Some properties of k-step exclusion processes

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Abstract

We introduce k-step exclusion processes as generalizations of the simple exclusion process. We state their main equilibrium properties when the underlying stochastic matrix corresponds to a random walk or is positive recurrent and reversible. Finally we prove laws of large numbers for tagged and second class particles.

Keywords: Interacting particle systems; Invariant measures; Tagged particle; second class particle

1 Introduction

Let $\mathbf{S}$ be a finite or countable set, and $p(.,.)$ a transition matrix of a discrete Markov chain on $\mathbf{S}$.

For $k \in \mathbf{N}$, a k-step exclusion process is a natural generalization of simple exclusion process. It is a continuous time Markov process on state space $\mathbf{X} = \{0,1\}^{\mathbf{S}}$, the set of configurations of particles distributed on $\mathbf{S}$ with at most one particle per site.

Informally this process can be describe in the following way: on each site of $\mathbf{S}$ we have one clock that rings (independently of others clocks) at random exponential times with parameter 1. If a particle is present at site x when the clock associated to this site rings, then the particle occupies the first vacant site encountered in the sequence $(X_n)_{1\leq n\leq k}$, where $\{X_n\}_n$ is the Markov chain with probability transition $p(.,.)$ starting at x (site x itself is consider vacant for the attempt, *i.e.*, if the chain returns to x before encountering an empty site then the particle stays at x). If no empty site is encountered during these k attempts

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then the movement is cancelled and the particle stays at x (waiting for the next of its clock ring).

When $k = 1$ the process is exactly the simple exclusion process introduced by Spitzer (1970) [20]. This process has been intensively studied since then; we refer the reader to Liggett's book (1985) [17] Chapter VIII to an overview of its principal properties.

When k goes to infinity it is plausible that these processes will approach the Long Range Exclusion Process (see Guiol (1995)[8] or Andjel & al (1998)[3]). In the Long Range Exclusion Process a particle that moves, follows the chain X_n until it finds an empty site, such that particles can travel long distance in short time. This process was also introduced by Spitzer (1970) in [20] and studied systematically by Liggett (1980) in [16] and in a lesser proportion by Zheng (1988)[21] and Guiol (1997)[9]. A version of the k-step exclusion processes was used by Liggett in [16] to approximate the long range exclusion process, but in that case particles disappear if they do not encounter a vacant site in their attempt to move.

Recent developments in the studies of long range processes such as the Long Range Exclusion Process [9, 3], the Hammersley Process (see Aldous & Diaconis (1995)[1]), Self-organizing Particle Systems (see Carlson & al (1993)[5]), the Toom model (see Lebowitz & al (1996)[14]) shows the existence of a real interest to have (Feller) approximations for these processes.

The aim of this paper is to give an overview of the principal properties of k-step exclusion processes.

After a short review in section 2 of definitions and properties, we describe in section 3 some invariant measures for the k-step exclusion process.

We denote by $\mathcal{I}_k$ the set of invariant measure of the k-step Exclusion Process.

Let $\rho : \mathbf{S} \to [0, 1]$, we denote by ν_ρ the product measure such that

$$\nu_\rho\{\eta(x_i) = 1, 1 \leq i \leq n\} = \prod_{i=1}^{n} \rho(x_i)$$

for all $n \in \mathbf{N}$, $x_i \in \mathbf{S}$, $1 \leq i \leq n$.

Suppose that $p(.,.)$ is reversible. Then we have

Theorem 1.1 *Let $p(.,.)$ be reversible with π i.e.*

$$\pi(x)p(x, y) = \pi(y)p(y, x) \textit{ for all } x, y \in \mathbf{S},$$

and

$$\rho(x) = \frac{\pi(x)}{1 + \pi(x)}. \tag{1}$$

Then for all $k \geq 1$

$$\nu_\rho \in \mathcal{I}_k.$$

This result is absolutely similar to the one in the simple exclusion case (see Liggett (1985) Chapter VIII, p. 380). We recall that for this last process the double stochasticity of the transition matrix $p(.,.)$ was also a condition to show that Bernoulli product measures with constant densities were invariant measures too. At the beginning of section 3 we give a counter example for which this result is no more true for 2-step exclusion process.

Nevertheless, for a random walk on $\mathbf{Z}^d$ we have the (familiar) results.

Theorem 1.2 *Suppose $p(x,y) = p(0, y-x)$ for all $x, y \in \mathbf{Z}^d$ then*

$$\nu_\alpha \in \mathcal{I}_k,$$

for every constant $\alpha \in [0,1]$.

Furthermore in this case Bernoulli measures with constant densities are the only (ergodic) invariant measures that are translation invariant. We denote by $\mathcal{S}$ the set of translation invariant measure, $(\mathcal{S})_e$ the set of ergodic elements of $\mathcal{S}$ and $(\mathcal{I}_k)_e$ the set of extremal elements of $\mathcal{I}_k$.

Theorem 1.3 *Suppose $p(x,y) = p(0, y-x)$ for all $x, y \in \mathbf{Z}^d$ and $p(.,.)$ irreducible then*

$$(\mathcal{I}_k \cap \mathcal{S})_e = \{\nu_\alpha : \alpha \in [0,1]\}.$$

When the transition matrix $p(.,.)$ is reversible and positive recurrent (ergodic), as for the simple exclusion process, we are able to characterize all the invariant measures. In this case there is a unique reversible probability stationary measure (for the chain $p(.,.)$), π, so that, according to theorem 1.1, $\nu_\rho \in \mathcal{I}_k$, with ρ defined as in (1). Denote by $\nu^{(n)}$ the measure

$$\nu^{(n)}(.) = \nu_\rho \left(. \middle| \sum_x \eta(x) = n \right).$$

It is not difficult to prove that if $p(.,.)$ is irreducible, then these measures are invariants and extremal.

Theorem 1.4 *Suppose that $p(x,y)$ is a positive recurrent, reversible, and irreducible matrix on $\mathbf{S}$. Then*

(a) $(\mathcal{I}_k)_e = \{\nu_n, 0 \le n \le \infty\}$;

(b) If $\mu\{\eta : \sum_x \eta(x) = n\} = 1$, then $\lim_{t\to\infty} \mu S_k(t) = \nu_n$ for $0 \le n \le \infty$, where ν_∞ is the measure concentrated on $\eta \equiv 1$.

This result is also a generalization to the k-step exclusion of a simple exclusion result (see Liggett (1985) [17], p. 384).

In section 4 we state some properties for a tagged particle a the second class particle involve in a k-step exclusion processes. We suppose $\mathbf{S} = \mathbf{Z}^d$ and that $p(x,y) = p(0, y-x)$ (*i.e.* corresponds to a random walk on $\mathbf{Z}^d$).

Informally a tagged particle behaves exactly like a regular particle of the system. A second class particle moves like other particles (with the exclusion rule) except that it has to exchange position when a regular (first class) particle tries to jump over it.

We first concentrate on the k-step exclusion process as seen from a tagged particle. Tagged particle for the simple exclusion process was introduced by Spitzer (1970)[20], and was studied by Arratia (1983) [4], Ferrari (1986)[6] and Saada (1987)[19].

We generalize to k-step exclusion some results of invariance and ergodicity of product Bernoulli measure as seen from a tagged particle obtain from [6]. For the notation of the following results we refer the reader to section 4.1.

Theorem 1.5 *(a) The Palm measure $\widehat{\nu_\rho}$ of ν_ρ is invariant for the tagged particle process,*

$$i.e.\ \widehat{\nu_\rho}\widehat{S}_k(t) = \widehat{\nu_\rho}.$$

(b) Furthermore if $p(.,.)$ is irreducible

$$(\widehat{\mathcal{I}}_k \cap \widehat{\mathcal{S}})_e = \{\widehat{\nu}_\rho :\ 0 \le \rho \le 1\}.$$

The second class particle was introduced to analyze the shock structure and its fluctuation for the simple exclusion process (see Ferrari & Fontes (1993)[7] for a review on the topic). We show that all product Bernoulli measures with constant densities $\rho \in]0,1]$ as seen from a second class particle are invariant if and only if $p(.,.)$ is symmetric.

Theorem 1.6 *For all $\rho \in]0,1]$, $\widehat{\nu}_\rho \in \overline{\mathcal{I}}_k$ if and only if $p(.,.)$ is symmetric.*

In section 5 we prove laws of large numbers for the tagged particle and for the second class particle showing that X_t/t (where X_t is the position of the Tagged particle at time t) and Y_t/t (where Y_t is the position of the second class particle at time t) converge *a.s.* and in L_1 (theorems 5.1 and 5.4).

Although there are strong links and similarities between simple and k-step exclusion they do not always behave in the same way. For the symmetric case duality relations fail for the k-step exclusion processes. This technique was a key ingredient for the study of the symmetric simple exclusion process (see [17] Chapter VIII Section 1). Furthermore, results obtained in the symmetric case have also some relevance for asymmetric case. For instance the method to prove that Bernoulli product measures with constant densities are extremal for the asymmetric simple exclusion process (see the end of the proof of Proposition 2 p. 377 in Saada (1987)[19]) relies highly on the result for the symmetric case. In particular this last fact was one ingredient to prove the strong laws of large numbers for the tagged particle for the simple exclusion case in Saada(1987) [19].

2 Definitions and Remarks

Let $k \in \mathbf{N}^*$, under the mild hypothesis :

$$\sup_{y \in \mathbf{S}} \sum_{x \in \mathbf{S}} p(x,y) < +\infty$$

one can defined a continuous semi-group $S_k(t)$ on $C(\mathbf{X})$ (see [17] page 30) with infinitesimal generator Ω_k given by :

for all cylindric function f,

$$\Omega_k f(\eta) = \sum_{\eta(x)=1, \eta(y)=0} q_k(x,y,\eta) \left[f(\eta_{xy}) - f(\eta) \right], \tag{2}$$

where $q_k(x,y,\eta) = \mathbf{E}^x \left[\prod_{i=1}^{\sigma_y - 1} \eta(X_i), \sigma_y \leq \sigma_x, \sigma_y \leq k \right]$ is the intensity for moving from x to y on configuration η and $\sigma_y = \min\{n \geq 1 : X_n = y\}$ is the first (non zero) arrival time at site y of the chain starting at site x.

By Hille-Yosida's theorem the closure of Ω_k corresponds to a continuous Markov semi-group, which corresponds to the k-step exclusion process.

Remark 2.1 *It is also possible to construct this process via the graphical construction due to Harris (1978)[12].*

Let $\mathcal{P}$ be the set of probability measures on $\mathbf{X}$. When $\mathbf{S} = \mathbf{Z}^d$ we will denote by $\mathcal{S}$ the set of elements of $\mathcal{P}$ which are translation invariants.

For every $\mu \in \mathcal{P}$ we denote $\mu S_k(t)$ the probability measure given by

$$\int f \, d\mu S_k(t) = \int S_k(t) f \, d\mu,$$

for all $f \in C(\mathbf{X})$.

Let $\mathcal{D}$ be the set of cylindric functions on $\mathbf{X}$, and let $\mathcal{I}_k$ denotes the set of elements of $\mathcal{P}$ which are invariant by the k-step exclusion process, *i.e.*

$$\begin{aligned} \mathcal{I}_k &= \{\mu \in \mathcal{P} : \mu S_k(t) = \mu \text{ for all } t\} \\ &= \{\mu \in \mathcal{P} : \textstyle\int \Omega_k f \, d\mu = 0 \text{ for all } f \in \mathcal{D}\}. \end{aligned} \tag{3}$$

Remark 2.2 *For all $k \in \mathbf{N}^*$, k-step exclusion process has the Feller property:*

$$\textit{for all } f \in C(\mathbf{X}) \textit{ and all } t > 0, \textit{ we have } S_k(t) f \in C(\mathbf{X}).$$

Second equality in (3) is due to that fact.

Remark 2.3

One can write the rates in the following way.

For all $k \in \mathbf{N}$, $x, y \in \mathbf{S}$ and $\eta \in \mathbf{X}$ we have

$$q_k(x,y,\eta) = p(x,y) + \sum_{z \neq x,y} p(x,z)p(z,y)\eta(z) + ...$$

$$... + \sum_{z_1 \neq x,y,\ldots,z_{k-1} \neq x,y} p(x,z_1)...p(z_{k-1},y)\eta(z_1)...\eta(z_{k-1}).$$

In particular we have for any k, x, y, η

$$q_k(x,y,\eta) \geq p(x,y).$$

Remark 2.4 *Last remark is the lack of monotonicity in k of these processes. i.e $f \in C(\mathbf{X})$ monotone increasing function does not imply $S_{k+1}(t)f \geq S_k(t)f$.*

This can be seen in the following simple example.

Let $\mathbf{S} = \{0,1,2\}$, $p(0,1) = p(1,2) = p(2,0) = 1$. Denote by η the initial configuration such that $\eta(0) = \eta(1) = 1$, $\eta(2) = 0$. Then construct a coupling $(\xi_t^\eta, \zeta_t^\eta)$ with ξ_t^η and ζ_t^η be respectively the 2-step exclusion process and the simple exclusion process starting from configuration η. Then $\xi_0^\eta = \zeta_0^\eta = \eta$, but with a positive probability particles on site 0 will try to jump (before particle at 1) saying at time δ^- then $\xi_\delta^\eta \not\leq \zeta_\delta^\eta$ and $\xi_\delta^\eta \not\geq \zeta_\delta^\eta$. ■

3 Equilibrium results

Remark 3.1 *The correspondence between simple exclusion process and k-step exclusion processes is not as easy as expected. For instance the condition $p(.,.)$ doubly stochastic does not assure anymore the invariance of the product measure with constant densities when $k \geq 2$ as one can see in the following example.*

Let $\mathbf{S} = \{-1,0,1\}$ suppose that $p(-1,0) = p(-1,1) = p(0,1) = 1/2$, $p(0,-1) = p(1,0) = 1/3$, $p(-1,-1) = p(1,1) = 0$, $p(0,0) = 1/6$, $p(1,-1) = 2/3$. Then $p(.,.)$ is doubly stochastic so that the product measure ν_α is invariant for the simple exclusion process for every constant $\alpha \in [0,1]$ (cf. theorem 2.1 (a) Chap. VIII in [17]). But a simple calculation using the generator of the 2-step exclusion process gives

$$\int \Omega_2 \mathbf{1}_{\{}\eta \in \mathbf{X} : \eta(-1) = \eta(0) = 1\} \, d\nu_\alpha = \frac{\alpha^2(1-\alpha)}{36} \neq 0$$

for every constant $\alpha \in]0,1[$. This means that these measures are not invariant for the 2-step exclusion process for that particular $p(.,.)$.

Nevertheless characterize all the invariant and translation invariant measures most of the time it is an easy generalization of the simple exclusion case.

The following properties are some generalization of well known results for the simple exclusion process.

3.1 Proofs of Theorem 1.1 and 1.2

We will prove the following proposition which contains both results.

Proposition 3.2 *For all $k \geq 1$,*
(a) Suppose it exists $\pi(.) : \mathbf{S} \to \mathbf{R}^+$ such that

$$\pi(x)p(x,y) = \pi(y)p(y,x) \qquad \textit{for all } x, y \in \mathbf{S}. \tag{4}$$

Then,

$$\nu_\rho \in \mathcal{I}_k,$$

where $\rho(x) = \pi(x)/\left[1+\pi(x)\right]$.
(b) If $\{p(.,.)\}$ is translation invariant then for all constant $\alpha \in [0,1]$, we have

$$\nu_\alpha \in \mathcal{I}_k.$$

Proof :
For all $A \subset \mathbf{Z}^d, |A| = n$ finite and $f_A(\eta) = \prod_{u \in A} \eta(u)$ we need to show that

$$\int \Omega_k f_A \; d\nu_\alpha = 0.$$

Applying (2) to f_A :

$$\int \Omega_k f_A(\eta) \; d\nu_\alpha = \sum_{y \in A} \sum_{x \in A^c} \int q_k(x,y,\eta)\eta(x)\left[1-\eta(y)\right] \prod_{u \in A, u \neq y} \eta(u) \; d\nu_\alpha$$

$$- \sum_{y \in A} \sum_{x \in A^c} \int q_k(y,x,\eta) f_A(\eta)\left[1-\eta(x)\right] \; d\nu_\alpha \; . \tag{5}$$

Note that for all $1 \leq l \leq k-1$

$$\alpha(x)\left[1-\alpha(y)\right] p(x,z_1)p(z_1,z_2)...p(z_l,y)$$

$$= \left[1-\alpha(y)\right] \frac{\pi(z_1)}{1+\pi(x)} p(z_1,x)p(z_1,z_2)...p(z_l,y)$$

$$...$$

$$= \left[1-\alpha(y)\right]\left[1-\alpha(x)\right] p(z_l,z_{l-1})...p(z_1,x)\pi(z_l)p(z_l,y)$$

$$= \alpha(y)\left[1-\alpha(x)\right] p(y,z_l)p(z_l,z_{l-1})...p(z_1,x) \tag{6}$$

Therefore using remark 2.3, integrating in (5) and summing terms with the same power in $\alpha(u)$ for $u \neq x, y$ we obtain $2^k - 1$ terms of the form :

for any $1 \leq l \leq k-1$ and $(i_1, i_2, ..., i_l)$ permutation of $1, ..., l$

$$\sum_{\substack{y \in A, z_{i_1}, ..., z_{i_j} \in A \setminus \{y\} \\ x \in A^c, z_{i_{j+1}}, ..., z_{i_l} \in A^c \setminus \{x\}}} \left(\prod\nolimits_{y \neq u \in A \cup \{z_{i_{j+1}}, \ldots, z_{i_l}\}} \alpha(u)\right) \times$$
$$\{\alpha(x)\,(1-\alpha(y))\,p(x,z_1)...p(z_l,y) - \alpha(y)\,[1-\alpha(x)]\,p(y,z_l)...p(z_1,x)\}$$

The term in brackets is zero by (6) this gives (a).

We prove (b) for $k = 2$. The general case consists in gathering terms that annihilates each other. As it is quite long and tedious no more detail will be given.

Let $\alpha(x) \equiv \alpha$ we the have the sum of three following terms (multiplied by $\alpha^n(1-\alpha)$)

$$\left\{ \sum_{y \in A} \sum_{x \in A^c} p(x,y) - \sum_{y \in A^c} \sum_{x \in A} p(x,y) \right\} \tag{7}$$

$$\left\{ \sum_{y \in A} \sum_{z \in A \setminus \{y\}} \sum_{x \in A^c} p(x,z)p(z,y) - \sum_{x \in A} \sum_{z \in A \setminus \{x\}} \sum_{y \in A^c} p(x,z)p(z,y) \right\} \tag{8}$$

$$\alpha \left\{ \sum_{x \in A^c} \sum_{z \in A^c \setminus \{x\}} \sum_{y \in A} p(x,z)p(z,y) - \sum_{y \in A^c} \sum_{z \in A^c \setminus \{y\}} \sum_{x \in A} p(x,z)p(z,y) \right\}. \tag{9}$$

Those three terms are zero :

for (7) we only have to add and subtract the same quantity in the brackets $\sum_{y \in A} \sum_{x \in A} p(x,y)$

$$\sum_{y \in A} \sum_{x \in \mathbf{S}} p(x,y) = |A| = \sum_{y \in \mathbf{S}} \sum_{x \in A} p(x,y).$$

For (8) we add and subtract $\sum_{y \in A} \sum_{z \in A} \sum_{x \in A} p(x,z)p(z,y)$, and since

$$\begin{aligned} &\textstyle\sum_{y \in A} \sum_{z \in A} \sum_{x \in \mathbf{S}} p(x,z)p(z,y) = \sum_{y \in A} \sum_{z \in A} p(z,y) \\ &= \textstyle\sum_{z \in A} \sum_{x \in A} p(x,z) = \sum_{y \in \mathbf{S}} \sum_{z \in A} \sum_{x \in A} p(x,z)p(z,y); \end{aligned}$$

denoting $p = p(x,x)$ we have

$$p \left\{ \sum_{y \in A} \sum_{x \in A^c} p(x,y) - \sum_{y \in A^c} \sum_{x \in A} p(x,y) \right\}$$

which is null by the preceding argument.

Finally for (9) we add and subtract
$\sum_{y\in A}\sum_{z\in A^c}\sum_{x\in A} p(x,z)p(z,y)$ and since

$$\begin{aligned}&\textstyle\sum_{y\in A}\sum_{z\in A^c}\sum_{x\in \mathbf{S}} p(x,z)p(z,y) = \sum_{y\in A}\sum_{z\in A^c} p(z,y)\\ &= \textstyle\sum_{z\in A^c}\sum_{x\in A} p(x,z) = \sum_{y\in \mathbf{S}}\sum_{z\in A^c}\sum_{x\in A} p(x,z)p(z,y).\end{aligned}$$

it remains

$$p\left\{\sum_{y\in A^c}\sum_{x\in A} p(x,y) - \sum_{y\in A}\sum_{x\in A^c} p(x,y)\right\}$$

which is null as in the first step. This concludes (b). ■

3.2 Ergodicity

We will only sketch the proof of theorem 1.3 as it is very standard (see Liggett's book Chapter VIII, for instance).

As these processes are Feller it is possible to show that if μ_1 and μ_2 are in $(\mathcal{I}_k \cap \mathcal{S})_e$ then one can find a coupled measure $\tilde{\nu} \in (\widetilde{\mathcal{I}}_k \cap \widetilde{\mathcal{S}})_e$ with marginals μ_1 and μ_2.

Using basic coupling then one as to show that if $\tilde{\nu} \in (\widetilde{\mathcal{I}}_k \cap \widetilde{\mathcal{S}})$ then $\tilde{\nu}\{\eta \geq \xi$ or $\eta \leq \xi\} = 1$.

Those two points are sufficient to conclude (see Andjel (1981)[2] for instance).

(A complete detailed proof can be found in Guiol (1995)[8]).

3.3 Reversible cases

In this paragraph we will suppose that $p(.,.)$ is reversible (*i.e.* there exists a positive $\pi(.)$ on $\mathbf{S}$ such that $\pi(x)p(x,y) = \pi(y)p(y,x)$ for all $x, y \in \mathbf{S}$).

We have shown in Proposition 3.2 (a) that (changing notations a little) $\nu^{(\alpha)} \in \mathcal{I}_k$ for $0 \leq \alpha \leq \infty$ and all $k \geq 1$, where $\nu^{(\alpha)}$ is the product measure on U with $\nu^{(\alpha)}\{\eta(x) = 1\} = \alpha\pi(x)/(1+\alpha\pi(x))$ for all $x \in \mathbf{S}$.

Proof of Theorem 1.4

Let

$$\nu^{(n)}(.) = \nu^{(\alpha)}\left(.|\sum_x \eta(x) = n\right).$$

Proof :

Suppose $p(.,.)$ positive recurrent, reversible with stationary measure π :

$$\pi(x)p(x,y) = \pi(y)p(y,x), \qquad \sum_x \pi(x) = 1.$$

Hence by proposition 3.2 (a) we have $\nu^{(\alpha)} \in \mathcal{I}_k$ where $\nu^{(\alpha)}\{\eta(x) = 1\} = \alpha\pi(x)/\left[1+\alpha\pi(x)\right]$ for $\alpha > 0$.

As $\sum_x \pi(x) = 1$, these measures are concentrated on $\{\eta : \sum_x \eta(x) < \infty\}$ because

$$\int \sum_x \eta(x)\ d\nu^{(\alpha)} = \sum_x \nu^{(\alpha)}\{\eta(x) = 1\} = \sum_x \frac{\alpha\pi(x)}{1+\alpha\pi(x)} < \infty$$

and $\nu^{(\alpha)}\{\eta :\ \sum_x \eta(x) = n\} > 0$ for all $0 \leq n < \infty$.

Since $p(.,.)$ is irreducible by hypothesis the restriction of the k-step exclusion process to $\{\eta : \sum_x \eta(x) = n\}$ is a positive recurrent, irreducible Markov chain, with stationary measure

$$\nu^{(n)}(.) = \nu^{(\alpha)}(.|\sum_x \eta(x) = n)$$

which is independent of α. Then we have

$$\begin{aligned}\frac{\alpha\pi(x)}{1+\alpha\pi(x)} &= \nu^{(\alpha)}\{\eta :\ \eta(x) = 1\} \\ &= \textstyle\sum_{n=0}^{\infty} \nu^{(n)}\{\eta :\ \eta(x) = 1\}\nu^{(\alpha)}\{\eta :\ \sum_x \eta(x) = n\}.\end{aligned}$$

Since $\eta_n \leq \eta_{n+1}$ (from monotonicity of η_t), letting $\alpha \uparrow \infty$ in the last equality we obtain

$$\lim_{n\to\infty} \nu_n = \nu^{\infty}, \tag{10}$$

where $\nu^{\infty} = \delta_1$ is the Dirac mass on $\varpi_1 \equiv 1$.

Now let μ be such that

$$\mu\{\eta :\ \sum_x \eta(x) = \infty\} = 1$$

and for $1 \leq n < \infty$ choose a probability measure $\mu^{(n)}$ such that $\mu^{(n)} \leq \mu$ and $\mu^{(n)}\{\eta :\ \sum_x \eta(x) = n\} = 1$.

We have

$$\mu^{(n)} S_k(t) \leq \mu S_k(t) \tag{11}$$

for all $t > 0$. Therefore by the convergence theorem of positive recurrent Markov chains :

$$\lim_{n\to\infty} \mu^{(n)} S_k(t) = \nu^{(n)} \tag{12}$$

for all n. From this we deduce (b) using (10), (11) and (12).

(a) is a consequence of (b). ■

4 Tagged and Second Class particles in the k-step exclusion process

We are interested in the k-step exclusion process as seen from a tagged particle (resp. a second class particle).

For all these processes the origin will always be occupied.

For the tagged particle, after a random exponential time with mean 1, a transition of the system will occur moving the origin to X_τ, where $\{X_n\}$ is a Markov chain with probability transition $p(.,.)$, $X_0 = 0$ and $\tau = \inf\{1 \leq n \leq k : \eta(X_n) = 0\}$. When $\{1 \leq n \leq k : \eta(X_n) = 0\} = \emptyset$ we set $X_\infty = 0$ and the system does not move. The other particles move as in the k-step exclusion process.

For the second class particle we add to the preceding rules that after a random exponential time mean 1, the system is translated putting the origin on y if $\eta(y) = 1$ with probability $q_k(y, 0, \eta)$. So the second particle moves like a standard particle but when an other particle arrive on its site (here the origin) then the two particles exchange their positions.

4.1 Notations and definitions

We will suppose $\mathbf{S} = \mathbf{Z}^d$ and $p(x,y) = p(0, y-x)$.

The state space will be a little modified :

$$\widehat{\mathbf{X}} = \mathbf{X} \cap \{\eta : \ \eta(0) = 1\}.$$

The k-step exclusion process as seen from a tagged particle has for state space $\widehat{\mathbf{X}}$ and for generator :

$$\widehat{\Omega}_k f(\eta) = \sum_{\eta(x)=1,\eta(y)=0 x,y\neq 0} q_k(x,y,\eta)\,[f(\eta_{xy}) - f(\eta)]$$

$$+ \sum_{\eta(y)=0 y\neq 0} q_k(0,y,\eta)\,[f(\eta_{0y} - y) - f(\eta)]\,.$$

Where configuration $\eta - y$ corresponds to $(\eta - y)(z) = \eta(z+y)$.

We will also consider the k-step exclusion process as seen from a second class particle on $\widehat{\mathbf{X}}$ with generator :

$$\overline{\Omega}_k f(\eta) = \widehat{\Omega}_k f(\eta) + \breve{\Omega}_k f(\eta)$$

where

$$\breve{\Omega}_k f(\eta) = \sum_{\eta(x)=1 x\neq 0} q_k(x,0,\eta)\,[f(\eta_{0x} - x) - f(\eta)]\,.$$

By Liggett's criterion :

$$\sum_{\eta(x)=1,\eta(y)=0x,y\neq 0} q_k(x,y,\eta)\left[f(\eta_{xy})-f(\eta)\right]$$

is a generator of a contractive semi-group and

$$\sum_{\eta(y)=0y\neq 0} q_k(0,y,\eta)\left[f(\eta_{0y}-y)-f(\eta)\right]$$

(resp.

$$\sum_{\eta(x)=1x\neq 0} q_k(x,0,\eta)\left[f(\eta_{0x}-x)-f(\eta)\right])$$

is a bounded operator. Hence by theorem 2 of Gustafson [10] $\widehat{\Omega}_k$ and $\overline{\Omega}_k$ are infinitesimal generator of contractive semi-group that we denote respectively $\widehat{S}(t)$ et $\overline{S}(t)$. To each of these semi-group corresponds a unique Markov process on $\widehat{\mathbf{X}}$ with generators $\widehat{\Omega}_k$ and $\overline{\Omega}_k$.

Let $\mathcal{P}(\widehat{\mathbf{X}})$ be the set of probability measure on $\widehat{\mathbf{X}}$, and $\widehat{\mathcal{I}}_k = \{\mu \in \mathcal{P}(\widehat{\mathbf{X}}) : \mu\widehat{S}_k(t) = \mu\}$ (resp. $\overline{\mathcal{I}}_k = \{\mu \in \mathcal{P}(\widehat{\mathbf{X}}) : \ \mu\overline{S}_k(t) = \mu\}$) the set of invariant measures for the k-step exclusion process as seen from the tagged particle (resp. the second class particle).

4.2 Invariant measures

Let $\mu \in \mathcal{S}$ such that $\beta(\mu) = \mu\{\eta(0) = 1\} > 0$. We define the Palm measure of μ, by $\widehat{\mu}$ a measure on $\widehat{\mathbf{X}}$ such that

$$\widehat{\mu} = \mu(.|\eta(0) = 1).$$

(Let us recall that if μ puts all its mass on empty configuration (*i.e.* $\mu = \delta_0$) one define $\widehat{\mu} = \delta_{\{\eta=\{0\}\}}$ as the measure that puts all its mass on configuration with only one particle at 0).

We denote

$$\widehat{\mathcal{S}} = \{\widehat{\mu} : \ \mu \in \mathcal{S}\}.$$

The next property will be useful :

Lemma 4.1 *Let $\mu, \nu \in \mathcal{S}$. If $\widehat{\mu} = \widehat{\nu}$ and $\beta = \beta(\mu) = \beta(\nu)$ then*

$$\mu = \nu.$$

Proof :
Suppose $0 < \beta < 1$.
Note that for all measurable A

$$\mu(A) = \beta\mu\left\{A|\eta(0) = 1\right\} + (1-\beta)\mu\left\{A|\eta(0) = 0\right\}$$

and since $\widehat{\mu} = \widehat{\nu}$

$$= \beta\nu\left\{A|\eta(0) = 1\right\} + (1-\beta)\mu\left\{A|\eta(0) = 0\right\}.$$

Furthermore

$$\mu\left\{A|\eta(0) = 0\right\} = \frac{\mu\left\{A, \eta(0) = 0\right\}}{1-\beta},$$

we only have to show that

$$\mu\left\{A, \eta(0) = 0\right\} = \nu\left\{A, \eta(0) = 0\right\} \tag{13}$$

to conclude $\mu = \nu$.

Now take a one to one mapping :

$$\begin{aligned} \mathbf{N} \setminus \{0\} &\longrightarrow \mathbf{Z}^d \setminus \{0\} \\ i &\longmapsto x_i \end{aligned}$$

we have

$$\begin{gathered} \mu\left\{A, \eta(0) = 0\right\} \\ = \textstyle\sum_{i=1}^{\infty} \mu\left\{A, \eta(0) = 0, \eta(x_1) = \ldots = \eta(x_{i-1}) = 0, \eta(x_i) = 1\right\} \\ +\mu(A, \eta \equiv 0). \end{gathered} \tag{14}$$

But by translation invariance of μ one has for all i

$$\mu\left\{A, \eta(0) = 0, \eta(x_1) = \ldots = \eta(x_{i-1}) = 0, \eta(x_i) = 1\right\}$$

$$= \mu\left\{\tau_{x_i}A, \eta(-x_i) = 0, \eta(x_1 - x_i) = \ldots = \eta(x_{i-1} - x_i) = 0, \eta(0) = 1\right\}$$

$$= \mu\left\{\tau_{x_i}A, \eta(-x_i) = 0, \eta(x_1 - x_i) = \ldots = \eta(x_{i-1} - x_i) = 0 | \eta(0) = 1\right\}\beta$$

and since $\widehat{\mu} = \widehat{\nu}$

$$= \nu\left\{\tau_{x_i}A, \eta(-x_i) = 0, \eta(x_1 - x_i) = \ldots = \eta(x_{i-1} - x_i) = 0 | \eta(0) = 1\right\}\beta$$

$$= \nu\left\{\tau_{x_i}A, \eta(-x_i) = 0, \eta(x_1 - x_i) = \ldots = \eta(x_{i-1} - x_i) = 0, \eta(0) = 1\right\}$$

and by ν's translation invariance

$$= \nu\left\{A, \eta(0) = 0, \eta(x_1) = \ldots = \eta(x_{i-1}) = 0, \eta(x_i) = 1\right\}.$$

On the other hand as

$$\mu(\eta \equiv 0) = 1 - \sum_{i=0}^{\infty} \mu\left\{\eta(x_0) = \eta(x_1) = \ldots = \eta(x_{i-1}) = 0, \eta(x_i) = 1\right\},$$

we deduce by the same type of argument that

$$\mu(\eta \equiv 0) = \nu(\eta \equiv 0).$$

Therefore (14)

$$\mu\left\{A, \eta(0) = 0\right\} = \nu\left\{A, \eta(0) = 0\right\}$$

Which allows us to conclude. ∎

Proof of theorem 1.5

The proof is decomposed into the two following theorems.

Theorem 4.2 *(a) If $\nu \in \mathcal{S}$ then for all $t \geq 0$ we have*

$$\widehat{\nu}\widehat{S}_k(t) = \nu\widehat{S_k(t)}.$$

(b) The Palm measure $\widehat{\nu_\rho}$ of ν_ρ is invariant for the tagged particle process.

$$i.e.\ \widehat{\nu_\rho}\widehat{S}_k(t) = \widehat{\nu_\rho}.$$

Proof:

(b) is an immediate consequence of (a) and theorem 1.2.

Define $T : C(\widehat{\mathbf{X}}) \to C(U)$ an operator by

$$Tf(\eta) = \begin{cases} \eta(0)f(\eta) & \\ 0 & \text{if } \eta(0) = 0. \end{cases}$$

Note that if $\mu \in \mathcal{S}$ and $f \in C(\widehat{\mathbf{X}})$

$$\int Tf\ d\mu = \beta(\mu) \int f\ d\widehat{\mu}.$$

Then for all measure $\mu \in \mathcal{S}$ and all $f \in C(\widehat{\mathbf{X}})$ we have

$$\int T\widehat{\Omega}_k f\ d\mu = \int \Omega_k Tf\ d\mu \tag{15}$$

for all cylindric f. Indeed :

$$\Omega_k Tf(\eta) = \sum_{\eta(x)=1\eta(y)=0} q_k(x,y,\eta)\left[\eta_{xy}(0)f(\eta_{xy}) - \eta(0)f(\eta)\right]$$

$$= \sum_{\eta(x)=1,\eta(y)=0x,y\neq 0} q_k(x,y,\eta)\eta(0)\left[f(\eta_{xy}) - f(\eta)\right]$$

$$+ \sum_{\eta(y)=0\eta(0)=1} q_k(0,y,\eta)\left[\eta(y)f(\eta_{0y}) - \eta(0)f(\eta)\right].$$

$$+\sum_{\eta(x)=1\eta(0)=0} q_k(x,0,\eta)\left[\eta(x)f(\eta_{0x})-\eta(0)f(\eta)\right].$$

$$=\sum_{\eta(x)=1,\eta(y)=0x,y\neq 0} q_k(x,y,\eta)\eta(0)\left[f(\eta_{xy})-f(\eta)\right]$$

$$+\sum_{\eta(0)=0x\neq 0} q_k(x,0,\eta)\eta(x)f(\eta_{0x})-\sum_{\eta(y)=0y\neq 0} q_k(0,y,\eta)\eta(0)f(\eta).$$

In the same way we write

$$T\widehat{\Omega}_k f(\eta)=\sum_{\eta(x)=1,\eta(y)=0x,y\neq 0} q_k(x,y,\eta)\eta(0)\left[f(\eta_{xy})-f(\eta)\right]$$

$$+\sum_{\eta(y)=0y\neq 0} q_k(0,y,\eta)\eta(0)\left[f(\eta_{0y}-y)-f(\eta)\right].$$

The first terms and the end of the second terms of $\Omega_k Tf(\eta)$ and $T\widehat{\Omega}f(\eta)$ are corresponding, then we only have to show that

$$\sum_{x\neq 0}\mu\{q_k(x,0,\eta)\left[1-\eta(0)\right]\eta(x)f(\eta_{0x})\}$$

$$=\sum_{y\neq 0}\mu\{q_k(0,y,\eta)\eta(0)\left[1-\eta(y)\right]f(\eta_{0y}-y)\}$$

Which is true because of the translation invariance of μ :

$$\mu\{q_k(x,0,\eta)\left[1-\eta(0)\right]\eta(x)f(\eta_{0x})\}$$

$$=\mu\{q_k(0,-x,\eta)\left[1-\eta(-x)\right]\eta(0)f(\eta_{-x0}+x)\}$$

$$=\mu\{q_k(0,y,\eta)\eta(0)\left[1-\eta(y)\right]f(\eta_{0y}-y)\}$$

letting $y=-x$.

Now compute for $f\in C(\widehat{\mathbf{X}})$

$$T\widehat{S}_k(t)f-S_k(t)Tf=\int_0^t\frac{d}{ds}S_k(t-s)T\widehat{S}_k(s)f\ ds$$

$$=\int_0^t S_k(t-s)[T\widehat{\Omega}_k-\Omega_k T]\widehat{S}_k(s)f\ ds$$

It follows

$$\int[T\widehat{S}_k(t)f-S_k(t)Tf]\ d\mu=\int_0^t\int[T\widehat{\Omega}_k-\Omega_k T]\widehat{S}_k(s)f\ d\mu S_k(t-s)\ ds$$

which is zero by (15) because $\mu S_k(t)\in\mathcal{S}$ for all $t>0$.

Then

$$\int T\widehat{S}_k(t)f\ d\mu = \int S_k(t)Tf\ d\mu \tag{16}$$

for all measure $\mu \in \mathcal{S}$. Therefore

$$\beta(\mu)\int f\ d\widehat{\mu}\widehat{S}_k(t) = \beta(\mu S_k(t))\int f\ d\mu\widehat{S_k(t)}.$$

Since $\mu \in \mathcal{S}$ we have $\beta(\mu) = \beta(\mu S_k(t))$ (see Lemma 2.3 section 1) and the result follows. ∎

Remark 4.3 *This proof is a very general one, we never use explicitly the expression of the coefficients $q_k(x, y, \eta)$.*

As quoted in [6] part (a) is and extension of theorems of Harris [11] and Port and Stone [18].

The next result is similar to [6]

Theorem 4.4 *Suppose $p(.,.)$ is irreducible, then*

$$(\widehat{\mathcal{I}}_k \cap \widehat{\mathcal{S}})_e = \{\widehat{\nu}_\rho :\ 0 \leq \rho \leq 1\}.$$

Proof : The proof is the same as in [6] page 1282. ∎

This ends the proof of theorem 1.5.

Proof of theorem 1.6

For this we prove the following proposition that contains the theorem.

Proposition 4.5 *For all $k \geq 1$,*

(a) If $p(.,.)$ is symmetric and $\mu \in \mathcal{S}$ then $\widehat{\mu} \in \overline{\mathcal{I}}_k$.

(b) Let $\rho \in]0, 1]$, $\widehat{\nu}_\rho \in \overline{\mathcal{I}}_1$ if and only if $p(.,.)$ is symmetric.

(c) Let $\rho \in]0, 1]$, for $\mathbf{S} = \mathbf{Z}$ if $p(x, x+1) = 1 - p(x, x-1) = p$ then $\widehat{\nu}_\rho \in \overline{\mathcal{I}}_k$ if and only if $p(.,.)$ is symmetric (i.e. $p = 1/2$).

(d) If f $\widehat{\nu}_\rho \in \overline{\mathcal{I}}_k$ then $p(.,.)$ is symmetric except eventually for a finite number ($\leq k$) of ρ.

Note that if $\mu \in \mathcal{S}$ then

$$\int T\overline{\Omega}_k f(\eta)\ d\mu = \int \Omega_k Tf(\eta)\ d\mu + \int T\breve{\Omega}_k f(\eta)\ d\mu \tag{17}$$

where

$$\breve{\Omega}_k f(\eta) = \sum_{\eta(x)=1 x\neq 0} q_k(x, 0, \eta)\left[f(\eta_{0x} - x) - f(\eta)\right].$$

We try to find conditions for which for all $f \in \mathcal{D}$

$$\int T\breve{\Omega}_k f(\eta)\ d\mu = 0. \tag{18}$$

$$\int T\breve{\Omega}_k f(\eta)\ d\mu = \sum_{x\neq 0}\int q_k(x,0,\eta)\eta(0)\eta(x)\left[f(\eta_{0x}-x)-f(\eta)\right]\ d\mu$$

and since $\eta_{0x}=\eta$ when $\eta(x)=\eta(0)$, by translation invariance of μ we obtain

$$=\sum_{x\neq 0}\left[\int q_k(0,-x,\eta)\eta(-x)\eta(0)f(\eta)\ d\mu - \int q_k(x,0,\eta)\eta(0)\eta(x)f(\eta)\ d\mu\right]$$

$$=\sum_{x\neq 0}\int \left[q_k(0,x,\eta)-q_k(x,0,\eta)\right]\eta(0)\eta(x)f(\eta)\ d\mu \tag{19}$$

which is zero if $p(.,.)$ is symmetric and shows part (a) applying the same ideas as for the end of the proof of theorem 4.2.

Now let $\mu=\nu_\rho$, for $\rho\in]0,1[$. For all $y\in\mathbf{S}\setminus\{0\}$ let $f_y=\eta(y)$. Then writing (19) for f_y and developing coefficients $q_k(.,.,.)$ we obtain the three following cases :

- **1st case** if $k=1$

$$\int T\breve{\Omega}_k f_y(\eta)\ d\mu = \rho^2\left([p(0,y)-p(y,0)]+\rho\sum_{x\neq 0 x\neq y}[p(0,x)-p(x,0)]\right)$$

and as $p(.,.)$ is doubly stochastic

$$=\rho^2(1-\rho)\left[p(0,y)-p(y,0)\right].$$

This last expression is null for all y only if $p(.,.)$ is symmetric. This ends part (b);

- **2nd case** if $\mathbf{S}=\mathbf{Z}$ et $p(x,x+1)=1-p(x,x-1)=p$ and $q=1-p$; without loss of generality we take y such that $|y|>k$, then (19) is a polynomial in ρ degree $k+2$ whose order $2l+1$ coefficients, $3\leq 2l+1\leq k+2$ are

$$(p^{2l+1}-q^{2l+1})+C_{2l+1}^{2l}pq(p^{2l-1}-q^{2l-1})+...+C_{2l+1}^{l+1}p^l q^l(p-q),$$

et and order $2l$, $3\leq 2l\leq k+2$ are

$$(p^{2l}-q^{2l})+C_{2l}^{2l-1}pq(p^{2l-2}-q^{2l-2})+...+C_{2l}^{l+2}p^l q^l(p^2-q^2).$$

Since these coefficients have the same sign then (19) is null only if $p=q=1/2$. This ends part (c);

- **3rd case** For part (d) without loss of generality we may suppose that $p(x,x)=0$, we then obtain a polynomial in ρ of degree $k+2$ for which 0 is a root of order two and so it has at most k non null roots. Its lowest degree term is

$$\rho^2\left[p(0,y)-p(y,0)\right].$$

This term is constantly null only if $p(.,.)$ is symmetric. ■

5 Weak law of large numbers

5.1 Introduction

We still suppose that $p(.,.)$ is translation invariant and $\mathbf{S} = \mathbf{Z}^d$.

We are interested in the asymptotic behavior of a tagged particle (resp. a second class particle) originally at $x \in \mathbf{S}$, when the other particles of the system are distributed according to ν_ρ the Bernoulli product measure, $(0 \leq \rho \leq 1)$.

For $k = 1$, supposing that $p(.,.)$ has a finite first moment, F. Spitzer [20] has compute $\mathbf{E}X_t$, where X_t is the position of the tagged particle at time t, and has shown the existence of an almost sure limit for X_t/t. C. Kipnis [13] (in the nearest neighbors case in dimension one) and E. Saada [19] in the other cases have shown that this limit is constant and equals to

$$\lim_{t\to\infty} \frac{X_t}{t} = \sum_{y \in \mathbf{Z}^d} y p(0,y) \ \ a.s.$$

5.2 Tagged Particle

For the tagged particle in the k-step exclusion process we have

Theorem 5.1 *Suppose $p(.,.)$ with bounded first moment. If the k-step exclusion process as seen from the tagged particle has for initial law $\widehat{\nu}_\rho$ then,*

$$\mathbf{E}X_t = x + (1-\rho)t \sum_y y p(0,y) + \ldots$$

$$\ldots + (1-\rho)t \sum_y \sum_{z_1,\ldots,z_{k-1} \neq 0,y} \rho^{\gamma} y p(0,z_1)\ldots p(z_{k-1},y)$$

$$\textit{where } \gamma(z_1,\ldots,z_{k-1}) = Card\{z_1,\ldots,z_{k-1}\}$$

$$\textit{and}$$

$$\lim_{t\to\infty} \frac{X_t}{t} \ \ \textit{exists a.s and in } L^1.$$

Proof : It's a corollary of theorem 4.2 (see Liggett [17] p. 396). By this theorem X_t is stationary increasing. Then $\mathbf{E}X_t$ is a linear function of t.

On the other hand

$$\lim_{t\downarrow 0} \frac{\mathbf{E}X_t - x}{t} = (1-\rho)\sum_y y p(0,y) + \ldots$$

$$\ldots + (1-\rho)\sum_y \sum_{z_1,\ldots,z_{k-1}} \rho^{\gamma(z_1,\ldots,z_{k-1})} y p(0,z_1)\ldots p(z_{k-1},y)$$

we have the first result.

The *a.s.* convergence of X_n/n is a consequence of the Ergodic theorem. Noting that

$$\mathbf{E} \sup_{0 \leq t \leq 1} \|X_t\| < \infty,$$

and using a Borel-Cantelli argument we deduce the convergence for all t. ■

We then state a natural conjecture :

Conjecture 5.2 *For $k \geq 2$,*

Under the hypothesis of theorem 5.1

$$\lim_{t \to \infty} \frac{X_t}{t} = (1-\rho) \sum_y yp(0,y) + ...$$

$$... + (1-\rho) \sum_y \sum_{z_1,\ldots,z_{k-1}} \rho^{\gamma(z_1,\ldots,z_{k-1})} yp(0,z_1)...p(z_{k-1},y) \ \ a.s.$$

Remark 5.3 *The major difficulty to prove this is to show that*

$$\nu_\rho \in (\mathcal{I}_k)_e$$

when $k \geq 2$.

Using coupling arguments of Liggett (1976) in [15] it is possible, at least for the nearest neighbor asymmetric case (i.e. $d = 1$, $p(x, x+1) = p$, $p(x, x-1) = 1-p$, $p \neq 1/2$), to show that $\nu_\rho \in (\mathcal{I}_k)_e$.

5.3 Second class particle

Let Y_t the position at time t of the second class particle originally at $x \in \mathbf{S}$.

Theorem 5.4 *With the hypothesis of theorem 5.1 and if $p(.,.)$ is symmetric then*

$$\mathbf{E}Y_t = x \quad \textit{and}$$

$$\lim_{t \to \infty} \frac{Y_t}{t} \textit{ exists a.s. and in } L^1.$$

Proof : The proof parallels the one for the tagged particle. ■

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References

[1] Aldous, D., Diaconis, P., Hammersley's interacting particle process and longest increasing subsequences, *Probab. Theory Related Fields*, **103** (1995), No. 2, 199–213.

[2] Andjel, E.D., The asymmetric simple exclusion process on $\mathbf{Z}^d$, *Z. Wahrsch. Verw. Gebiete*, **58** (1981), 423-432.

[3] Andjel, E.D., Guiol, H., *In Preparation.*

[4] Arratia, R., The motion of a tagged particle in the simple symmetric exclusion on $\mathbf{Z}$, *Ann. Probab.*, **11** (1983), 362-373.

[5] Carlson, J.M., Grannan, E.R., Swindle, G.H., Tour, J., Singular diffusion limits of a class of reversible self-organizing particle systems, *Ann. Probab.*, **21** (1993), No. 3, 1372–1393.

[6] Ferrari, P.A., The simple exclusion process as seen from a Tagged Particle, *Ann. Probab.*, **14** (1986), 1277-1290.

[7] Ferrari, P.A., Fontes, L.R.G. Shocks in asymmetric one-dimensional exclusion processes. Resenhas 1 (1993), No. 1, 57-68.

[8] Guiol, H., Thèse de doctorat de l'Université de Provence, France, 1995.

[9] Guiol, H., Un résultat pour le processus d'exclusion à longue portée, *Ann. Inst. Henri Poincaré*, Vol **33**, No 4, (1997), 387-405.

[10] Gustafson, K., A perturbation lemma, *Bull. Amer. Math. Soc.*, **72** (1966), 334-338.

[11] Harris, T.E., Random measures and motions of point processes, *Z. Wahrsch. verw. Gebiete*, **9** (1967), 36-58.

[12] Harris, T.E., Additive set-valued Markov processes and graphical methods, *Ann. Probab.*, **6** (1978), 355-378.

[13] Kipnis, C., Central limit theorems for infinite series of queues and applications to simple exclusion, *Ann. Probab.*, **14** (1986), 397-408.

[14] Lebowitz, J.L., Neuhauser, C., Ravishankar, K., Dynamics of a spin-exchange model *Stochastic Process. Appl.*, **64** (1996), No. 2, 187-208.

[15] Liggett, T.M., Coupling the simple exclusion process, *Ann. Probab.*, **4** (1976), 339-356.

[16] Liggett, T.M., Long-range exclusion processes, *Ann. Probab.*, **8** (1980), 861-889.

[17] Liggett, T.M., *Interacting Particle Systems*, Springer, New York, 1985.

[18] Port, S.C. and Stone, C.J., Infinite particle systems, *Trans. Amer. Math. Soc.*, **178** (1977), 307-340.

[19] Saada, E., A limit theorem for the position of a tagged particle in a simple exclusion process, *Ann. Probab.*, **15** (1987), 375-381.

[20] Spitzer, F., Interaction of Markov processes, *Adv. Math.*, **5** (1970), 247-290.

[21] Zheng, X., Ergodic theorem for generalized long-range exclusion processes with positive recurrent transition probabilities, *Acta Math. Sinica*, **4** (1988), 193-209.