Critical strange attractor and universality in a renormalization scheme for three frequency Hamiltonian systems

C. Chandre and H.R. Jauslin
Laboratoire de Physique, CNRS, Université de Bourgogne, BP 400, F-21011 Dijon, France

We study an approximate renormalization-group transformation to analyze the breakup of invariant tori for three degrees of freedom Hamiltonian systems. The scheme is implemented for the spiral mean torus. We find numerically that the critical surface is the stable manifold of a critical strange attractor. We compute scaling exponents associated with this fixed set, and find that they can be expected to be universal.

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The breakup of invariant tori is one of the key mechanism of the transition to chaos in Hamiltonian dynamics. For two dimensional systems and for frequencies like the golden mean, it has been observed that, at the transition, a sequence of periodic orbits approaches geometrically a torus of the given frequency, with a nontrivial scaling behavior [1–3]. This self-similarity has been described in terms of a nontrivial fixed point of a renormalization-group transformation [4–8]. The sequence of periodic orbits responsible for the breakup of the invariant torus is generated by the convergents (the truncations of the continued fraction expansion) of the frequency of the torus. For the extension to systems with three degrees of freedom (d.f.) involving three incommensurate frequencies, we lack a theory that generalizes the continued fractions. Numerically, three d.f. Hamiltonian systems (or equivalently four dimensional volume-preserving maps) have been studied with an extension of Greene's criterion [9–12]. The conclusion of these analysis was that there is no geometrical accumulation of periodic orbits around the critical torus, and thus absence of universality.

The aim of this Letter is to show that one can still expect *universal* behavior in the breakup of invariant tori, described by a critical strange attractor of a renormalization-group transformation. The idea is that all Hamiltonians attracted by renormalization to this strange set will display sequences of scaling factors that appear in a different order but with a universal statistical distribution. The dominant unstable Lyapunov exponent characterizes the approach to criticality of the universality class.

Strange attractors for renormalization maps have been conjectured and observed in statistical mechanics [13,14] and dynamical systems [15]. In Ref. [16], a strange attractor was found for renormalization of circle maps, and in Ref. [17] similar evidence was found for area-preserving maps in two dimensions, from the scaling analysis of periodic orbits. The origin of randomness in these latter studies is due to the randomness of the sequence of continued fraction approximants for an ensemble of the considered frequencies. In contrast, in the present three d.f. case, the rational approximants are a regular sequence, obtained by iteration of a *single* unimodular matrix, which allows us to define a renormalization transformation with a *fixed* frequency vector.

The idea is to set up a transformation $\mathcal{R}$ that maps a Hamiltonian H into a rescaled Hamiltonian $\mathcal{R}(H)$, such that irrelevant degrees of freedom are eliminated. The transformation $\mathcal{R}$ should have roughly the following properties: $\mathcal{R}$ has an attractive integrable fixed point H_0 (trivial fixed point) that has a smooth invariant torus of a given frequency $\boldsymbol{\omega}_0$. Every Hamiltonian in its domain of attraction $\mathcal{D}$ has a smooth invariant torus with frequency vector $\boldsymbol{\omega}_0$. The aim is to show that there is another fixed set Λ which lies on the boundary $\partial\mathcal{D}$ (the critical surface) and that is attractive for every Hamiltonian on $\partial\mathcal{D}$.

The numerical implementation for two d.f. [5–7] gives support to this picture. This is by no means trivial since the construction of the renormalization iteration is based on properties that are valid close to H_0. The numerical results indicate that the domain of convergence of the iteration $\mathcal{R}$ indeed extends up to the critical surface. A mathematical justification of this observation, and the formulation of conditions for its validity are completely open problems.

In this paper, we study the extension of these ideas to systems with three frequencies, by analyzing an approximate renormalization transformation based on the work of Escande and Doveil [18–20]. Its properties are found in agreement with the general picture described above. We find universal exponents associated with the fixed set Λ. They are universal in the sense that they only depend on the frequency vector considered, and not on the chosen one-parameter family. These results give new insights for the set-up of an exact renormalization scheme in the spirit of Refs. [5–8].

Similar types of systems were first studied in Ref. [21] with an approximate renormalization scheme that kept less terms than needed to detect the strange attractor. Instead, their approximation yields a rotation (a center) which is structurally unstable, as mentioned in Ref. [21]. It is thus necessary to include the next relevant terms in the approximation, as we do in the following construction.

The transformation we define acts on the following class of Hamiltonians with three d.f., quadratic in the actions $\boldsymbol{A} = (A_1, A_2, A_3)$, and described by three even scalar functions of the angles $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$:

$$H(\boldsymbol{A}, \boldsymbol{\varphi}) = \frac{1}{2}(1 + m(\boldsymbol{\varphi}))(\boldsymbol{\Omega} \cdot \boldsymbol{A})^2 + [\boldsymbol{\omega}_0 + g(\boldsymbol{\varphi})\boldsymbol{\Omega}] \cdot \boldsymbol{A} + f(\boldsymbol{\varphi}), \qquad (1)$$

where m, g, and f are of zero average. The vector $\boldsymbol{\omega}_0$ is the frequency vector of the considered torus and $\boldsymbol{\Omega}$ is a vector not parallel to $\boldsymbol{\omega}_0$, with norm one: $\|\boldsymbol{\Omega}\| = \left(|\Omega_1|^2 + |\Omega_2|^2 + |\Omega_3|^2\right)^{1/2} = 1$. This model is the simplest one involving invariant tori with three frequencies. It can be thought as an intermediate case between two and three d.f. Its behavior is more complex than in two d.f.: For instance, the geometric Aubry-Mather theory is, to our knowledge, not available [22].

The transformation $\mathcal{R}$ is defined for a fixed frequency vector $\boldsymbol{\omega}_0$ with three incommensurate components (i. e., there is no nonzero integer vector $\boldsymbol{\nu}$ satisfying $\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu} = 0$). We choose a frequency vector widely studied for three dimensional systems [23,24] $\boldsymbol{\omega}_0 = (\sigma^2, \sigma, 1)$, where σ satisfies $\sigma^3 = \sigma + 1$, and is known as the spiral mean ($\sigma \approx 1.3247$). Since $\boldsymbol{\omega}_0$ is Diophantine, the KAM theorem applies to Hamiltonians (1) (although they are isoenergetically degenerate, see Ref. [25]) and shows the existence of a torus $\mathcal{T}$ with frequency vector $\boldsymbol{\omega}_0$ for a sufficiently small and smooth perturbation consisting of the three scalar functions m, g and f. The domain of existence of $\mathcal{T}$ corresponds to a neighborhood of the trivial fixed point

$$H_0(\boldsymbol{A}) = (\boldsymbol{\Omega} \cdot \boldsymbol{A})^2/2 + \boldsymbol{\omega}_0 \cdot \boldsymbol{A}, \qquad (2)$$

for which $\mathcal{T}$ is located at $\boldsymbol{A} = 0$.

From some of its properties, σ plays a similar role as the golden mean in the two d.f. case [23]. The analogy comes from the fact that one can generate rational approximants by iterating a *single* unimodular matrix N. In what follows, we denote *resonance* an element of the sequence $\{\boldsymbol{\nu}_k = N^{k-1}\boldsymbol{\nu}_1, k \geq 1\}$ where $\boldsymbol{\nu}_1 = (1, 0, 0)$ and

$$N = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}.$$

The word resonance refers to the fact that the small denominators $\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_k$ that appear in the perturbation series or in the KAM iteration, tend to zero geometrically as k increases ($\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_k = \sigma^{2-k} \to 0$ as $k \to \infty$). Our hypothesis (which is also the starting point of a generalization of Greene's criterion in Ref. [10]) is that this sequence plays a leading role in the breakup of the invariant torus with frequency vector $\boldsymbol{\omega}_0$. We build an approximate scheme by considering the three main resonances $\boldsymbol{\nu}_1$, $\boldsymbol{\nu}_2$, and $\boldsymbol{\nu}_3$; the renormalization focuses on the next smaller scale represented by the resonances $\boldsymbol{\nu}_2$, $\boldsymbol{\nu}_3$, together with $\boldsymbol{\nu}_4 = N\boldsymbol{\nu}_3 = \boldsymbol{\nu}_1 - \boldsymbol{\nu}_3$. It includes a partial elimination of the perturbation (the part which can be considered nonresonant on the smaller scale, namely the mode $\boldsymbol{\nu}_1$), a shift of the resonances, a rescaling of the actions and of the energy, and a translation in the action variables. It is, in spirit, close to the type of transformation considered in Refs. [7,8].

The approximations involved in this scheme are the two main ones used by Escande and Doveil:

(a) A quadratic approximation in the actions: the rescaled Hamiltonian $\mathcal{R}(H)$ is in general, higher than quadratic in the actions; in order to remain in the same family of Hamiltonians (1), we neglect these higher order terms.

(b) A three-resonance approximation: we only keep the three main resonances at each iteration of the transformation.

Renormalization transformation.— The approximate transformation we define acts on a reduced family of Hamiltonians (1). We consider the three most relevant Fourier modes $\{\boldsymbol{\nu}_i, i = 1, 2, 3\}$, for the perturbation at each step of the transformation. We denote h_i the Fourier mode of H with frequency vector $\boldsymbol{\nu}_i$:

$$h_i(\boldsymbol{A}) = m_{\nu_i}(\boldsymbol{\Omega} \cdot \boldsymbol{A})^2/2 + g_{\nu_i}\boldsymbol{\Omega} \cdot \boldsymbol{A} + f_{\nu_i}.$$

The Hamiltonian can be written as

$$H(\boldsymbol{A}, \boldsymbol{\varphi}) = H_0(\boldsymbol{A}) + \sum_{i=1}^{3} h_i(\boldsymbol{A}) \cos(\boldsymbol{\nu}_i \cdot \boldsymbol{\varphi}),$$

where H_0 is given by Eq. (2). Our transformation combines thus five steps:

(1) A canonical transformation that eliminates the first main resonance $\boldsymbol{\nu}_1$ to the order $O(\varepsilon)$. This is performed by a Lie transformation $\mathcal{U}_S : (\boldsymbol{\varphi}, \boldsymbol{A}) \mapsto (\boldsymbol{\varphi}', \boldsymbol{A}')$, generated by $S(\boldsymbol{A}, \boldsymbol{\varphi}) = S_1(\boldsymbol{A}) \sin(\boldsymbol{\nu}_1 \cdot \boldsymbol{\varphi})$. The Hamiltonian expressed in the new coordinates is given by:

$$H' = \exp(\hat{S})H \equiv H + \{S, H\} + \{S, \{S, H\}\}/2! + \cdots,$$

where $\{\ ,\ \}$ is the Poisson bracket between two scalar functions of the actions and angles. The generating function S is determined by the requirement that the order $O(\varepsilon)$ of the mode $\boldsymbol{\nu}_1$ vanishes:

$$\{S, H_0\} + h_1(\boldsymbol{A}) \cos(\boldsymbol{\nu}_1 \cdot \boldsymbol{\varphi}) = 0.$$

This equation has the following solution:

$$S(\boldsymbol{A}, \boldsymbol{\varphi}) = -h_1(\boldsymbol{A}) \sin(\boldsymbol{\nu}_1 \cdot \boldsymbol{\varphi})/\boldsymbol{\omega}(\boldsymbol{A}) \cdot \boldsymbol{\nu}_1,$$

where $\boldsymbol{\omega}(\boldsymbol{A}) = \boldsymbol{\omega}_0 + (\boldsymbol{\Omega} \cdot \boldsymbol{A})\boldsymbol{\Omega}$. This step generates arbitrary orders in the action variables. In order to map the family of Hamiltonians (1) into itself, we expand H' to quadratic order in the actions, and we neglect higher orders. The justification for this approximation is that as the torus is located at $\boldsymbol{A} = 0$ for H_0, one can expect that for small ε, it is close to $\boldsymbol{A}=0$. We notice that $h_2(\boldsymbol{A})$ and $h_3(\boldsymbol{A})$ are not changed to the order $O(\varepsilon^3)$. Furthermore, we neglect all the Fourier modes except $\mathbf{0}$, $\boldsymbol{\nu}_2$, $\boldsymbol{\nu}_3$, and $\boldsymbol{\nu}_4$. We expand the Hamiltonian to the order $O(\varepsilon^3)$. This leads to the expression of H':

$$H' = H_0 + h_2\cos(\boldsymbol{\nu}_2\cdot\boldsymbol{\varphi}) + h_3\cos(\boldsymbol{\nu}_3\cdot\boldsymbol{\varphi}) \\ +\langle\{S,h_1\}\rangle/2 + \{S,h_3\}, \tag{3}$$

where $\langle\,\rangle$ denotes the mean value defined as

$$\langle h\rangle(\boldsymbol{A}) = \int_{T^3}\frac{d^3\varphi}{(2\pi)^3}h(\boldsymbol{A},\boldsymbol{\varphi}).$$

The last term of Eq. (3) contains the Fourier mode $\boldsymbol{\nu}_4$ of amplitude

$$h_4(\boldsymbol{A}) = (S_1\boldsymbol{\nu}_1\cdot\boldsymbol{\partial}h_3 + h_3\boldsymbol{\nu}_3\cdot\boldsymbol{\partial}S_1)/2, \tag{4}$$

where $\boldsymbol{\partial}$ denotes the derivative with respect to $\boldsymbol{A}$.

(2) A shift of the resonances $\boldsymbol{\nu}_k\mapsto\boldsymbol{\nu}_{k-1}$: a linear transformation $(\boldsymbol{A},\boldsymbol{\varphi})\mapsto(N^{-1}\boldsymbol{A},\tilde{N}\boldsymbol{\varphi})$, where $\tilde{N}$ denotes the transposed matrix of N. This step changes the frequency $\boldsymbol{\omega}_0$ into $\tilde{N}\boldsymbol{\omega}_0=\sigma^{-1}\boldsymbol{\omega}_0$ (since $\boldsymbol{\omega}_0$ is an eigenvector of $\tilde{N}$ by construction).

(3) We rescale the energy (or equivalently the time) by a factor σ, in order to keep the frequency fixed at $\boldsymbol{\omega}_0$.

The vector $\boldsymbol{\Omega}$ is changed into $\tilde{N}\boldsymbol{\Omega}$. We define the image $\boldsymbol{\Omega}'$ of $\boldsymbol{\Omega}$ by $\boldsymbol{\Omega}'=\tilde{N}\boldsymbol{\Omega}/\|\tilde{N}\boldsymbol{\Omega}\|$, in order to have $\boldsymbol{\Omega}'$ of unit norm. We rewrite the mean-value term of Eq. (3) as

$$\langle\{S,h_1\}\rangle = \mu(\boldsymbol{\Omega}\cdot\boldsymbol{A})^2 + a\boldsymbol{\Omega}\cdot\boldsymbol{A} + \text{const}.$$

(4) In order to keep the image of g with zero mean-value, we eliminate a by a translation in the action variables $\boldsymbol{A}\mapsto\boldsymbol{A}+\boldsymbol{a}$ (of order $O(\varepsilon^2)$ in the $\boldsymbol{\Omega}$ direction).

The constant part of the quadratic term of the resulting Hamiltonian is $\frac{1}{2}\sigma\|\tilde{N}\boldsymbol{\Omega}\|^2(1+\mu)(\boldsymbol{\Omega}'\cdot\boldsymbol{A})^2$.

(5) In order to map this Hamiltonian back into the form (1), we rescale the actions $\hat{H}(\boldsymbol{A},\boldsymbol{\varphi})=\lambda H(\boldsymbol{A}/\lambda,\boldsymbol{\varphi})$, with $\lambda=\sigma\|\tilde{N}\boldsymbol{\Omega}\|^2(1+\mu)$.

In summary, the transformation is equivalent to a mapping acting on a 11-dimensional space

$$(m_{\nu_1},g_{\nu_1},f_{\nu_1},m_{\nu_2},g_{\nu_2},f_{\nu_2},m_{\nu_3},g_{\nu_3},f_{\nu_3},\boldsymbol{\Omega}) \\ \mapsto(m'_{\nu_1},g'_{\nu_1},f'_{\nu_1},m'_{\nu_2},g'_{\nu_2},f'_{\nu_2},m'_{\nu_3},g'_{\nu_3},f'_{\nu_3},\boldsymbol{\Omega}'),$$

defined by the following relations

$$m'_{\nu_i} = m_{\nu_{i+1}}/(1+\mu), \tag{5}$$

$$g'_{\nu_i} = \sigma\|\tilde{N}\boldsymbol{\Omega}\|g_{\nu_{i+1}}, \tag{6}$$

$$f'_{\nu_i} = \lambda\sigma f_{\nu_{i+1}}, \tag{7}$$

for $i=1,2$, and

$$m'_{\nu_3} = 2h_4^{(2)}/(1+\mu), \tag{8}$$

$$g'_{\nu_3} = \sigma\|\tilde{N}\boldsymbol{\Omega}\|h_4^{(1)}, \tag{9}$$

$$f'_{\nu_3} = \lambda\sigma h_4^{(0)}, \tag{10}$$

$$\boldsymbol{\Omega}' = \tilde{N}\boldsymbol{\Omega}/\|\tilde{N}\boldsymbol{\Omega}\|, \tag{11}$$

where $h_4^{(i)}$ is the coefficient in $(\boldsymbol{\Omega}\cdot\boldsymbol{A})^i$ of h_4 given by Eq. (4).

Critical surface of the transformation.— The numerical implementation of this scheme shows that there are two main domains separated by the *critical surface*: one where the iteration converges to H_0 and the other where it diverges to infinity. The renormalization-group picture for two-dimensional systems with golden mean frequency showed that this surface is the stable manifold of a nontrivial fixed point (or nontrivial fixed set related to this nontrivial fixed point by symmetries [6]). Here, we cannot expect any nontrivial fixed point: this has been highlighted in Refs. [10,21] and can be explained by considering the map (11). This map has no stable point on the sphere $\mathcal{S}^2$ (there is only one hyperbolic fixed point which corresponds to $\boldsymbol{\omega}_0$). The eigenvalues of $\tilde{N}$ are σ^{-1} and $-\sqrt{\sigma}e^{\pm i\alpha}$ where $\alpha\approx 2\pi\times 0.1120$. The map (11) leads asymptotically to a rotation of angle α. As α is close to $2\pi/9$, we expect the results to oscillate *approximately* with period 9 as it has been observed in Ref. [10]. We iterate the transformation on the critical surface. Figure 1 shows the scaling factor λ_{k+9} as a function of λ_k after k iterations. The points near the diagonal correspond to approximate period 9 behavior. There are however strong deviations from this behavior. Figure 2 shows the statistical distribution of the scaling factors.

The iterations on the critical surface converge to a nonperiodic bounded set Λ. Figure 3 shows the projection of Λ on the plane (g_{ν_1},m_{ν_1}). This strange set has a codimension 1 stable manifold, i. e., one expansive direction transverse to the critical surface. This set plays, for the system we consider, the same role as the nontrivial fixed point of the renormalization-group transformation for quadratic irrational frequencies in two d.f. Hamiltonian systems. In particular, its existence implies *universality* for one-parameter families crossing the critical surface. We define exponents that characterize the universality class associated with the spiral mean. The mean-rescaling is defined by

$$\lambda = \lim_{n\to\infty}\left(\prod_{k=1}^{n}\lambda_k\right)^{n},$$

where λ_k is the value of the rescaling after k iterations on the critical surface. We also calculate the largest Lyapunov exponent κ. The result we found is that these limits do not depend on the point on the critical surface where we start the iteration nor on the initial choice of $\boldsymbol{\Omega}$. The coefficients κ and λ depend only on $\boldsymbol{\omega}_0$. Numerically, we find $\kappa\approx 0.5870$ and $\lambda\approx 2.6640$.

In conclusion, the numerical results indicate that for the spiral mean, the critical surface of the approximate renormalization transformation is the stable manifold of a critical strange set instead of a nontrivial fixed point. This feature depends strongly on the characteristics of the eigenvalues of N, and consequently we do not exclude a priori that some specific choices of the frequency vector (or equivalently of N) could lead to a nontrivial fixed point of a renormalization-group transformation.

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FIG. 1. λ_{k+9} as a function of λ_k.

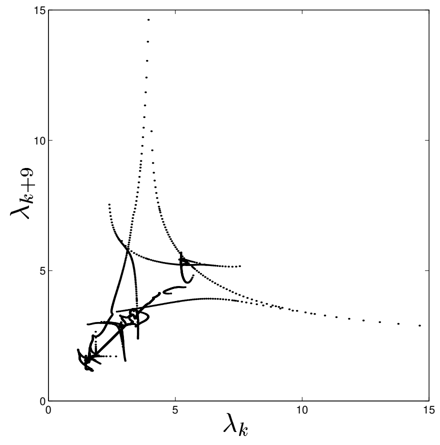

FIG. 2. Distribution of the values of the rescalings λ of the critical strange attractor Λ.

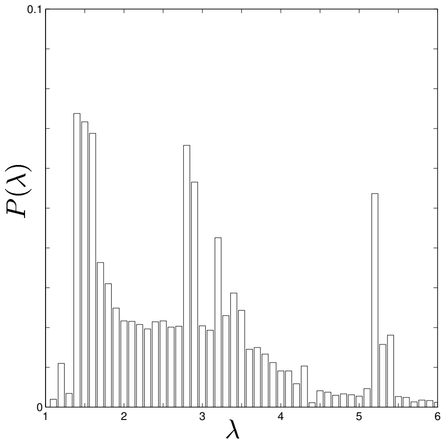

FIG. 3. Projection on the plane (g_{ν_1}, m_{ν_1}) of the critical strange attractor Λ