EXACT SPECTRAL VALUES FOR DISCRETE QUANTUM PENDULUM-INTEGRALS

JOHANNES KELLENDONK

ABSTRACT. For specific choice of parameters the spectrum of the discrete Quantum Pendulum-Integral contains the eigenvalues of a finite matrix which depends analytically on the flux. Under natural continuity assumptions these eigenvalues include the spectral values which may be obtained by the algebraic Bethe ansatz.

INTRODUCTION

Discrete Quantum Pendulum-integrals (QP-integrals) are integrals of motion of a 1+0 dimensional discrete integrable Quantum Field Theory. This Quantum Field Theory is discrete Sine-Gordon Field Theory with shortest possible periodicity in space. With space compatified in that way to one point, the model becomes that of the discrete quantum (mathematical) pendulum [BKP93]. Coincidently, QP-integrals arrise in solid state physics as well: they may be interpreted as magnetic Schrödinger operators on $\mathbb{Z}^2$ generalizing the Hofstadter Hamiltonian [KS96] which describes an elementary Quantum-Hall-system [RSar]. QP-integrals so arrise as self adjoint elements of the discrete Weyl-Heisenberg-algebra, which is the C^*-algebra generated by two unitary elements u and v which are subject to the relation $uv - q^{-1}vu$, $q = e^{i\gamma}$. γ plays here the rôle of the flux of the magnetic field. To define QP-integrals in this context, consider the family

$$\begin{aligned} H(a,b,k) \quad &:= \quad au + \bar{a}u^* + bv + \bar{b}v^* + k(q^{\frac{1}{2}}uv + q^{-\frac{1}{2}}v^*u^*) \\ &\quad\quad +k^{-1}(q^{-\frac{1}{2}}uv^* + q^{\frac{1}{2}}vu^*) \end{aligned} \tag{1}$$

parametrized by complex continuous functions $a = a(q^{\frac{1}{2}})$ and $b = b(q^{\frac{1}{2}})$ and a strictly positive real number k. We will right away focuss our attention on particular choices for a and b and write more briefly

$$\tilde{H}(z,n,k) := H(z + z^{-1}, q^{\frac{n}{2}} + q^{-\frac{n}{2}}, k), \tag{2}$$

$z = z(q^{\frac{1}{2}})$, $n \in \mathbb{N}$. In case $z(q^{\frac{1}{2}})$ is of modulus 1, $\tilde{H}(z,n,k)$ arrises as a reduction of an integral of motion of discrete Sine-Gordon Field Theory with shortest possible periodicity in space [Kut94, KKSar]. For $z(q^{\frac{1}{2}}) = q^{\frac{n}{2}}$, i.e. $a = b = 2\cos\frac{n\gamma}{2}$, $\tilde{H}(z,n,k)$ is referred to as QP-integral. $\tilde{H}(q^{\frac{n}{2}},n,k)$ was also called (doubly) discrete Quantum Pendulum, e.g. in [KS96], but this terminology can be missleading

Date: June 16, 1998.

since $\tilde H(q^{\frac{n}{2}},n,k)$ cannot be regarded as the Hamiltonian of the (doubly) discrete Quantum Pendulum. We shall proof here that, for $z+z^{-1}\in\mathbb{R}$, the spectrum of $\tilde H(z,n,k)$ contains the eigenvalues of the tridiagonal matrix

$$\tilde A(z,n,k) := \begin{pmatrix} \tilde x_0 & \tilde y_0 & & & \\ \tilde z_0 & \tilde x_1 & \tilde y_1 & & \\ & \ddots & \ddots & \ddots & \\ & & & & \\ & & & \tilde z_{n-2} & \tilde x_{n-1} \end{pmatrix} \tag{3}$$

where

$$\tilde x_j = -(z+z^{-1})(kq^{\frac{1-n}{2}+j}+k^{-1}q^{\frac{n-1}{2}-j}), \tag{4}$$

$$\tilde y_j = q^{\frac{n}{2}}+q^{-\frac{n}{2}}-(q^{j+1-\frac{n}{2}}+q^{\frac{n}{2}-j-1}), \tag{5}$$

$$\tilde z_j = q^{\frac{n}{2}}+q^{-\frac{n}{2}}-(k^2q^{j+1-\frac{n}{2}}+k^{-2}q^{\frac{n}{2}-j-1}). \tag{6}$$

Figure 1 shows a plot of the spectrum of $\tilde H(q^{\frac{3}{2}},3,2)$ against γ (for rational $\frac{\gamma}{2\pi}$ with small denominator) together with the three curves of eigenvalues of $\tilde A(q^{\frac{3}{2}},3,2)$.

Although diagonalization of the above matrix furnishes only finitely many spectral values we find this result surprising, since one expects the spectrum of $H(a,b,k)$ to be a Cantor set for $\frac{\gamma}{2\pi}\notin\mathbb{Q}$ and generic a and b, and not much is known about spectral values of operators with Cantor spectrum.

As already mentionned, in case $z(q^{\frac{1}{2}})$ is of modulus 1, $\tilde H(z,n,k)$ is an integral of motion of an integrable discrete Quantum Field Theory. The integrability means in particular that one may determine (generalized) eigenvalues and -functions of its integrals of motion, among which are the QP-integrals. This is achieved by use of the algebraic Bethe ansatz [TF79], which was adapted to the present case by Nadja Kutz [Kut94]. It leads to the result [Kut94, KKSar]:

Theorem 1. *Let $\delta\in S^1=\mathbb{R}/\mathbb{Z}$, $n\in\mathbb{N}$, and $k>0$. If the set of $n-1$ spectral parameters $\eta_1,\dots,\eta_{n-1}$, which ought to be complex, pairwise different numbers, satisfies the so-called Bethe ansatz-equations*

$$\prod_{j\neq i=1}^{n-1}\frac{\eta_j q-\eta_i}{\eta_j-\eta_i q} = e^{2i\delta}\frac{q^{-\frac{1}{2}}\eta_j^2+\eta_j(k+k^{-1})+q^{\frac{1}{2}}}{q^{\frac{1}{2}}\eta_j^2+\eta_j(k+k^{-1})+q^{-\frac{1}{2}}} \tag{7}$$

then, with $q=e^{i\gamma}$,

$$\begin{aligned} E(\eta_1,\dots,\eta_{n-1}) &= -2(k+k^{-1})\cos\left(\tfrac{n-1}{2}\gamma+\delta\right) \\ &\quad +4\sin\left(\tfrac{n-1}{2}\gamma+\delta\right)\,\sin\tfrac{\gamma}{2}\textstyle\sum_{j=1}^{n-1}\eta_j^{-1}(\gamma). \end{aligned}$$

lies in the spectrum of $\tilde H(e^{i\delta},n,k)$.

A formula for eigenfunctions (in the generalized sense, i.e. not neccessarily normalizable) corresponding to these so-called Bethe ansatz-eigenvalues will be recalled further down.

We expect that the so-called Bethe ansatz eigenvalues (1) coincide with the eigenvalues of $\tilde{A}(e^{i\delta}, n, k)$. Indeed, we shall derive that the Bethe ansatz eigenvalues are among the eigenvalues of $\tilde{A}(e^{i\delta}, n, k)$ in case $k = 1$ and extend this result to $k \neq 1$ under natural assumptions on the behaviour of the Bethe ansatz eigenvalues on k, z and γ. Therefore, our result may also be understood as a simpler approach to the determination of the Bethe ansatz eigenvalues and we expect that a similar simplification occurrs in other models, too, in particular those coming from the Sine-Gordon model. In fact, the Bethe ansatz-equations are rather difficult to solve. Already for $n = 3$ they lead to polynomial equations which have orders to high to be tractable by analytic means. In [KKSar], solutions for $k > 1$ were for that reason only obtained in case $n \leq 2$.

The key to compare the Bethe ansatz eigenvalues with the eigenvalues of $\tilde{A}(e^{i\delta}, n, k)$ is to look first at $k = 1$, a case, in which the solution of the Bethe ansatz-eigenvector simplifies enormously. It is shown in [KKSar] that the Bethe ansatz-eigenvector is then not only a true eigenvector, or, more precisly, a square summable vector in a particular representation on $\ell^2(\mathbb{Z})$, but it even vanishes in that representation outside $\{0, \cdots, n-1\} \subset \mathbb{Z}$. This leads us to a remark on yet another approach to determine spectral values of $\tilde{H}(z, n, k)$ which we will, however, not follow here. That approach is essentially an ansatz for eigenfunctions of $\tilde{H}(z, n, k)$ which have finite support. Formulated in a representation on $L^2(S^1)$, the Fourier space of $\ell^2(\mathbb{Z})$, the above ansatz is one for polynomial eigenfunctions. It may then also be reformulated in terms of Bethe ansatz-equations [WZ94, KKSar] (different from (7)) which, however, are again rather difficult to solve.

After recalling some important facts about the discrete Weyl-Heisenberg-algebra, which we do in the next section following mainly [KKSar], we prove in Section 2 that the eigenvalues of $\tilde{A}(z, n, k)$ are spectral values of $\tilde{H}(z, n, k)$ by showing that the characteristic polynomial factorizes in the appropriate way. In Section 3 we relate the Bethe ansatz-eigenvalues to these eigenvalues.

1. Preliminaries

The discrete Weyl-Heisenberg- or rotation algebra $\mathcal{A}_\gamma$ with angle γ is the C^*-envelope of the $*$-algebra generated by two elements u and v which are subject to the relations [Rie81]

$$uu^* - 1, \quad u^*u - 1, \quad vv^* - 1, \quad v^*v - 1, \quad uv - q^{-1}vu,$$
$$q = e^{i\gamma}.$$

Any element of the discrete Weyl-Heisenberg-algebra $\mathcal{A}_\gamma$ may be approximated in norm by finite sums of the form $\sum c_n u^{n_1} v^{n_2}$, $n \in \mathbb{Z}^2$, $c_n \in \mathbb{C}$. A faithful representation of $\mathcal{A}_\gamma$ on the Hilbert space $\ell^2(\mathbb{Z}^2)$ is given by

$$u \cdot \psi(n) = e^{iA_1(n)} \psi(n_1 - 1, n_2), \tag{8}$$

$$v \cdot \psi(n) = e^{iA_2(n)} \psi(n_1, n_2 - 1), \tag{9}$$

where $A : \mathbb{Z}^2 \to \mathbb{R}^2$ is a gauge potential with discrete rotation γ:

$$(A_2(n) - A_2(n_1 - 1, n_2)) - (A_1(n) - A_1(n_1, n_2 - 1)) = \gamma,$$

and we have used the convention to denote the action of an element $x \in \mathcal{A}_\gamma$ in some representation space simply by $\cdot$. In other words, $\mathcal{A}_\gamma$ is isomorphic to the norm-closure of the subalgebra of bounded operators of $\ell^2(\mathbb{Z}^2)$ generated by u and v in the above representation. The above representation yields the framework for the discretized version of the Landau model which describes an electric particle in the discretized plane $\mathbb{Z}^2$ to which a constant magnetic field is perpendicularly applied. More precisely, $\mathcal{A}_\gamma$ is the algebra of observables for that model, and the famous Hofstadter Hamiltonian, which plays the rôle of the discrete Landau Hamiltonian, equals $u + v + u^* + v^*$ acting in the above representation. The angle γ is proportional to the magnetic field or the flux per unit cell. The function A is for given γ unique up to a discrete gradient. Different gauges lead in general to different representations which are, however, unitarily equivalent. For simplicity we will call $\frac{\gamma}{2\pi}$ the flux.

The above representation is faithful but not irreducible. Its decomposition into irreducible components leads, for rational $\frac{\gamma}{2\pi}$, to a family of representations labelled by a 2-torus, the 2-torus of quasimomenta, or, Bloch-parameters. The first step of this decomposition, which applies as well to irrational $\frac{\gamma}{2\pi}$, leads to the family of Weyl-Schrödinger representations, labelled by an angle θ, with representation space $\ell^2(\mathbb{Z})$. They are given by

$$u \cdot \psi(n) = e^{i\theta} q^{-n} \psi(n) \tag{10}$$

$$v \cdot \psi(n) = \psi(n-1). \tag{11}$$

For irrational $\frac{\gamma}{2\pi}$ the Weyl-Schrödinger representations are irreducible and still faithful, for rational $\frac{\gamma}{2\pi}$ this is not the case. In fact, for $\frac{\gamma}{2\pi} = \frac{M}{N}$, $(M, N) = 1$ (meaning that M and N are coprime), the Weyl-Schrödinger representations are neither irreducible nor faithful. In that case, the Weyl-Schrödinger representation with angle θ decomposes into a direct integral of N-dimensional irreducible representations parametrized by a second angle. Denoting the second angle by φ, u has in the irreducible representation labelled by (θ,φ) matrix representation

$$u_{(\theta,\varphi)} = e^{i\theta} \begin{pmatrix} q^{-1} & & & & \\ & \ddots & & & \\ & & q^{-l} & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}, \tag{12}$$

which in particular is independent of φ, and v

$$v_{(\theta,\varphi)} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & \ddots & 1 \\ e^{i\varphi} & 0 & \cdots & & 0 \end{pmatrix}. \tag{13}$$

Two such matrix representations, one for (θ,φ) and one for (θ',φ'), are unitarily equivalent whenever $\theta-\theta' = n\gamma$ for some $n\in\mathbb{Z}$ and $\varphi=\varphi'$, that is if $u_\theta^N = u_{\theta'}^N$ and $v_\varphi^N = v_{\varphi'}^N$. The asymmetry appearing in the parameters (θ,φ) labelling the irreducible components is, from a physical point of view, artificial. In the (discrete) Landau model, u^N and v^N are translation operators along N sites in the 1- and 2-direction, respectively, and the parameters $N\theta$ and φ have the physical interpretation of quasimomentum in 1- and 2-direction, respectively. We denote by $x_{(\theta,\varphi)}$ the matrix representation of $x\in\mathcal{A}_\gamma$ in the N-dimensional representation labelled by (θ,φ).

It follows from the above consideration that the spectrum $\sigma(x)$ of any element $x\in\mathcal{A}_\gamma$ is, for irrational $\frac{\gamma}{2\pi}$ equal to the spectrum of x in any Weyl-Schrödinger representation, and for rational $\frac{\gamma}{2\pi}$ given by

$$\sigma(x) = \bigcup_{(\theta,\varphi)\in S^1\times S^1} \sigma(x_{(\theta,\varphi)}).$$

Since $x_{(\theta,\varphi)}$ is a finite-dimensional matrix its spectrum consists of eigenvalues, the zeroes of its characteristic polynomial. It turns out that, for our family of operators, the characteristic polynomial has a very special form, namely, with $\frac{\gamma}{2\pi}=\frac{M}{N}$, $(M,N)=1$,

$$\det(H_{(\theta,\varphi)}(a,b,k)-E) = p(a,b,k;E) + h(a,b,k;\theta,\varphi)$$

where $p(a,b,k;E)$ is a polynomial in E of degree N which is independent of (θ,φ), and $h(a,b,k;\theta,\varphi)$, the so-called off-set function, is independent of E. Such a relation is called a Chambers relation. While the polynomial p may look rather complicated the off-set function h is explicitely known [KS96, KKSar]. With $T_N(\frac{z+z^{-1}}{2}) = \frac{z^N+z^{-N}}{2}$ for any complex $z \neq 0$ it is

$$\begin{aligned} h(a,b,k;\theta,\varphi) &= -2\left(T_N(-\frac{a}{2})e^{iN\theta} + T_N(-\frac{b}{2})e^{i\varphi}\right) \\ &\quad +(-1)^M\left(k^N e^{iN\theta+\varphi} + k^{-N}e^{iN\theta-\varphi}\right) + c.c.. \end{aligned}$$

We also write

(14)

$$\tilde{h}(z,n,k;\zeta_1,\zeta_2) = h(z+z^{-1}, 2\cos\frac{n\gamma}{2}, k;\theta,\varphi), \quad \zeta_1 = e^{iN\theta}, \quad \zeta_2 = e^{i\varphi}.$$

For the above result we have normalized $p(a,b,k;0) = 0$ and c.c. stands for complex conjugate. T_N is, up to constants, the Nth Chebychev polynomial of the second kind.

Despite the irregular behaviour of $p(a,b,k;E)$ on γ, the spectrum of $H(a,b,k)$ depends in a certain sense continuously on the flux. The collection of all $\mathcal{A}_\gamma$, $\gamma \in S^1$ can be combined into a continuous field of C^*-algebras over S^1 [Rie89]. Its continuous sections, which are in particular functions $S^1 \to \bigcup_\gamma \mathcal{A}_\gamma$: $\gamma \mapsto x_\gamma \in \mathcal{A}_\gamma$ for which $\gamma \mapsto \|x_\gamma\|$ is continuous, form a C^*-algebra, and it is shown [Ell82] that, under the hypothesis that all x_γ are normal, the following continuity property holds: for each open $U \subset \mathbb{C}$, the set of $\gamma \in S^1$ for which $\sigma(x_\gamma) \cap U \neq \emptyset$ and the set of $\gamma \in S^1$ for which $\sigma(x_\gamma) \in U$ are both open. In the present case, $H(a,b,k) = H_\gamma(a(\gamma), b(\gamma), k)$, which depends on γ also through the relation between u and v, is a continuous section of selfadjoint elements provided we remove one point (let's say π) from S^1. We are forced to remove one point, because $H_\gamma(a(\gamma), b(\gamma), k)$ contains square roots of $q = e^{i\gamma}$. An immediate consequence of the above property is therefore that any continuous function $E : S^1\backslash\{\pi\} \to \mathbb{C}$ which satisfies $E(\gamma) \in \sigma(H_\gamma(a(\gamma), b(\gamma), k))$ on a dense subset of S^1 must already satisfy $E(\gamma) \in \sigma(H_\gamma(a(\gamma), b(\gamma), k))$ for all $\gamma \in S^1\backslash\{\pi\}$.

2. Factorization of the characteristic polynomial

As already mentionned in the introduction, if $k = 1$, a different ansatz to determine the eigenvalues (1) of $H(a,b,k)$ is one for eigenfunctions of finite support in a Weyl-Schrödinger representation of $\mathcal{A}_\gamma$. Denoting by $H_\theta(a,b,k)$ the operator $H(a,b,k)$ acting in such a representation with angle θ we have

$$H_\theta(a,b,k)\psi(m) = z_{m-1}\psi(m-1) + x_m\psi(m) + y_m\psi(m+1)$$

where

$$x_m = aq^{-m}e^{i\theta} + \bar{a}q^m e^{-i\theta}, \tag{15}$$

$$y_m = b + (kq^{-m-\frac{1}{2}}e^{i\theta} + inv.), \tag{16}$$

$$z_m = \bar{b} + (k^{-1}q^{-m-\frac{1}{2}}e^{i\theta} + inv.), \tag{17}$$

$x + inv.$ standing for $x + x^{-1}$ and θ labelling the representation. Note that $H_\theta(a,b,k)$ is selfadjoint only if $k \in \mathbb{R}$. We have required this above and in [KKSar] for the definition of QP-integrals but below we allow k to be complex.

Lemma 1. *Let $\theta \in \mathbb{C}$ and $n \in \mathbb{N}$. If $k = -q^{\frac{n-1}{2}} e^{-i\theta}$ and $b = 2\cos\frac{n\gamma}{2}$ then $y_{-1} = y_{n-1} = 0$.*

Proof: Setting $\kappa = kq^{\frac{1}{2}} e^{i\theta}$ one obtains that $y_{-1} = y_{n-1} = 0$ is equivalent to $-b = \kappa + \kappa^{-1} = \kappa q^{-n} + \kappa^{-1} q^n$. This, in turn, is equivalent to $\kappa = \epsilon q^{\frac{n}{2}}$ and $b = -\epsilon 2\cos\frac{n\gamma}{2}$, $\epsilon = \pm 1$, the statement of the lemma being one choice of sign. q.e.d.

In view of the above lemma we shall from now on restrict the parameter b to be

$$b = 2\cos\frac{n\gamma}{2}.$$

Let P_n the orthogonal projection onto the subspace of $\ell^2(\mathbb{Z})$ of all wavefunctions which vanish outside $\{0, \cdots, n-1\}$. The operator

$$A_\theta(z,n,k) := P_n \tilde{H}_\theta(z,n,k) P_n$$

has, when restricted to the image of P_n, (tridiagonal) matrix representation

$$A_\theta(z,n,k) = \begin{pmatrix} x_0 & y_0 & & & \\ z_0 & x_1 & y_1 & & \\ & \ddots & \ddots & \ddots & \\ & & & & \\ & & & z_{n-2} & x_{n-1} \end{pmatrix}.$$

The following result was also obtained in [KKSar].

Theorem 2. *If $k = 1$ and $\theta = \pi + \frac{n-1}{2}\gamma$ then the eigenvalues of $\tilde{A}(z,n,1)$ coincide with those eigenvalues of $H_\theta(z,n,1)$ whose corresponding eigenvectors vanish outside $\{0, \cdots, n-1\} \subset \mathbb{Z}$.*

Proof: If $k = 1$ and $\theta = \pi + \frac{n-1}{2}\gamma$ then $\tilde{A}(z,n,k) = A_\theta(z,n,k)$ and $y_{-1} = y_{n-1} = 0$ and $y_j = z_j$. This implies that $H_{\pi+\frac{n-1}{2}\gamma}(z,n,1)$ commutes with P_n from which the statement follows. q.e.d.

Let

(18)
$$\tilde{\mathcal{B}}(z,n,k) := \{(\zeta_1,\zeta_2) \in \mathbb{C}\times\mathbb{C} | \zeta_2 = -z^N \frac{(-1)^{Mn}\zeta_1 - (-1)^{M+N}k^N}{1-(-1)^{M(n-1)+N}\zeta_1 k^N}\}$$

and

(19)
$$\mathcal{B}(z,n,k) := \{(\theta,\varphi) \in S^1 \times S^1 | (e^{iN\theta}, e^{i\varphi}) \in \tilde{\mathcal{B}}\}.$$

In [KKSar] it was shown that $\mathcal{B}(z,n,k)$ furnishes, for stictly positive k and $|z| = 1$, the set of labels (Bloch-parameters) for the irreducible representations of $\mathcal{A}_\gamma$ in which the Bethe ansatz-eigenvalues (1) are spectral values of $\tilde{H}_{(\theta,\varphi)}(z,n,k)$. The following lemma, which follows for stictly positive k and $|\zeta_1| = |\zeta_2| = |z| = 1$ from that result, is straightforwardly verified:

Lemma 2. *The off-set function $\tilde{h}(z,n,k;\zeta_1,\zeta_2)$ is for $z+z^{-1}\in\mathbb{R}$, non-zero $k\in\mathbb{C}$, and $n\in\mathbb{N}$ constant on $\tilde{\mathcal{B}}(z,n,k)$.*

In the irreducible representation of $\mathcal{A}_{\frac{2\pi M}{N}}$ labelled by (θ,φ), $\tilde{H}(z,n,k)$ is represented by the matrix (tridiagonal apart from the entries in the corners)

$$\tilde{H}_{(\theta,\varphi)}(z,n,k) = \begin{pmatrix} x_0 & y_0 & & & z_{-1}e^{-i\varphi} \\ z_0 & x_1 & y_1 & & \\ & \ddots & \ddots & \ddots & \\ & & & & \\ y_{-1}e^{i\varphi} & & & z_{N-2} & x_{N-1} \end{pmatrix}.$$

Theorem 3. *Let $z+z^{-1}\in\mathbb{R}$ and, for $\frac{\gamma}{2\pi}=\frac{M}{N}$, $(M,N)=1$,*

$$\chi(z,n,k,E) := p(z+z^{-1}, 2\cos\frac{n\gamma}{2}, k; E) + h(z,n,k;\zeta_1,\zeta_2)$$

where $(\zeta_1,\zeta_2)\in\tilde{\mathcal{B}}(z,n,k)$. Then

$$\chi(z,n,k,E) = \det(\tilde{A}(z,n,k)-E)\ \det(\tilde{B}(z,n,k)-E), \tag{20}$$

where

$$\tilde{B}(z,n,k) := \begin{pmatrix} \tilde{x}_n & \tilde{y}_n & & & \\ \tilde{z}_n & \tilde{x}_{n+1} & \tilde{y}_{n+1} & & \\ & \ddots & \ddots & \ddots & \\ & & & & \\ & & & \tilde{z}_{N-2} & \tilde{x}_{N-1} \end{pmatrix}.$$

In particular, the eigenvalues of $\tilde{A}(z,n,k)$ are, even for arbitary $\frac{\gamma}{2\pi}$, spectral values of $\tilde{H}(z,n,k)$, their corresponding (generalized) eigenvectors occurring in irreducible representations which, for $\frac{\gamma}{2\pi}=\frac{M}{N}$, $(M,N)=1$, have Bloch parameters in $\mathcal{B}(z,n,k)$.

Proof: Consider first the rational case $\frac{\gamma}{2\pi}=\frac{M}{N}$, $(M,N)=1$, in which $\chi(z,n,k,E)$ is the characteristic polynomial of $\tilde{H}_{(\theta,\varphi)}(z,n,k)$ with $(\theta,\varphi)\in\mathcal{B}(z,n,k)$. Since $\chi(z,n,k,E)$ is independent on the value of $(e^{iN\theta},e^{i}\varphi)$ as long as the latter belongs to $\tilde{\mathcal{B}}(z,n,k)$ we may consider complex $k\neq 0$ and complex θ such that $k=-q^{\frac{n-1}{2}}e^{-i\theta}$. Then, using Lemma 1, the characteristic polynomial can be written as

$$\begin{aligned}\chi(z,n,k,E) &= \det(A_\theta(z,n,k)-E)\ \det(B_\theta(z,n,k)-E)\\ &\quad -(-1)^N e^{-i\varphi}\prod_{j=0}^{N-1} z_j,\end{aligned}$$

where $B_\theta(z,n,k) := (1-P_n)\tilde{H}_{(\theta,\varphi)}(z,n,k)(1-P_n)$ and $(e^{iN\theta},e^{i}\varphi)\in\tilde{\mathcal{B}}(z,n,k)$. (Here P_n has to be understood as the projection onto the first n components of the vectors in $\mathbb{C}^N$.) But $(e^{iN\theta}, e^{i}\varphi) \in \tilde{\mathcal{B}}(z,n,k)$ implies

$$e^{-i\varphi} = -z^{-N} \frac{1-(-1)^{M(n-1)+N} e^{iN\theta}(-e^{-i\theta} q^{\frac{n-1}{2}})^N}{(-1)^{Mn} e^{iN\theta} - (-1)^{M+N}(-e^{-i\theta} q^{\frac{n-1}{2}})^N} = 0.$$

Furthermore, at the above value for θ we have $A_\theta(z,n,k) = \tilde{A}(z,n,k)$ and $B_\theta(z,n,k) = \tilde{B}(z,n,k)$. This proves equation (20). Thus all eigenvalues of $\tilde{A}(z,n,k)$ are spectral values of $\tilde{H}(z,n,k)$, at least if $\frac{\gamma}{2\pi}$ is rational. By the continuity property of the spectrum with respect to γ, which we mentionned in the last section, this is also the case for irrational $\frac{\gamma}{2\pi}$. q.e.d.

3. Comparison with Bethe ansatz-eigenvalues

As already mentionned in the introduction, the algebraic Bethe ansatz can be used to determine exact spectral values of $\tilde{H}(z,n,k)$ as well. In this section we argue that the spectral values determined in Section 2 contain the Bethe ansatz-eigenvalues, the argumentation being based on natural continuity assumptions which we cannot proof yet. Much of the work has already been done, because Theorem 3 states in particular that the Bloch-parameters, which label, for rational flux, the irreducible representations in which spectral values of $\tilde{H}(z,n,k)$ coincide with the eigenvalues of $\tilde{A}(z,n,k)$, belong to $\mathcal{B}(z,n,k)$. On the other hand it has been shown in [KKSar] that the Bethe ansatz-eigenvalues (1) are spectral values of $\tilde{H}(z,n,k)$ in these representations as well. Therefore, to assert our claim in the rational case we just need to show that the spectral values of $\tilde{H}(z,n,k)$ coinciding with the eigenvalues of $\tilde{A}(z,n,k)$ on the one hand and its Bethe ansatz-eigenvalues on the other belong to the same h-bands. By an h-band we mean the closure of a connected component of $p^{-1}((h_{min}, h_{max}))$, the preimage under p understood as a polynomial in E (i.e. $p(E) = p(z+z^{-1}, 2\cos\frac{n\gamma}{2}, k; E)$) of the open interval (h_{min}, h_{max}), h_{min} and h_{max} denoting the absolute minimum and maximum, respectively, of the off-set function as a function of (θ, φ). The strategy is, to show this for $k=1$ by comparison of the eigenfunctions, and then to extend this result to arbitrary $k \neq 0$ by topological arguments assuming continuity.

First, we recall from [KKSar] a few basic results about the algebraic Bethe ansatz for the discrete Quantum Pendulum and more general reductions of integrals of motion of discrete Sine Gordon Field Theory.

At the heart of the algebraic Bethe ansatz lies the monodromy matrix $\mathcal{M}(\eta)$ which depends on a spectral parameter η and satisfies the Yang-Baxter-equation with the fundamental R-matrix of $U_{q^{\frac{1}{2}}}(sl_2)$. We will not explain here how to obtain this matrix but refer the reader to [BKP93, KKSar]. The monodromy matrix is a 2×2-matrix, and the algebraic Bethe ansatz is an ansatz to obtain eigenvalues of its trace $\mathcal{M}_{11}(\eta) + \mathcal{M}_{22}(\eta)$ which is an integral of motion of the discrete Sine-Gordon Field Theory. Starting point is the construction of a so-called Bethe ansatz-groundstate, which is a nullvector of $\mathcal{M}_{21}(\eta)$. One proceeds to construct "excited" states by applying the "ladder operator" $\mathcal{M}_{12}(\eta)$ one or several times. The Yang-Baxter-equation gives rise to a commutation relation between the trace $\mathcal{M}_{11}(\eta) + \mathcal{M}_{22}(\eta)$ and $\mathcal{M}_{12}(\eta)$ which may be used to determine more eigenvalues of the trace in a purely algebraic way, i.e., just upon using these relations. Their corresponding eigenvectors arise upon application of products like $\mathcal{M}_{12}(\eta_1)\cdots\mathcal{M}_{12}(\eta_n)$ to the Bethe ansatz-groundstate, however, due to the nature of the above commutation relations, only with a special choice of spectral parameters $\eta_1, \cdots, \eta_n$. The system of equations which determines these spectral values is called Bethe ansatz-equations. This Bethe ansatz is called algebraic since the determination of the eigenvalues does not require the use of a representation. In particular, to obtain these so-called Bethe ansatz-eigenvalues, one just has to know that a Bethe ansatz-groundstate exists but not what it looks like. For the present case, the Bethe ansatz-equations and -eigenvalues are stated in Theorem 1. But this theorem says nothing about the representations in which the Bethe ansatz-eigenvalues may be found.

In the present case, the entries of the monodromy matrix $\mathcal{M}(\eta)$ lie in a C^*-algebra. Let $\mathcal{C}$ be the C^*-algebra generated by four unitaries $\{u, v, w, A^{(2)}\}$ of which u and v satisfy $uv = q^{-1}vu$, i.e. they generate $\mathcal{A}_\gamma$, and w and $A^{(2)}$ commute with every other element. Hence $\mathcal{C}$ is isomorphic to $\mathcal{A}_\gamma \otimes C(S^1 \times S^1)$. We may describe the monodromy matrix as follows. It satisfies $\mathcal{M}_{22}(\eta)^* = \mathcal{M}_{11}(\eta)$, where we treat the spectral paramater as a variable which is independent under the $*$-operation, and

$$\mathcal{M}_{11}(\eta) = A^{(0)} - \eta A^{(1)} + \eta^2 A^{(2)}$$

with

$$A^{(0)} = w^2 A^{(2)}$$

and

$$A^{(1)} + {A^{(1)}}^* = H(wA^{(2)} + (wA^{(2)})^*, w + w^*, k).$$

(The precise form of $A^{(1)}$ is not of importance here.) $H(wA^{(2)} + (wA^{(2)})^*, w + w^*, k)$, which is (1) with operator valued a and b, should now be understood as an element of $\mathcal{C}$. In particular, there are three integrals of motion for the discrete sine Gordon theory with shortest possible periodicity, namely $A^{(0)} + {A^{(0)}}^*$, $A^{(1)} + {A^{(1)}}^*$, and $A^{(2)} + {A^{(2)}}^*$. The spectra of $A^{(0)}$ and of $A^{(2)}$ are both $\{z \in \mathbb{C} | \, |z| = 1\}$, and our task is to detemine the spectrum of $A^{(1)} + {A^{(1)}}^*$.

The off-diagonal entries of the monodromy matrix satisfy $\mathcal{M}_{21}(\eta)^* = -\mathcal{M}_{12}(\eta)$, and one may write

$$\mathcal{M}_{21}(\eta) = SC(\eta)$$

where $C(\eta) \in \mathcal{C}$ and S is a unitary element which satisfies the following relations with the generators of $\mathcal{C}$

$$\begin{aligned}
A^{(2)}S &= q^{\frac{1}{2}}SA^{(2)}, && (21)\\
wS &= q^{-\frac{1}{2}}Sw, && (22)\\
uS &= q^{\frac{1}{2}}Su, && (23)\\
vS &= qSv. && (24)
\end{aligned}$$

Expansion of $C(\eta)$ in powers of the spectral parameter results in

$$C(\eta) = \eta^{\frac{1}{2}}(C^{(\frac{1}{2})} - \eta C^{(\frac{3}{2})})$$

with

$$\begin{aligned}
C^{(\frac{1}{2})} &= q^{-\frac{1}{2}}\left((k^{-\frac{1}{2}} + k^{\frac{1}{2}}q^{-\frac{1}{2}}wu^*)w^2A^{(2)} + (k^{-\frac{1}{2}}u^* + k^{\frac{1}{2}}q^{\frac{1}{2}}w)v\right),\\
C^{(\frac{3}{2})} &= q^{-\frac{3}{2}}wu^*\left((k^{-\frac{1}{2}} + k^{\frac{1}{2}}q^{-\frac{1}{2}}wu^*)qA^{(2)} + (k^{-\frac{1}{2}}u^* + k^{\frac{1}{2}}q^{\frac{1}{2}}w)v\right).
\end{aligned}$$

In distinction to [KKSar] we have formulated here the operators in terms of the generators of $\mathcal{C}$ and written u and v for P_1 and P_2, respectively. What should be kept in mind is that $C(\eta)$ has the form

$$C(\eta) = f_1(u, w, A^{(2)}, \eta) + f_2(u, w, A^{(2)}, \eta)\, v \tag{25}$$

with two continuous functions f_1 and f_2.

3.1. **Bethe ansatz-groundstate.** The Bethe ansatz ground state is a null-vector Ω of $\mathcal{M}_{21}(\eta)$ and therefore a solution of the two equations

$$\begin{aligned}
\left((k^{-\frac{1}{2}} + k^{\frac{1}{2}}q^{-\frac{1}{2}}wu^*)w^2A^{(2)} + (k^{-\frac{1}{2}}u^* + k^{\frac{1}{2}}q^{\frac{1}{2}}w)v\right)\Omega &= 0 && (26)\\
(w^2 - q)A^{(2)}\Omega &= 0. && (27)
\end{aligned}$$

To solve for such a nullvector one has to choose a representation for $\mathcal{C}$. But since w is central in $\mathcal{C}$, we may as well take a representation of the quotient algebra which is $\mathcal{C}$ modulo the ideal generated by the element $w^2 - q$. We choose a representation with representation space $\ell^2(\mathbb{Z})$, labelled by two angles θ and δ:

$$\begin{aligned}
u \cdot \phi(n) &= e^{i\theta}q^{-n}\phi(n), && (28)\\
v \cdot \phi(n) &= \phi(n-1), && (29)\\
A^{(2)} \cdot \phi(n) &= q^{-\frac{1}{2}}e^{i\delta}\phi(n). && (30)
\end{aligned}$$

Note that this representation extends the Weyl-Schrödinger representation for the algebra generated by u and v. Upon substituting $q^{\frac{1}{2}}$ for w equation (26) looks in the above representation, where Ω is a function over $\mathbb{Z}$,

$$(k^{-\frac{1}{2}} + k^{\frac{1}{2}}e^{-i\theta}q^n)q^{\frac{1}{2}}e^{i\delta}\Omega(n) + (k^{-\frac{1}{2}}e^{-i\theta}q^n + k^{\frac{1}{2}}q)\Omega(n-1) = 0. \tag{31}$$

As already mentionned, for the derivation of Theorem 1 one only needs to show that a solution of the above equation in form of a bounded function over $\mathbb{Z}$ exists. But in order to relate the Bethe ansatz-eigenvalues with the eigenvalues of $\tilde{A}(k)$, we will relate their corresponding eigenvectors. For this reason, the structure of the solution for Ω is important for us. To solve (31) there are two cases to distinguish. In case that, for all $n \in \mathbb{Z}$, $ke^{-i\theta}q^n \neq -1$, (26) gives rise to a recursion relation for Ω which is easily seen to have a bounded solution. It also gives rise to the equation for the Bloch-parameters (θ, φ) at which, for rational flux, the Bethe ansatz-eigenvalue corresponding to the groundstate exists, or stated differently, which label the irreducible representation to which Ω belongs. That consideration, which has been carried out in [KKSar] – though in a different representation, leads ultimately to the result that, for rational flux, the Bethe ansatz-eigenvalues (1) occurr in irreducible representations which are labelled by $(\theta, \varphi) \in \mathcal{B}(e^{i\delta}, n, k)$. But in this section we are interested in the case $k = 1$ and $\theta = \pi$. Then $\Omega(n) = \delta_{n0}$ is a solution of (31), where δ_{mn} is the Kronecker symbol, $\delta_{mn} = 1$ if $n = m$ and 0 otherwise.

3.2. **"Excited" Bethe ansatz-eigenstates.** Since $w+w^*$ and $wA^{(2)}+(wA^{(2)})^*$ are central elements in $\mathcal{C}$ they act as scalar operators in irreducible representations of $\mathcal{C}$. These scalars are integrals of motion and consequently good quantum numbers. They may be interpreted as labels for a super selection sector. Since the "ladder" operator $\mathcal{M}_{12}(\eta)$ has the form $\mathcal{M}_{12}(\eta) = -C^*(\eta)S^*$ and S does not commute with the elements of $\mathcal{C}$ it cannot be implemented as an operator in the above representation and its application to a state in a specific sector will change the sector. It is convenient to combine all these sectors into one large representation. As representation space we take $\mathcal{H} = \bigoplus_{n\in\mathbb{Z}} \mathcal{H}_n$ where each $\mathcal{H}_n$ is a copy of $\ell^2(\mathbb{Z})$ which is preserved by $\mathcal{C}$ and plays the rôle of a sector. Identifying $\mathcal{H}$ with $\ell^2(\mathbb{Z}^2)$ by denoting the component of $\Psi \in \mathcal{H}$ which belongs to $\mathcal{H}_n$ by $\Psi(\cdot, n)$ we represent the algebra generated by $\mathcal{C}$ and the element S as

$$u \cdot \Psi(n_1, n_2) = e^{i\vartheta} q^{\frac{n_2}{2} - n_1} \Psi(n_1, n_2), \tag{32}$$

$$v \cdot \Psi(n_1, n_2) = \Psi(n_1 - 1, n_2), \tag{33}$$

$$w \cdot \Psi(n_1, n_2) = q^{\frac{n_2}{2}} \Psi(n_1, n_2), \tag{34}$$

$$A^{(2)} \cdot \Psi(n_1, n_2) = e^{i\delta} q^{-\frac{n_2}{2}} \Psi(n_1, n_2), \tag{35}$$

$$S \cdot \Psi(n_1, n_2) = q^{-n_1} \Psi(n_1, n_2 - 1). \tag{36}$$

The angles ϑ and δ label the representation. It is straightforward to check that this definition is compatible with the relations among the elements of $\mathcal{C}$ and S.

Lemma 3. *Consider the operator $H(wA^{(2)} + (wA^{(2)})^*, w + w^*, 1)$ acting in the above representation with $\vartheta = \pi - \frac{\gamma}{2}$. It preserves the subspaces $\mathcal{H}_n$ and its restriction to $\mathcal{H}_1$ coincides with $\tilde{H}_\pi(e^{i\delta}, 1, 1)$. Moreover, a Bethe ansatz-groundstate for $\tilde{H}_\pi(e^{i\delta}, 1, 1)$ may be identified with the vector $\Omega(n_1, n_2) = \delta_{n_1 0}\delta_{n_2 1}$ in $\mathcal{H}$ and any vector of the form $\mathcal{M}_{12}(\eta_1)\cdots\mathcal{M}_{12}(\eta_{n-1})\Omega$ vanishes outside the subset $\{(0, n), \ldots, (n-1, n)\} \subset \mathbb{Z}^2$.*

Proof: That $H(wA^{(2)} + (wA^{(2)})^*, w + w^*, 1)$ preserves the subspaces $\mathcal{H}_n$ is straightforwardly verified. Its restriction to $\mathcal{H}_1$ clearly coincides with $\tilde{H}(e^{i\delta}, 1, 1)$ acting in the Weyl-Schrödinger representation with angle $\theta = \vartheta + \frac{\gamma}{2}$. Hence, if $\vartheta = \pi - \frac{\gamma}{2}$, then $\Omega(n) = \delta_{n0}$ is a Bethe ansatz-groundstate for $\tilde{H}(e^{i\delta}, 1, 1)$ in that Weyl-Schrödinger representation. In representation (32-36) Ω has to be identified with $\Omega(n_1, n_2) = \delta_{n_1 0}\delta_{n_2 1}$, i.e. it has support $\{(0, 1)\}$. The lemma follows therefore from the special form (25) of $C(\eta)$ and Eqns. (33,36). q.e.d.

Now consider a Bethe ansatz-eigenvector in the above representation on $\mathcal{H}$. It is a vector of the form $\mathcal{M}_{12}(\eta_1)\cdots\mathcal{M}_{12}(\eta_{n-1})\Omega$, $n \in \mathbb{N}$, "excited" when $n > 1$, where $\eta_1, \ldots, \eta_{n-1}$ satisfy the Bethe ansatz-equations (7). Although there are, in general, infinitely many such vectors for $H(wA^{(2)} + (wA^{(2)})^*, w + w^*, k)$, their corresponding eigenvalues do not exhaust the spectrum of this operator. In particular, (34) and Lemma 3 imply that the eigenvalue of w on the above Bethe ansatz-eigenvector $\mathcal{M}_{12}(\eta_1)\cdots\mathcal{M}_{12}(\eta_{n-1})\Omega$ has to be $q^{\frac{n}{2}}$. Thus, such a Bethe ansatz-eigenvector is a (generalized) eigenvector of the restriction of $H(wA^{(2)} + (wA^{(2)})^*, w + w^*, k)$ to $\mathcal{H}_n$, and this restriction coincides with the operator $H(2\cos\delta, 2\cos\frac{n\gamma}{2}, k) = \tilde{H}(e^{i\delta}, n, k)$ acting in the Weyl-Schrödinger representation with angle $\theta + \frac{n}{2}\gamma$.

Theorem 4. *Suppose that there exist an open interval I of the real line containing 1 and a neighborhoud $U \subset S^1$ of δ such that the Bethe ansatz-eigenvalues (1) depend continuously on k and δ when (k, δ) is varied inside $I \times U$. Then, for rational values of the flux, the Bethe ansatz-eigenvalues (1) for $\tilde{H}(e^{i\delta}, n, k)$, $\delta \in S^1$, $n \in \mathbb{N}$, $k \in I$, are eigenvalues of $\tilde{A}(e^{i\delta}, n, k)$.*

Proof: First, let $k = 1$ and consider $H(wA^{(2)} + (wA^{(2)})^*, w + w^*, 1)$ in a representation (32-36) with $\theta = \pi - \frac{\gamma}{2}$. Then, by the above lemma, Bethe ansatz-eigenvectors of the form $\mathcal{M}_{12}(\eta_1)\cdots\mathcal{M}_{12}(\eta_{n-1})\Omega$ belong to $\mathcal{H}_n$ and have support contained in $\{(0, n), \ldots, (n-1, n)\} \subset \mathbb{Z}^2$. Restricting $H(wA^{(2)} + (wA^{(2)})^*, w + w^*, 1)$ to $\mathcal{H}_n$ we obtain the operator $\tilde{H}(e^{i\delta}, n, 1)$ acting in the Weyl-Schrödinger representation with $\theta = \pi + \frac{n-1}{2}\gamma$, and we have just shown that the Bethe ansatz-eigenvectors are among its eigenvectors belonging to the image of P_n. For $k = 1$, the statement of the theorem follows therefore from Theorem 2. To extend this statement to $k \in I$ recall that we only have to show, that the eigenvalues of which we want to show equality lie in the same h-bands. To see that this is the case, first note that the eigenvalues of $\tilde{A}(e^{i\delta}, n, k)$ depend continuously on k and δ. By assumption, the Bethe ansatz-eigenvalues depend continuously on (k, δ) when varied inside $I \times U$. Therefore, if the offset function does not have an absolute extremum at $(\theta, \varphi) \in \mathcal{B}(z, n, k)$ then the eigenvalues in question are spectral values in the interior of h-bands and hence cannot change bands under variation of k inside I. This is the case for an open dense set of values for δ. Using continuity in δ the theorem follows. q.e.d.

Continuity of the Bethe ansatz eigenvalues in $\gamma \in S^1 \backslash \{-1\}$ would imply the above result to hold true for all values of γ.

Acknowledgement. I thank Ruedi Seiler and Alexander Dranov for stimulating discussions. This work was supported by the Deutsche Forschungsgemeinschaft.

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Fachbereich Mathematik, Technische Universität Berlin, 10623 Berlin, Germany, Email: kellen@math.tu-berlin.de

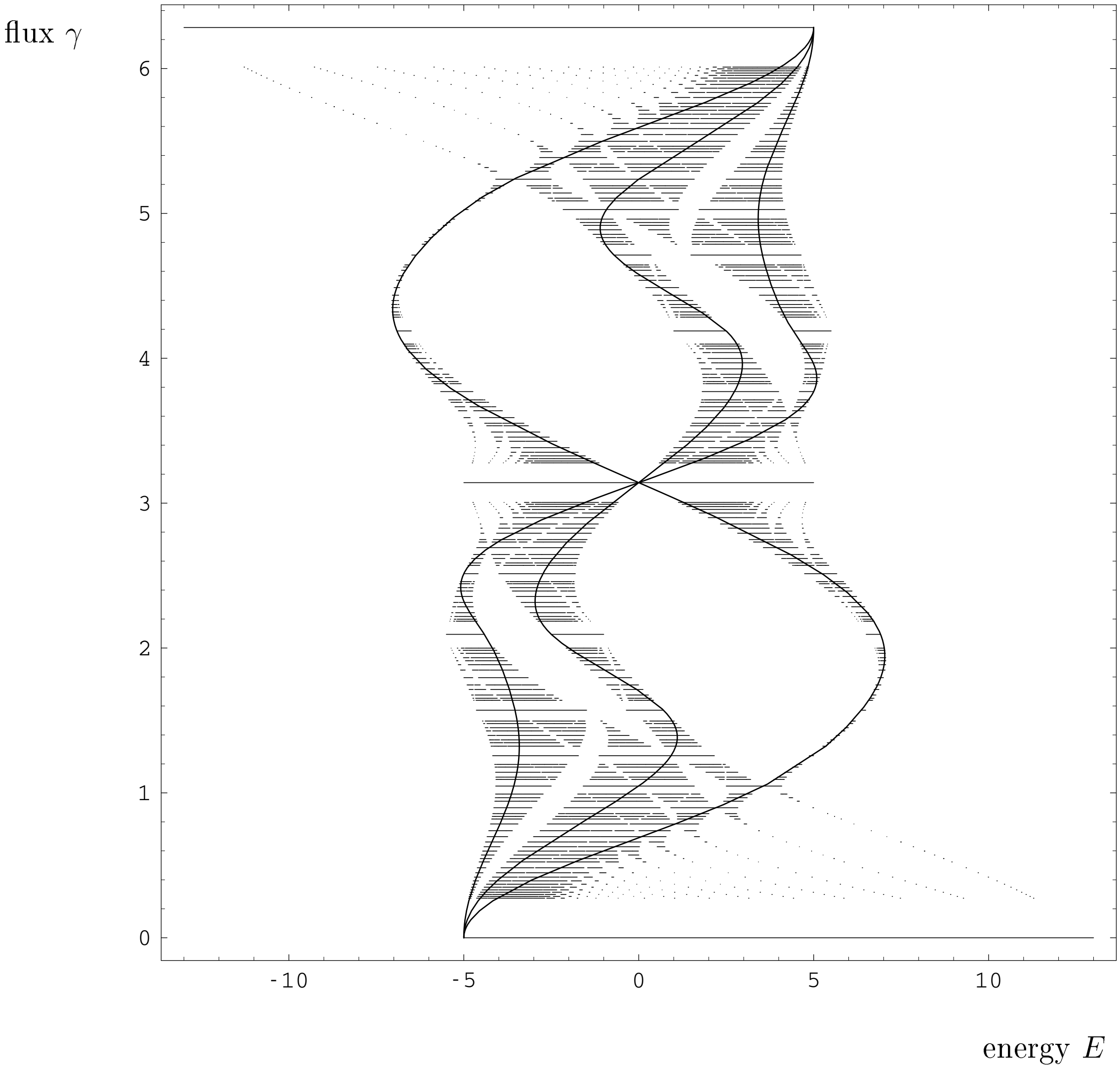

FIGURE 1. Flux-dependence of the spectrum of the QP-integral $\tilde{H}(e^{i\frac{3\gamma}{2}},3,2)$ and eigenvalues of $\tilde{A}(e^{i\frac{3\gamma}{2}},3,2)$