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Monogenic Functions and Representations of Nilpotent Lie Groups in Quantum Mechanics

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Abstract

We describe several different representations of nilpotent step two Lie groups in spaces of monogenic Clifford valued functions. We are inspired by the classic representation of the Heisenberg group in the Segal-Bargmann space of holomorphic functions. Connections with quantum mechanics are described.

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1 Introduction

It is well known, by the celebrated Stone-von Neumann theorem, that all models for the canonical quantisation [**?**] are isomorphic and provide us with equivalent representations of the Heisenberg group [**?**, Chap. 1]. Nevertheless it is worthwhile to look for some models which can act as alternatives for the Schrödinger representation. In particular, the Segal-Bargmann representation [**?**, **?**] serves to

- give a geometric representation of the dynamics of harmonic oscillators;
- present a nice model for the creation and annihilation operators, which is important for quantum field theory;
- allow applying tools of analytic function theory.

The huge abilities of the Segal-Bargmann (or Fock [**?**]) model are not yet completely employed, see for example new ideas in a recent preprint [**?**].

We look for similar connections between nilpotent Lie groups and spaces of monogenic [**?**, **?**] Clifford valued functions. Particularly we are interested in a third possible representation of the Heisenberg group, acting on monogenic functions on $\mathbb{R}^n$. There are several reasons why such a model can be of interest. First of all the theory of monogenic functions is (at least) as interesting as several complex variable theory, so the monogenic model should share many pleasant features with the Segal-Bargmann model. Moreover, monogenic functions take their value in a Clifford algebra, which is a natural environment in which to represent internal degrees of freedom of elementary particles such as spin. Thus from the very beginning it has a structure which in the Segal-Bargmann model has to be added, usually by means of the second quantization procedure [**?**]. So a monogenic representation can be even more relevant to quantum field theory than the Segal-Bargmann one (see Remark 2.2).

¿From the different aspects of the Segal-Bargmann space $F_2(\mathbb{C}^n)$ we select the one giving a unitary representation of the Heisenberg group $\mathbb{H}^n$. The representation is unitary equivalent to the Schrödinger representation on $L_2(\mathbb{R}^n)$ and the Segal-Bargmann transform is precisely the intertwining operator between these two representations (see Appendix A.3). Monogenic functions can be introduced in this scheme in two ways, as either $L_2(\mathbb{R}^n)$ or $F_2(\mathbb{C}^n)$ can be substituted by a space of monogenic functions.

In the first case one defines a new unitary irreducible representation of the Heisenberg group on a space of monogenic functions and constructs an analogue of the Segal-Bargmann transform as the intertwining operator of the new representation and the Segal-Bargmann one. We examine this possibility in section 2. In a certain sense the representation of the Heisenberg group constructed here lies between Schrödinger and Segal-Bargmann ones, taking properties both of them.

In the second case we first select a substitute for the Heisenberg group, so we can replace the Segal-Bargmann space by a space of monogenic functions. The space $\mathbb{C}^n$ underlying $F_2(\mathbb{C}^n)$ is intimately connected with the structure of the Heisenberg group $\mathbb{H}^n$ in the sense that $\mathbb{C}^n$ is the quotient of $\mathbb{H}^n$ with respect to its centre. In order to define a space of monogenic functions, say on $\mathbb{R}^{n+1}$, we have to construct a group playing a similar rôle with respect to this space. We describe an option in section 3.

Finally we give the basics of coherent states from square integrable group representations and an interpretation of the classic Segal–Bargmann space in terms of these in Appendix A.

This paper is closely related to [?], where connections between analytic function theories and group representations were described. Representations of another group ($SL(2,\mathbb{R})$) in spaces of monogenic functions can be found in [?]. We hope that the present paper make only few first steps towards an interesting function theory and other steps will be done elsewhere.

2 The Heisenberg group and spaces of analytic functions

2.1 The Schrödinger representation of the Heisenberg group

We recall here some basic facts on the Heisenberg group $\mathbb{H}^n$ and its Schrödinger representation, see [?, Chap. 1] and [?, Chap. 1] for details.

The Lie algebra of the Heisenberg group is generated by the $2n+1$ elements $p_1, \ldots, p_n, q_1, \ldots, q_n, e$, with the well-known Heisenberg commutator relations:

$$[p_i, q_j] = \delta_{ij} e. \tag{2.1}$$

All other commutators vanish. In the standard quantum mechanical interpretation the operators are momentum and coordinate operators [?, § 1.1].

It is common practice to switch between real and complex Lie algebras. Complexify $\mathfrak{h}^n$ to obtain the complex algebra $\mathbb{C}\mathfrak{h}^n$, and take four complex numbers a, b, c and d such that $ad - bc \neq 0$. The *real* $2n+1$-dimensional subspace spanned by

$$A_k = ap_k + bq_k \qquad B_k = cp_k + dq_k$$

and the commutator $[A_k, B_k] = (ad - bc)e$, where $e = [p_k, q_k]$ is of course isomorphic to $\mathfrak{h}^n$, and exponentiating will give a group isomorphic to the Heisenberg group.

An example of this procedure is obtained from the construction of the so-called creation and annihilation operators of Bose particles in the k-th state, a_k^+ and a_k^- (see [?, § 1.1]). These are defined by:

$$a_k^{\pm} = \frac{q_k \mp \imath p_k}{\sqrt{2}}, \tag{2.2}$$

giving the commutators $[a_i^+, a_j^-] = (-\imath)\delta_{ij}e$. Putting $-\imath e = \ell$, the real algebra spanned by $a_k^\pm$ and ℓ is an alternative realization of $\mathfrak{h}^n$, $\mathfrak{h}_a^n$.

An element g of the Heisenberg group $\mathbb{H}^n$ (for any positive integer n) can be represented as $g = (t, \mathbf{z})$ with $t \in \mathbb{R}$, $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$. The group law in coordinates $(t, \mathbf{z})$ is given by

$$g * g' = (t, \mathbf{z}) * (t', \mathbf{z}') = (t + t' + \frac{1}{2}\sum_{j=1}^{n} \Im(\bar{z}_j z_j'), \mathbf{z} + \mathbf{z}'), \qquad (2.3)$$

where $\Im z$ denotes the imaginary part of the complex number z. Of course the Heisenberg group is non-commutative.

The relation between the Heisenberg group and its Lie algebra is given by the exponentiation $\exp : \mathfrak{h}_a^n \to \mathbb{H}^n$. We define the formal vector $\mathbf{a}^+$ as being $(a_1^+, \ldots, a_n^+)$ and $\mathbf{a}^-$ as $(a_1^-, \ldots, a_n^-)$, which allows us to use the formal inner products

$$\begin{aligned} \mathbf{u} \cdot \mathbf{a}^+ &= \sum_{k=1}^{n} u_k a_k^+ \\ \mathbf{v} \cdot \mathbf{a}^- &= \sum_{k=1}^{n} v_k a_k^- . \end{aligned}$$

With these we define, for real vectors $\mathbf{u}$ and $\mathbf{v}$, and real s

$$\exp(\mathbf{u} \cdot (\mathbf{a}^+ + \mathbf{a}^-)) = (0, \sqrt{2}\mathbf{u}) \qquad (2.4)$$

$$\exp(\mathbf{v} \cdot (\mathbf{a}^- - \mathbf{a}^+)) = (0, \imath\mathbf{v}) \qquad (2.5)$$

$$\exp(s\ell) = (e^{-2s}, 0). \qquad (2.6)$$

Possible Schrödinger representations are parameterized by the non-zero real number $\hbar$ (the Planck constant). As usual, for considerations where the correspondence principle between classic and quantum mechanics is irrelevant, we consider only the case $\hbar = 1$. The Hilbert space for the Schrödinger representation is $L_2(\mathbb{R}^n)$, where elements of the complex Lie algebra $\mathbb{C}\mathfrak{h}^n$ are represented by the unbounded operators

$$\sigma(a_k^\pm) = \frac{1}{\sqrt{2}}\left(x_k I \mp \frac{\partial}{\partial x_k}\right). \qquad (2.7)$$

¿From which it follows, using any j, that

$$\sigma(\ell) = [a_j^+, a_j^-] = -2I.$$

The corresponding representation π of the Heisenberg group is given by exponentiation of the $\sigma(a_k^+)$ and $\sigma(a_k^-)$, but this is most readily expressed by using p_k and q_k, and so is generated by shifts and multiplications $s_{\mathbf{c}} : f(\mathbf{x}) \mapsto f(\mathbf{x}+\mathbf{c})$ and $m_{\mathbf{b}} : f(\mathbf{x}) \mapsto e^{\imath \mathbf{x}\cdot\mathbf{b}} f(\mathbf{x})$, with the Weyl commutation relation

$$s_{\mathbf{c}} m_{\mathbf{b}} = e^{\imath \mathbf{c}\cdot\mathbf{b}} m_{\mathbf{b}} s_{\mathbf{c}}.$$

There is an orthonormal basis of $L_2(\mathbb{R}^n)$ on which the operators $\sigma(a_k^{\pm})$ act in an especially simple way. It consists of the functions:

$$\phi_m(\mathbf{y}) = [2^m m! \sqrt{\pi}]^{-1/2} e^{-\mathbf{x}\cdot\mathbf{x}/2} H_m(\mathbf{y}), \tag{2.8}$$

where $\mathbf{y} = (y_1, \ldots, y_n)$, $m = (m_1, \ldots, m_n)$, and $H_m(\mathbf{y})$ is the generalized Hermite polynomial

$$H_m(\mathbf{y}) = \prod_{i=1}^{n} H_{m_i}(y_i).$$

For these

$$a_k^+ \phi_m(\mathbf{y}) = \sqrt{m_k+1}\, \phi_{m'}(\mathbf{y}), \qquad a_k^- \phi_m(\mathbf{y}) = \sqrt{m_k}\, \phi_{m''}(\mathbf{y})$$

where

$$\begin{aligned} m' &= (m_1, m_2, \ldots, m_{k-1}, m_k + 1, m_{k+1}, \ldots, m_n) \\ m'' &= (m_1, m_2, \ldots, m_{k-1}, m_k - 1, m_{k+1}, \ldots, m_n). \end{aligned}$$

This is the most straightforward way to express the creation or annihilation of a particle in the k-th state.

Let us now consider the generating function of the $\phi_m(\mathbf{x})$,

$$A(\mathbf{x}, \mathbf{y}) = \sum_{j=0}^{\infty} \frac{x^j}{\sqrt{j!}} \phi_k(\mathbf{y}) = \exp(-\frac{1}{2}(\mathbf{x}\cdot\mathbf{x} + \mathbf{y}\cdot\mathbf{y}) + \sqrt{2}\mathbf{x}\cdot\mathbf{y}). \tag{2.9}$$

We state the following elementary fact in Dirac's bra-ket notation.

Lemma 2.1 *Let H and H' be two Hilbert spaces with orthonormal bases $\{\phi_k\}$ and $\{\phi'_k\}$ respectively. Then the sum*

$$U = \sum_{j=0}^{\infty} \mid \phi'_j \rangle \langle \phi_j \mid \tag{2.10}$$

defines a unitary operator $U : H \to H'$ with the following properties:

1. *$U\phi_k = \phi'_k$;*

2. *If an operator $T : H \to H$ is expressed, relative to the basis ϕ_k, by the matrix (a_{ij}) then the operator $UTU^{-1} : H' \to H'$ is expressed relative to the basis ϕ'_k by the same matrix.*

Now, if we take the function $A(\mathbf{x}, \mathbf{y})$ from (2.9) as a kernel for an integral transform,

$$[Af](\mathbf{y}) = \int_{\mathbb{R}^n} A(\mathbf{y}, \mathbf{x}) f(\mathbf{x})\, dx$$

we can consider it subject to the Lemma above. However, for this we need to define the space H' and an orthonormal basis $\{\phi'_k\}$ (we already identified H with $L_2(\mathbb{R}^n)$ and the $\{\phi_k\}$ are given by (2.8)). There is some freedom in doing this.

For example it is possible to take the holomorphic extension $A(\mathbf{z}, \mathbf{y})$ of $A(\mathbf{x}, \mathbf{y})$ with respect to the first variable. Then

1. H' is the Segal-Bargmann space of analytic functions over $\mathbb{C}^n$ with scalar product defined by the integral with respect to Gaussian measure $e^{-|\mathbf{z}|^2}\, d\mathbf{z}$;

2. The Heisenberg group acts on the Segal-Bargmann space as follows:

$$[\beta_{(t,\mathbf{z})} f](\mathbf{u}) = f(\mathbf{u} + \mathbf{z}) e^{\imath t - \bar{\mathbf{z}}\cdot\mathbf{u} - |\mathbf{z}|^2/2}. \tag{2.11}$$

   This action generates the set of coherent states $f_{(0,\mathbf{v})}(\mathbf{u}) = e^{-\bar{\mathbf{v}}\mathbf{u} - |\mathbf{v}|^2/2}$, $\mathbf{u}$, $\mathbf{v} \in \mathbb{C}^n$ from the vacuum vector $f_0(\mathbf{u}) \equiv 1$;

3. The operators of creation and annihilation are $a_k^+ = z_k I$, $a_k^- = \frac{\partial}{\partial z_k}$.

4. The Segal-Bargmann space is spanned by the orthonormal basis $\phi'_k = \frac{1}{\sqrt{m!}} z^n$ or by the set of coherent states $f_{(0,\mathbf{v})}(\mathbf{u}) = e^{-\bar{\mathbf{v}}\mathbf{u} - |\mathbf{v}|^2/2}$, $\mathbf{u}$, $\mathbf{v} \in \mathbb{C}^n$

5. The intertwining kernel for $\sigma_{(t,\mathbf{z})}$ (2.7) and $\beta_{(t,\mathbf{z})}$ (2.11) is

$$A(\mathbf{z}, \mathbf{y}) = e^{-(\mathbf{z}\cdot\mathbf{z} + \mathbf{x}\cdot\mathbf{x})/2 - \sqrt{2}\mathbf{z}\cdot\mathbf{x}} = \sum_{k=0}^{\infty} \frac{\mathbf{z}^m}{\sqrt{m!}} \cdot \frac{1}{\sqrt{2^m m! \sqrt[4]{\pi}}} e^{-\mathbf{x}\cdot\mathbf{x}/2} H_m(\mathbf{y})$$

6. The Segal-Bargmann space has a reproducing kernel

$$K(\mathbf{u},\mathbf{v}) = e^{\mathbf{u}\cdot\bar{\mathbf{v}}} = \sum_{k=1}^{\infty} \phi_k(\mathbf{u})\bar{\phi}_k(\mathbf{v}) = \int e^{\mathbf{u}\cdot\bar{\mathbf{z}}} e^{\mathbf{z}\cdot\bar{\mathbf{v}}} e^{-|\mathbf{z}|^2}\, d\mathbf{z}.$$

Details can be found in [**?**, **?**], see also Appendix A.3.

The Segal-Bargmann space is an interesting and important object, but there are also other options. In particular we can consider an alternative representation of the Heisenberg group, this time acting on monogenic functions, an action we introduce in the next subparagraph.

2.2 Representation of $\mathbb{H}^n$ in spaces of monogenic functions

We consider the real Clifford algebra $\mathcal{C}\ell(n)$, i.e. the algebra generated by $e_0 = 1$, e_j, $1 \leq j \leq n$, using the identities:

$$e_i e_j + e_j e_i = -2\delta_{ij}, \qquad 1 \leq i, j \leq n.$$

For a function f with values in $\mathcal{C}\ell(n)$, the action of the Dirac operator of $\mathbb{R}^{n+1}$ is defined by (here $x = x_0 + \mathbf{x}$ is the $n+1$ dimensional variable)

$$Df(x) = \sum_{i=0}^{n} \partial_i f(x).$$

A function f satisfying $Df = 0$ in a certain domain is called monogenic there; later on we shall use the term 'monogenic' for solutions of more general Dirac operators. Obviously the notion of monogenicity is closely related to the one of holomorphy on the complex plane. As a matter of fact $D^2 = -\Delta$, and monogenic functions are solutions of the Laplacian. The Clifford algebra is not commutative, and so it is necessary to introduce a symmetrized product. For k elements a_i, $1 \leq i \leq k$ of the algebra it is defined by

$$a_1 \times a_2 \times \ldots \times a_k = \frac{1}{k!} \sum_{\sigma} a_{\sigma(1)} a_{\sigma(2)} \ldots a_{\sigma(n)},$$

where the sum is taken over all possible permutations of k elements. If the same element appears several times, we use an exponent notation, e.g. $a^2 \times b^3 = a \times a \times b \times b \times b$.

Let now V_k be the symmetric power monomial defined by the expression

$$V_k(\mathbf{x}) = \frac{1}{\sqrt{k!}}(e_1x_0 - e_0x_1)^{k_1} \times (e_2x_0 - e_0x_2)^{k_2} \times \cdots \times (e_nx_0 - e_0x_n)^{k_n}. \quad (2.12)$$

It can be proved that these monomials are all monogenic (see e.g. [**?**]), and even that they constitute a basis for the space of monogenic polynomials (as a module over $\mathcal{C}\ell(n)$). In general the symmetrized product is not associative, and manipulating it can become quite formal. However, if we restrict the monomials defined above to the hyperplane $x_0 = 0$, we obtain

$$V_k(x) = \frac{1}{\sqrt{k!}}x_1^{k_1}x_2^{k_2}\ldots x_n^{k_n},$$

and so we have the multiplicative property

$$\sqrt{\frac{k!k'!}{(k+k)!}}V_kV_{k'} = V_{k+k'}, \quad x_0 = 0.$$

Another important function is the monogenic exponential function which is defined by

$$E(\mathbf{u}, x) = \exp(\mathbf{u}\cdot\mathbf{x})\left(\cos(\|\mathbf{u}\|\, x_0) - \frac{\mathbf{u}}{\|\mathbf{u}\|}\sin(\mathbf{u}x_0)\right).$$

It is not hard to check [**?**, § 14] that this function is monogenic, and of course its restriction to the hyperplane $x_0 = 0$ is simply the exponential function, $E(\mathbf{u}, \mathbf{x}) = \exp(\mathbf{u}\cdot\mathbf{x})$.

We can therefore extend the symmetric product by the so-called Cauchy-Kovalevskaya product [**?**, § 14]: If f and g are monogenic in $\mathbb{R}^{n+1}$, then $f \times g$ is the monogenic function equal to fg on $\mathbb{R}^n$. Introducing the monogenic functions $\mathbf{x}_i = e_ix_0 - e_0x_i$ we can then write

$$V_k(x) = \frac{1}{\sqrt{k!}}x_1^{k_1} \times x_2^{k_2} \times \ldots \times x_n^{k_n}.$$

It is fairly easy to check the V_k form an orthonormal set with respect to the following inner product (see [**?**, § 3.1] on Clifford valued inner products):

$$\langle V_k, V_{k'}\rangle = \int_{\mathbb{R}^{n+1}} \bar{V}_k(x)V_{k'}(x)\, e^{-|x|^2}\, dx. \quad (2.13)$$

Let M_2 be closure of the linear span of $\{V_k\}$, using complex coefficients.

The creation and annihilation operators a_k^+ and a_k^- can be represented by symmetric multiplication (see [?]) with the monogenic variable $\mathbf{x}_j$, which will be written $\mathbf{x}_k I_\times$, and by the (classical) partial derivative $\frac{\partial}{\partial \mathbf{x}_j} = \frac{\partial}{\partial x_j}$ with respect to $\mathbf{x}_j$, which appear in the definition of hypercomplex differentiability. On basis elements they act as follows:

$$\begin{aligned}
\mathbf{x}_j I_\times V_{(k_1,\ldots,k_j,\ldots,k_n)} &= \sqrt{k_j+1}V_{(k_1,\ldots,k_j+1,\ldots,k_n)},\\
\frac{\partial}{\partial \mathbf{x}_j} V_{(k_1,\ldots,k_j,\ldots,k_n)} &= \sqrt{k_j}V_{(k_1,\ldots,k_j-1,\ldots,k_n)},
\end{aligned}$$

It can be checked that this really is a representation of $a_k^\pm$, and that a_k^+ and a_k^- are each other's adjoint. We use the equalities $a_j^- = \frac{1}{\sqrt{2}}(a_j^+ + a_j^-)$ and $a_j^+ = \frac{\imath}{\sqrt{2}}(a_j^- - a_j^+)$, and the commutation relations $[a_i^+, a_j^-] = e\delta_{ij}$ to obtain a representation of the Heisenberg group. Thus an element $(t, \mathbf{z})$, $\mathbf{z} = \mathbf{u} + i\mathbf{v}$ of the Heisenberg group can be written as

$$\begin{aligned}
(t,\mathbf{z}) &= \left(t+\frac{\mathbf{u}\cdot\mathbf{u}-\mathbf{v}\cdot\mathbf{v}}{4},0\right)\left(0,\frac{(1+\imath)(\mathbf{u}+\mathbf{v})}{2}\right)\left(0,\frac{(1-\imath)(\mathbf{u}-\mathbf{v})}{2}\right)\\
&= \exp\left(\left(t+\frac{\mathbf{u}^2-\mathbf{v}^2}{4}\right)e\right)\exp\left(\frac{(\mathbf{u}+\mathbf{v})q}{\sqrt{2}}\right)\exp\left(\frac{(\mathbf{u}-\mathbf{v})\imath p}{\sqrt{2}}\right).
\end{aligned}$$

It is therefore represented by the operator

$$\begin{aligned}
\pi_{(t,\mathbf{z})} &= \exp\left(-\left(t+\frac{\mathbf{u}\cdot\mathbf{u}-\mathbf{v}\cdot\mathbf{v}}{4}\right)\right) \textit{nonumber} \qquad (2.14)\\
&\quad \exp\left(\frac{((\mathbf{u}+\mathbf{v})\cdot\mathbf{x})I_\times}{\sqrt{2}}\right)\exp\left(\frac{(\mathbf{u}-\mathbf{v})\cdot(\partial_\mathbf{x})}{\sqrt{2}}\right), \qquad (2.15)
\end{aligned}$$

where obviously for a monogenic function f we have

$$\begin{aligned}
\exp\left(\frac{(\mathbf{u}-\mathbf{v})\imath p}{\sqrt{2}}\right)f(x) &= f\left(x+\frac{\mathbf{u}-\mathbf{v})}{\sqrt{2}}\right)\\
\exp\left(\frac{((\mathbf{u}+\mathbf{v})\cdot\mathbf{x})I_\times}{\sqrt{2}}\right)f(x) &= E\left(\frac{\mathbf{u}+\mathbf{v}}{\sqrt{2}},\cdot\right)\times f(x)
\end{aligned}$$

Therefore it is easy to calculate the image of the constant function $f_0(\mathbf{x}) = V_0(\mathbf{x}) \equiv 1$, and we obtain the set of functions

$$f_{(t,\mathbf{z})}(\mathbf{x}) = \pi_{(t,\mathbf{z})} f_0(\mathbf{x})$$

$$
\begin{aligned}
&= \exp\left(-\left(t+\frac{\mathbf{u}\cdot\mathbf{u}-\mathbf{v}\cdot\mathbf{v}}{4}\right)\right) E\left(\frac{\mathbf{u}+\mathbf{v}}{\sqrt{2}},\cdot\right)\times f_0(x)\\
&\quad \exp\left(-\left(t+\frac{\mathbf{u}\cdot\mathbf{u}-\mathbf{v}\cdot\mathbf{v}}{4}\right)\right) E\left(\frac{\mathbf{u}+\mathbf{v}}{\sqrt{2}},x\right). \qquad (2.16)
\end{aligned}
$$

In the language of quantum physics $f_0(\mathbf{x})$ is the *vacuum vector* and functions $f_{(t,\mathbf{z})}(\mathbf{x})$ are *coherent states* (or *wavelets*) for the representation of $\mathbb{H}^n$ we described. We can summarize the properties of the representation:

1. All functions in M_2 are complex-vector valued, monogenic in $\mathbb{R}^{n+1}$, and square integrable with respect to the measure $e^{-|x|^2}dx$.

2. The representation of the Heisenberg group is given by (2.15). This representation generates a set of coherent states $f_{(0,z)}(\mathbf{x})$ (2.16) as shifts of the vacuum vector $f_0(\mathbf{x})\equiv 1$.

3. The creation and annihilation operators a_k^+ and a_k^- are represented by symmetric (Cauchy-Kovalevskaya) multiplication by $\mathbf{x}_j$ and by derivation of monogenic functions. They are adjoint with respect to the inner product (2.13).

4. M_2 is generated as a closed linear space by the orthonormal basis $V_k(\mathbf{x}) = \frac{1}{\sqrt{k!}}(e_1x_0-e_0x_1)^{k_1}\times(e_2x_0-e_0x_2)^{k_2}\times\cdots\times(e_nx_0-e_0x_n)^{k_n}$, and also by the set of coherent states $f_{(t,\mathbf{z})}(\mathbf{x})$ of (2.16).

5. The kernel of the operator intertwining the model constructed here and the Segal-Bargmann one is given by

$$B(\mathbf{z},x)\sum_{j=0}^{\infty}V_j(x)\frac{\mathbf{z}^j}{\sqrt{j!}} = \exp(\sum_{k=1}^{n}\mathbf{x}_k\bar{z}_k),$$

which is the holomorphic extension in $\mathbf{z}=\mathbf{u}+\imath\mathbf{v}$ of $E(\mathbf{u},x)$. The transformation pair is given by

$$
\begin{aligned}
\mathcal{B}f(x) &= \int_{\mathbb{C}^n} B(\mathbf{z},x)f(\mathbf{z})\exp\left(\frac{-|\mathbf{z}|^2}{2}\right)d\mathbf{z}\\
\mathcal{B}^{-1}\phi(\mathbf{z}) &= \int_{\mathbb{R}^{n+1}} \overline{B(\mathbf{z},x)}\phi(x)\exp\left(\frac{-|x|^2}{2}\right)dx
\end{aligned}
$$

6. The space M_2 has a reproducing kernel
$$K(x,y)=\sum_{k=0}^{\infty}V_k(x)\bar{V}_k(y)=\int_{\mathbb{C}^n}B(\mathbf{z},x)\overline{B(\mathbf{z},y)}\,e^{-|z|^2}dz.$$
Notice that $\overline{K(x,y)}$ is monogenic in y; it is the monogenic extension of $\overline{E(\mathbf{y},x)}$.

One can see that some properties of M_2 are closer to those of the Segal-Bargmann space than to those of the space $L_2(\mathbb{R}^n)$ it replaces. It should be noted that the representation of the Heisenberg group we obtained here is new and quite unexpected.

Remark 2.2 We construct M_2 as a space of complex-vector valued functions. We can also consider an extended space $\widetilde{M_2}$ being generated by the orthonormal basis $V_k(\mathbf{x})$ or coherent states $f_{(0,z)}(\mathbf{x})$ with Clifford valued coefficients multiplied from the right hand side. Such a space will share many properties of $\mathbb{M}^2$ and have an additional structure: there is a natural representation $s : f(\mathbf{x}) \mapsto s^* f(s\mathbf{x}s^*)s$ of Spin (n) group in $\widetilde{M_2}$. Thus this space provides us with a representation of two main symmetries in quantum field theory: the Heisenberg group of quantized coordinate and momentum (external degrees of freedom) and Spin (n) group of quantified inner degrees of freedom. Another composition of the Heisenberg group and Clifford algebras can be found in [**?**].

3 Another nilpotent Lie group and its representation

3.1 Clifford algebra and complex vectors

Starting from the real Clifford algebra $\mathcal{C}\ell(n)$, we consider complex n-vector valued functions defined on the real line $\mathbb{R}^1$ with values in $\mathbb{C}^n$. Moreover we will look at the j-th component of $\mathbb{C}^n$ as being spanned by the elements 1 and e_j of the Clifford algebra. For two vectors $\mathbf{u}=(u_1,\ldots,u_n)$ and $\mathbf{v}=(v_1,\ldots,v_n)$ we introduce the Clifford vector valued product (see [**?**, § 3.1] on Clifford valued inner products):
$$\mathbf{u}\cdot\mathbf{v}=\sum_{j=1}^{n}\bar{u}_jv_j=\sum_{j=1}^{n}(u'_j-u''_je_j)(v'_j+v''_je_j), \tag{3.1}$$

where $u_j = u'_j + u''_j e_j$ and $v_j = v'_j + v''_j e_j$. Of course, $u \cdot u$ coincides with $\|u\|^2 = \sum_1^n (u'^2_j + u''^2_j)$, the standard norm in $\mathbb{C}^n$. So we can introduce the space $R_2(\mathbb{R}^1)$ of $\mathbb{C}^n$ -valued functions on the real line with the product

$$\langle f, f' \rangle = \int_{\mathbb{R}}^1 f(x) \cdot f'(x)\, dx. \tag{3.2}$$

Again $\langle f, f \rangle^{1/2}$ gives us the standard norm in the Hilbert space of L_2 integrable $\mathbb{C}^n$ valued functions.

3.2 A nilpotent Lie group

We introduce a nilpotent Lie group, $\mathbb{G}^n$. As a C^∞-manifold it coincides with $\mathbb{R}^{2n+1}$. Its Lie algebra has generators P, Q_j, T_j, $1 \le j \le n$. The non-trivial commutators between them are

$$[P, Q_j] = T_j; \tag{3.3}$$

all others vanish. Particularly $\mathbb{G}^n$ is a step two nilpotent Lie group and the T_j span its centre. It is easy to see that $\mathbb{G}^1$ is just the Heisenberg group $\mathbb{H}^1$.

We denote a point g of $\mathbb{G}^n$ by $2n+1$-tuple of reals $(t_1, \ldots, t_n; p; q_1, \ldots, q_n)$. These are the exponential coordinates corresponding to the basis of the Lie algebra $T_1, \ldots, T_n, P, Q_1, \ldots, Q_n$. The group law is given in exponential coordinates by the formula

$$\begin{aligned}
&(t_1, \ldots, t_n; p; q_1, \ldots, q_n) * (t'_1, \ldots, t'_n; p'; q'_1, \ldots, q'_n) = \\
&= \ (t_1 + t'_1 + \frac{1}{2}(p'q_1 - pq'_1), \ldots, t_n + t'_n + \frac{1}{2}(p'q_n - pq'_n); \\
&\qquad p + p'; q_1 + q'_1, \ldots, q_n + q'_n).
\end{aligned} \tag{3.4}$$

We consider the homogeneous space $\Omega = \mathbb{G}^n/\mathbb{Z}$. Here $\mathbb{Z}$ is the centre of $\mathbb{G}^n$; its Lie algebra is spanned by T_j, $1 \le j \le n$. It is easy to see that $\Omega \sim \mathbb{R}^{n+1}$. We define the mapping $s : \Omega \to \mathbb{G}^n$ by the rule

$$s : (a_0, a_1, \ldots, a_n) \mapsto (0, \ldots, 0; a_0; a_1, \ldots, a_n). \tag{3.5}$$

It is the "inverse" of the natural projection $s^{-1} : \mathbb{G}^n \to \Omega = \mathbb{G}^n/\mathbb{Z}$.

It easy to see that the mapping $\Omega \times \Omega \to \Omega$ defined by the rule $s^{-1}(s(a) * s(a'))$ is just Euclidean (coordinate-wise) addition $a + a'$.

To introduce the Dirac operator we will need the following set of left-invariant differential operators, which generate right shifts on the group:

$$\begin{aligned} T_j &= \frac{\partial}{\partial t_j}, && (3.6)\\ P &= \frac{\partial}{\partial p} + \frac{1}{2}\sum_1^n q_j \frac{\partial}{\partial t_j}, && (3.7)\\ Q_j &= -\frac{\partial}{\partial q_j} + \frac{1}{2} p \frac{\partial}{\partial t_j}. && (3.8) \end{aligned}$$

The corresponding set of right invariant vector fields generating left shifts is

$$\begin{aligned} T_j^* &= \frac{\partial}{\partial t_j}, && (3.9)\\ P^* &= \frac{\partial}{\partial p} - \frac{1}{2}\sum_1^n q_j \frac{\partial}{\partial t_j}, && (3.10)\\ Q_j^* &= -\frac{\partial}{\partial q_j} - \frac{1}{2} p \frac{\partial}{\partial t_j}. && (3.11) \end{aligned}$$

A general property is that any left invariant operator commutes with any right invariant one.

3.3 A representation of $\mathbb{G}^n$

We introduce a representation ρ of $\mathbb{G}^n$ in the space $R_2(\mathbb{R})$ by the formula:

$$[\rho_g f](x) = (e^{e_1(2t_1+q_1(\sqrt{2}x-p))} f_1(x-\sqrt{2}p), \ldots, e^{e_n(2t_n+q_n(\sqrt{2}x-p))} f_n(x-\sqrt{2}p)), \tag{3.12}$$

where $f(x) = (f_1(x), \ldots, f_n(x))$ and the meaning of $R_2(\mathbb{R})$ was discussed in Subsection 3.1. We note that the generators e_j of Clifford algebras do not interact with each other under the representation just defined. One can check directly that (3.12) defines a representation of $\mathbb{G}^n$. Indeed:

$$\begin{aligned} [\rho_g \rho_{g'} f](x) &= \rho_g(e^{e_1(2t'_1+q'_1(\sqrt{2}x-p'))} f_1(x-\sqrt{2}p'), \ldots,\\ &\qquad e^{e_n(2t'_n+q'_n(\sqrt{2}x-p'))} f_n(x-\sqrt{2}p'))\\ &= (e^{e_1(2t_1+q_1(\sqrt{2}x-p))} e^{e_1(2t'_1+q'_1(\sqrt{2}(x-\sqrt{2}p)-p'))} f_1(x-\sqrt{2}p-\sqrt{2}p'),\\ &\quad \ldots, \end{aligned}$$

$$
\begin{aligned}
& e^{e_n(2t_n+q_n(\sqrt{2}x-p))}e^{e_n(2t'_n+q'_n(\sqrt{2}(x-\sqrt{2}p)-p'))}f_n(x-\sqrt{2}p-\sqrt{2}p')) \\
= \; & (e^{e_1(2(t_1+t'_1+\frac{1}{2}(p'q_1-pq'_1))+(q_1+q'_1)(\sqrt{2}x-(p+p')))}f_1(x-\sqrt{2}(p+p')), \\
& \ldots, \\
& (e^{e_n(2(t_n+t'_n+\frac{1}{2}(p'q_n-pq'_n))+(q_n+q'_n)(\sqrt{2}x-(p+p')))}f_n(x-\sqrt{2}(p+p')) \\
= \; & [\rho_{gg'}f](x),
\end{aligned}
\tag{3.13}
$$

where gg' is defined by (3.4).

ρ_g has the important property that it preserves the product (3.2). Indeed:

$$
\begin{aligned}
\langle \rho_g f, \rho_g f' \rangle &= \int_{\mathbb{R}} [\rho_g f](x)\cdot[\rho_g f'](x)\,dx \\
&= \int_{\mathbb{R}} \sum_{j=1}^{n} \bar{f}_j(x-\sqrt{2}p)e^{-e_j(2t_j+q_j(\sqrt{2}x-p))} \\
&\qquad\qquad e^{e_j(2t_j+q_j(\sqrt{2}x-p))}f'_j(x-\sqrt{2}p)\,dx \\
&= \int_{\mathbb{R}} \sum_{j=1}^{n} \bar{f}_j(x-\sqrt{2}p)f'_j(x-\sqrt{2}p)\,dx \\
&= \int_{\mathbb{R}} \sum_{j=1}^{n} \bar{f}_j(x)f'_j(x)\,dx \\
&= \langle f, f' \rangle .
\end{aligned}
$$

Thus ρ_g is *unitary* with respect to the Clifford valued inner product (3.2). Notice this notion is stronger than unitarity for the scalar valued inner product, as the latter is the trace of the Clifford valued one. A proof of unitarity could also consist of proving the action of the Lie algebra is skew-symmetric, i.e. that for an element b of the Lie algebra and f arbitrary

$$
\langle d\rho_b f, f \rangle = \langle f, -d\rho_b f \rangle .
$$

Here $d\rho_b$ is derived representation of d for an element $b \in \mathfrak{g}_n$ of the Lie algebra of $\mathbb{G}^n$. In the next subsection we will need the explicit form of it. For the selected basis of $\mathfrak{g}_n$ we have:

$$
\begin{aligned}
[d\rho(T_j)f](x) &= (0,0,\ldots,0,2e_1f_j(x),0,\ldots,0,0); \\
[d\rho(P)f](x) &= (-\sqrt{2}\frac{\partial}{\partial x}f_1(x),\ldots,-\sqrt{2}\frac{\partial}{\partial x}f_j(x),\ldots,-\sqrt{2}\frac{\partial}{\partial x}f_n(x)); \\
[d\rho(Q_j)f](x) &= (0,0,\ldots,0,\sqrt{2}e_jxf_j(x),0,\ldots,0,0).
\end{aligned}
$$

Particularly $d\rho(Q_j)d\rho(Q_k) = 0$ for all $j \neq k$. This does not follow from the structure of $\mathbb{G}$ but is a feature of the described representation.

REMARK 3.1 The group $\mathbb{G}^n$ is called as "a generalized Heisenberg group" in [**?**] where its induced representations are considered.

3.4 The wavelet transform for $\mathbb{G}^n$

In $R_2(\mathbb{R})$ we have the $\mathbb{C}^n$ -valued function

$$f_0(x) = (e^{-x^2/2}, \ldots, e^{-x^2/2}), \tag{3.14}$$

which which is the *vacuum vector* in this case. It is a zero eigenvector for the operator

$$a^- = d\rho(P) - \sum_{j=1}^{n} e_j d\rho(Q_j), \tag{3.15}$$

which is *the only annihilating operator* in this model. But we still have n creation operators:

$$a_k^+ = d\rho(P) - \sum_{j=1}^{n}(1 - 2\delta_{jk})e_j d\rho(Q_j) = a^- + 2e_k d\rho(Q_k). \tag{3.16}$$

While a^- and a_k^+ look a little bit exotic for $\mathbb{G}^1 = \mathbb{H}^1$ they are exactly the standard annihilation and creation operators. Another feature of the representation is that the a_k^+ do not commute with each other and have a non-trivial commutator with a^-:

$$[a_j^+, a_k^+] = 2e_k d\rho(T_k) - 2e_j d\rho(T_j), \qquad [a_j^+, a^-] = -2e_j d\rho(T_j)$$

We need the transforms of $f_0(x)$ under the action (3.12), i.e. the *coherent states* $f_g(x) = [\rho_g f_0](x)$ in this model:

$$\begin{aligned} f_g(x) &= (\ldots, e^{e_j(2t_j + q_j(\sqrt{2}x - p))} e^{-(x-\sqrt{2}p)^2/2}, \ldots) \\ &= (\ldots, e^{2e_j r_j - (p^2+q_j^2)/2} e^{-((p-e_j q_j)^2 + x^2)/2 + \sqrt{2}(p - e_j q_j)x}, \ldots) \\ &= (\ldots, e^{2e_j t_j - z_j \bar{z}_j/2} e^{-(\bar{z}_j^2 + x^2)/2 + \sqrt{2}\bar{z}_j x}, \ldots) \end{aligned} \tag{3.17}$$

where $z_j = p + e_j q_j$, $\bar{z}_j = p - e_j q_j$.

Having defined coherent states we can introduce the *wavelet transform* $\mathcal{W} : R_2(\mathbb{R}) \to L_\infty(\mathbb{G}^n)$ by the standard formula:

$$\mathcal{W}f(g) = \langle f, f_g \rangle\,. \tag{3.18}$$

Calculations completely analogous to those of the complex case allow us to find the images $\mathcal{W}f_{(t',\mathbf{a})}(t,\mathbf{z})$ of coherent states $f_{(t',\mathbf{a})}(x)$ under (3.18) as follows:

$$\begin{aligned}
\mathcal{W}f_{(t',\mathbf{a})}(t,\mathbf{z}) &= \left\langle f_{(t',\mathbf{a})}, f_{(t,\mathbf{z})}\right\rangle \\
&= \int_{\mathbb{R}} \sum_{j=1}^{n} \exp\left(-2e_j t_j - \frac{z_j\bar{z}_j}{2} - \frac{z_j^2 + x^2}{2} + \sqrt{2} z_j x\right) \\
&\qquad \exp\left(+2e_j t_j' - \frac{a_j\bar{a}_j}{2} - \frac{\bar{a}_j^2 + x^2}{2} + \sqrt{2}\bar{a}_j x\right)\, dx \\
&= \sum_{j=1}^{n} \exp\left(-2e_j t_j - \frac{z_j\bar{z}_j}{2} + 2e_j t_j' - \frac{a_j\bar{a}_j}{2} + \bar{a}_j z_j\right) \\
&\qquad \times \int_{\mathbb{R}} \exp(-x^2 + 2x\frac{z_j + \bar{a}_j}{\sqrt{2}} - \frac{(z_j + \bar{a}_j)^2}{2})\ , dx \\
&= \sum_{j=1}^{n} \exp\left(-2e_j (t_j - t_j') - \frac{z_j\bar{z}_j + a_j\bar{a}_j}{2} + \bar{a}_j z_j\right) \\
&\qquad \times \int_{\mathbb{R}} \exp(-(x - \frac{z_j + \bar{a}_j}{\sqrt{2}})^2)\, dx \\
&= \sum_{j=1}^{n} \exp\left(-2e_j (t_j - t_j') - \frac{z_j\bar{z}_j + a_j\bar{a}_j}{2} + \bar{a}_j z_j\right)
\end{aligned} \tag{3.19}$$

Here $a_j = a_0 + e_j a_j$, $\bar{a}_j = a_0 - e_j a_j$; z_j, $\bar{z}_j$ were defined above.

In this case all $\mathcal{W}f_{(t',\mathbf{a})}(t,\mathbf{z})$ are *monogenic* functions with respect to the following Dirac operator:

$$\frac{\partial}{\partial p} - \sum_{j=1}^{n} e_j \frac{\partial}{\partial q_j} + \frac{1}{2}\sum_{j=1}^{n} (e_j p + q_j)\frac{\partial}{\partial t_j}, \tag{3.20}$$

with z_j related to p and q_j as above. This can be checked by the direct calculation or follows from the observation: the Dirac operator (3.20) is the image of the annihilation operator a^- (3.15) under the wavelet transform (3.18).

The situation is completely analogous to the Segal-Bargmann case, where holomorphy is defined by the operators $\frac{\partial}{\partial \bar{z}_k}$, which are the images of the annihilation operators a_k^-. Actually, it is the Dirac operator associated with the unique left invariant metric on $\mathbb{G}^n/\mathbb{Z}$ for which P together with the Q_k forms an orthonormal basis in the origin, and therefore everywhere.

The operator (3.20) is a realization of a generic Dirac operator constructed for a nilpotent Lie group, see [?]. Indeed the operator (3.20) is defined by the formula $D = P + \sum_1^n e_j Q_j$, where P and Q_j are the left invariant vector fields in (3.6)–(3.8). So the operator (3.20) is left invariant and one has only to check the monogenicity of $\mathcal{W}f_{(0,0)}(t,\mathbf{z})$—all other functions $\mathcal{W}f_{(t',\mathbf{a})}(t,\mathbf{z})$ are its left shifts.

Of course all linear combinations of the $\mathcal{W}f_{(t',\mathbf{a})}(t,\mathbf{z})$ are also monogenic. So if we define two function spaces, R_2 and M_2, as being the closure of the linear span of all $f_g(x)$ and $\mathcal{W}f_{(t',\mathbf{a})}(t,\mathbf{z})$ respectively, then

1. M_2 is a space of monogenic function on $\mathbb{G}^n$ in the sense above.
2. $\mathbb{G}^n$ has representations both in R_2 and in M_2. On the second space the group acts via left regular representation.
3. These representation are intertwining by the integral transformation with the kernel $T(t',\mathbf{a},x) = f_{(t',\mathbf{a})}(x)$.
4. The space M_2 has a reproducing kernel $K(t',\mathbf{a},t,z) = \mathcal{W}f_{(t',\mathbf{a})}(t,\mathbf{z})$.

The standard wavelet transform can be processed as expected.

For the reduced wavelet transform associated with the mapping $s:\Omega\to\mathbb{G}^n$ in particular we have

$$\widehat{\mathcal{W}}f_{\mathbf{a}}(\mathbf{z}) = \mathcal{W}f_{(0,\mathbf{a})}(0,\mathbf{z}) = \sum_{j=1}^{n} \exp \bar{a}_j z_j.$$

However the reduced wavelet transform cannot be constructed from a single vacuum vector. We need exactly n linearly independent vacuum vectors and the corresponding multiresolution wavelet analysis (wavelet transform with several independent vacuum vectors) which is outlined in [?] (see also M.G. Krein's works [?] on "directing functionals"). Indeed we have n different vacuum vectors $(\ldots,0,e^{-x^2/2},0,\ldots)$ each of which is an eigenfunction for the action of the centre of $\mathbb{G}^n$. All functions $\widehat{\mathcal{W}}f_{\mathbf{a}}(\mathbf{z})$ are *monogenic* with respect to the Dirac operator

$$D = \frac{\partial}{\partial p} + \sum_{j=1}^{n} e_j \frac{\partial}{\partial q_j}. \tag{3.21}$$

For details on the reduced wavelet transform in a more general setting we refer to the second Appendix.

A Appendices

A.1 The wavelet transform and coherent states

Let X be a topological space and G be a group of transformations $g : x \mapsto g \cdot x$ acting from the left on X, i.e. $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$. Moreover, assume G acts transitively on X. Let there exist a measure dx on X and a representation $\pi_g : f(x) \mapsto m(g,x) f(g^{-1} \cdot x)$ (where $m(g,x)$ is a function), such that π is unitary with respect to the scalar product $\langle f_1, f_2 \rangle_{L_2(X)} = \int_X f_1(x) \bar{f}_2(x)\, dx$, i.e.

$$\langle \pi_g f_1, \pi_g f_2 \rangle_{L_2(X)} = \langle f_1, f_2 \rangle_{L_2(X)} \qquad \forall f_1, f_2 \in L_2(X).$$

We shall work with the Hilbert space $L_2(X)$ where each π_g is a unitary action.

Let H be a closed compact[1] subgroup of G and let $f_0(x)$ be a function on which each element h of H acts as multiplication with a constant $\chi(h)$,

$$\pi_h f_0(x) = \chi(h) f_0(x) \qquad , \forall h \in H. \tag{A.1}$$

This means χ is a character of H and f_0 is a common eigenfunction for all operators π_h. Equivalently f_0 is a common eigenfunction for the operators corresponding under π to a basis of the Lie algebra of H. Note also that $|\chi(h)|^2 = 1$ because π is unitary. f_0 is called *vacuum vector* (with respect to the subgroup H). We introduce $F_2(X)$, the closed linear subspace of $L_2(X)$ uniquely defined by the conditions:

1. $f_0 \in F_2(X)$;
2. $F_2(X)$ is G-invariant;

[1] While the compactness will be explicitly used during our abstract consideration, it is not crucial in fact. Appendix A.3 will show how to deal with non-compact H.

3. $F_2(X)$ is G-irreducible.

f_0 is then called a *cyclic vector* for this space. 3 puts an extra condition upon f_0: there could be functions f_0 the orbit of which spans a reducible space. The theory can be extended to this case without much difficulty, but we will restrict ourselves to irreducible spaces here, and the restriction of π on $F_2(X)$ is an irreducible unitary representation. The transforms of f_0 will be called *coherent states*. They will be written down as w_g, with

$$w_g(x) = f_0(g^{-1} \cdot x).$$

The *wavelet transform* $\mathcal{W}$ can be defined for square-integrable unitary representations π by the formula [**?**]

$$\begin{aligned} \mathcal{W} &: F_2(X) \to L_\infty(G) \\ &: f(x) \mapsto \mathcal{W}f(g) = \langle f, w_g \rangle_{L_2(X)} \end{aligned} \tag{A.2}$$

The main advantage of the wavelet transform $\mathcal{W}$ is that it expresses π in geometrical terms in the sense that it *intertwines* π and the left regular representation λ on G defined by $\lambda_g F(g') = F(g^{-1}g')$:

$$\lambda_g \mathcal{W}f(g') = \mathcal{W}f(g^{-1}g') = \langle f, w_{g^{-1}g'} \rangle = \langle \pi_g f, w_{g'} \rangle = \mathcal{W}\pi_g f(g'), \tag{A.3}$$

i.e., $\lambda\mathcal{W} = \mathcal{W}\pi$. Applying this for $f = f_0$ gives

$$\mathcal{W}f_0(g^{-1}g') = \mathcal{W}w_g(g'). \tag{A.4}$$

Another important feature of $\mathcal{W}$ is that it does not lose information: the function f can *be recovered* as a linear combination of the coherent states w_g from its wavelet transform $\mathcal{W}f(g)$ [**?**]:

$$f(x) = \int_G \mathcal{W}f(g) w_g(x)\, dg\, dg. \tag{A.5}$$

Here dg is the Haar measure on G which is normalized in such a way that $\int_G |\mathcal{W}f_0(g)|^2\, dg = 1$. One also has the orthogonal *projection* $\mathcal{P}$ from $L_2(G, dg)$ onto the image $F_2(G, dg) = \mathcal{W}F_2(X)$, which is just the convolution on G with the image $\mathcal{W}f_0(g)$ of the vacuum vector [**?**]:

$$\mathcal{P}\phi(g') = \int_G \phi(g)\mathcal{W}f_0(g^{-1}g')\, dg. \tag{A.6}$$

A.2 The reduced wavelet transform

Our main observation will be that one can be much more economical (if the subgroup H is non-trivial) with the help of (A.1): in this case one does not need to know $\mathcal{W}f(g)$ on the whole group G, but only on the homogeneous space G/H.

Let $s : G \to G$ be a mapping such that $[s(b)] = [b]$ (square brackets denoting equivalence classes in G/H), and such that $s(a) = s(b)$ if $[a] = [b]$. Let Ω be the image of G under s. Any $g \in G$ has a unique decomposition of the form $g = s(g)h$, $a \in \Omega$, and we will write $h = r(g) = s(g)^{-1}g$. G/H is a left G-homogeneous space for the action defined by $g : [a] \mapsto [ga]$. Therefore Ω can be considered to be a G-homogeneous space by the action $t_g : a \mapsto s(ga)$. Due to (A.1) we have

$$\begin{aligned} w_g(x) &= \pi_g f_0(x) = \pi_{s(g)}(\pi_{r(g)} f_0)(x) \\ &= \pi_{s(g)}(\chi(r(g)) f_0)(x) = \chi(r(g))\pi_{s(g)} f_0(x) \\ &= \chi(r(g)) w_{s(g)}(x). \end{aligned} \tag{A.7}$$

Therefore

$$\mathcal{W}f(g) = \langle f, w_g \rangle_{L_2(X)} = \overline{\chi}(r(g)) \left\langle f, w_{s(g)} \right\rangle_{L_2(X)} = \overline{\chi}(r(g))\mathcal{W}f(r(g)).$$

Thus $\mathcal{W}f(g)$ is known once its restriction to Ω is known or, in more abstract sense, once the wavelet transform is known on G/H. Therefore the restriction of $\mathcal{W}f$ to Ω merits a new notation, $\widehat{\mathcal{W}}f$. The mapping $\widehat{\mathcal{W}} : F_2(X) \to L_\infty(\Omega)$ will be called *reduced wavelet transform* and we shall denote by $F_2(\Omega)$ the image of $\widehat{\mathcal{W}}$ equipped with the inner product induced by $\widehat{\mathcal{W}}$ from $F_2(X)$.

It follows from (A.3) that $\widehat{\mathcal{W}}$ intertwines π with the representation ρ given for a function ϕ on Ω by

$$\rho_g \phi(a) = \chi(r(g^{-1}a))\phi(s(g^{-1}a)).$$

Indeed, for ϕ of the form $\phi = \widehat{\mathcal{W}}f$ we have that

$$\rho_g \widehat{\mathcal{W}}f(a) = \chi(r(g^{-1}a))\widehat{\mathcal{W}}f(s(g^{-1}a)) = \mathcal{W}f(g^{-1}a),$$

and so

$$\rho_g \widehat{\mathcal{W}}f(a) = \mathcal{W}f(g^{-1}a) = \widehat{\mathcal{W}}\pi_g f(a). \tag{A.8}$$

While ρ is not as geometrical as λ, in applications it is still has a more geometrical nature than the original π. If the Haar measure dh on H is taken in such a way that $\int_H |\chi(h)|^2\, dh = 1$ and $dg = dh\, da$ we can rewrite (A.5) as follows:

$$\begin{aligned} f(x) &= \int_G \mathcal{W}f(g)w_g(x)\, dg \\ &= \int_\Omega \int_H \mathcal{W}f(ah)w_{ah}(x)\, dh\, da \\ &= \int_\Omega \int_H \widehat{\mathcal{W}}f(a)\overline{\chi}(h)\chi(h)w_a(x)\, dh\, da \\ &= \int_\Omega \widehat{\mathcal{W}}f(a)w_a(x)\, da \end{aligned}$$

We define an integral transformation $\mathcal{F}$ according to the last formula:

$$\mathcal{F}\phi(x) = \int_\Omega \phi(a)w_a(x)\, da. \tag{A.9}$$

This has the property $\mathcal{F} \circ \widehat{\mathcal{W}} = I$ on $F_2(X)$. One can then consider the integral transform $\mathcal{K} = \mathcal{F} \circ \widehat{\mathcal{W}}$, explicitly

$$\mathcal{K}f(x) = \int_\Omega \langle f, w_a\rangle_{L_2(X)}\, w_a(x)\, da, \tag{A.10}$$

which is defined on the whole of $L_2(X)$ (not only $F_2(X)$). It is known that $\mathcal{K}$ *is an orthogonal projection* $L_2(X) \to F_2(X)$ [**?**]. If we formally use linearity of the scalar product $\langle\cdot,\cdot\rangle_{L_2(X)}$ (i.e., assume that Fubini's Theorem holds) we obtain from (A.10)

$$\begin{aligned} \mathcal{K}f(x) &= \int_\Omega \langle f, w_a\rangle_{L_2(X)}\, w_a(x)\, da \\ &= \int_X f(\mathbf{y})K(y,x)\, d\mu(\mathbf{y}), \end{aligned} \tag{A.11}$$

where

$$K(y,x) = \int_\Omega \bar{w}_a(\mathbf{y})w_a(x)\, da$$

Sometimes a reduced form $\widehat{\mathcal{P}} : L_2(\Omega) \to F_2(\Omega)$ of the projection $\mathcal{P}$ (A.6) is of interest in itself. It is an extension of the integral operator $\widehat{\mathcal{W}} \circ \mathcal{F}$, and it is an easy calculation using (A.4) that

$$[\widehat{\mathcal{P}}\phi](a') = \int_{\Omega} \phi(a)\mathcal{W}f_0(a^{-1}a')\bar{\chi}(r(a^{-1}a'))\,da. \tag{A.12}$$

As we shall see its explicit form can be calculated easily in practical cases.

Observe that, from (A.3), the image of $\mathcal{W}$ is invariant under action of the left but not right regular representations. $F_2(\Omega)$ is invariant under the representation (A.8), which is a pullback of the left regular representation on G, but not its right counterpart, and so in general there is no way to define an action of left-invariant vector fields on Ω, which are infinitesimal generators of right translations, on $L_2(\Omega)$. But there is an exception. Let $\mathfrak{X}_j$ be a maximal set of left-invariant vector fields on G such that

$$\mathfrak{X}_j\mathcal{W}f_0(g) = 0.$$

Because the $\mathfrak{X}_j$ are left invariant and (A.4) we have $\mathfrak{X}_j\mathcal{W}w_{g'}(g) = 0$ for all g' and thus the image of $\mathcal{W}$, being the linear span of $\mathcal{W}w_{g'}$, is part of the intersection of kernels of $\mathfrak{X}_j$. The same remains true if we consider the pullback $\widehat{\mathfrak{X}}_j$ of $\mathfrak{X}_j$ to Ω. Note that in general there are fewer linearly independent $\widehat{\mathfrak{X}}_j$ than there are $\mathfrak{X}_j$. We call $\widehat{\mathfrak{X}}_j$ *Cauchy-Riemann-Dirac* operators because of the property that

$$\widehat{\mathfrak{X}}_j\widehat{\mathcal{W}}f(a) = 0 \qquad \forall\mathcal{W}f \in F_2(\Omega). \tag{A.13}$$

Explicit constructions of the Dirac type operator for a discrete series representation can be found in [**?**, **?**].

A.3 The Segal-Bargmann space

We consider a representation of the Heisenberg group $\mathbb{H}^n$ (see Section 2) on $L_2(\mathbb{R}^n)$ by shift and multiplication operators [**?**, § 1.1]:

$$g = (t,\mathbf{z}) : f(\mathbf{x}) \to [\pi_{(t,\mathbf{z})}f](\mathbf{x}) = e^{\imath(2t-\sqrt{2}\mathbf{q}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{p})}f(\mathbf{x}-\sqrt{2}\mathbf{p}), \qquad \mathbf{z} = \mathbf{p}+\imath q, \tag{A.14}$$

This is the Schrödinger representation with parameter $\hbar = 1$. As a subgroup H we select the centre of $\mathbb{H}^n$ consisting of elements $(t,0)$. It is non-compact but using the special form of representation (A.14) we can consider the cosets[2]

[2] $\widetilde{G}$ is sometimes called the *reduced* Heisenberg group. It seems that $\widetilde{G}$ is a virtual object, which is important in connection with a selected representation of G.

$\widetilde{G}$ and $\widetilde{H}$ of G and H by the subgroup with elements $(\pi m, 0)$, $m \in \mathbb{Z}$. Then (A.14) also defines a representation of $\widetilde{G}$ and $\widetilde{H} \sim \Gamma$. We consider the Haar measure on $\widetilde{G}$ such that its restriction on $\widetilde{H}$ has total mass equal to 1.

As "vacuum vector" we will select the original *vacuum vector* of quantum mechanics—the Gauss function $f_0(\mathbf{x}) = e^{-\mathbf{x}\cdot\mathbf{x}/2}$. Its transformations are defined as follows:

$$\begin{aligned} w_g(\mathbf{x}) = \pi_{(t,\mathbf{z})} f_0(\mathbf{x}) &= e^{\imath(2t-\sqrt{2}\mathbf{q}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{p})}\, e^{-(\mathbf{x}-\sqrt{2}\mathbf{p})^2/2} \\ &= e^{2\imath t-(\mathbf{p}\cdot\mathbf{p}+\mathbf{q}\cdot\mathbf{q})/2} e^{-((\mathbf{p}-\imath\mathbf{q})^2+\mathbf{x}\cdot\mathbf{x})/2+\sqrt{2}(\mathbf{p}-\imath\mathbf{q})\cdot\mathbf{x}} \\ &= e^{2\imath t-\mathbf{z}\cdot\bar{\mathbf{z}}/2} e^{-(\bar{\mathbf{z}}\cdot\bar{\mathbf{z}}+\mathbf{x}\cdot\mathbf{x})/2+\sqrt{2}\bar{\mathbf{z}}\cdot\mathbf{x}}. \end{aligned}$$

In particular $w_{(t,0)}(\mathbf{x}) = e^{-2it} f_0(\mathbf{x})$, i.e. it really is a vacuum vector with respect to $\widetilde{H}$ in the sense of our definition. Of course $\widetilde{G}/\widetilde{H}$ is isomorphic to $\mathbb{C}^n$. Embedding $\mathbb{C}^n$ in G by the identification of $(0, \mathbf{z})$ with $\mathbf{z}$, the mapping $s : \widetilde{G} \to \widetilde{G}$ is defined simply by $s((t, \mathbf{z})) = (0, \mathbf{z}) = \mathbf{z}$; Ω then is identical with $\mathbb{C}^n$.

The Haar measure on $\mathbb{H}^n$ coincides with the standard Lebesgue measure on $\mathbb{R}^{2n+1}$ [**?**, § 1.1] and so the invariant measure on Ω also coincides with Lebesgue measure on $\mathbb{C}^n$. Note also that the composition law sending $\mathbf{z}_1$ $\mathbf{z}_2$ to $s((0, \mathbf{z}_1)(0, \mathbf{z}_2))$ reduces to Euclidean shifts on $\mathbb{C}^n$. We also find $s((0, \mathbf{z}_1)^{-1} \cdot (0, \mathbf{z}_2)) = \mathbf{z}_2 - \mathbf{z}_1$ and $r((0, \mathbf{z}_1)^{-1} \cdot (0, \mathbf{z}_2)) = (\frac{1}{2}\Im\bar{\mathbf{z}}_1 \cdot \mathbf{z}_2, 0)$.

The reduced wavelet transform takes the form of a mapping $L_2(\mathbb{R}^n) \to L_2(\mathbb{C}^n)$ and is given by the formula

$$\begin{aligned} \widehat{\mathcal{W}} f(\mathbf{z}) &= \langle f, w_{(0,\mathbf{z})} \rangle \\ &= \pi^{-n/4} \int_{\mathbb{R}}^{n} f(\mathbf{x})\, e^{-\mathbf{z}\cdot\bar{\mathbf{z}}/2}\, e^{-(\mathbf{z}\cdot\mathbf{z}+\mathbf{x}\cdot\mathbf{x})/2+\sqrt{2}\mathbf{z}\cdot\mathbf{x}}\, dx \\ &= e^{-|\mathbf{z}|^2/2} \pi^{-n/4} \int_{\mathbb{R}}^{n} f(\mathbf{x})\, e^{-(\mathbf{z}\cdot\mathbf{z}+\mathbf{x}\cdot\mathbf{x})/2+\sqrt{2}\mathbf{z}\cdot\mathbf{x}}\, dx, \end{aligned} \qquad \text{(A.15)}$$

where $\mathbf{z} = \mathbf{p} + \imath\mathbf{q}$. Then $\widehat{\mathcal{W}} f$ belongs to $L_2(\mathbb{C}^n, dg)$. This can better be expressed by saying that the function $\breve{f}(\mathbf{z}) = e^{|\mathbf{z}|^2/2} \widehat{\mathcal{W}} f(\mathbf{z})$ belongs to $L_2(\mathbb{C}^n, e^{-|\mathbf{z}|^2} dg)$ because $\breve{f}(\mathbf{z})$ is analytic in $\mathbf{z}$. These functions constitute the *Segal-Bargmann space* [**?**, **?**] $F_2(\mathbb{C}^n, e^{-|\mathbf{z}|^2} dg)$ of functions analytic in $\mathbf{z}$ and square-integrable with respect the Gaussian measure $e^{-|\mathbf{z}|^2} d\mathbf{z}$. Analyticity of $\breve{f}(\mathbf{z})$ is equivalent to the condition that $(\frac{\partial}{\partial \bar{\mathbf{z}}_j} + \frac{1}{2}\mathbf{z}_j I)\mathcal{W} f(\mathbf{z})$ equals zero.

The integral in (A.15) is the well-known Segal-Bargmann transform [?, ?]. Its inverse is given by a realization of (A.9):

$$\begin{aligned} f(\mathbf{x}) &= \int_{\mathbb{C}^n} \widehat{\mathcal{W}f(\mathbf{z})} w_{(0,\mathbf{z})}(\mathbf{x})\, d\mathbf{z} \\ &= \int_{\mathbb{C}^n} \breve{f}(\mathbf{z}) e^{-(\bar{\mathbf{z}}^2+\mathbf{x}\cdot\mathbf{x})/2+\sqrt{2}\bar{\mathbf{z}}x}\, e^{-|\mathbf{z}|^2}\, d\mathbf{z}. \end{aligned} \tag{A.16}$$

This gives (A.9) the name of Segal-Bargmann inverse. The corresponding operator $\mathcal{P}$ (A.10) is the identity operator $L_2(\mathbb{R}^n) \rightarrow L_2(\mathbb{R}^n)$ and (A.10) gives an integral presentation of the Dirac delta.

Meanwhile the orthoprojection $L_2(\mathbb{C}^n, e^{-|\mathbf{z}|^2} dg) \rightarrow F_2(\mathbb{C}^n, e^{-|\mathbf{z}|^2} dg)$ is of interest and is a principal ingredient in Berezin quantisation [?, ?]. We can easy find its kernel from (A.12). Indeed, $\widehat{\mathcal{W}} f_0(\mathbf{z}) = e^{-|\mathbf{z}|^2}$, and the kernel is

$$\begin{aligned} K(\mathbf{z},\mathbf{w}) &= \widehat{\mathcal{W}} f_0(\mathbf{z}^{-1}\cdot\mathbf{w})\bar{\chi}(r(\mathbf{z}^{-1}\cdot\mathbf{w})) \\ &= \widehat{\mathcal{W}} f_0(\mathbf{w}-\mathbf{z})\exp(\imath\Im(\bar{\mathbf{z}}\cdot\mathbf{w}) \\ &= \exp(\frac{1}{2}(-|\mathbf{w}-\mathbf{z}|^2 + \mathbf{w}\cdot\bar{\mathbf{z}} - \mathbf{z}\cdot\bar{\mathbf{w}})) \\ &= \exp(\frac{1}{2}(-|\mathbf{z}|^2 - |\mathbf{w}|^2) + \mathbf{w}\cdot\bar{\mathbf{z}}). \end{aligned}$$

To obtain the reproducing kernel for functions $\breve{f}(\mathbf{z}) = e^{|\mathbf{z}|^2}\widehat{\mathcal{W}} f(\mathbf{z})$ in the Segal-Bargmann space we multiply $K(\mathbf{z},\mathbf{w})$ by $e^{(-|\mathbf{z}|^2+|\mathbf{w}|^2)/2}$ which gives the standard reproducing kernel, $\exp(-|\mathbf{z}|^2+\mathbf{w}\cdot\bar{\mathbf{z}})$ [?, (1.10)].

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