q}} \le C_{4.2}(q,m,d,g)\|F\|_{l_{2q}(L_2)}, \quad m > d/(4q). \tag{4.2}$$

Proof The desired result was proved for $H_0 = H_0(\mathbf{1}, \mathbf{A}, 0)$ in [2, Theorems 2.11, 2.12] and for $H_0 = H_0(\mathbf{1}, 0, U)$ (with some negative part of U allowed) in [31, Theorem B.9.2]. Our proof literally follows the same pattern.

1) Let $q = 1$. Due to the representation

$$(H_0 + E)^{-m} = (\Gamma(m))^{-1} \int_0^\infty e^{-H_0 t} e^{-Et} t^{m-1} dt,$$

the estimate (4.1) implies

$$|R_0^m(-1)\psi| \le M(\beta(-\triangle) + I)^{-m}|\psi|, \quad \forall m > 0,\ \psi \in L_2(\mathbf{R}^d). \tag{4.3}$$

It follows that

$$\|FR_0^m(-1)\|_{\mathbf{S}_2} \le M\|F(\beta(-\triangle) + I)^{-m}\|_{\mathbf{S}_2} \le C(m,d,g)\|F\|_{L_2}, \quad m > d/4.$$

2) Let $q = 1/2$. As in [2, Theorem 2.12], we take $F \in L_2(\mathbf{R}^d)$ with support in some cube $\mathbf{Q}^d + j$, $\mathbf{Q}^d = (0,1)^d$, $j \in \mathbf{Z}^d$, and write

$$FR_0^m(-1) = [FR_0^{m/2}(-1)(1 + |x|^2)^{m/2}][(1 + |x|^2)^{-m/2} R_0^{m/2}(-1)], \quad m > d/2. \tag{4.4}$$

Each of the two factors in the right hand side of (4.4) is, by (4.3), pointwise dominated by the same object with $M(\beta(-\triangle)+I)^{-m}$ instead of $R_0^m(-1)$, in which case they are Hilbert–Schmidt operators (see, e.g., [31] or [32] for the first factor). Summing over $j \in \mathbf{Z}^d$, we get (4.2) for $q = 1/2$.

3) Interpolating between $q = 1/2$ and $q = 1$, we obtain the desired result. □

4.2 Proof of Proposition 1.4

1) Let $d \le 3$. Then, by (1.24) and Proposition 4.2 (for $q = 1$), for any $\lambda_0 \in \rho(H_0)$ one has $|V|^{1/2}R_0(\lambda_0) \in \mathbf{S}_2$. Thus, in the identity

$$\begin{aligned} R(\lambda_0;\alpha) - R_0(\lambda_0) = -(\operatorname{sign} V|V|^{1/2}R_0(\lambda_0))^*(I + |V|^{1/2}R_0(\lambda_0)|V|^{1/2}\operatorname{sign} V)^{-1} \\ \times(|V|^{1/2}R_0(\lambda_0)), \quad \lambda_0 \in \rho(H_0) \cap \rho(H_0(\alpha)), \end{aligned} \tag{4.5}$$

the right hand side is a trace class operator. This gives (1.26) for $k = 1$.

2) Let $d \ge 4$. By (1.25) and Proposition 4.2 (for $q = 1/2$), one has $VR_0^m(\lambda_0) \in \mathbf{S}_1$ for all $\lambda_0 \in \rho(H_0)$ and $m > d/2$. From here, by [23, Theorem XI.12], follows (1.26) for any integer $k > m - \frac{1}{2}$ and any $\lambda_0 < 0$ with large enough $|\lambda_0|$. □

5 Pointwise asymptotics

The aim of this section is to prove Theorems 1.5 and 1.8. First we state and prove two abstract results (Theorems 5.1 and 5.2) on the asymptotics of the functions $\mathcal{N}_+(\lambda; H_0, G; \alpha)$ and $\mathcal{N}_-(\lambda; H_0, G; \alpha)$ for $\alpha \to \infty$. Then, applying Proposition 3.4, we obtain the desired results for the SSF of the Schrödinger operator.

5.1 Statement of abstract results

Let $\mathcal{H}$ be a "basic" and $\mathcal{K}$ — an "auxiliary" Hilbert space, $H_0 \geq 0$ be a selfadjoint operator in $\mathcal{H}$ and G be a closed operator which acts from $\mathcal{H}$ into $\mathcal{K}$ so that

$$GR_0^{1/2}(-1) \in \mathbf{\Sigma}_{2\kappa}(\mathcal{H}, \mathcal{K}), \quad \kappa > 0, \tag{5.1}$$

$$GE_{H_0}([0, R)) \in \mathbf{S}_{2q}(\mathcal{H}, \mathcal{K}), \quad q \leq 1, \quad \forall R > 0. \tag{5.2}$$

By Proposition 3.2, for a.e. $\lambda \in \mathbf{R}$ the pair H_0, G satisfies Condition 3.1 and thus the functions $\mathcal{N}_\pm(\lambda; H_0, G; \alpha)$ are well-defined.

Theorem 5.1 *Let the conditions* (3.1), (5.1), (5.2) *be satisfied with* $\kappa \geq q$ *if* $q < 1$ *and* $\kappa > 1$ *if* $q = 1$. *Then, for a.e.* $\lambda \in \mathbf{R}$, *the quantities*

$$\limsup_{\alpha \to \infty} \alpha^{-\kappa} \mathcal{N}_+(\lambda; H_0, G; \alpha), \quad \liminf_{\alpha \to \infty} \alpha^{-\kappa} \mathcal{N}_+(\lambda; H_0, G; \alpha) \tag{5.3}$$

are finite and do not depend on λ.

Note that in [3], in the framework of a similar abstract scheme, but under broader conditions on G, H_0, it was proved that the quantities (5.3) do not depend on $\lambda \in \mathbf{R} \setminus \sigma(H_0)$. Our construction borrows some ideas of [3].

Next, in order to discuss $\mathcal{N}_-(\lambda; H_0, G; \alpha)$, we relax the condition (5.1); it is sufficient to assume (3.2). Below for a number $0 < q \leq 1$ we introduce the notation

$$\begin{array}{ll} q_* = q & \text{if } q < 1; \\ q_* > 1 \text{ — any number} & \text{if } q = 1. \end{array}$$

Theorem 5.2 *Let the conditions* (3.1), (3.2), (5.2) *be satisfied. Then, for a.e.* $\lambda \in \mathbf{R}$*:*

$$\lim_{\alpha \to \infty} \alpha^{-q_*} \mathcal{N}_-(\lambda; H_0, G; \alpha) = 0. \tag{5.4}$$

5.2 Preliminary results.

Below we use the notations (3.3), (3.4).

Proposition 5.3 [5, 19] *Let* H_0 *be a selfadjoint operator in* $\mathcal{H}$ *and* $G \in \mathbf{S}_{2q}(\mathcal{H}, \mathcal{K})$, $q \leq 1$. *Then for a.e.* $\lambda \in \mathbf{R}$ *the operator* $T(\lambda + i\varepsilon; H_0, G)$ *has limit values as* $\varepsilon \to +0$ *in* $\mathbf{S}_{q_*}(\mathcal{K})$ *and* $K(\lambda + i0; H_0, G) \in \mathbf{S}_q(\mathcal{K})$.

Proposition 5.2 for $q = 1$, $q_* = 2$ was proved in [5]; the general case was studied in [19]. Further information can be found in [20].

Lemma 5.4 *Let the operators* A, K *be as in* (3.17). *Then, for any* $p \geq q$, *the quantities* (3.18) *obey*

$$\limsup_{s \to +0} s^p \eta_\pm(s; A, K) = \Delta_p^{(\pm)}(A), \tag{5.5}$$

$$\liminf_{s \to +0} s^p \eta_\pm(s; A, K) = \delta_p^{(\pm)}(A). \tag{5.6}$$

Proof Let us fix $0 < \tau < 1$ and use (3.19):

$$s^p \eta_\pm(s; A, K) \le s^p n_\pm(s(1-\tau), A) + s^{p-q} \tau^{-q} \mu_q(\tau s, K) \|K\|_{\mathbf{S}_q}.$$

Passing to the upper limits as $\varepsilon \to +0$ and taking into account (3.21), we get

$$\limsup_{s \to +0} s^p \eta_\pm(s; A, K) \le (1-\tau)^{-p} \Delta_p^{(\pm)}(A).$$

Since τ is arbitrary, this implies

$$\limsup_{s \to +0} s^p \eta_\pm(s; A, K) \le \Delta_p^{(\pm)}(A).$$

Similarly, using (3.20) instead of (3.19), we find

$$\limsup_{s \to +0} s^p \eta_\pm(s; A, K) \ge \Delta_p^{(\pm)}(A),$$

which gives (5.5). In a similar way one proves (5.6). □

Lemma 5.5 *Conditions* (5.1) *and* (5.2) *for* $q \le \kappa$ *imply*

$$GR_0(-1) \in \mathbf{\Sigma}^0_{2\kappa}(\mathcal{H}, \mathcal{K}). \tag{5.7}$$

Proof By (5.2), for any $R > 0$ one has

$$GR_0(-1) E_{H_0}([0, R)) \in \mathbf{S}_{2q}(\mathcal{H}, \mathcal{K}) \subset \mathbf{\Sigma}^0_{2\kappa}(\mathcal{H}, \mathcal{K}).$$

Next,

$$\begin{aligned} \|GR_0(-1) - GR_0(-1) E_{H_0}([0, R))\|_{\mathbf{\Sigma}_{2\kappa}} &= \|GR_0(-1) E_{H_0}([R, \infty))\|_{\mathbf{\Sigma}_{2\kappa}} \\ &\le \|GR_0^{1/2}(-1)\|_{\mathbf{\Sigma}_{2\kappa}} (R+1)^{-1/2} \to 0, \quad R \to \infty. \end{aligned}$$

Since $\mathbf{\Sigma}^0_{2\kappa}$ is closed in $\mathbf{\Sigma}_{2\kappa}$, this gives (5.7). □

Lemma 5.6 *Assume the hypothesis of Theorem 5.1. Then, for a.e.* $\lambda \in \mathbf{R}$*:*

(i) $T(\lambda + i\varepsilon; H_0, G)$ *has limit values as* $\varepsilon \to +0$ *in* $\mathbf{\Sigma}_\kappa(\mathcal{K})$*;*

(ii) $K(\lambda + i0; H_0, G) \in \mathbf{S}_q(\mathcal{K})$.

For a.e. λ, $\mu \in \mathbf{R}$*:*

(iii) $A(\lambda + i0; H_0, G) - A(\mu + i0; H_0, G) \in \mathbf{\Sigma}^0_\kappa(\mathcal{K})$.

Proof Fix an open bounded interval $\delta \in \mathbf{R}$. Below we check (i), (ii) for a.e. $\lambda \in \delta$ and (iii) for a.e. $\lambda, \mu \in \delta$.

1. One has

$$T(z; H_0, G) = T(z; H_0, GE_{H_0}(\delta)) + T(z; H_0, GE_{H_0}(\mathbf{R} \setminus \delta)). \tag{5.8}$$

By (5.2), $GE_{H_0}(\delta) \in \mathbf{S}_{2q}(\mathcal{H}, \mathcal{K})$. Hence, by Proposition 5.3, the operator $T(\lambda + i\varepsilon; H_0, GE_{H_0}(\delta))$ for a.e. $\lambda \in \mathbf{R}$ has limit values as $\varepsilon \to +0$ in $\mathbf{S}_{q_*}(\mathcal{K}) \subset \mathbf{\Sigma}_\kappa(\mathcal{K})$ and

$K(\lambda+i0;H_0,GE_{H_0}(\delta))\in\mathbf{S}_q(\mathcal{K})$. On the other hand, the limit $T(\lambda+i0;H_0,GE_{H_0}(\mathbf{R}\setminus\delta))$ exists in $\mathbf{\Sigma}_\kappa(\mathcal{K})$ for all $\lambda\in\delta$ and $K(\lambda+i0;H_0,GE_{H_0}(\mathbf{R}\setminus\delta))=0$. This proves (i), (ii).

2. Let us check (iii). For λ, $\mu\in\delta$ and $\varepsilon>0$ one has

$$A(\lambda+i\varepsilon;H_0,G)-A(\mu+i\varepsilon;H_0,G)=A(\lambda+i\varepsilon;H_0,GE_{H_0}(\delta))$$
$$-A(\mu+i\varepsilon;H_0,GE_{H_0}(\delta))+(\lambda-\mu)\mathrm{Re}\,(GR_0(\lambda+i\varepsilon)E_{H_0}(\mathbf{R}\setminus\delta)(GR_0(\mu-i\varepsilon))^*). \quad (5.9)$$

The first two terms in the right hand side of (5.9) for a.e. λ, $\mu\in\delta$ have limits as $\varepsilon\to+0$ in $\mathbf{S}_{q_*}(\mathcal{K})\subset\mathbf{\Sigma}^0_\kappa(\mathcal{K})$ (by Proposition 5.3). The third term for a.e. λ, $\mu\in\delta$ has a limit as $\varepsilon\to+0$, which can be written as $(GR_0(-1))M(GR_0(-1))^*$ with some $M\in\mathbf{B}(\mathcal{H})$. From here, taking into account (5.7), we see that the third term is in $\mathbf{\Sigma}^0_\kappa$, which gives (iii). $\Box$

Lemma 5.7 *Assume the hypothesis of Theorem 5.2. Then, for a.e.* $\lambda\in\mathbf{R}$*:*

(i) $T(\lambda+i\varepsilon;H_0,G)$ *has limit values*[5] *as* $\varepsilon\to+0$ *in* $\mathbf{S}_\infty(\mathcal{K})$*;*

(ii) $K(\lambda+i0;H_0,G)\in\mathbf{S}_q(\mathcal{K})$*;*

(iii) $(A(\lambda+i0;H_0,G))_-\in\mathbf{S}_{q_*}(\mathcal{K})$.

Proof Fix arbitrary $R>0$ and an interval $\delta=(-1,R)$. It suffices to check (i)–(iii) for a.e. $\lambda\in\delta$. As in the proof of Lemma 5.6, we write the representation (5.8) and see that for a.e. $\lambda\in\mathbf{R}$ the first summand in the r.h.s. has limit values in $\mathbf{S}_{q_*}(\mathcal{K})$ and for all $\lambda\in\delta$ the second summand has limit values in $\mathbf{S}_\infty(\mathcal{K})$, which are nonnegative selfadjoint operators. This proves (i), (iii). The statement (ii) follows from the inclusion $K(\lambda+i0;H_0,GE_{H_0}(\delta))\in\mathbf{S}_q(\mathcal{K})$, a.e. $\lambda\in\mathbf{R}$. $\Box$

5.3 Proof of Theorems 5.1, 5.2

Proof of Theorem 5.1 By Lemmas 5.4, 5.6(ii), for a.e. $\lambda\in\mathbf{R}$,

$$\limsup_{\alpha\to\infty}\alpha^{-\kappa}\mathcal{N}_+(\lambda;H_0,G;\alpha)=\Delta^{(+)}_\kappa(A(\lambda+i0)),$$
$$\liminf_{\alpha\to\infty}\alpha^{-\kappa}\mathcal{N}_+(\lambda;H_0,G;\alpha)=\delta^{(+)}_\kappa(A(\lambda+i0)).$$

By Lemma 5.6(i), the quantities in the right hand sides are finite. By Lemma 5.6(iii), they are independent of λ (for the functionals δ_κ, Δ_κ do not change if their argument changes by an operator of the class $\mathbf{\Sigma}^0_\kappa$.) $\Box$

Proof of Theorem 5.2 By Lemmas 5.4, 5.7(ii), for a.e. $\lambda\in\mathbf{R}$:

$$\limsup_{\alpha\to\infty}\alpha^{-q_*}\mathcal{N}_-(\lambda;H_0,G;\alpha)=\Delta^{(-)}_{q_*}(A(\lambda+i0)).$$

By Lemma 5.7(iii), $(A(\lambda+i0))_-\in\mathbf{S}_{q_*}(\mathcal{K})\subset\mathbf{\Sigma}^0_{q_*}(\mathcal{K})$ and thus $\Delta^{(-)}_{q_*}(A(\lambda+i0))=0$. $\Box$

5.4 Proof of Theorems 1.5, 1.8

Proof of Theorem 1.5

1. Let us check the hypothesis of Theorem 5.1 for $\mathcal{H}=\mathcal{K}=L_2(\mathbf{R}^d)$, $H_0=H_0(g,\mathbf{A},U)$, $G=\sqrt{V}$, $\kappa=d/2$. We choose q in (5.2) as follows: $q=1/2$ if $d=1$; $q=\max\{1/2,\tau\}$ if $d=2$ (remind that τ is an exponent from (1.28)); $q=1$ if $d\geq 3$. Now, by Proposition 4.2,

$$GR_0^m(-1)\in\mathbf{S}_{2q}(\mathcal{H},\mathcal{K}) \quad (5.10)$$

[5] In fact, this statement follows from Proposition 3.2

for large enough m, where q is as specified above. Obviously, (5.10) implies (5.2). Finally, the inclusion (5.1) holds by (2.6)–(2.8).

2. For $\lambda < 0$, due to Proposition 1.3 and formula (3.12), the quantities (5.3) coincide and are equal to $C_{1.23}$. From here and Theorem 5.1 we see that there exists a set $M_0 \subset \mathbf{R}$, meas $(\mathbf{R} \setminus M_0) = 0$, such that

$$\lim_{\alpha\to\infty} \alpha^{-d/2}\mathcal{N}_+(\lambda; H_0, \sqrt{V}; \alpha) = C_{1.23}, \quad \forall \lambda \in M_0. \tag{5.11}$$

3. Let a sequence $\alpha_n \to \infty$. By (5.1), (5.10) and Proposition 1.4, for every $n \in \mathbf{N}$ the hypothesis of Proposition 3.4 holds for the operators H_0, $H(\alpha_n)$. Thus, there exist such sets $M_n \subset \mathbf{R}$, meas $(\mathbf{R} \setminus M_n) = 0$, that

$$\xi(\lambda; H(\alpha_n), H_0) = -\mathcal{N}_+(\lambda; H_0, \sqrt{V}; \alpha_n), \quad \forall \lambda \in M_n,\ n \in \mathbf{N}. \tag{5.12}$$

Combining (5.11) and (5.12), we see that (1.30) holds true for $M = M_0 \cap (\cap_{n\in\mathbf{N}} M_n)$. □

Proof of Theorem 1.8

Let us check the hypothesis of Theorem 5.2 for $\mathcal{H} = \mathcal{K} = L_2(\mathbf{R}^d)$, $H_0 = H_0(g, \mathbf{A}, U)$, $G = \sqrt{V}$ and $q = \tau$. Indeed, (3.1) and (3.2) hold by hypothesis. By Proposition 4.2,

$$GR_0^m(-1) \in \mathbf{S}_{2\tau}(\mathcal{H}, \mathcal{K}) \tag{5.13}$$

for large enough $m > 0$; (5.13) implies (5.2). Theorem 5.2 says that for a.e. $\lambda \in \mathbf{R}$

$$\lim_{\alpha\to\infty} \alpha^{-\rho}\mathcal{N}_-(\lambda; H_0, \sqrt{V}; \alpha) = 0,$$

where ρ is as specified in the statement of Theorem 1.8. Finally, as in part 3 of the proof of Theorem 1.5, taking into account Proposition 3.4, we obtain (1.33). □

Remark 5.8 Theorems 5.1 and 5.2 allow one to consider some other operators H_0, as long as it is possible to establish some analogues of Propositions 2.1 and 4.2. In particular, one can take $H_0 = (-\triangle)^l$, $l > 0$. In this case, the estimates of the type (2.1)–(2.3) and (4.2) are well-known — see, e.g., [8, 9, 32]. Here one can also consider perturbations V by differential operators of an order lower than l with decaying coefficients.

Another possibility is to consider the relativistic magnetic Schrödinger operator

$$\tilde{H}_0 = (H_0(\mathbf{1}, \mathbf{A}, 0) + I)^{1/2} + U(x). \tag{5.14}$$

Here the bound similar to (2.1)–(2.3) can be obtained by using the Heintz inequality. In order to get the bounds of the type (4.2), one can exploit the pointwise inequality

$$|e^{-t\tilde{H}_0}\psi| \le e^{-t(-\triangle+I)^{1/2}}|\psi|, \quad t > 0,\ \psi \in L_2(\mathbf{R}^d),$$

in the spirit of §4.1. In this connection, see also [25, 6].

6 Integral asymptotics

The aim of this section is to prove Theorem 1.7. First we prove an abstract result (Theorem 6.2) and then apply it to the Schrödinger operator.

6.1 Statement of an abstract result

Let $H_0 \geq 0$ be a selfadjoint operator in a "basic" Hilbert space $\mathcal{H}$. Next, let $\mathcal{K}_+$ and $\mathcal{K}_-$ be "auxiliary" Hilbert spaces, $G_+ : \mathcal{H} \to \mathcal{K}_+$ and $G_- : \mathcal{H} \to \mathcal{K}_-$ be closed operators and

$$G_- R_0^{1/2}(-1) \in \mathbf{S}_\infty(\mathcal{H}, \mathcal{K}_-), \tag{6.1}$$

$$G_+ R_0^{1/2}(-1) \in \mathbf{\Sigma}_{2\kappa}(\mathcal{H}, \mathcal{K}_+), \quad \kappa > 0. \tag{6.2}$$

Let $V_+ = G_+^* G_+$, $V_- = G_-^* G_-$ and define the operators (1.18) as the form sums; denote

$$R_+(\lambda; \alpha) = (H_+(\alpha) - \lambda I)^{-1}; \quad R_-(\lambda; \alpha) = (H_-(\alpha) - \lambda I)^{-1}; \quad R(\lambda; \alpha) = (H(\alpha) - \lambda I)^{-1}.$$

Before stating the result on the SSF, we discuss the asymptotics of a discrete spectrum below $\lambda = 0$. Introduce the notation (1.19) and for $\lambda < 0$ denote

$$\Omega_\kappa(\lambda) := \limsup_{\alpha \to \infty} \alpha^{-\kappa} N(\lambda; \alpha), \quad \omega_\kappa(\lambda) := \liminf_{\alpha \to \infty} \alpha^{-\kappa} N(\lambda; \alpha). \tag{6.3}$$

Proposition 6.1 *Let the conditions* (3.1), (6.1), (6.2) *hold and suppose that for any* $R > 0$

$$G_+ E_{H_0}([0, R)) \in \mathbf{\Sigma}_{2\kappa}^0(\mathcal{H}, \mathcal{K}_+). \tag{6.4}$$

Then the quantities Ω_κ, ω_κ *in* (6.3) *do not depend on* $\lambda < 0$.

The proof is given in the next subsection. Note that the condition (6.1) in the hypothesis of Proposition 6.1 can be relaxed. In [27], in the framework of a similar abstract scheme but under slightly different conditions on H_0, G_+, G_-, the stability of the leading term of the asymptotics of $N(\lambda; H_0, G_+, G_-; \alpha)$ (see (0.7)) has been established for $\lambda \in \mathbf{R} \setminus \sigma(H_0)$. In contrast to Proposition 6.1, the proof of the main result of [27] requires the use of a fairly complicated technique.

The main result of this section (Theorem 6.2) deals with the SSF for the pair H_0, $H(\alpha)$. Its proof uses also the SSF for the pairs H_0, $H_+(\alpha)$ and $H_+(\alpha)$, $H(\alpha)$. In order to define these SSF, suppose that for all $\alpha > 0$ and some $k > 0$, $\lambda_0 < \inf(\sigma(H_0) \cup \sigma(H(\alpha)))$, the following inclusions hold:

$$R_+^k(\lambda_0, \alpha) - R_0^k(\lambda_0) \in \mathbf{S}_1(\mathcal{H}), \tag{6.5}$$

$$R^k(\lambda_0, \alpha) - R_0^k(\lambda_0) \in \mathbf{S}_1(\mathcal{H}). \tag{6.6}$$

The numbers k, λ_0 may depend on α but must be the same in (6.5) and (6.6). From (6.5) and (6.6) it follows that

$$R_+^k(\lambda_0, \alpha) - R^k(\lambda_0, \alpha) \in \mathbf{S}_1(\mathcal{H}). \tag{6.7}$$

Theorem 6.2 *Let the conditions* (3.1), (6.1), (6.2) *for* $\kappa > 1$, *and* (6.5), (6.6) *hold true. Next, suppose that for some* $m > 1/2$*:*

$$G_- R_0^m(-1) \in \mathbf{S}_2(\mathcal{H}, \mathcal{K}_-), \tag{6.8}$$

$$\sup_{\alpha>0} \|G_+ R_+^m(-1;\alpha)\|_{\mathbf{S}_2} =: C_{6.9} < \infty. \tag{6.9}$$

Then for any $E > 0$*:*

$$\lim_{\alpha\to\infty} \int_{-E}^{\infty} (\alpha^{-\kappa}\xi(\lambda; H(\alpha), H_0) + \Omega_\kappa)_-(\lambda + 2E)^{-2m} d\lambda = 0, \tag{6.10}$$

$$\lim_{\alpha\to\infty} \int_{-E}^{\infty} (\alpha^{-\kappa}\xi(\lambda; H(\alpha), H_0) + \omega_\kappa)_+(\lambda + 2E)^{-2m} d\lambda = 0, \tag{6.11}$$

where Ω_κ, ω_κ *are defined by* (6.3) *(and do not depend on* λ *by Proposition 6.1* [6]*). In particular, if* $\Omega_\kappa = \omega_\kappa$*, then*

$$\lim_{\alpha\to\infty} \int_{-E}^{\infty} |\alpha^{-\kappa}\xi(\lambda; H(\alpha), H_0) + \Omega_\kappa|(\lambda + 2E)^{-2m} d\lambda = 0. \tag{6.12}$$

6.2 Proof of Proposition 6.1

1. Obviously, $N(\lambda_1, \alpha) \le N(\lambda_2, \alpha)$ if $\lambda_1 \le \lambda_2 < 0$. It follows that

$$\Omega_\kappa(\lambda_1) \le \Omega_\kappa(\lambda_2), \quad \omega_\kappa(\lambda_1) \le \omega_\kappa(\lambda_2), \quad \lambda_1 \le \lambda_2 < 0.$$

Thus, it suffices to check the opposite inequalities

$$\Omega_\kappa(\lambda_1) \ge \Omega_\kappa(\lambda_2), \quad \omega_\kappa(\lambda_1) \ge \omega_\kappa(\lambda_2), \quad \lambda_1 \le \lambda_2 < 0.$$

2. Denote

$$X_\pm(\lambda) := G_\pm R_0^{1/2}(\lambda), \quad Z_\pm(\lambda) := X_\pm^*(\lambda) X_\pm(\lambda), \quad \lambda < 0. \tag{6.13}$$

Obviously, $Z_\pm(\lambda_1) \le Z_\pm(\lambda_2)$ for $\lambda_1 \le \lambda_2 < 0$. Besides, below we will show that

$$Z_+(\lambda_1) - Z_+(\lambda_2) =: M \in \mathbf{\Sigma}_\kappa^0(\mathcal{H}). \tag{6.14}$$

From here by (2.5) it follows that

$$\begin{aligned} N(\lambda_1, \alpha) = n_+(\alpha^{-1}, Z_+(\lambda_1) - Z_-(\lambda_1)) &\ge n_+(\alpha^{-1}, Z_+(\lambda_1) - Z_-(\lambda_2)) \\ &= n_+(\alpha^{-1}, Z_+(\lambda_2) - Z_-(\lambda_2) + M), \\ \Omega_\kappa(\lambda_1) \ge \Delta_\kappa^{(+)}(Z_+(\lambda_2) - Z_-(\lambda_2) + M) &= \Delta_\kappa^{(+)}(Z_+(\lambda_2) - Z_-(\lambda_2)) = \Omega_\kappa(\lambda_2). \end{aligned}$$

Similarly, it follows that $\omega_\kappa(\lambda_1) \ge \omega_\kappa(\lambda_2)$.

[6] Condition (6.4) in the hypothesis of Proposition 6.1 follows from (6.9): (6.9) $\Rightarrow$ $G_+ R_0^m(-1) \in \mathbf{S}_2 \subset \mathbf{\Sigma}_{2\kappa}^0$ $\Rightarrow$ (6.4).

3. Now we need to check (6.14). Clearly, (6.14) will follow if we prove

$$X_+(\lambda_1) - X_+(\lambda_2) \in \mathbf{\Sigma}^0_{2\kappa}(\mathcal{H}, \mathcal{K}_+). \tag{6.15}$$

One can write the operator from (6.15) as

$$X_+(\lambda_1) - X_+(\lambda_2) = G_+ R_0(-1) B, \quad B \in \mathbf{B}(\mathcal{H}). \tag{6.16}$$

As in Lemma 5.5, we find that (6.4) and (6.2) imply $G_+ R_0(-1) \in \mathbf{\Sigma}^0_{2\kappa}(\mathcal{H}, \mathcal{K}_+)$. From here and (6.16) follows (6.15). □

6.3 Proof of Theorem 6.2

Lemma 6.3 *Assume the hypothesis of Proposition 6.1. Denote*

$$L(\lambda, \alpha, a) := \alpha^{-\kappa} N_+(\lambda; H_+(\alpha), G_+; a\alpha), \quad \lambda < 0,\ \alpha > 0,\ a > 0. \tag{6.17}$$

Then the quantities

$$B(a) := \limsup_{\alpha \to \infty} L(\lambda, \alpha, a), \quad b(a) := \liminf_{\alpha \to \infty} L(\lambda, \alpha, a) \tag{6.18}$$

do not depend on $\lambda < 0$ and depend continuously on $a > 0$. In particular,

$$\lim_{a \to 1} B(a) = \Omega_\kappa, \quad \lim_{a \to 1} b(a) = \omega_\kappa. \tag{6.19}$$

Proof For $\alpha > 0$ and $\beta > 0$ define the operators $H(\alpha, \beta) = H_0 + \alpha V_- - \beta V_+$ as the form sums. Clearly,

$$N_+(\lambda; H_+(\alpha), G_+; \beta) = \operatorname{rank} E_{H(\alpha,\beta)}((-\infty, \lambda)), \quad \lambda < 0.$$

Introducing the notations (6.13) and using (2.5), we find

$$N_+(\lambda; H_+(\alpha), G_+; a\alpha) = n_+(\alpha^{-1}; aZ_+(\lambda) - Z_-(\lambda)),$$

and thus

$$B(a) = \Delta^{(+)}_\kappa(aZ_+(\lambda) - Z_-(\lambda)), \quad b(a) = \delta^{(+)}_\kappa(aZ_+(\lambda) - Z_-(\lambda)).$$

Now the continuity of $L(a)$, $l(a)$ follows from the continuity of $\Delta^{(+)}_\kappa$, $\delta^{(+)}_\kappa$ in $\mathbf{\Sigma}_\kappa$. The fact that $L(a)$, $l(a)$ do not depend on λ, follows from Proposition 6.1 (after the substitution $G_+ \mapsto \sqrt{a} G_+$). □

Proof of Theorem 6.2

1. First note that, by (1.8) and (6.5)–(6.7),

$$\xi(\lambda; H(\alpha), H_0) = \xi(\lambda; H(\alpha), H_+(\alpha)) + \xi(\lambda; H_+(\alpha), H_0). \tag{6.20}$$

Next, by Proposition 3.4,

$$\xi(\lambda; H(\alpha), H_+(\alpha)) = -\mathcal{N}_+(\lambda; H_+(\alpha), G_+; \alpha), \tag{6.21}$$

$$\xi(\lambda; H_+(\alpha), H_0) = \mathcal{N}_-(\lambda; H_0, G_-; \alpha). \tag{6.22}$$

Denote

$$\begin{aligned}
F(\lambda,\alpha) &:= -\alpha^{-\kappa}\xi(\lambda; H(\alpha), H_0),\\
Q(\lambda,\alpha) &:= \alpha^{-\kappa}\mathcal{N}_+(\lambda; H_+(\alpha), G_+;\alpha),\\
J(\lambda,\alpha) &:= \alpha^{-\kappa}\mathcal{N}_-(\lambda; H_0, G_-;\alpha).
\end{aligned}$$

Write the equality (6.20) as

$$F(\lambda,\alpha) = Q(\lambda,\alpha) - J(\lambda,\alpha). \tag{6.23}$$

Finally, we denote for brevity $d\mu(\lambda) := (\lambda+2E)^{-2m}d\lambda$ for $\lambda \geq -E$ and $d\mu(\lambda) = 0$ for $\lambda < -E$. In these notations, we need to prove that

$$\lim_{\alpha\to\infty}\int (F(\lambda,\alpha) - \Omega_\kappa)_+ d\mu(\lambda) = 0, \tag{6.24}$$

$$\lim_{\alpha\to\infty}\int (F(\lambda,\alpha) - \omega_\kappa)_- d\mu(\lambda) = 0. \tag{6.25}$$

2. Let us prove (6.24). In (3.22), take $H_+(\alpha)$ for H_0 and G_+ for G. Using the notation (6.17), we obtain:

$$\begin{aligned}
\int (Q(\lambda,\alpha) - (1-\theta)^{-2\kappa}L(-E,\alpha,(1-\theta)^{-2}))_+ d\mu(\lambda) &\leq \alpha^{1-\kappa}\theta^{-1-m}\|G_+R_+^m(-E,\alpha)\|^2_{\mathbf{S}_2}\\
&\leq \max\{1, E^{-2m}\}C_{6.9}\theta^{-1-m}\alpha^{1-\kappa}.
\end{aligned}$$

Next, taking into account (6.23) and the inequality $J(\lambda,\alpha)\geq 0$,

$$\begin{aligned}
(Q(\lambda,\alpha) - (1-\theta)^{-2\kappa}L(-E,\alpha,(1-\theta)^{-2}))_+ &\geq (F(\lambda,\alpha) - (1-\theta)^{-2\kappa}L(-E,\alpha,(1-\theta)^{-2}))_+\\
&\geq (F(\lambda,\alpha)-\Omega_\kappa)_+ - ((1-\theta)^{-2\kappa}L(-E,\alpha,(1-\theta)^{-2}) - \Omega_\kappa)_+.
\end{aligned}$$

From here we find

$$\begin{aligned}
\int (F(\lambda,\alpha) - \Omega_\kappa)_+ d\mu(\lambda) \leq{}& \max\{1,E^{-2m}\}C_{6.9}\theta^{-1-m}\alpha^{1-\kappa}\\
&+((1-\theta)^{-2\kappa}L(-E,\alpha,(1-\theta)^{-2}) - \Omega_\kappa)_+\int d\mu(\lambda).
\end{aligned}$$

Passing to the limit as $\alpha\to\infty$, we find (using the notations (6.18)):

$$\limsup_{\alpha\to\infty}\int (F(\lambda,\alpha)-\Omega_\kappa)_+ d\mu(\lambda) \leq ((1-\theta)^{-2\kappa}B((1-\theta)^{-2}) - \Omega_\kappa)_+\int d\mu(\lambda).$$

By (6.19), the right hand side of the last inequality goes to 0 as $\theta\to+0$, and we arrive at (6.24).

3. Let us prove (6.25). In (3.23), take $H_+(\alpha)$ for H_0 and G_+ for G. Using the notation (6.17), we obtain:

$$\begin{aligned}
\int (Q(\lambda,\alpha) - (1+\theta)^{-\kappa}L(-2E,\alpha,(1+\theta)^{-1}))_- d\mu(\lambda) &\leq \alpha^{1-\kappa}\theta^{-1}(2/(2m-1)+4)\\
\times\|G_+R_+^m(-2E,\alpha)\|^2_{\mathbf{S}_2} &\leq (2/(2m-1)+4)\max\{1,(2E)^{-2m}\}C_{6.9}\theta^{-1}\alpha^{1-\kappa}.
\end{aligned}$$

Next, taking into account (6.23),

$$
\begin{aligned}
&(Q(\lambda,\alpha)-(1+\theta)^{-\kappa}L(-2E,\alpha,(1+\theta)^{-1}))_-\\
&=(F(\lambda,\alpha)-\omega_\kappa+J(\lambda,\alpha)+(\omega_\kappa-(1+\theta)^{-\kappa}L(-2E,\alpha,(1+\theta)^{-1})))_-\\
&\geq(F(\lambda,\alpha)-\omega_\kappa)_- - J(\lambda,\alpha)-(\omega_\kappa-(1+\theta)^{-\kappa}L(-2E,\alpha,(1+\theta)^{-1}))_+.
\end{aligned}
$$

From here we find:

$$
\int(F(\lambda,\alpha)-\omega_\kappa)_-d\mu(\lambda)\leq(2/(2m-1)+4)\max\{1,(2E)^{-2m}\}C_{6.9}\theta^{-1}\alpha^{1-\kappa}+\int J(\lambda,\alpha)d\mu(\lambda)
$$

$$
+(\omega_\kappa-(1+\theta)^{-\kappa}L(-2E,\alpha,(1+\theta)^{-1}))_+\int d\mu(\lambda). \tag{6.26}
$$

By (3.24),

$$
\limsup_{\alpha\to\infty}\int J(\lambda,\alpha)d\mu(\lambda)\leq\limsup_{\alpha\to\infty}\alpha^{1-\kappa}\|G_-R_0^m(-2E)\|^2_{\mathbf{S}_2}=0.
$$

Passing to the limit as $\alpha\to\infty$ in (6.26) and taking into account the last relation, we obtain:

$$
\limsup_{\alpha\to\infty}(F(\lambda,\alpha)-\omega_\kappa)_-d\mu(\lambda)\leq(\omega_\kappa-(1+\theta)^{-\kappa}b((1+\theta)^{-1}))_+\int d\mu(\lambda).
$$

By (6.19), the right hand side of the last inequality goes to 0 as $\theta\to+0$, and we obtain (6.25). □

Remark 6.4 Under the hypothesis of Theorem 6.2, formula (0.13) is valid. Indeed, using (0.10), (6.20)–(6.22), for $\lambda\in\mathbf{R}\setminus\sigma(H_0)$ we obtain:

$$
\begin{aligned}
&\xi(\lambda;H(\alpha),H_0)=-\mathcal{N}_+(\lambda;H_+(\alpha),G_+;\alpha)+\mathcal{N}_-(\lambda;H_0,G_-;\alpha)\\
&=-N_+(\lambda;H_+(\alpha),G_+;\alpha)+N_-(\lambda;H_0,G_-;\alpha)=-N(\lambda;H_0,G_+,G_-;\alpha).
\end{aligned}
$$

6.4 Proof of Theorem 1.7

Let us check the hypothesis of Theorem 6.2 for $H_0=H_0(g,\mathbf{A},U)$, $G_\pm=\sqrt{V_\pm}$, $\mathcal{H}=\mathcal{K}_+=\mathcal{K}_-=L_2(\mathbf{R}^d)$ and $\kappa=d/2$. Inclusions (6.1), (6.2) follow from (2.6). Conditions (6.8), (6.9) follow from (4.2) (for $q=1$); while checking (6.9), we include the term $+\alpha V_-$ into the background potential U and use the fact that the constant $C_{4.2}$ does not depend on $U\geq0$. Conditions (6.5), (6.6) follow from Proposition 1.4.

Next, by Proposition 1.3, $\Omega_\kappa=\omega_\kappa=C_{1.23}$. Thus, (6.12) gives (1.31) with $p=2m$. □

Remark 6.5 Let us comment on the possibility of applying Theorem 6.2 to other differential operators H_0. Here the main difficulty is in the check of the condition (6.9), which in our case required the use of rather specific technique of pointwise estimates for the Gaussian kernels. This technique is applicable to the elliptic differential operators of the order $l\leq2$. Thus, one can apply Theorem 6.2 to the relativistic magnetic Schrödinger operator (5.14).

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