Break–down of librational invariant surfaces

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Abstract

A numerical investigation of the stability of invariant librational tori is presented. The method has been developed for a model describing the spin–orbit coupling in Celestial Mechanics. Periodic orbits approaching the librational torus are computed by means of a Newton's method. According to Greene's criterium, their stability is strictly related to the survival of invariant tori.

We consider librational tori around the main spin–orbit resonances (1:1, 3:2). Their existence provides the stability of the resonances, due to confinement properties in the 3–dimensional phase space associated to our model. The results are consistent with the actual observations of the eccentricity and of the oblateness parameter. A different behaviour of the Moon and Mercury around the main resonances is evidentiated, providing interesting suggestions about the different probabilities of capture in a resonance.

Keywords

Spin–orbit problem, break–down threshold, librational tori.

1 INTRODUCTION

The stability and break–down of invariant surfaces in nearly integrable Hamiltonian systems has been widely investigated, both from a theoretical and numerical point of view. The Kolmogorov–Arnold–Moser theorem ([11], [1], [14]) provides an explicit constructive algorithm to determine a value of the perturbing parameter ensuring the existence of an invariant torus with non–resonant rotation number. Several applications and developments of KAM theory have shown that it is possible to prove the existence of invariant tori for values of the perturbing parameter close to the numerical evidence of the *survival* threshold ([2], [3], [4], [5]).

Methods for computing the numerical threshold have been worked out since many years ([9], [7], [12], [15]). One of the most reliable techniques has been developed by Greene in [9] and it is based on the conjecture that the break–down of an invariant torus is related to the stability character of the nearby periodic orbits. Greene's method was originally applied to a simple model, the standard mapping, in order to establish the existence of invariant curves of *rotational* type. More precisely, we assume that the phase space structure is composed by *librational* tori, which surround the resonances. A chaotic separatrix divides the librational regime from the region in which *rotational* invariant tori can be found.

In this paper, we address the problem of determining the numerical break–down threshold of librational–type invariant tori, using an extension of Greene's method. Theoretical estimates on the existence of librational tori in the spin–orbit problem were developed in [3] using KAM theory. Here, we consider a simple model derived from Celestial Mechanics, the *spin–orbit* problem. Let S be a triaxial satellite (with principal moments of inertia $A < B < C$), moving on a Keplerian orbit around a central planet P. Under suitable simplifying assumptions, the equations of motion can be derived from Hamilton's equations associated to a one–dimensional, time–dependent, nearly–integrable Hamiltonian system, whose perturbing parameter is proportional to the oblateness coefficient $\frac{B-A}{C}$. We remark that the equations of motion depend also on the orbital eccentricity of the satellite. A "spin–orbit resonance" occurs whenever the ratio between the period of rotation of the satellite (say, T_{rot}) around an internal axis and its period of revolution around the planet (say, T_{rev}) is a rational number, namely

$$\frac{T_{rev}}{T_{rot}} = \frac{p}{q} , \tag{1}$$

for some $p, q \in \mathbf{Z}_+$, $q \neq 0$. When $p = q = 1$, we speak of a 1:1 or synchronous resonance.

As it is well known, most of the evolved planets or satellites of the solar system are trapped in a 1:1 resonance, with the only exception of Mercury which is observed to move in a 3:2 resonance. According to evolutionary theories, these bodies were rotating fast in the past and they were slowed down by the internal friction toward their ending resonant states. Probably due to its high eccentricity, Mercury privileged the 3:2 rather than the 1:1 resonance. A question remains open: why

Mercury privileged the 3:2 rather than the 1:1 resonance? A plausible explanation relies on the high orbital eccentricity of this planet, with respect to those of the other bodies trapped in a synchronous resonance. In particular, a larger librational region around the 3:2 resonance provides a bigger probability of capture ([10]).
The exploration of the phase–space structure provides interesting suggestions about the stability of the resonances. In particular, since the phase–space associated to our model is 3–dimensional, the existence of librational tori around the resonances implies their stability, since the trajectories are definitely confined in the region of librational motion. It is therefore a crucial point to determine the break–down threshold of the librational tori. To this end, we compute numerically the rotation number associated to a given librational torus and according to Greene's criterium we look for the periodic orbits approaching this torus. The frequencies of the periodic orbits are determined as the successive truncations of the continued fraction expansion of the rotation number of the librational torus. The location of the periodic orbits is obtained applying a Newton's method, combined with a *continuation* technique. The equations of motion have been integrated using a Runge–Kutta–Fehlberg 7/8 method. The stability of the periodic orbits has been determined computing the corresponding *residuals*, which are related to the eigenvalues of the monodromy matrix.

The method is quite efficient and shows the stability of the main resonances (1:1, 3:2) for physically relevant values of the perturbing parameter. Precisely, the results are consistent with the actual observations of the oblateness coefficient, both for the Moon and for Mercury. Moreover, for the Moon the librational region around the synchronous resonance is larger than that around the 3:2, while for Mercury the librational regions around the main resonances have comparable sizes. This remark suggests a greater probability of capture in the 1:1 resonance for the Moon and in the 3:2 resonance for Mercury.

This paper is organized as follows: a spin–orbit model is introduced in §2; the algorithm to determine the stability of the librational invariant tori is discussed in §3; the results and conclusions are presented, respectively, in §4 and §5.

2 A MODEL OF SPIN–ORBIT INTERACTION

Let S be a *satellite* of a *planet* P; we assume that S is a triaxial ellipsoid with principal moments of inertia $A < B < C$. The satellite is supposed to orbit around P and to rotate about an internal spin–axis. Let T_{rev} and T_{rot} be the periods of revolution and rotation; a *spin–orbit resonance* occurs whenever the ratio between T_{rev} and T_{rot} is a rational number, i.e. there exist $p, q \in \mathbf{Z}_+$ $(q \neq 0)$ such that

$$\frac{T_{rev}}{T_{rot}} = \frac{p}{q} . \tag{2}$$

As in [2], [6], we consider a simplified model for the spin–orbit interaction assuming that *a*) the orbit of S around P is Keplerian (i.e., no secular perturbations are considered);

b) the spin–axis coincides with the axis whose moment of inertia is maximum;

c) the spin–axis is perpendicular to the orbit plane;

d) dissipative forces as well as interactions with other bodies of the solar system are neglected.

The above assumptions allow to reduce the problem to the study of the following second–order differential equation:

$$\ddot{x} \; + \; \varepsilon \; (\frac{a}{r})^3 \; \sin(2x - 2f) \; = \; 0 \; , \tag{3}$$

where $\varepsilon = \frac{3}{2}\frac{B-A}{C}$, a is the semimajor axis, r and f are the instantaneous radius and the true anomaly of the Keplerian ellipse, x denotes the angle between the longest axis of the ellipsoid and the pericenter direction. We refer to ε as the "perturbing parameter" (since the system is obviously integrable as $\varepsilon = 0$), which is proportional to the oblateness coefficient $\frac{B-A}{C}$. Equation (3) should be completed with the well–known Keplerian relations:

$$\begin{aligned} \mathrm{tg}\frac{f}{2} &= \sqrt{\frac{1+e}{1-e}}\; \mathrm{tg}\frac{E}{2} \; , \\ r &= 1 - e\cos E, \\ t &= E - e\sin E \; , \end{aligned}$$

where in the above formulae we selected the units of measure so that the period of revolution is 2π and the semimajor axis is one (e is the *orbital eccentricity* of the Keplerian orbit).

For each value of the time t, solving the last of the above equations by, e.g., a Newton's approach allows to find the corresponding true anomaly and instantaneous radius. We also keep in mind that r and f depend on the orbital eccentricity.

3 BREAK–DOWN OF LIBRATIONAL INVARIANT SURFACES

We intend to determine the break–down threshold of librational invariant tori. To this end, we follow the idea developed by J. Greene ([9]) for the determination of the stochasticity threshold of *rotational* invariant tori. More precisely, Greene's method is based on the conjecture that the break–down of an invariant torus is related to a change, from stability to instability, of the nearby periodic orbits. The frequency of these periodic orbits is selected as the successive truncations of the continued fraction expansion of the rotation number of the invariant torus. We remark that Greene's method has been widely tested on a variety of dynamical systems and part of the method was rigorously proved in [8], [13].

Let P be the Poincaré map (for t mod. 2π) corresponding to given values of the eccentricity and of the parameter ε_0. We want to investigate the existence of librational tori around a fixed point $(\overline{x},\ \overline{y})$, which corresponds to the location of the synchronous resonance. The case of librational tori around a resonance $r : s$ can be reduced to the fixed point case considering the map P^s instead of P. We arbitrarily select an initial condition $x = x_0,\ y = y_0,\ t = 0$, close to the fixed point. The corresponding rotation number is estimated as follows. Define polar coordinates $(\rho,\ \theta)$ on the Poincaré map by means of a transformation T as

$$\begin{pmatrix} x \\ y \end{pmatrix} = T \begin{pmatrix} \rho \\ \theta \end{pmatrix} = \begin{pmatrix} \overline{x} + \rho\cos\theta \\ \overline{y} + \rho\sin\theta \end{pmatrix} .$$

The rotation number is provided by the formula

$$\omega \;=\; \frac{1}{2\pi} \lim_{N\to\infty} \sum_{k=1}^{N} \frac{\theta_k - \theta_{k-1}}{N} \ , \tag{4}$$

whose approximation is given by

$$\tilde{\omega} \;=\; \frac{1}{2\pi} \sum_{k=1}^{N} \frac{\theta_k - \theta_{k-1}}{N} \ , \tag{5}$$

for some (large) integer $N > 0$, for instance $N = 10000$.
Let $\{a_k\}_{k\geq 0}$ be the continued fraction sequence associated to $\tilde{\omega}$, i.e.

$$\tilde{\omega} \;=\; [a_0, a_1, a_2, ...] \ . \tag{6}$$

In order to deal with strongly irrational rotation numbers (e.g., *noble* numbers) we slightly modify $\tilde{\omega}$ adding an infinite sequence of 1's after the jth partial quotient of its continued fraction representation. Let $\overline{\omega}$ be the corresponding number, i.e.

$$\overline{\omega} \;=\; [a_0, a_1, a_2, ..., a_j; 1^\infty] \ . \tag{7}$$

Remark: Notice that any *noble* number $\overline{\omega}$ satisfies a diophantine condition of the type $|\overline{\omega} - \frac{p}{q}|^{-1} \leq Cq^2$, for any $p,\ q \in \mathbf{Z},\ q \neq 0$ and for some positive constant C. This condition is an essential requirement for the applicability of KAM theory.

The periodic orbits approaching $\overline{\omega}$ are given by the successive truncations of the continued fraction associated to $\overline{\omega}$. Therefore we consider the periodic orbits $\mathcal{P}_\varepsilon(\frac{p_n}{q_n})$, $n = 1, 2, ...$, with frequencies given by

$$\frac{p_n}{q_n} \;=\; [a_0, a_1, a_2, ..., a_n] \ . \tag{8}$$

In order to look for the exact location of the periodic orbits, we use a Newton approach combined with a continuation method. More precisely, fix $\varepsilon = \varepsilon_0$; for a periodic orbit with frequency $\frac{p}{q}$, we go back from the polar coordinates (ρ, θ) to the cartesian coordinates (x, y). Next we compute q iterations of the Poincaré map determining the transformed variables, say (x', y'), with associated polar coordinates (ρ', θ'). In summary, we determine a map $F : (\rho, \theta) \to (\rho', \theta')$, where $F = T^{-1} \circ P^q \circ T$, and we require that the transformed variables satisfy the periodicity conditions:

$$\begin{aligned} \rho' &= \rho \\ \theta' &= \theta + 2\pi p \; . \end{aligned}$$

By a Newton's method we look for the initial conditions of the periodic orbit $\mathcal{P}_{\varepsilon_0}(\frac{p_n}{q_n})$, taking as initial guess the polar coordinates (ρ_0, θ_0) corresponding to (x_0, y_0).

In order to determine the stability of the periodic orbits, we introduce the tangent space trajectory $(\delta x_j,\ \delta y_j)$, related to the initial conditions $(\delta x_0,\ \delta y_0)$ at the point (x_0, y_0) by a matrix M:

$$(\delta x_j,\ \delta y_j) \ = \ M \ (\delta x_0,\ \delta y_0) \ ; \tag{9}$$

the matrix M is given by the product of the jacobian of the Poincaré map over the full cycle of the periodic orbit. The eigenvalues of M are the associated Floquet multipliers. According to [9], we define the *residue* by the relation

$$R \ = \ \frac{1}{4} \ (2 - \mathrm{Tr} M) \tag{10}$$

(where the factors 2 and 4 are introduced for convenience). When $0 < R < 1$ the eigenvalues of M are complex conjugated with unitary modulus and $\mathcal{P}(\frac{p}{q})$ is stable. As $R < 0$ or $R > 1$ the periodic orbit is unstable, since there exists at least one expanding direction. Therefore the value of the residue R provides the stability character of the periodic orbit.

Remark: We experienced that if the eigenvalues of M are close to 1, Newton's method aimed to look for the periodic orbits fails. However, it is possible to overcome the problem using a modified Newton's method.

Once we have a periodic orbit $\mathcal{P}_{\varepsilon_0}(\frac{p_n}{q_n})$, it is possible to follow the curve of periodic orbits in the parameter space using a continuation method ([16]). The goal is to find the critical value $\varepsilon_c^{(n)}(\overline{R})$ where the residue reaches a fixed value, say $|R| = \overline{R}$. According to Greene's criterium, the sequence $\{\varepsilon_c^{(n)}(\overline{R})\}_n$ tends to the break–down threshold of the torus. A sharp evaluation requires that the sequence $\{\varepsilon_c^{(n)}(\overline{R})\}_n$ alternates from above and below around the critical value. As Greene suggested in [9], it is better to use the sequence given by $\overline{R} = \frac{1}{4}$ which provides fast convergence to the break-down threshold.

4 RESULTS

By means of the method presented in §3, we construct numerically librational invariant surfaces and we explore their stability computing the corresponding residuals, according to Greene's method. We focus our attention on the two most widely studied cases of the solar system: the Moon and Mercury. As we know, the Moon

is actually observed to move in a synchronous resonance, while Mercury seems to be trapped in a 3:2 spin–orbit resonance. In order to investigate the behaviour of these objects in dynamical configurations eventually different from the observed ones, we explore the stability of librational islands around the main resonances, i.e. 1:1, 3:2. This analysis will possibly lead to provide evidence of the stability of the main resonances and to explain the different ending states in which the Moon and Mercury are actually observed. More precisely, as pointed out in the introduction, the existence of librational tori around a resonance provides the stability of that resonance, due to the confinement property in phase–space. Moreover, the size of the librational zone is proportional to the probability of capture into the resonance. Having fixed the orbital eccentricity e, we let the parameter ε vary. More specifically, since it is sometimes difficult to locate the periodic orbits approximating a given invariant torus (especially when approaching the chaotic separatrix), we select a suitably small value of the parameter ε and look for the periodic orbit, when ε increases, by a continuation method. Using the actual values of the eccentricity of the Moon and Mercury (i.e., $e = 0.0549$ and $e = 0.2056$, respectively), we let the oblateness parameter ε vary and compute the maximal value of ε for which librational tori around the 1:1, 3:2 resonances exist. Notice that astronomical observations yield $\varepsilon = 3.45 \cdot 10^{-4}$ for the Moon and $\varepsilon = 1.5 \cdot 10^{-4}$ for Mercury.
As for the rotation numbers (see Table 1), we selected noble numbers of the form $\omega = (0, 1, a, 1^\infty)$, with $a \in \mathbf{Z}_+$ chosen so that the size of the librational torus around the resonance is as small as possible. Notice that the value of the rotation number increases monotonically going away from the resonance up to the separatrix. The rotation numbers are smaller than one, since we are confined in the librational regime. In fact, these rotation numbers are defined modulus one. A value close to one means that the orbit rotates around the fixed point slowly clockwise direction.

Table 1.

	Resonance	Rotation Number
Moon	1:1	$\omega_{11} = (0, 1, 37, 1^\infty) = 0.974105362269573$
	3:2	$\omega_{32} = (0, 1, 42, 1^\infty) = 0.977073703040858$
Mercury	1:1	$\tilde{\omega}_{11} = (0, 1, 60, 1^\infty) = 0.983770984965496$
	3:2	$\tilde{\omega}_{32} = (0, 1, 35, 1^\infty) = 0.972691051619340$

For convenience, we discuss separately the results concerning the Moon and Mercury as follows.

4.1 Moon

We compute the periodic orbits approaching the tori with rotation numbers ω_{11}, ω_{32}, taking successive truncations of their continued fraction expansions. Table 2 reports the values of the frequencies and the corresponding initial conditions (x_0, y_0). The last column shows the value $\varepsilon_c^{(n)}(\overline{R})$ of the perturbing parameter at which $\overline{R} = 0.25$.

Table 2.

Reson.	Rot. Num.	p	q	x_0	y_0	$\varepsilon_c^{(n)}(\overline{R})$
1:1	ω_{11}	38	39	1.55428748001770	1.00035126514094	0.00417711
		75	77	1.55210324036282	1.00196468948276	0.00426337
		113	116	1.55345064856468	1.00025263102704	0.00422229
		188	193	1.55347422774804	1.00024647568483	0.00423966
3:2	ω_{32}	43	44	1.56952248398490	1.49156712158839	0.00891844
		85	87	1.56903487897801	1.49205067420178	0.00844578
		128	131	1.56945792726568	1.49155703664094	0.00896453
		213	218	1.56930360797141	1.49164283204691	0.00889058

In order to have a graphical inspection of the motion, we draw in Figures 1 and 2 the Poincaré maps of some orbits around the main resonances 1:1, 3:2. More precisely, in Figure 1 the innermost curve corresponds to the rotation number ω_{11}, whereas the perturbing parameter is set equal to the *true* value, i.e. $\varepsilon = 3.45 \cdot 10^{-4}$. The second librational curve corresponds to the frequency of table 2 closer to ω_{11}, i.e. $\omega = \frac{p}{q} = \frac{188}{193}$, while the perturbing parameter is equal to the *critical* value $\varepsilon_c^{(4)}(\overline{R}) = 0.00423966$. The outermost *rotational* curves are drawn for reference (again for $\varepsilon_c^{(4)}(\overline{R}) = 0.00423966$). Figure 2 refers to the 3:2 resonance and it is obtained according to the same choices of Figure 1. These pictures show that the ratio between the size of the librational region around the 1:1 resonance and that around the 3:2 resonance is 2 even if for different values of epsilon; in fact the value of this ratio for the true value of epsilon is 2.25, thus suggesting a bigger opportunity for the Moon to end up in the synchronous resonance.

4.2 Mercury

Analogously to the case of the Moon, we compute the periodic orbits approaching the invariant tori with rotation numbers $\tilde{\omega}_{11}$, $\tilde{\omega}_{32}$, whose frequencies and corresponding initial conditions are provided in table 3. The critical break–down values are reported in the last column of table 3.

Table 3.

Reson.	Rot. Num.	p	q	x_0	y_0	$\varepsilon_c^{(n)}(\overline{R})$
1:1	$\tilde{\omega}_{11}$	61	62	1.56325002908316	1.00234077521868	0.00287710
		121	123	1.56151428212756	1.00342954593302	0.00287773
		182	185	1.55306215799641	1.00406328447291	0.00287153
		303	308	1.55310016120454	1.00406361271904	0.00287424
3:2	$\tilde{\omega}_{32}$	36	37	1.56854391952542	1.49865901820654	0.00317528
		71	73	1.56548708659463	1.49898285440097	0.00310717
		107	110	1.56739991666779	1.49879410612854	0.00326879
		178	183	1.55278507181504	1.49980755856118	0.00325630

Figures 3 and 4 show the Poincaré mappings around the 1:1 and 3:2 resonances obtained similarly to the case of the Moon. In this case the size of the librational region around the two resonances is about the same for similar values of epsilon and the ratio between the two size for the true value of epsilon is 1.2 .

5 CONCLUSIONS

We have derived a method for computing the break–down threshold of librational invariant tori. In particular, we considered invariant tori around the main resonances (1:1, 3:2) in the spin–orbit problem. The results show that in both cases (i.e., Moon and Mercury) the *main* resonances are definitely stable for values of the perturbing parameter ε which are bigger than the actual astronomical observations.

The discrepancy of the behaviour of the Moon and Mercury might be due to the values of the eccentricity ($e = 0.0549$ for the Moon and $e = 0.2056$ for Mercury), eventually leading to different capture probabilities into a resonance.

We believe that interesting informations about the dynamics of the resonances might be derived from an extension of the analysis performed for the spin–orbit problem using a more general model, taking into account the obliquity as well as perturbations by other bodies of the solar system.

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