SPLITTING OF SEPARATRICES FOR THE CHIRIKOV'S STANDARD MAP

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Abstract. This is a revised English version of my earlier paper [L1] deposited in VINITI in 1984. An asymptotic formula for the exponentially small angle of intersection of the stable and unstable manifolds of the fixed hyperbolic point for the standard map is derived provided the parameter ε tends to zero.

§1. Introduction. The formula for the separatrices splitting

This paper is a revised English version of my earlier paper [L1] deposited in VINITI in 1984. Here I changed some notations, changed the formulations of the conjectures to make them more realistic (the first one now is a theorem, the second one remains open), and added recent references. I keep the numbers for old references while the new ones acquired letter labels.

It is known that a big part of Liouville invariant tori persists under small perturbations of an integrable Hamiltonian system. The corresponding perturbation theory is called KAM theory (theory of Kolmogorov– Arnold–Moser) [1],[2], [3]. In a system which differs slightly from an integrable one, the invariant tori constitute a smooth family diffeomorfic to the product of a torus and a Cantor set of positive Lebesgue measure. Let us call these tori Kolmogorov tori. The measure of the complement to the set of Kolmogorov tori intersected with a bounded set has the order $\mathcal{O}(\sqrt{\varepsilon})$ provided $\varepsilon \to 0$, where ε is a perturbation parameter. This estimate and the results concerning the smoothness are contained in the papers [4,5,6]. In the complement to the set of Kolmogorov tori the system behaves stochastically [7]. Now there is now in any sense satisfactory qualitative theory describing the behaviour of a Hamiltonian system in the complement to the set of Kolmogorov tori. It is known that at least for two–dimensional area–preserving diffeomorphisms the remnants of destroyed Liouville tori are p[resent in the phase space as invariant Cantor sets [8,9,15].

A good model for the perturbation theory of completely integrable dynamical systems is the standard map SM introduced by B.V.Chirikov [10,11,C,G] which is defined by the formulas

$$\begin{aligned} SM :&(x, y) \longmapsto (x_1, y_1), \\ &x_1 = x + y_1 (\mod 2\pi) \\ &y_1 = y + \varepsilon \sin x \end{aligned} \tag{1.1}$$

If $\varepsilon = 0$, the transformation (1.1) is integrable, the variable y is an integral. If $\varepsilon > 0$, the transformation has a fixed point $(0, 0)$. The stable W^s and the unstable

W^u manifolds (the separatrices) of the point $(0,0)$ have an intersection at a point (π, y_Γ) where $y_\Gamma \approx 2\sqrt{\varepsilon}$, the first intersection of W^u with symmetry line $x = \pi$.

The aim of this work is to derive the following asymptotic as $\varepsilon \to 0$ formula for the angle φ of the intersection of W^u and W^s at (π, y_Γ):

$$\varphi = \frac{\pi|\Theta_1|}{\varepsilon} \mathrm{e}^{-\frac{\pi^2}{\sqrt{\varepsilon}}} \left[1 + \mathcal{O}\left(\varepsilon^{\frac{1}{8}-\delta}\right)\right]. \tag{1.2}$$

The number $|\Theta_1|$ in (1.2) is a constant which is to be defined in the process of solution of some auxiliary nonlinear problem not containing ε (see §3). The first numerical evaluation of this number made by I.G.Shachmannski in 1984 gives the value $|\Theta_1| \approx 1040$. Further more precise calculation ([LST] yielded $|\Theta_1| = 1118.82770595$. The number $\delta > 0$ in (1.2) is arbitrary, the constant in the estimate $\mathcal{O}\left(\varepsilon^{\frac{1}{8}-\delta}\right)$ depending on δ. The formula in question shows that for small ε the angle φ is exponentially small. The exponential smallness was proved earlier by A.I.Neishtadt [12]. Some estimates for the angle were obtained in unpublished paper [CS]. Nonnullity of φ implies transversality of the intersection of the separatrices at the point (π, y_Γ). The existence of such a point generates a very complicated picture of intersections of the separatrices W^s and W^u, which was described by H.Poincaré [13]. The separatrix W^u is an injective immersion of $\mathbb{R}$ into the phase space $(\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}$. The image of this injection is a noncompact curve passing through the point $(0,0)$, oscillating with the amplitude growing as the distance from $(0,0)$ grows, and winding onto itself. The stable separatrix W^s behaves in the same way. Intersections of these two curves constitute a complicated entangled web whose closure is the stochastic layer near the separatrix. Numerical experiments [10,11] show the presence this stochastic layer, its growth as ε grows, and its fusion with other similar stochastic layers which pervade the phase space of the standard Chirikov map (see also [14,15,16,FS]).

One can not consider the derivation and proof of formula (1.2) presented below as quite rigorous. One needs to supplement it with the proof of two conjectures concerning the existence of the time–energy variables formulated in §3 and §4.

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The author thanks I.G.Shachmannski who has undertaken the first numerical computations of the constant $|\Theta_1|$.

§2. An approximation of the Chirikov's map in the complex domain

Effects connected with the creation of the stochastic layer are exponentially small for small ε, as it was mentioned in §1. It is difficult to distinguish them on the background of approximations having the error of order of a power of ε. Professor A.I.Shnirelman communicated to the author the idea that one could better to investigate these effects in the complex domain where they grew up enough to be of the same order as the mentioned powerlike approximations.

It is convenient to pass in (1.1) from the variables (x, y) to variables (X, Y) by formulas

$$X = x, \quad Y = \frac{1}{\sqrt{\varepsilon}} y. \tag{2.1}$$

The standard map reads in this variables as $(X, Y) \longmapsto (X_1, Y_1)$, where

$$\frac{X_1 - X}{\sqrt{\varepsilon}} = Y_1, \quad \frac{Y_1 - Y}{\sqrt{\varepsilon}} = \sin X. \tag{2.2}$$

We will consider (2.2) in a complex phase space $\mathbb{C}/2\pi\mathbb{Z} \times \mathbb{C}$.

If $\mathrm{Im}X$ and $|Y|$ is not very large, the system (2.2), for small ε, can be well approximated by the pendulum differential equation:

$$X' = Y, \quad Y' = \sin X. \tag{2.3}$$

To investigate the standard map for large positive $\mathrm{Im}X$, it is convenient to make the change of variables

$$X = -i \ln \frac{\varepsilon}{2} + iu, \quad \sqrt{\varepsilon}Y = iv. \tag{2.4}$$

The variable u and v are called *semistandard variables.* Our map reads in these variables:

$$\begin{aligned} u_1 &= u + v_1\,, \\ v_1 &= v + \mathrm{e}^u - \frac{\varepsilon^2}{4}\mathrm{e}^{-u}\,. \end{aligned} \tag{2.5}$$

The transformation (2.5) is defined on the phase space $(\mathbb{C}/2i\pi\mathbb{Z}) \times \mathbb{C}$. The term $-\frac{\varepsilon^2}{4}\mathrm{e}^{-u}$ in (2.5) is small in comparison with the term e^u provided $\mathrm{Re}u < const$. Therefore the transformation (2.5) in that domain can be well approximated by a transformation not depending on ε: $SSM : (u, v) \longmapsto (u_1, v_1)$,

$$u_1 = u + v_1\,, \quad v_1 = v + \mathrm{e}^u\,, \tag{2.6}$$

which we call *semistandard map* (it was introduced in [GP], see also [P]). The statement "well approximated " will be precisely formulated later in concrete situations.

In the subsequent sections we will study invariant curves with respect to the transformations in question. By an *invariant curve with step h* we mean an analytic map $W : \mathbb{C} \longrightarrow (\mathbb{C}/2i\pi\mathbb{Z}) \times \mathbb{C}$, $W(t) = (X(t), Y(t))$, satisfying the following system of difference equations:

$$\begin{aligned} \Delta_h X(t) &= \sqrt{\varepsilon} Y(t+h), \\ \Delta_h Y(t) &= \sqrt{\varepsilon} \sin X(t)\,. \end{aligned} \tag{2.7}$$

Here

$$\Delta_h f(t) = f(t+h) - f(t)\,. \tag{2.8}$$

One can give an analogous definition also for the transformation (2.6). To study W^u in the domain where an approximation by differential equations is suitable, it is convenient to take the expression (4.2) for h. To investigate integral curves of the transformations (2.5) and (2.6) in the semistandard variables, it is convenient to set $h = 1$. To pass from (2.7) to corresponding equations in the semistandard variables, one needs to pass to a "semistandard time " x by a formula

$$t = a + hx\,. \tag{2.9}$$

When investigating the behaviour of separatrices in the complex domain $\mathrm{Im}X \geq 1$, it is convenient to set $a = i\frac{\pi}{2}$ in (2.9).

§3. The investigation of the semistandard map

In this section we formulate necessary assertions about properties of the semistandard map (2.6). By *invariant curve* of the semistandard map we mean an analytic map

$$\Gamma\colon \mathbb{C} \longrightarrow (\mathbb{C}/2i\pi\mathbb{Z}) \times \mathbb{C}, \Gamma(x) = (u(x), v(x))\,,$$

obeying to the equations

$$\begin{aligned} u(x+1) - u(x) &= v(x+1)\,, \\ v(x+1) - v(x) &= \mathrm{e}^{u(x)}\,. \end{aligned} \tag{3.1}$$

It is easy to verify that, if $(u(x), v(x))$ is an invariant curve, then $(u_1(x), v_1(x))$ with

$$\begin{aligned} u_1(x) &= u(-x)\,, \\ v_1(x) &= -v(-x+1) \end{aligned} \tag{3.2}$$

is an invariant curve too.

Theorem 1 (on the existence and the unicity of the unstable manifold). *There exists the unique invariant curve $\Gamma_-(x) = (u_-(x), v_-(x))$ of the semistandard map with the following asymptotics as $x \to \infty$:*

$$\begin{aligned} u_-(x) &= -\ln\frac{x^2}{2} + \mathcal{O}\left(\frac{1}{x^2}\right)\,, \\ v_-(x) &= -\frac{2}{x} + \mathcal{O}\left(\frac{1}{x^2}\right)\,. \end{aligned} \tag{3.3}$$

In (3.3) $\ln\frac{x^2}{2} > 0$ if $x < 0$.

The following asymptotic expansion for $x \to \infty$ is valid uniformlly in a sector $\delta_0 \le \arg x \le 2\pi - \delta_0$, $\delta_0 \in]0, \pi/2[$ being an arbitrary fixed number:

$$u_-(x) = -\ln\frac{x^2}{2} + \sum_{k=1}^{\infty} a_k x^{-2k}, \tag{3.4}$$

where a_k are real numbers. The first three values of a_k are $a_1 = -\frac{1}{4}$,$a_2 = \frac{91}{864}$,$a_3 = -\frac{319}{2880}$.

This theorem will be proven in §7 (see also [L2]

Applying the transformation (3.2) to the curve Γ_- yields the stable manifold $\Gamma_+ = (u_+, v_+)$.

It follows from (3.4) that $\Gamma_+(x)$ has the same asymptotic expansion as $\Gamma_-(x)$ but in the sector defined by the inequality

$$-\pi + \delta_0 \le \arg x \le -\delta_0\ . \tag{3.5}$$

As a consequence, we have in this sector for arbitrary positive N

$$|u_+(x) - u_-(x)| \le \mathrm{const} \cdot |x|^{-N}, \tag{3.6}$$

where the const depends on N and on the choice of δ_0. In fact, a stronger estimate is valid.

Theorem 2. *The following estimates*

$$
(3.7)\qquad \begin{aligned} |u_+(x) - u_-(x)| &\leq const \cdot |x|^2 \exp(-2\pi|Imx|), \\ |v_+(x) - v_-(x)| &\leq const \cdot |x| \exp(-2\pi|Imx|), \end{aligned}
$$

are valid in the sector $-\pi + \delta_0 \leq \arg x \leq -\delta_0$, the const depending only on the choice of δ_0.

For large positive Reu the semistandard map can be well approximated by a system of differential equations:

$$
(3.8)\qquad u' = v\,, \quad v' = \mathrm{e}^u\,,
$$

the latter possessing an integral

$$
(3.9)\qquad \mathcal{E} = \frac{1}{2}v^2 + \mathrm{e}^u\,.
$$

The following Theorem provides us with an analytical integral for SSM near Γ_-. Given numbers $A > 1$ and $\delta_0 \in]0, \pi/2[$ define the domains: $D_A \subset \mathbb{C}$ by inequalities

$$
(3.10)\qquad \begin{gathered} -\pi + \delta_0 \leq \arg x \leq -\delta_0, \\ \mathrm{Im}x \leq -A, \end{gathered}
$$

and $\Omega_A \subset \mathbb{C}^2$ by inequalities

$$
(3.11)\qquad \begin{aligned} &x \in D_A, \\ &|y|^2 \leq \exp(-2\pi|\mathrm{Im}x|), \\ &|y|^2 < \exp(-4\pi A). \end{aligned}
$$

Define D'_A in the same way as D_A but with $2A$ and $2\delta_0$ standing for A and δ_0 respectively.

Theorem 3. *There exist a constant $A > 1$ and an analytical symplectic injective immersion $\Phi : \Omega_A \to \mathbb{C}^2$ such that*

1. *$\Phi(x, 0) = \Gamma_-(x)$ if $x \in D_A$;*
2. *Φ conjugates SSM with the shift $(x, y) \mapsto (x + 1, y)$;*
3. *if $x \in D'_A$ then the ball in $\mathbb{C}^2$ with the center $(u_-(x), v_-(x))$ and of radius $r = \exp(-\frac{4\pi}{3}|Imx|)$ is contained in $\Phi(\Omega_A)$;*
4. *$pr_2 \circ \Phi^{-1}$ has the derivatives up to the second order bounded by a constant depending only on δ_0;*
5. *$pr_1 \circ \Phi^{-1}$ has the first derivatives bounded by const$|x|^2$ with const depending only on δ_0.*

The theorems 2,3 are proven in [L2] except the assertion (5), but the latter can be easily deduced from the estimates of the cited paper. Another more stronger version of this theorem was formulated in [L1] as a conjecture, probably false.

Denote $\mathcal{E}_- = \mathrm{pr}_2 \circ \Phi^{-1}$. It follows from (3.7) and the assertion (3) of Theorem 2 that the image of $D_A^{'}$ under the map Γ_+ is contained in $\Phi(\Omega_A)$ provided A is sufficiently large. So for these values of A there is defined the superposition

$$\Theta = \mathcal{E}_- \circ \Gamma_+\big|_{D_A'} . \tag{3.12}$$

It follows from the assertion (2) of Theorem 3 that $\Theta(x)$ is periodic in the variable x with a period 1. Due to the assertion (1) of Theorem 3 and Theorem 2 the function Θ tends to zero if $\mathrm{Im}x$ tends to $-\infty$. Therefore one can expand $\Theta(x)$ in the following Fourier series converging in a lower halfplane:

$$\Theta(x) = \sum_{n=1}^{\infty} \Theta_n \mathrm{e}^{-in2\pi x} . \tag{3.13}$$

It is the first coefficient Θ_1 of this expansion that enters the formula (1.2) for the angle of separatrices intersection. Yu.B.Suris has proven that Θ_1 is purely imaginary. Numerical calculations show that the imaginary part of Θ_1 is negative and give the value $|\Theta_1| = 1118.82770595$ (see [LST]).

§4. The separatrices of the standard map in the complex domain

We will consider the standard map $SM : (X, Y) \longmapsto (X_1, Y_1)$,

$$X_1 = X + \sqrt{\varepsilon} Y_1(\mod 2\pi) , \quad Y_1 = Y + \sqrt{\varepsilon} \sin X \tag{4.1}$$

in the space $(\mathbb{C}/2\pi\mathbb{Z}) \times \mathbb{C}$. The transformation (4.1) for positive ε h as a hyperbolicfixed point $(0, 0)$. Eigenvalues of the linear part of the map at this point are $\lambda_1 = \mathrm{e}^h$, $\lambda_2 = \mathrm{e}^{-h}$, where

$$h = \ln(1 + \sqrt{\varepsilon}\sqrt{1 + \varepsilon/4} + \varepsilon/2) = \sqrt{\varepsilon} - \frac{1}{24}\varepsilon^{3/2} + \frac{3}{640}\varepsilon^{5/2} + \mathcal{O}\big(\varepsilon^{7/2}\big) . \tag{4.2}$$

In this section we will consider invariant curves of the transformation (4.1) (see §2) with the step h given by the expression (4.2).

Theorem 4. *Given a constant $C > 0$, there exists the unique invariant curve $W^u(C, t) = \big(X_-(C, t), Y_-(C, t)\big)$ of the map (4.1) possessing the following asymptotics at $t \to -\infty$:*

$$X_-(C, t) = Ce^t + o(e^t) . \tag{4.3}$$

If ε is sufficiently large, one can choose the constant C in the interval $[3, 5]$ *so that*

$$X_-(C, 0) = \pi . \tag{4.4}$$

We suppose in our further considerations that the constant C for W^u is chosen from the condition (4.4). For small ε it admits the asymptotics (8.12). Denote the invariant curve of Theorem 4 satisfying the condition (4.4) by $W^u(t) = \big(X_-(t), Y_-(t)\big)$.

We will study for small ε the behaviour of the curve W^u in the domain $\mathrm{Re}t \le h$, $|\arg t - i\frac{\pi}{2}(2k+1)|ge\delta_0$, $k \in \mathbb{Z}$. It follows from the formulas (8.1)–(8.3), that

$$X_-(t + i\pi) = -X_-(t) \quad X_-(\bar{t}) = \overline{X_-(t)} . \tag{4.5}$$

So it is sufficient to consider W^u in the domain

$$\mathcal{D} = \Big\{ t \in \mathbb{C}\colon\ 0 \le \mathrm{Im}t \le \frac{\pi}{2}\ \ \mathrm{Re}t \le h\ \ \arg t \le -\delta_0 < 0 \Big\} .$$

Theorem 5. *The followig estimate is valid in the domain $\mathcal{D}$:*

$$|X_-(t) - X_0(t)| \le const \cdot \varepsilon \left(1 + \frac{1}{|t - i\pi/2|^2}\right) , \tag{4.6}$$

where

$$X_0(t) = 4 \arctan e^t , \tag{4.7}$$

const depending only on δ_0 .

Remark. *The function (4.7) is the first component of a solution of the differential equations (2.3) which approximate the difference equations (2.2).*

For further convenience we divide the domain $\mathcal{D}$ into the following four pieces: $\mathcal{D} = \bigcup_{i=0}^{3} \mathcal{D}_i$. Here $\mathcal{D}_0$ is the halfstrip defined by the inequalities $\mathrm{Re}t \le 1$, $0 \le \mathrm{Im}t \le \pi/2$. The domain $\mathcal{D}_3$ is the intersection with $\mathcal{D}$ of the rectangle

$$\Pi_3 = \left\{ t\colon\ -\sqrt{\varepsilon}\frac{\sigma}{2\pi} \ln\frac{1}{\varepsilon} \le \mathrm{Re}t \le h\,,\ 0 \le \frac{\pi}{2} - \mathrm{Im}t \le \sqrt{\varepsilon}\frac{\sigma}{2\pi}\ln\frac{1}{\varepsilon} \right\} ,$$

where

$$\sigma = \frac{1}{8} + \frac{\delta}{2} . \tag{4.8}$$

One gets the domain $\mathcal{D}_2$ by throwing out the rectangl Π_3 from the rectangle

$$\Pi_2 = \left\{ t\colon\ -\varepsilon^{1/4} \le \mathrm{Re}t \le h\,,\ 0 \le \frac{\pi}{2} - \mathrm{Im}t \le \varepsilon^{1/4} \right\} ,$$

and the domain $\mathcal{D}_1$ by throwig out the rectangle Π_2 from the rectangle $\Pi_1 = \left\{ t\colon\ -1 \le \mathrm{Re}t \le h\,,\ 0 \le \mathrm{Im}t \le \frac{\pi}{2} \right\}$.

When deriving the formula for the separatrices splitting, we need in the asymptotics of $W^u(t)$ for $t \in \mathcal{D}_2$. In this domain the approximation by $X_0(t)$ is not valid. Instead we will take the function

$$-i \ln \frac{h^2}{2} + i u_- \left(\frac{t - i\pi/2}{h}\right) ,$$

where $u_-(x)$ is the first component of the unstable manifold of the semistandard map. It is convenient to pass to the semistandard variables by setting

$$X_- \left(i\frac{\pi}{2} + hx\right) = -i \ln \frac{h2}{2} + i\tilde{u}_-(x) . \tag{4.9}$$

Since for $hx \to 0$

$$X_0 \left(i\frac{\pi}{2} + hx\right) = -i \ln \frac{h^2}{2} - \ln\frac{x^2}{2} + i\tilde{u}_-(x) + \mathcal{O}\left(\varepsilon x^2\right) , \tag{4.10}$$

taking into account the asymptotics (3.4) and the estimate (4.6), we obtain that the function $u_-(x)$ approximates $\tilde{u}_-(x)$ with the same accuracy at the boundary which divides the domains $\mathcal{D}_1$ and $\mathcal{D}_2$:

$$|\tilde{u}_-(x) - u_-(x)| \le \mathrm{const}\sqrt{\varepsilon} . \tag{4.11}$$

Theorem 6. *The following estimate is valid in the domain $\mathcal{D}_2$:*

$$|\tilde{u}_-(x) - u_-(x)| \leq const\varepsilon^{\frac{1}{4}-\frac{\delta}{2}} . \tag{4.12}$$

where $\delta > 0$ is arbitrary, const does not depend on the choice of δ.

The theorems 4 and 5 will be proven in §.

One obtains the stable separatrix W^s from the unstable one W^u by the transformation

$$\begin{aligned} X_+(t) &= 2\pi - X_-(-t) , \\ Y_+(t) &= Y_-(-t+h) . \end{aligned} \tag{4.13}$$

Set

$$\tilde{\mathcal{D}} = \left\{ t \in \mathbb{C}\colon \ |\mathrm{Re}t| \leq h , \ |\mathrm{Im}t| \leq \frac{\pi}{2} - \sqrt{\varepsilon}\frac{\sigma}{2\pi} \ln \frac{1}{\varepsilon} \right\} . \tag{4.14}$$

Let $\tilde{\Omega}$ is a subset in $\mathbb{C}^2$ obtained by joining all rectilinear segments with ends at points $X_-(t), Y_-(t)$ and $X_+(t), Y_+(t)$, the variable t running over $\tilde{\mathcal{D}}$.

Conjecture B. *There exist analytic functions $t(X,Y)$ and $E(X,Y)$ defined in a neighbourhood of $\tilde{\Omega}$ satisfying the conditions:*

(1) *they are real at real X and Y;*
(2) $E(X_-(t), Y_-(t)) = 0 , \ t(X_-(t), Y_-(t)) = t$*;*
(3) $dt \wedge dE = dX \wedge dY$;
(4) *if (X_1, Y_1) is connected with (X, Y) by the formulas (4.1) then $E(X_1, Y_1) = E(X, Y)$ and $t(X_1, Y_1) = t(X, Y) + h$;*
(5) *if one passes to the semistandard variables and sets*

$$E(X,Y) = -\frac{1}{h^2}\tilde{\mathcal{E}}(u,v) , \tag{4.15}$$

then the derivatives of the function $\tilde{\mathcal{E}}$ with respect to the variables u and v up to the second orderd are bounded by constants depending but on δ.

I beleive that this conjecture can be proven in the same way as Theorem 3. The proof of the formula (1.2) depends essentially on the validity of Conjecture B.

§5. Deduction of the formula for the splitting of the separatrices

The separatrices $W^s(t)$ and $W^u(t)$ have an intersection at a point corresponding to the value $t = 0$ as it follows from symmetry reasons. It is at this point where we are going to calculate the value of the angle of intersection of separatrices.

Define a function ψ by the formula

$$\psi(t) = E(X_+(t), Y_+(t)) . \tag{5.1}$$

The function $\psi(t)$ possesses the following properties:

(1) it is periodic with period h due to (4) of Conjecture B;
(2) $\psi(0) = 0$ due to (2) of Conjecture B;
(3) it is defined and analytic in the strip $|\mathrm{Im}t| \leq \frac{\pi}{2} - \frac{\sigma}{2\pi}\sqrt{\varepsilon}\ln\frac{1}{\varepsilon}$ again due to Conjecture B and due to (1) of this list.

Let us calculate its derivative at the homoclinic point, *i.e.* at $t=0$:

$$\psi'(0) = X'_-(0)Y'_+(0) - Y'_-(0)X'_+(0)\,. \tag{5.2}$$

To obtain (5.2) we used the formulas

$$\begin{aligned} X'_-(t) &= \frac{\partial E}{\partial Y}\left(X_-(t), Y_-(t)\right)\,,\\ Y'_-(t) &= -\frac{\partial E}{\partial X}\left(X_-(t), Y_-(t)\right)\,, \end{aligned} \tag{5.3}$$

which one easily deduce from (1) and (2) of Conjecture B. The following formula for the angle of separatrices intersection is a consequence of (5.2):

$$\begin{aligned} \varphi \approx \sin\varphi &= -\frac{\sqrt{\varepsilon}}{\sqrt{X'_-(0)^2 + Y'_-(0)^2}\sqrt{X'_+(0)^2 + Y'_+(0)^2}} \\ &= -\frac{\sqrt{\varepsilon}}{4}\psi'(0)\left(1 + \mathcal{O}(\sqrt{\varepsilon})\right)\,. \end{aligned} \tag{5.4}$$

Here we used the estimate of Theorem 5 and the equalities (4.13):

$$\begin{aligned} &X'_-(0) = 2 + \mathcal{O}(\sqrt{\varepsilon})\,, \quad X'_+(0) = 2 + \mathcal{O}(\sqrt{\varepsilon})\,,\\ &Y'_-(0) = \mathcal{O}(\sqrt{\varepsilon})\,, Y'_+(0) = \mathcal{O}(\sqrt{\varepsilon})\,. \end{aligned} \tag{5.5}$$

The multiplier $\sqrt{\varepsilon}$ has appeared in (5.4) due to the change of variables from the initial ones (x,y) in (1.1) to (X,Y) by formulas (2.1).

To calculate $\psi(t)$ let us compare it with the function

$$\chi(t) = -\frac{1}{h^2}\Theta\left(\frac{t - i\frac{\pi}{2}}{h}\right) - \overline{\frac{1}{h^2}\Theta\left(\frac{\bar{t} - i\frac{\pi}{2}}{h}\right)}\,, \tag{5.6}$$

where $\Theta(x)$ is the function defined in \$ 3. Note that $\Theta(x)$ is defined for $\mathrm{Im}x \le -\mathrm{const} < 0$ where const does not depend on ε. It follows that for ε sufficiently small the strip where the function (5.6) is defined contains the strip

$$|\mathrm{Im}t| \le \frac{\pi}{2} - \frac{\sigma}{2\pi}\sqrt{\varepsilon}\ln\frac{1}{\varepsilon}$$

where we are going to compare $\psi(t)$ and $\chi(t)$.

Note that both the functions are real for real$t$. It is sufficient therefore to compare $\psi(t)$ and $\chi(t)$ in the rectangle

$$\Pi = \left\{t \in \mathbb{C}\colon\ |\mathrm{Re}t| \le \frac{h}{2}\,, \quad 0 \le \mathrm{Im}t \le \frac{\pi}{2} - \frac{\sigma}{2\pi}\sqrt{\varepsilon}\ln\frac{1}{\varepsilon}\right\}\,.$$

In the domain $\Pi\cap\mathcal{D}_1$ (see §4) the function $\chi(t)$ is exponentially small for small ε, and $\psi(t)$ admits, due to Conjecture B and the assertion of Theorem 5, the estimate

$$|\psi(t)| \le \mathrm{const}(|X_+(t) - X_-(t)| + |Y_+(t) - Y_-(t)|) \le \mathrm{const}\,.$$

Hence in that domain

$$|\psi(t) - \chi(t)| \le \text{const}\,.$$

Let us estimate the difference $\psi(t) - \chi(t)$ in the domain $\Pi \cap \mathcal{D}_2$. For this purpose it is convenient to pass to the semi standard variables:

$$\text{(5.8)}\qquad \begin{cases} X_-\left(i\frac{\pi}{2} + hx\right) = -i\log\frac{h^2}{2} + i\tilde{u}_-(x)\,, \\ Y_-\left(i\frac{\pi}{2} + hx\right) = \frac{i}{\sqrt{\varepsilon}}\tilde{v}_-(x)\,, \\ X_+\left(i\frac{\pi}{2} + hx\right) = -i\log\frac{h^2}{2} + i\tilde{u}_+(x)\,, \\ Y_+\left(i\frac{\pi}{2} + hx\right) = \frac{i}{\sqrt{\varepsilon}}\tilde{v}_+(x)\,. \end{cases}$$

One may neglect the second term in the right side of (5.6) in the domain $\Pi \cap \mathcal{D}_2$. It follows from (4.15) that

$$\text{(5.9)}\qquad \psi(t) = -\frac{1}{h^2}\tilde{\mathcal{E}}\Big(\tilde{u}_+(x), \tilde{v}_+(x)\Big)\,.$$

We have in the domain $\Pi \cap \mathcal{D}_2$, taking into account (3.12), (5.3) and the boundness of the second derivatives of the functions $\tilde{\mathcal{E}}$ and $\mathcal{E}_-$:

$$\text{(5.10)}\qquad \begin{aligned} &\tilde{\mathcal{E}}\left(\tilde{u}_+(x), \tilde{v}_+(x)\right) - \Theta(x) \\ &= -\tilde{v}'_-(x)[\tilde{u}_+(x) - \tilde{u}_-(x)] + \tilde{u}'_-(x)[\tilde{v}_+(x) - \tilde{v}_-(x)] \\ &+ v'_-(x)[u_+(x) - u_-(x)] - u'_-(x)[v_+(x) - v_-(x)] \\ &+ \mathcal{O}\left(|\tilde{u}_+ - \tilde{u}_-|^2 + |\tilde{v}_+ - \tilde{v}_-|^2 + |u_+ - u_-|^2 + |v_+ - v_-|^2\right)\,. \end{aligned}$$

¿From (3.13) and (4.8) it follows that in this domain

$$\text{(5.11)}\qquad |\Theta(x)| \le \text{const}e^{-\sigma\ln\frac{1}{\varepsilon}} = \text{const}\varepsilon^{\frac{1}{8}+\frac{\delta}{2}}\,,$$

and from analogous considerations

$$\text{(5.12)}\qquad |\text{pr}_1 \circ \Phi^{-1} \circ \Gamma_+(x) - x| \le= \text{const}\cdot\varepsilon^{\frac{1}{8}+\frac{\delta}{2}}\,.$$

It is not difficult to deduce from (3.11), (3.12) and Theorem 3 the following estimate:

$$\text{(5.13)}\qquad \begin{aligned} &|u_+ - u_-| + |v_+ - v_-| \le \text{const}\left(\ln\frac{1}{\varepsilon}\right)^2 \varepsilon^{\frac{1}{8}+\frac{\delta}{2}} \\ &\le \text{const}\cdot\varepsilon^{\frac{1}{8}}\,. \end{aligned}$$

Using the estimate of Theorem 6 one gets that (5.13) is true also for $\tilde{u}$, $\tilde{v}$:

$$\text{(5.14)}\qquad |\tilde{u}_+ - \tilde{u}_-| + |\tilde{v}_+ - \tilde{v}_-| \le \text{const}\cdot\varepsilon^{\frac{1}{8}}\,.$$

So the quadratic term in (5.10) has the order $\mathcal{O}\left(\varepsilon^{1/4}\right)$. Regroupping linear terms in (5.10) and using the asymptotics (3.4) yields the following estimate for the right part of (5.10) :

$$
\begin{aligned}
&\left|\tilde{\mathcal{E}}\left(\tilde{u}_+(x), \tilde{v}_+(x)\right) - \Theta(x)\right| \le \mathrm{const}\left[|\tilde{u}_+(x) - u_+(x)|\right. \\
&+ |\tilde{v}_+(x) - v_+(x)| + |\tilde{u}_+^{'}(x) - u_+^{'}(x)| + |\tilde{v}_+^{'}(x) - v_+^{'}(x)| \\
&+ |\tilde{u}_-(x) - u_-(x)| + |\tilde{v}_-(x) - v_-(x)| + |\tilde{u}_-^{'}(x) - u_-^{'}(x)| \\
&\left. + |\tilde{v}_+^{'}(x) - v_+^{'}(x)|\right] + \mathcal{O}\left(\varepsilon^{1/4}\right) \le \mathrm{const}\cdot\varepsilon^{\frac{1}{4}-\frac{\delta}{2}} .
\end{aligned}
\tag{5.15}
$$

It follows from (5.15) and (5.7) that in the strip $|\mathrm{Im}t| \le \frac{\pi}{2} - \frac{\sigma}{2\pi}\sqrt{\varepsilon}\ln\frac{1}{\varepsilon}$:

$$
|\psi(t) - \chi(t)| \le \mathrm{const}\cdot\varepsilon{-1} + \frac{1}{4} - \frac{\delta}{2} . \tag{5.16}
$$

The use of Fourier expansion results in the following lemma:

Lemma 5.1. *Let a function $f(t)$ be analytic in a strip $|Imt| \le b$, be continuous in the clousure of that strip, and be periodic with a period $a > 0$. Let*

$$
e^{-\frac{2\pi b}{a}} \le \frac{1}{2} \qquad \textit{and} \qquad \int_0^a f(t)\,dt = 0 \quad (*) .
$$

Then the following estimate is true for real values of t:

$$
|f(t)| \le 4 \max_{|\mathrm{Im}\tau| \le b} |f(\tau)| e^{-\frac{2\pi b}{a}} . \tag{5.17}
$$

Applying this lemma to the function $\psi(t) - \chi(t)$ yields on the real axis

$$
\psi(t) = \frac{2|\Theta_1|}{\varepsilon} \mathrm{e}^{-\frac{\pi^2}{\sqrt{\varepsilon}}} \sin\frac{2\pi t}{h} + |\left(\mathrm{e}^{-\frac{\pi^2}{\sqrt{\varepsilon}}}\varepsilon^{-1+\frac{1}{8}-\delta}\right) . \tag{5.18}
$$

Here we took into account that Θ_1 is purely imaginary negative. Taking depivative of (5.18) at $t = 0$ and substituting the result into (5.4), we get the formula (1.2). Note that the property (*) for the function $\psi(t)$ follows from the assertion (3) of Conjecture B, the area preserving property of the standard map, and the existence of invariant curves [3].

§6. Operator L

In this section we investigate a linear operator which is a sufficiently good approximation of nonlinear operators occurring further.

The value of the operator L at a complex–valued function of the complex variable x is given by the expression

$$
(Lf)(x) = \Delta^2 f(x) - \frac{2}{x^2} f(x) , \tag{6.1}
$$

where the second difference $\Delta^2 f(x)$ is defined by the formula

$$\Delta^2 f(x) = f(x+1) + f(x-1) - 2f(x)\,. \tag{6.2}$$

We will associate to the expression (6.1) operators in different spaces and build inverse operators to them. For simplicity of notations we will use one letter L (and L^{-1} for an inverse operator) for denoting surely different operators. First we will consider L defined on the set of all, not necessarily defined everywhere, functions of the complex variable x.

Let us formulate two lemmas on solutions of the homogeneous equation

$$\Delta^2 u - \frac{2}{x^2} u = 0\,. \tag{6.3}$$

Lemma 6.1. *General solution of the homogeneous equation (6.1) has the form*

$$u(x) = a(x) u_0(x) + b(x) x^2\,. \tag{6.4}$$

where $a(x)$ and $b(x)$ are arbitrary periodic with the period 1 functions,

$$u_0(x) = \frac{1}{2} + x + x^2 \sum_{k=1}^{\infty} \frac{1}{(x-k)^2}\,. \tag{6.5}$$

Lemma 6.2. *The following asymptotic expansion is valid uniformly in any sector $|\arg x| \geq \alpha > 0$:*

$$u_0(x) = \sum_{m=1}^{\infty} (-1)^m B_m \frac{1}{x^{2m-1}}\,, \tag{6.6}$$

where B_m are the Bernoulli numbers: $B_1 = \frac{1}{5}$, $B_2 = 130$, $B_3 = 142$, $B_4 = \frac{1}{30}$, $B_5 = \frac{5}{66}$ ...

The function $u_0(x)$ satisfies the relation

$$u_0(x) - -u_0(-x) + \frac{\pi^2 x^2}{\sin^2 \pi x}\,. \tag{6.7}$$

Let us introduce the Wronski's determinant

$$W(f,g)(x) = (\Delta f(x)) \cdot g(x) - f(x) \cdot \Delta g(x)\,, \tag{6.8}$$

where $\Delta f(x) = f(x+1) - f(x)$. If f and g are solutions of the homogeneous equation (6.3) then $W(f,g)$ is a periodic function with period 1.

Corollary.

$$W(u_0, x^2) = \frac{1}{2}\,. \tag{6.9}$$

The inverse operator to the operator L is defined in this paper by the formula

$$(6.10) \qquad \left(L^{-1}f\right)(x) = 2\sum_{k=1} \infty \left[u_0(x)(x-k)^2 - u_0(x-k)x^2\right] f(x-k)\,.$$

It is not difficult to check that the right side of (6.10) satisfies the nonhomogeneous equation $Lu = f$ provided the series converges.

Given $R > 0\,,\ 0 < \alpha < \frac{\pi}{2}\,,$ define the domain $\mathcal{D}(R,\alpha)$ in the complex plane of the variable x by the inequality

$$\alpha \le \arg(x + R/\sin\alpha) \le 2\pi - \alpha\,,$$

and denote by the symbol $\mathcal{X}_\gamma\left(\mathcal{D}(R,\alpha)\right)\,,\ \gamma \ge 0$ the Banach space of complex valued continuous functions defined in $\mathcal{D}(R,\alpha)$ and analytical in interior points of $\mathcal{D}(R,\alpha)\,,$ possessing the finite norm

$$(6.11) \qquad \|f\|_\gamma = \sup_{x\in\mathcal{D}(R,\alpha)} |x|^\gamma |f(x)|$$

Lemma 6.3. *Let $\gamma > 3\,.$ The expression (6.10) defines a bounded operator*

$$L^{-1}\colon \mathcal{X}_\gamma\left(\mathcal{D}(R,\alpha)\right) \longrightarrow \mathcal{X}_\gamma - 2\left(\mathcal{D}(R,\alpha)\right)\,,$$

the norm of which can be estimated by a constant depending only on α and $\gamma\,.$

Proof. Using (6.6) we have:

$$(6.12) \qquad \left\|L^{-1}f\right\|_{\gamma-2} \le \text{const}\cdot \sup_{x\in\mathcal{D}(R,\alpha)} \sum_{k=1}^{\infty} \Big(|x|^{\gamma-3}\cdot|x-k|^{-\gamma+2} + |x|^\gamma\cdot|x-k|^{-\gamma-1}\Big)\,\|f\|_\gamma\,.$$

If $\beta > 1$ then

$$(6.13) \qquad \sum_{k=1}^{\infty} |x-k|^{-\beta} = \mathcal{O}\left(x^{-\beta+1}\right)\,.$$

One gets the assertion of Lemma by estimating the right side of (6.12) with use of (6.13). $\square$

Further we consider a Banach space $\mathcal{X}_\gamma(\ell)$ where ℓ is a segment in the complex plane parallel to the real axis. That space consists of all bounded complex–valued functions defined on ℓ and is endowed with the norm

$$(6.14) \qquad \|f\|_\gamma = \sup_{x\in\ell} |x|^\gamma |f(x)|\,.$$

Define the operator $L^{-1}\colon \mathcal{X}_{\gamma_1}(\ell) \longrightarrow \mathcal{X}_{\gamma_2}(\ell)$ by the formula (6.10), all functions being prolonged by zero onto the "negative continuation" of the segment $\ell\,,$ *i.e.* onto the ray parallel to the real axis and running to the left from the left end of the segment $\ell\,.$ Evidently all such operators are bounded provided the segment ℓ does not contain points of the ray $[1\,,+\infty[\,.$ The following lemma is true, one can prove it in the same way as Lemma 6.3.

Lemma 6.4. *If $\gamma > 3$ and the segment ℓ does not intersect with the ray $[1/2\,, +\infty[$ then the norm of the operator*

$$L^{-1}\colon \mathcal{X}_\gamma(\ell) \longrightarrow \mathcal{X}_{\gamma-2}(\ell)$$

is bounded by a constant depending only on γ and $\inf_{x\in\ell}|\arg x|$.

Lemma 6.5. *Let $1 \leq \gamma \leq 2$ and e segment ℓ is contained wholly in the domain $\mathcal{D}\setminus\mathcal{D}_0$ (see the definition in §4), then the following estimate is valid for the norm of the operator $L^{-1}\colon \mathcal{X}_\gamma(\ell) \longrightarrow \mathcal{X}_\gamma(\ell)$:*

$$\|L-1\| \leq const\cdot\varepsilon^{-1}\,, \tag{6.15}$$

where const depends only on γ and α. This operator is Volterra in the following sense. Given a function $\Phi\in\mathcal{X}_0(\ell)$ the operator $I-\varepsilon L^{-1}\Phi\cdot$ has a bounded inverse operator in the space $\mathcal{X}_\gamma(\ell)$, and its norm depends only of the norm of the function Φ γ and α. Here I is the identity operator, $\Phi\cdot$ is the operator of multiplication by Φ.

Proof. In the same manner as in the proof of Lemma 6.3, we have

$$\begin{aligned}\left\|L^{-1}f\right\|_\gamma \leq \mathrm{const}\cdot\sup_{x\in\ell}\sum_{1\leq k\leq|\mathrm{Re}x+\frac{1}{\sqrt{\varepsilon}}}&\big(|x|^{\gamma-1}\cdot|x-k|^{-\gamma+2}\\ &+|x|^\gamma+2\cdot|x-k|^{-\gamma-1}\big)\,\|f\|_\gamma\,.\end{aligned} \tag{6.16}$$

Let us introduce variables

$$t = 1+\sqrt{\varepsilon}\mathrm{Re}x\,,\qquad \tilde{t} = 1+\sqrt{\varepsilon}(\mathrm{Re}x-k)\,. \tag{6.17}$$

Then the sum in the right side of (6.16) can be evaluated by the integral

$$\frac{1}{\varepsilon}\int_0^t\Big[(1-t+\frac{\pi}{2})^{\gamma-1}(1-\tilde{t})^{2-\gamma}+(1-t+\frac{\pi}{2})^{2+\gamma}(1-\tilde{t})^{-1-\gamma}\Big]\,d\tilde{t}\,, \tag{6.18}$$

which implies immediately the first assertion of Lemma. Since the integrand in (6.18) is a bounded function,one obtains, applying usual Volterra estimates of the terms of the sequence $f_0 = f$, $f_{n+1} = \varepsilon L^{-1}\Phi\cdot f_n$, the second assertion of Lemma. □

§7. Proof of Theorem 1

It is convenient to pass from a system of difference equation of the first order (3.1) to one equation of the second order

$$\Delta^2 u(x) = \mathrm{e}^{u(x)}\,. \tag{7.1}$$

It is sufficient to prove the existence an uniqueness of an analytical solution of the equation (7.1) which admits for $\mathrm{Re}x \leq -R < 0$ the form

$$u(x) = -\ln\frac{x^2}{2}+\xi(x)\,, \tag{7.2}$$

where

$$\xi(x) = \mathcal{O}\left(x^2\right) . \tag{7.3}$$

Note that, if we find the existence and uniqueness of such a function $\xi(x)$ for $\mathrm{Re}x \le -R < 0$, then $u(x)$ can be restored by means of (7.1) on the whole complex plane.

One can rewrite the equation (7.1) in terms of the function $\xi(x)$ in the form

$$\Delta^2\xi(x) - \frac{2}{x^2}\xi(x) = \Phi(x, \xi(x)) , \tag{7.4}$$

where

$$\Phi(x,\xi) = -2\ln\left(1 - \frac{1}{x^2}\right) - \frac{2}{x^2} + \frac{2}{x^2}\left(\mathrm{e}^{\xi} - 1 - \xi\right) . \tag{7.5}$$

The equation (7.4) together with condition (7.3) posed on the behaviour of $\xi(x)$ as $x \to \infty$ is equivalent to one equation

$$\xi = F(\xi) , \tag{7.6}$$

where

$$F(\xi) = L^{-1}\Phi(\xi) , \qquad \Phi(\xi)(x) = \Phi(x, \xi(x)) . \tag{7.7}$$

We will consider the equation (7.6) in $\mathcal{X}_2\left(\mathcal{D}(R,\alpha)\right)$ (see §6). Denote by B the unit closed ball with the center at 0 in that space. If R in the definition of the domain $\mathcal{D}(R,\alpha)$ is sufficiently large, ξ , ξ_1 , $\xi_2 \in B$, then the following estimates are true:

$$\|\Phi(\xi)\|_4 \le \mathrm{const}\frac{\|\xi\|_2}{R^2} + \frac{2}{3} , \tag{7.8}$$

$$\|\Phi(\xi_1) - \Phi(\xi_2)\|_4 \le \frac{\mathrm{const}}{R^2}\|\xi_1 - \xi_2\|_2 . \tag{7.9}$$

The estimate (7.8) and Lemma 6.3 imply that, for sufficiently large R, the sequence ξ_n , $n = 0, 1, 2, \ldots$, $\xi_0 = 0$, $\xi_{n+1} = F(\xi_n)$, is contained in B . The estimate (7.9) shows its convergence in B for sufficiently large R .

One can easily prove the assertion about an asymptotic expansion following the same lines. Fix an arbitrary natural number N and introduce a function η by setting

$$u(x) = -\ln\frac{x^2}{2} + \sum_{k=0}^{N}\frac{a_k}{x^{2k}} + \eta(x) . \tag{7.10}$$

We have an equation for defining $\eta(x)$:

(7.11)

$$L\eta(x) = 2\ln\left(1 - \frac{1}{x^2}\right) + \sum_{k=1}^{N+1}\frac{2}{k}\cdot\frac{1}{x^{2k}}$$
$$+ \frac{2}{x^2}\left[\mathrm{e}^{\sum_{k=1}^{N} a_k x^{-2k} + \eta(x)} - 1 - \eta(x) - \sum_{k=1}^{N+1}\frac{2}{k}\cdot\frac{1}{x^{2k}} + \frac{x^2}{2}\sum_{k=1}^{N+1} a_k\Delta^2\frac{1}{x^{2k}}\right]$$

We choose the numbers a_k so that the expression in brackets in the right side of (7.11) admits the estimate

$$\mathcal{O}\left(\eta^2\right) + \mathcal{O}\left(\frac{1}{x^2}\eta\right) + \mathcal{O}\left(\frac{1}{x^{2N+2}}\right) . \tag{7.12}$$

Then one has to reverse the operator L and to resolve the obtained equation in the space $\mathcal{X}_{2N+2}\left(\mathcal{D}(R,\alpha)\right)$.

The uniqueness of a solution of the equation (7.6) with the condition (7.3) follows also from the estimates (7.8) and (7.9). The first one shows that, for R sufficiently large, a solution belongs to B . The second one that the nonlinear operator F defined by the formula (7.7) is a contracting operator.

§8. Proof of Theorem 4

Fix a constant $C > 0$. The system (2.7) together with the condition (4.3) is equivalent to one equation

$$X_-(t) = C\mathrm{e}^t + F\big(X_-\big)(t) , \tag{8.1}$$

where

$$F\big(X\big)(t) = \frac{\sqrt{\varepsilon}}{\sqrt{1+\varepsilon/4}} \sum_{k=1}^{\infty} \Phi\big(X(t-kh)\big) \sinh kh , \tag{8.2}$$

$$\Phi(x) = \sin x - x . \tag{8.3}$$

Consider the equation (8.1) in the ball of radius 10 with center at the origin in Banach space of functions analytic in a halfplane $\mathrm{Re}t < -R < 0$, continuous in the closure of that halfplane endowed with a norm

$$\|X\| = \sup_{\mathrm{Re}t<R} \left|\mathrm{e}^{-t}X(t)\right| . \tag{8.4}$$

The nonlinear operator F in the said ball admits obvious estimates

$$\|F(X)\| \le \mathrm{const}\cdot \mathrm{e}^{-2R}\|X\| + 5 , \tag{8.5}$$

$$\|F(X_1) - F(X_2)\| \le \mathrm{const}\cdot \mathrm{e}^{-R}\|X_1 - X_2\| . \tag{8.6}$$

It follows from the estimates (8.5) and (8.6) that the equation (8.1) and consequently the system (2.7) with the condition (4.3) has unique solution for any C in the mentioned space provided R is sufficiently large. The solution can be prolonged to the whole complex plane of the variable t uniquely by means of (2.7). Denote the obtained solution by the symbol $X_-(C,t)$.

Note that in view of proved uniqueness the following equality is true

$$X_-(C_1,t) = X_-\left(C_2, t + \ln C_1/C_2\right) \tag{8.7}$$

for any positive C_i , $i = 1 , 2$.

Consider $X_-(4,t)$. From Lemma 9.1 which will be proven in §9 it follows that for $\mathrm{Re} \le 1$, $|\mathrm{Im}t| \le 1$

$$|X_-(4.t) - 4\arctan \mathrm{e}^t| \le \text{const} \cdot \varepsilon \,. \tag{8.8}$$

Therefore for real $t \le 1/2$

$$\left| \frac{\partial}{\partial t} X_-(4,t) - \frac{2}{cosht} \right| \le \text{const} \cdot \varepsilon \,. \tag{8.9}$$

The (8.7) and (8.8) yield the estimate

$$\left| \frac{\partial}{\partial C} X_-(C,t) - \frac{2}{cosht + \ln \frac{C}{4}} \right| \le \text{const} \cdot \varepsilon \,, \tag{8.10}$$

which is valid uniformly for the values $C \in [3,5]$, $t \le \frac{1}{2} - \ln \frac{5}{4} \approx 0.277$. It follows from (8.8) and (8.10) that, for ε small enough, the equation

$$X_-(C,0) = \pi \tag{8.11}$$

has the unique solution in the interval [3.5], the estimate

$$C = 4 + \mathcal{O}(\varepsilon) \,. \tag{8.12}$$

being valid for that solution.

§9. Proof of Theorem 5

First we prove the estimate (4.6) in the domain $\mathcal{D}_0$. It is of use to prove it in a wider domain $\tilde{\mathcal{D}}_0$ which is obtained by adding to $\mathcal{D}_0$ rectangles

$$|\mathrm{Re}t| \le 1\,, \; 0 \le Imt \le 1 \qquad \text{and} \qquad -1 \le Ret \le -\frac{1}{2}\,, \; 0 \le \mathrm{Im}t \le \frac{\pi}{2}\,.$$

Lemma 9.1. *The following estimate is valid in the domain $\tilde{\mathcal{D}}_0$*

$$|X_-(4,t) - X_0(t)| \le const \cdot \varepsilon \,, \tag{9.1}$$

where const does not depend on ε. Here $X_0(t)$ is defined by (4.7), $X_-(4,t)$ was defined in §8.

The estimate (4.6) in the domain $\mathcal{D}_0$ follows from (9.1), (8.7), and (8.12).

Proof of Lemma 9.1. Set

$$Z(t) = X_-(4,t) - X_0(t) \,. \tag{9.2}$$

Since the function $X_0(t) = 4\arctan \mathrm{e}^t$ satisfies the integral equation

$$X_)(t) = 4\mathrm{e}^t + \int_{-\infty}^{t} \Phi\bigl(X_0(\tau)\bigr) \sinh(t-\tau)\, d\tau \tag{9.3}$$

where the function $\Phi(x)$ is defined by the formula (8.3), one may rewrite the equation for $Z(t)$ in the form

$$Z = F(X_0 + Z) - F(X_0) + f\,. \tag{9.4}$$

In (9.4)

$$f(t) = \frac{\sqrt{\varepsilon}}{\sqrt{1+\varepsilon/4}} \sum_{k=1}^{\infty} \Phi\big(X_0(t-kh)\big)\,\sinh kh - \int_0^{\infty} \Phi\big(X_0(t-\kappa)\big)\,\sinh\kappa\,d\kappa\,, \tag{9.5}$$

and the operator F is defined by the formula (8.2).

The following estimate is true

$$|f(t)| \le \mathrm{const}\cdot\varepsilon \mathrm{e}^{\mathrm{Re}t}\,. \tag{9.6}$$

Indeed

$$\begin{aligned} |f(t)| &\le \mathrm{const}\cdot\varepsilon \mathrm{e}^{\mathrm{Re}t} \\ &+ \sum_{k=0}^{\infty}\sqrt{\varepsilon}\int_{kh}^{(k+1)h} \big|\Phi\big(X_0(t-\kappa)\big)\sinh\kappa - \Phi\big(X_0(t-(k+1)h)\big)\sinh(k+1)h\big|\,d\kappa \\ &\le \mathrm{const}\cdot\varepsilon \mathrm{e}^{\mathrm{Re}t} + \sum_{k=0}^{\infty}\sqrt{\varepsilon}\max_{\kappa\in[kh,(k+1)h]}|\psi''(\kappa)|h^2\,. \end{aligned} \tag{9.7}$$

In the right side 0f (9.7) the first term arose because of the change the multiplier $\frac{\sqrt{\varepsilon}}{\sqrt{1+\varepsilon/4}}$ by $\sqrt{\varepsilon}$, a function $\psi(\kappa)$ is defined by the formula

$$\psi(\kappa) = \Phi\big(X_0(t-\kappa)\big)\sinh\kappa\,.$$

It is evident that

$$\max_{\kappa\in[kh,(k+1)h]}|\psi''(\kappa)| \le \mathrm{const}\cdot \mathrm{e}^{\mathrm{Re}t}\mathrm{e}^{-kh}\,. \tag{9.8}$$

The estimate (9.6) follows from (9.7) and (9.8).

Let us solve the equation (9.4) by iterations:$Z_0 = 0\,,\ Z_{n+1} = F(X_0 + Z_n) - F(X_0) + f$. The estimate analogous to (8.6) enables us to prove by induction that in the domain $\tilde{\mathcal{D}}_0$ the following inequality is valid

$$|Z_{n+1}(t) - Z_n(t)| \le \varepsilon\cdot\frac{(\mathrm{const})^{n+1}}{n!}\cdot \mathrm{e}^{nRet}\,, \tag{9.9}$$

from which follows the assertion of the Lemma. □

It follows from (8.7) and (8.12) that

$$X_-(t) = X_-\big(4, t + \ln\frac{C}{4}\big) = X_-\big(4, t + \mathcal{O}(\varepsilon)\big)\,. \tag{9.10}$$

Hence in the domain $\mathcal{D}_0$

$$|X_(t) - X_-(4,t)| \le \text{const} \cdot \varepsilon \,. \tag{9.11}$$

When deriving (9.11) we took into account that in the domain Re$t \le -3/4$ the derivatives of the function $X_-(4,t)$ are bounded by constants not depending on ε. The latter follows from the estimate (9.1). The desired estimate (4.6) in the domain $\mathcal{D}_0$ follows from (9.1) and (9.11). Moreover we obtain that in the domain Re $\le -3/4$ the following inequality holds:

$$|X_0^{'}(t) - X_-^{'}(t)| \le \text{const} \cdot \varepsilon \,. \tag{9.12}$$

Let us pass to the proof of the estimate (4.6) in the domain $\mathcal{D} \setminus \mathcal{D}_0$. Set

$$\tilde{Z}(t) = X_-(t) - X_0(t) \,. \tag{9.13}$$

This function obeys the difference equation

$$\Delta^2 \tilde{Z} = \varepsilon \sin\bigl(X_0 + \tilde{Z}\bigr) - \Delta^2 X_0 \,. \tag{9.14}$$

We will consider (9.14) in the semistandard time x connected with t by the formula (2.9) where $a = i\frac{\pi}{2}$. Subtracting the term $2x - 2\tilde{Z}$ from the both sides of (9.14), one can represent it as

$$L\tilde{Z} = \tilde{F}\bigl(\tilde{Z}\bigr) \,, \tag{9.15}$$

where

$$
\begin{aligned}
\text{(9.16)}\quad \tilde{F}\bigl(\tilde{Z}\bigr) &= \varepsilon \sin\bigl(X_0 + \tilde{Z}\bigr) - \Delta^2 X_0 - \frac{2}{x^2}\tilde{Z} \\
\text{(A)}\quad &= \varepsilon \sin X_0 - \varepsilon X_0^{''} \\
\text{(B)}\quad &= \varepsilon X_0^{''} - \Delta^2 X_0 \\
\text{(C)}\quad &+ \left(\varepsilon \cos X_0 - \frac{2}{x^2}\right)\tilde{Z} \\
\text{(D)}\quad &+ \varepsilon \int_0^1 \sin\bigl(X_0 + \tau\tilde{Z}\bigr)(1-\tau)\tilde{Z}^2 \, d\tau \,.
\end{aligned}
$$

The term (A) in the right side of (9.16) is zero. Let us inverse the operator L in (9.15) and represent $\tilde{Z}$ in the form

$$\tilde{Z} = \tilde{Z}_{in} + L^{-1}\tilde{F}\bigl(\tilde{Z}\bigr) \,, \tag{9.17}$$

where $\tilde{F}\bigl(\tilde{Z}\bigr)$ in the right side is prolonged by zero to the left from the domain $\mathcal{D}_1$, and 9see Lemma 6.1)

$$\tilde{Z}_{in}(x) = a(x)u_0(x) + b(x)x^2 \,. \tag{9.18}$$

Periodic with period 1 functions $a(x)$ and $b(x)$ (not necessarily continuous) are defined by formulas

$$
\begin{aligned}
a(x) &= 2\begin{vmatrix} \tilde{Z}(x_{in}) & x_{in}^2 \\ \tilde{Z}(x_{in}-1) & (x_{in}-1)^2 \end{vmatrix}\,, \\
b(x) &= 2\begin{vmatrix} u_0(x_{in}) & \tilde{Z}(x_{in}) \\ u_0(x_{in}-1) & \tilde{Z}(x_{in}-1) \end{vmatrix}\,.
\end{aligned} \tag{9.19}
$$

Here

$$
x_{in} = x - \left[\mathrm{Re}x + \frac{1}{\sqrt{\varepsilon}}\right] \tag{9.20}
$$

are periodic functions of the variable x. The symbol $[\cdot]$ denotes the integer part of a number. Such a choice of the initial term in (9.17) is motivated by the demand for $\tilde{Z}$ to coincide with the previously obtained value (9.13) on contacting parts of $\mathcal{D}_0$ and $\mathcal{D}_1$.

Fix $\mathrm{Im}x = \lambda$ and consider the equation (9.17) on the segment $\ell = [-\frac{1}{\sqrt{\varepsilon}} + i\lambda, h + i\lambda] \cap \mathcal{D}$. It is clear that $\ell \subset \mathcal{D} \setminus \mathcal{D}_0$. The norms in the space $\mathcal{X}_4(\ell)$ (see §6) of the terms in (9.16) admit the following estimates provided $|\mathcal{Z}(x)| \le 1$:

$$
\|B\|_4 \le \mathrm{const}\,, \tag{9.21}
$$

$$
\|D\|_4 \le \mathrm{const} \cdot \sqrt{\varepsilon}\|\tilde{Z}\|_2^2\,. \tag{9.22}
$$

One obtains the estimate (9.21) by application of the Taylor formula to $\Delta^2 X_0$ with the remainder term containing the fourth derivative and the use of an estimate of $\frac{2}{sinh\tau}$ in the neighbourhood of $\tau = 0$. It occurs that $|B(x)| \le \mathrm{const}|x|^{-4}$. When estimating (D) one needs to take into account that in the domain in question $|\sin X_0| \le \mathrm{const}\frac{1}{\varepsilon|x|^2}$.

Let us consider the multiplier before $\tilde{Z}$ in the term (C):

$$
\begin{aligned}
\varepsilon\Phi(x) &= \varepsilon\cos X_0(x) - \frac{2}{x^2} + \varepsilon\left(1 + \frac{2}{\sinh^2\sqrt{\varepsilon}x - \frac{2}{\varepsilon x^2}}\right) \\
&= \varepsilon\left[1 + \frac{2(\varepsilon x^2 - \sinh^2\sqrt{\varepsilon}x)}{\varepsilon x^2\sinh^2\sqrt{\varepsilon}x}\right] = \varepsilon\mathcal{O}(1)\,.
\end{aligned} \tag{9.23}
$$

Let us estimate the term $\tilde{Z}_{in}(x)$ in (9.17) defined by formulas (9.18)–(9.20). By use of (9.12) we obtain that in the domain $\mathrm{Re}t \le -3/4$

$$
\left|\tilde{Z}(x) - \tilde{Z}(x-1)\right| \le \sqrt{\varepsilon}\left|X_0'(\tilde{t}) - X_-'(\tilde{t})\right| = \mathcal{O}(\varepsilon^{3/2})\,. \tag{9.24}
$$

Using (9.24), (9.18), (9.19) and the asymptotics (6.6) for the function $u_0(x)$, one obtains consecutively estimates

$$
\begin{aligned}
|a(x)| &= \mathrm{const}\left(|x_{in}|^2\left|\tilde{Z}(x_{in}) - \tilde{Z}(x_{in}-1)\right| + |x_{in}|\left|\tilde{Z}(x_{in})\right|\right) \le \mathrm{const}\cdot\varepsilon^{1/2}\,, \\
|b(x)| &= \frac{\mathrm{const}}{|x_{in}|}\left|\tilde{Z}(x_{in}) - \tilde{Z}(x_{in}-1)\right| + \frac{\mathrm{const}}{|x_{in}|^2}\left|\tilde{Z}(x_{in})\right| \le \mathrm{const}\cdot\varepsilon^2\,,
\end{aligned} \tag{9.25}
$$

$$(9.26)\qquad \left|\tilde Z_{in}(x)\right| \le \text{const}\left(\varepsilon^{1/2}\frac{1}{|x|}+\varepsilon^2|x|^2\right).$$

Consider the segments $\ell_n = \left[-\frac{1}{\sqrt{\varepsilon}}+i\lambda\,, -\frac{1}{\sqrt{\varepsilon}}+nh+i\lambda\right]$, $n=1,2,3,\ldots$, which constitute a part of the segment ℓ. By calculating the norms of $\tilde Z_{in}$ in the spaces $\mathcal{X}_2(\ell_n)$, the norms there being denoted $\|\cdot\|_{2,n}$, we have

$$(9.27)\qquad \|\tilde Z_{in}\|_{2,n} \le \text{const}\,,$$

where const does not depend on ε and n. Reversing the operator $I-\varepsilon L^{-1}\Phi$ in $\mathcal{X}_2(\ell)$ and removing by virtue of that the term (C), we obtain the following equation:

$$(9.28)\qquad \tilde Z = \left(I-\varepsilon L^{-1}\Phi\right)^{-1}\tilde Z_{in} + \left(I-\varepsilon L^{-1}\Phi\right)^{-1}L^{-1}(B+D)\,.$$

It follows from the estimate (9.23) and Lemma 6.5 that $\left(I-\varepsilon L^{-1}\Phi\right)^{-1}$ is a bounded operator from the space $\mathcal{X}_2(\ell_{n-1})$ to the space $\mathcal{X}_2(\ell_n)$ with the norm bounded by a constant independent of ε and n.

The (9.28), (9.21), and (9.22) yield the inequality

$$(9.29)\qquad \|\tilde Z\|_{2,n} \le \text{const} + \text{const}\cdot\varepsilon\|\tilde Z\|^2_{2,n-1}\,,$$

where const does not depend on ε and n.

Lemma 9.2. *Let y_n, $n=1,2,3\ldots$, be a sequence of positive numbers satisfying the inequality*

$$y_{n+1} \le a + by_n^2\,,$$

where $a>0$, $b>0$ $ab<1/4$, and

$$y_1 < y^* = \frac{1-\sqrt{1-4ab}}{2b} = a+\mathcal{O}\big(ab^2\big)\,.$$

Then $y_n<y^*$ for all n.

We can apply Lemma 9.2 to the sequence $\|\tilde Z\|_{2,n}$. This results in the estimate

$$(9.30)\qquad \|\tilde Z\|_2 \le \text{const}\,,$$

which is equivalent to the desired estimate (4.6) in the domain $\mathcal{D}\setminus\mathcal{D}_0$.

§10. Proof of Theorem 6

Denoting

$$(10.1)\qquad Z(x) = \tilde u_(x) - u_(x)$$

and using (2.7), (3.1), and (4.9), let us rewrite an equation for Z in the form

$$(10.2)\qquad LZ = \mathrm{e}^{u_-+Z} - \mathrm{e}^{u_-} - \frac{2}{x^2}Z - \frac{\varepsilon h^2}{4}\mathrm{e}^{-u_--Z}\,.$$

Consider the equation (10.2) in the space $\mathcal{X}_{1+\delta'}(\ell)$ where $\delta^{'} = \frac{2}{3}\delta$, $\ell = [i\lambda - \varepsilon^{1/4}, i\lambda + \frac{\sigma}{2\pi}\ln\frac{1}{\varepsilon}]$ is a segment in the plane of the variable x which lies wholly in $\mathcal{D}_2$.

Let us evaluate the norms in the right side of (10.2) in the space $\mathcal{X}_{3+\delta'}(\ell)$ provided $\|Z\|_{1+\delta'} \le 1$. We have

(10.3)
$$\left\| e^{u_-+Z} - e^{u_-} - \frac{2}{x^2}Z \right\|_{3+\delta'} \le \text{const}\left(\left\| \frac{1}{x^2}\big|Z(x)\big|^2 \right\|_{3+\delta'} + \left\| \frac{1}{x^4}\big|Z(x)\big| \right\|_{3+\delta'} \right),$$

$$(10.4)\qquad \left\| \frac{1}{x^2}\big|Z(x)\big|^2 \right\|_{3+\delta'} = \max_{x\in\ell}|x|^{1+\delta'}\big|Z(x)\big|^2 \le \text{const}\max_{x\in\ell}|x|^{-1-\delta'}\big\|Z\big\|^2_{1+\delta'} \le \frac{\text{const}}{\left(\ln\frac{1}{\varepsilon}\right)^{1+\delta'}}\big\|Z\big\|^2_{1+\delta'},$$

$$(10.5)\qquad \left\| \frac{1}{x^4}\big|Z(x)\big| \right\|_{3+\delta'} = \max_{x\in\ell}|x|^{-1+\delta'}\big|Z(x)\big|^2 \le \frac{\text{const}}{\ln^2\frac{1}{\varepsilon}}\big\|Z\big\|_{1+\delta'},$$

(10.6)
$$\left\| \frac{\varepsilon h^2}{4}e^{-u_--Z} \right\|_{3+\delta'} \le \text{const}\|x^2\varepsilon^2\|_{3+\delta'} \le \text{const}\max_{x\in\ell}|x|^{5+\delta'}\varepsilon^2 \le \text{const}\varepsilon^{2-(5+\delta'/4}.$$

Applying the inverse operator L^{-1} in (10.2) we obtain an equation for Z where $Z(x)$ is expressed in terms of $Z(x-k)$, $k = 1, 2, 3\ldots$:

$$(10.7)\qquad Z = Z_{in} + L^{-1}\big(\textit{evaluated terms}\big),$$

where Z_{in} is expressed by a formula analogous to (9.18). Instead of (9.24) we have, using (4.11)(the latter can be prove in a more wider domain with diameter of order $|x_{in}|$ in the plane of the variable x),

$$(10.8)\qquad \left|\tilde{Z}(x_{in}) - \tilde{Z}(x_{in}-1)\right| \le \text{const}|u^{'}_{-}(\tilde{x}) - \tilde{u}^{'}_{-}(\tilde{x})| \le \frac{const\cdot\sqrt{\varepsilon}}{|x_{in}|}.$$

Instead of (9.25) we have

(10.9)
$$|a(x)| = \text{const}\left(|x_{in}|^2\big|Z(x_{in}) - Z(x_{in}-1)\big| + |x_{in}|\big|Z(x_{in})\big|\right) \le \text{const}\cdot\varepsilon^{1/4},$$
$$|b(x)| = \frac{\text{const}}{|x_{in}|}\big|Z(x_{in}) - Z(x_{in}-1)\big| + \frac{\text{const}}{|x_{in}|^2}\big|Z(x_{in})\big| \le \text{const}\cdot\varepsilon,$$

Finally for Z_{in} we obtain the estimates

$$(10.10)\qquad |Z_{in}| \le \text{const}\left(\varepsilon^{\frac{1}{4}}\frac{1}{|x|} + \varepsilon|x|^2\right),$$

$$
(10.11) \quad \|Z_{in}\|_{1+\delta'} \leq \max_{x\in\ell} \mathrm{const}\left(\varepsilon^{1/4}|x|^{\delta'} + \varepsilon|x|^{3+\delta'}\right)
$$
$$
\leq \mathrm{const}\left(\varepsilon^{\frac14-\frac14\delta'} + \varepsilon^{1-\frac34-\frac34\delta'}\right) = \mathcal{O}\left(\varepsilon^{\frac14-\frac{\delta}{2}}\right).
$$

In the same manner as in §9 we consider the equation (10.7) on the sequence of segments

$$
\ell_n = \left[-\varepsilon^{-1/4} + i\lambda\,, -\varepsilon^{-1/4} + n + i\lambda\right].
$$

Denoting by the symbol $\|\cdot\|_{1+\delta',n}$ the norms in the spaces $\mathcal{X}_{1+\delta'}(\ell_n)$, we obtain from (10.7), (10.3), (10.4), (10.5), (10.6), (10.11), using Lemma 6.4 :

$$
(10.12) \quad \|Z\|_{1+\delta',n+1} \leq \mathrm{const}\cdot\varepsilon^{\frac14-\frac{\delta}{2}} + \frac{\mathrm{const}}{\ln^2\frac{1}{\varepsilon}}\|Z\|_{1+\delta',n} + \frac{\mathrm{const}}{\left(\ln\frac{1}{\varepsilon}\right)^{1+\delta'}}\|Z\|^2_{1+\delta',n}.
$$

Applying to the sequence $\|Z\|_{1+\delta',n}$ a lemma analogous to the Lemma 9.2 yields the desired estimate for Z.

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