STOCHASTIC SYMMETRY-BREAKING IN A GAUSSIAN HOPFIELD-MODEL

Anton Bovier[1]
Weierstraß–Institut
für Angewandte Analysis und Stochastik
Mohrenstrasse 39, D-10117 Berlin, Germany

Aernout C.D. van Enter[2]
Institute for Theoretical Physics
Rijksuniversiteit Groningen
Nijenborgh 4
NL-9747 AG Groningen, The Netherlands

Beat Niederhauser[3,4]
Fachbereich Mathematik
Technische Universität Berlin
Strasse des 17. Juni 136
D-10623 Berlin, Germany

Abstract: We study a "two-pattern" Hopfield model with Gaussian disorder. We find that there are infinitely many pure states at low temperatures in this model, and we find that the metastate is supported on an infinity of symmetric pairs of pure states. The origin of this phenomenon is the random breaking of a rotation symmetry of the distribution of the disorder variables.

Keywords: Hopfield model, Gaussian disorder, metastates, chaotic size-dependence, extrema of Gaussian processes.

[1]e-mail: bovier@wias-berlin.de
[2]e-mail: A.C.D.van.Enter@phys.rug.nl
[3]e-mail: niederha@math.tu-berlin.de
[4]supported by DFG in the Graduiertenkolleg "Stochastische Prozesse und probabilistische Analysis"

1. Introduction: Ising spins with a rotation symmetry

In this paper we will illustrate the notions of chaotic size dependence, metastates and their dispersal, and the chaotic pairs of states scenario, introduced as a possible description of the low temperature spin glass phase [N,NS2,NS3,NS4,NS6], on a simple model which is similar to the two-state Hopfield model. The fact that the model has site disorder makes it more tractable than the commonly considered bond-disorder spin glass models. The main difference with the standard Hopfield model of neural networks, is that instead of two i.i.d. Bernoulli random variables the disorder is described by two i.i.d. Gaussian random variables at every site. As a consequence, in the thermodynamic limit we obtain the existence, for a "two-pattern" model, of uncountably many (instead of two times two) pure states for this model, due to the existence of a continuous (rotation) symmetry of the distribution of the random variables describing the disorder. In any finite volume, however, this symmetry is necessarily randomly broken in a given realization. Intuitively, this means that there are only two pure ground states, and the low temperature Gibbs state is close to the symmetric mixture of two, out of a possible continuum, of pure Gibbs states, due to the fluctuations in the disorder.

The concepts we want to illustrate have their origin in the theory of spin-glasses. However, the most often considered spin-glass models, which have bond disorder, both in finite dimension (the Edwards-Anderson models) and the equivalent neighbour (Sherrington-Kirkpatrick) model, have turned out to be so complicated to analyze, that up till now it has not been possible to check which of the possible scenarios for the spin-glass phase applies to them.

We remind the reader that in the debate within the physics literature on the extreme sides there are the proposals of Fisher and Huse, [FH1,FH2,FH3,FH4] predicting the existence of only two pure states in any dimension higher or equal than 3, versus the proposal of Parisi and coworkers, in which an infinity of pure states is predicted [MPV, MPR]. This scenario has been claimed to apply down to the 3-dimensional Edwards-Anderson model. Intermediate scenarios have been discussed by [BF,NS1,NS2,NS3,NS4,NS5,NS6,N,vE].

Although of course lattice models with two pure states are common, our experience with models having an infinite number of pure states is a lot more limited. Therefore we hope that our discussion will be useful in illustrating various concepts, mostly introduced and studied in a systematic way by Newman and Stein (see in particular [N,NS2,NS3,NS4,NS6]), which have been introduced either in an abstract setting or via (in)formal arguments, by applying them to a concrete model.

The main idea in the approach of Newman and Stein is to classify the possible scenarios on the basis of first principles, using only general ergodic properties using the concept of "metastates", i.e. probability distributions on the space of Gibbs measures (first introduced apparently in [AW]; see [N,NS2,NS3,Ku1,Ku2,BG3] for more details, as well as applications of these concepts and extensions to equivalent neighbour or mean-field type models–to which our model also belongs).

In this context, in one of their most recent papers [NS6], they conjectured that in a disordered lattice system, in any approximate decomposition of a finite volume Gibbs states into "pure states", the weights in this decomposition should be mostly concentrated on a single subset of states that are related by an exact symmetry of the system, while other states would appear with a weight that tends to zero as the volume tends to infinity. The particular subset chosen could of course be random and could depend strongly on the volume. Applied to the Ising spin glass situation, this argument would predict the chaotic pairs picture.

Although a similar situation has been shown to occur in the usual Hopfield model with $M = \alpha N$ patterns if α is small in [BG3], we found it worthwhile to construct a simple model showing these features in order to see what is involved.

Let us state the definitions of our variant of the Hopfield model and the main quantities of interest. Let $\mathcal{S}_N = \{-1,+1\}^N$ denote the set of functions $\sigma : \{1,\dots,N\} \to \{-1,+1\}$, and the set $\mathcal{S} = \{-1,+1\}^{\mathbb{N}}$. We call σ a spin configuration and denote by σ_i the value of σ at i. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be an abstract probability space and let $\xi_i^\mu[\omega]$, $i \in \mathbb{N}$, $\mu = 1,2$, denote a family of i.i.d. standard Gaussian variables. We will write $\xi^\mu[\omega]$ for the N-dimensional vector whose ith component is given by $\xi_i^\mu[\omega]$; such a vector is called a *pattern*. On the other hand, we will write $\xi_i[\omega]$ for the two dimensional vector with the same components. When we write $\xi[\omega]$ without indices, we consider it as a $2 \times N$ matrix (its transpose will be denoted by ξ^t).

Throughout the paper, $(\cdot,\cdot)$ denotes the scalar product, without indication of the space where its arguments lie.

We define random maps $m_N^\mu[\omega](\sigma) : \mathcal{S}_N \to [-1,+1]$ (conventionally called *overlap parameters*) through

$$m_N^\mu[\omega](\sigma) \equiv \frac{1}{N}\sum_{i=1}^{N} \xi_i^\mu[\omega]\sigma_i. \tag{1.1}$$

The Hamiltonian is now defined as

$$\begin{aligned} H_N[\omega](\sigma) &\equiv -\frac{N}{2}\sum_{\mu=1,2}\Big(m_N^\mu[\omega](\sigma)\Big)^2 \\ &= -\frac{N}{2}\|m_N[\omega](\sigma)\|_2^2, \end{aligned} \tag{1.2}$$

where $\|\cdot\|_2$ denotes the l_2-norm in $\mathbb{R}^2$.

Note that if we rewrite $\xi'^1_i = \xi^{\theta}{}^1_i = \xi_i^1\cos(\theta)+\xi_i^2\sin(\theta)$ and $\xi'^2_i = \xi'^{\theta}_i = \xi_i^1\sin(\theta)-\xi_i^2\cos(\theta)$ the Hamiltonian has the same form in the primed variables. However, this transformation is a *statistical* symmetry, mapping one disorder realization of the model to another one, drawn from the same distribution, as opposed to for example the spin-flip symmetry which is an exact symmetry for any given realization of the disorder.

Through this Hamiltonian, finite volume Gibbs measures on $\mathcal{S}_N$ are defined by

$$\mu_{N,\beta}[\omega](\sigma) \equiv 2^{-N}\frac{e^{-\beta H_N[\omega](\sigma)}}{Z_{N,\beta}[\omega]}, \tag{1.3}$$

and the induced distribution of the overlap parameters

$$\mathcal{Q}_{N,\beta}[\omega] \equiv \mu_{N,\beta}[\omega]\circ m_N[\omega]^{-1}. \tag{1.4}$$

The normalizing factor in (1.3), called the *partition function*, is explicitly given by

$$Z_{N,\beta}[\omega] \equiv 2^{-N}\sum_{\sigma\in\mathcal{S}_N} e^{-\beta H_N[\omega](\sigma)} \equiv \mathbb{E}_\sigma\, e^{-\beta H_N[\omega](\sigma)}. \tag{1.5}$$

We are mainly interested in the concentration behaviour of $\mathcal{Q}_{N,\beta}$ as $N\to\infty$. It will be convenient to do this by considering the auxiliary measure $\widetilde{\mathcal{Q}}_{N,\beta} \equiv \mathcal{Q}_{N,\beta}\star\mathcal{N}_2(0,\frac{1}{\beta N}\mathbb{I})$ obtained by a convolution with a Gaussian measure, its so-called Hubbard-Stratonovich transform. Since, for N large, $\mathcal{N}_2(0,\frac{1}{\beta N}\mathbb{I})$ converges rapidly to the Dirac measure at zero, the two measures have asymptotically the same properties. For details see e.g. [BGP]. $\widetilde{\mathcal{Q}}_{N,\beta}$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^2$ and has the density

$$\frac{e^{-\beta N\Phi_{N,\beta}[\omega](z)}}{Z_{N,\beta}[\omega]}, \tag{1.6}$$

where $\Phi_{N,\beta}$ is given by

$$\Phi_{N,\beta}[\omega](\sigma) = \frac{1}{2}\|z\|_2^2 - \frac{1}{\beta N}\sum_{i=1}^N \ln\cosh\beta(\xi_i[\omega],z). \tag{1.7}$$

As usual in mean-field models, we construct the extremal Gibbs measures by *tilting* the Hamiltonian (1.2) with an *external magnetic field* (for a general discussion on the issue of limiting Gibbs states in mean field models, see [BG1], Sect. 2.4 or [BG3], Sect. 2). That is, we define a more general Hamiltonian

$$H_N^h[\omega](\sigma) \equiv -\frac{N}{2}\|m_N[\omega](\sigma)\|_2^2 - N(h,m), \tag{1.8}$$

where $h = (b\cos(\vartheta), b\sin(\vartheta)) \in \mathbb{R}^2$. The corresponding measures on the spins and on $\mathbb{R}^2$ are denoted by $\mu_{N,\beta}^h[\omega]$ and $\mathcal{Q}_{N,\beta}^h[\omega]$, respectively. We then take the limits $\lim_{b\to 0}\lim_{N\to\infty}$, for all values of $\vartheta \in [0, 2\pi)$. We distinguish the measures constructed from this Hamiltonian by an additional superscript h.

We are now able to give a precise formulation of our main results.

Theorem 1: *Let $h = (b\cos\vartheta, b\sin\vartheta)$. Then*

$$\lim_{b\to 0}\lim_{N\to\infty} \mathcal{Q}_{N,\beta}^h = \delta_{(r^*\cos\vartheta, r^*\sin\vartheta)}, \tag{1.9}$$

where r^* is the largest solution of the equation

$$r^* = \frac{1}{\sqrt{2\pi}}\int dx\, e^{-\frac{x^2}{2}} x\tanh(\beta x r^*). \tag{1.10}$$

Theorem 1 shows that there is an uncountable number of extremal limiting induced measures, indexed by the circle. The following Corollary shows that to each of them corresponds a distinct limiting Gibbs measure on the spins.

Corollary 2: *For any finite set $I \subset \mathbb{N}$, and $\mathbb{P}$-almost all ω,*

$$\mu_{\infty,\beta}^h[\omega]\Big(\{\sigma_I = s_I\}\Big) \equiv \lim_{b\to 0}\lim_{N\to\infty} \mu_{N,\beta}^h[\omega]\Big(\{\sigma_I = s_I\}\Big) = \prod_{i\in I} \frac{e^{\beta s_i(\xi_i[\omega],m)}}{2\cosh(\beta(\xi_i[\omega],m))}, \tag{1.11}$$

where $m = (r^*\cos(\vartheta), r^*\sin(\vartheta))$, and r^* as in (1.10).

In Theorem 1 and Corollary 2 convergence is almost sure due to the presence of the tilting field. The situation changes if we set $b = 0$ first and take the infinite volume limit later.

Theorem 3: *Let $\mathcal{Q}_{N,\beta}$ as in (1.4) and $m = m(\vartheta) = (r^*\cos\vartheta, r^*\sin\vartheta)$, where $\vartheta \in [0,\pi)$ is a uniformly distributed random variable. Then*

$$\mathcal{Q}_{N,\beta} \xrightarrow{\mathcal{D}} \frac{1}{2}\delta_{m(\vartheta)} + \frac{1}{2}\delta_{-m(\vartheta)} \equiv \mathcal{Q}_{\infty,\beta}[m]. \tag{1.12}$$

Furthermore, the (induced) AW-metastate is the image of the uniform distribution of ϑ under the measure-valued map $\vartheta \mapsto \mathcal{Q}_{\infty,\beta}[m(\vartheta)]$.

Corollary 4: *Let $I \subset \mathbb{N}$ be finite. Then the following holds:*

(i) Let $\{g_i\}_{i\in I}$ be a family of i.i.d. random variables, distributed as $\mathcal{N}(0, r^*)$. Then

$$\lim_{N\uparrow\infty} \mu_{N,\beta}(\sigma_I = s_I) \overset{\mathcal{D}}{\to} \frac{1}{2}\prod_{i\in I}\frac{e^{\beta s_i g_i}}{2\cosh\beta g_i} + \frac{1}{2}\prod_{i\in I}\frac{e^{-\beta s_i g_i}}{2\cosh\beta g_i}. \tag{1.13}$$

(ii) The AW-metastate is the image of the uniform distribution on ϑ under the measure-valued map $\vartheta \mapsto \mu_{\infty,m(\vartheta)}[\omega]$ where

$$\mu_{\infty,\beta,m}[\omega] = \frac{1}{2}\prod_{i\in I}\frac{e^{\beta s_i(\xi_i[\omega],m)}}{2\cosh\beta(\xi_i[\omega],m)} + \frac{1}{2}\prod_{i\in I}\frac{e^{-\beta s_i(\xi_i[\omega],m)}}{2\cosh\beta(\xi_i[\omega],m)}. \tag{1.14}$$

Statement (ii) of Corollary 4 motivates the notion of metastates. Whereas on the level of the induced measures $\mathcal{Q}_{N,\beta}$ one cannot see any influence by the conditioning, this is clearly the case on the level of the Gibbs measures on the spins.

The remainder of this paper is mainly devoted to the proofs of the two theorems (the corollaries are standard consequences (see e.g. [BGP1] or [BG3] for proofs of analogous statements in more complicated situation) and will not be given) is organized as follows. In Section 2 we prove the necessary concentration estimates on the measures $\mathcal{Q}_{N,\beta}$. This will yield immediately Theorem 1. In the case $h = 0$ we will show that the measure concentrates near the absolute minima of some random process, and in Section 3 we will analyse the properties of these minima. In particular we will prove that these converge in distribution to one-point sets. This will allow us to prove Theorem 3. In Section 4 we discuss some further consequences on the chaotic volume dependence, the empirical metastate and the superstate.

Remark: We consider the case of two patterns here in order to keep technicalities to a minimum. All our results can be extended without any novel difficulties to the case of any fixed finite number, M, of Gaussian patterns. In that case the set of extremal Gibbs measures will be indexed by the sphere in $\mathbb{R}^M$ and the metastate will be supported on pairs of mirror images on this sphere, with the position being uniformly distributed. Thus nothing really new will happen. The situation when the number of patterns grows with the volume may be more interesting and work in this direction is in progress.

2. Concentration

In this section we show the concentration properties of the measures $\widetilde{\mathcal{Q}}_N$ for large β. These imply the same concentration results for the measures $\mathcal{Q}_N$ by standard arguments that have been developed in much more complicated situations, see e.g. [BG2]. The estimates presented here are mostly similar, and often much simpler, to those that can be found e.g. in [BG2], but we decided to present some parts in detail where some care is required.

We start with the more delicate case $h = 0$ that will be relevant for the proof of Theorem 3 (which will be given at the end of Section 3). We are interested in the concentration behaviour of the measures $\widetilde{\mathcal{Q}}_{N,\beta}$. The following two lemmata each give a partial answer. The first one asserts that $\widetilde{\mathcal{Q}}_{N,\beta}$ is concentrated exponentially about a circle around the origin, whereas the second one tells us that even on this circle, only a small part really contributes to the total mass.

Lemma 2.1: *Let $\{\xi_i^\mu\}_{i\in\mathbb{N},\mu=1,2}$ be i.i.d. standard Gaussian variables, and define $\Phi_{N,\beta}(z)$ as*

$$\Phi_N(z) \equiv \frac{1}{2}\|z\|_2^2 - \frac{1}{\beta N}\sum_{i=1}^N \ln\cosh\beta(\xi_i, z). \tag{2.1}$$

Let furthermore $\delta_N = N^{-1/10}$. Then there exist strictly positive constants K, K', m, m' such that (r^* is the largest solution in (1.10))

$$\frac{\int_{|\,\|z\|-r^*|\geq\delta_N} e^{-\beta N\Phi_N(z)}\,dz}{\int_{|\,\|z\|-r^*|<\delta_N} e^{-\beta N\Phi_N(z)}\,dz} \leq K e^{-KN^m}, \tag{2.2}$$

on a set of $\mathbb{P}$-measure at least $1 - K'e^{-K'N^{m'}}$.

The second result needs an additional definition. Let

$$g_N(\vartheta) \equiv \frac{1}{\sqrt{N}}\sum_{i=1}^N \ln\cosh(\beta r^*\zeta_i\cos(\vartheta - \varphi_i)), \tag{2.3}$$

where (ζ_i, φ_i) are the polar coordinates of the two dimensional vector ξ_i.

Lemma 2.2: *Assume the hypotheses of Lemma 2.1. Let $a_N = N^{-1/25}$. Then there exist strictly positive constants K_1, K_2, C_1, C_2 such that on a set of $\mathbb{P}$-measure at least*

$$1 - K_1 e^{-N^{-1/25}} \tag{2.4}$$

the following bound holds,

$$\frac{\int_{A'_N} e^{-\beta N\Phi_N(z)}\,dz}{\int_{A_N} e^{-\beta N\Phi_N(z)}\,dz} \le C_1 e^{-N^{2/5}}, \tag{2.5}$$

where

$$\begin{aligned} A_N &= \Big\{(r,\vartheta)\in\mathbb{R}_0^+\times[0,2\pi)\Big|\,|r-r^*|<\delta_N, g_N(\vartheta)-\min_{\vartheta} g_N(\vartheta)<a_N\Big\},\\ A'_N &= \Big\{(r,\vartheta)\in\mathbb{R}_0^+\times[0,2\pi)\Big|\,|r-r^*|<\delta_N, g_N(\vartheta)-\min_{\vartheta} g_N(\vartheta)\ge a_N\Big\}. \end{aligned} \tag{2.6}$$

Combining these two lemmata and using the Borel-Cantelli lemma, we get immediately the following result.

Proposition 2.3: *Assume the hypotheses of Lemma 2.1. Then there exist strictly positive constants K, K', m, such that*

$$\mathbb{P}\left[\frac{\int_{A_N^c} e^{-\beta N\Phi_N(z)}\,dz}{\int_{A_N} e^{-\beta N\Phi_N(z)}\,dz} > Ke^{-K'N^m},\ i.o.\ in\ N\right]=0, \tag{2.7}$$

where A_N is as in Lemma 2.2.

To see why the preceding results should be expected, we must consider the function $\Phi_{N,\beta}$. Note that the expectation of this function,

$$\mathbb{E}\Phi_N(z) = \frac{1}{2}\|z\|_2^2 - \frac{1}{\beta}\mathbb{E}\ \ln\cosh\beta(\xi_1,z). \tag{2.8}$$

depends only on the modulus of its argument. It is useful to observe that if $z=(r\cos\theta, r\sin\theta)$, we can represent $\mathbb{E}\Phi_N(z)$ as

$$\mathbb{E}\Phi_N(z) = \frac{1}{2}r^2 - \mathbb{E}_\varphi\mathbb{E}_\zeta\ \ln\cosh(\beta r\zeta\cos(\varphi))\,d\varphi \tag{2.9}$$

where ζ, ϕ are the representation of the polar decomposition of a two dimensional normal vector, i.e. ζ is distributed with density $xe^{-x^2/2}$ on $\mathbb{R}^+$, and φ uniformly on the circle $[0,2\pi)$.

From this it follows that $\mathbb{E}\Phi_N(z)$ takes its minimum on the circle with radius $r^*(\beta)$, where r^* is defined in Theorem 1. It is easy to verify that there is $0<\beta^*<\infty$, such that $r^*(\beta)>0$ if and only if $\beta>\beta^*$.

It is also straightforward to check that $\mathbb{E}\Phi$ is sufficiently smooth to guarantee that it is bounded from above by a quadratic function (of $\|z\|$) in some neighbourhood containing r^*.

Proof of 2.1: We start with the numerator. We decompose the domain of integration into an "inner" part $\mathcal{I}$,and an "outer" part $\mathcal{O}$:

$$\begin{aligned}\left\{z \in \mathbb{R}^2 : |\|z\| - r^*| \geq \delta\right\} = \left\{z \in \mathbb{R}^2 : \|z\| - r^* \geq \delta\right\}& \\ \cup \left\{z \in \mathbb{R}^2 : \|z\| - r^* \leq -\delta\right\} = \mathcal{O} \cup \mathcal{I}.&\end{aligned} \tag{2.10}$$

Consider the integral on $\mathcal{O}$. We write it as

$$\int_{\mathcal{O}} e^{-N\Phi_N(z)}\, dz = \int_{\mathcal{O}} e^{-\beta N \mathbb{E}\,\Phi_N(z)} e^{-\beta N(\Phi_N(z) - \mathbb{E}\,\Phi_N(z))}\, dz, \tag{2.11}$$

and observe that $\mathbb{E}\Phi_N$ can be bounded below by a quadratic function $C(\|z\| - r^*)^2$. We are left with the task of estimating the term $\Phi_N(z) - \mathbb{E}\Phi_N(z)$. This is accomplished by the following Lemma.

Lemma 2.4: *Let* $f_N(z) = \frac{1}{\beta N}\sum_{i=1}^N \ln\cosh\beta(\xi_i, z)$ *and*

$$\mathcal{O} = \left\{z \in \mathbb{R}^2 : \|z\| > r^* + \delta\right\}. \tag{2.12}$$

Then, for δ *small enough, such that* $\delta^2/16 \leq \delta/2\sqrt{2}$*, there exist strictly positive constants* C_1, C_2, K_1, K_2 *such that*

$$\mathbb{P}\left[\sup_{z\in\mathcal{O}} |f_N(z) - \mathbb{E}f_N(z)| \geq \frac{C}{2}(\|z\| - r^*)^2\right] \leq K_1 e^{-K_2 N} + C_1\delta^{-2} e^{-C_2\delta^4 N} N^{-\frac{1}{2}}. \tag{2.13}$$

Proof: Define $\bar{f}_N(z) = f_N(z) - \mathbb{E}f_N(z)$. The left-hand side of (2.13) is bounded from above by

$$\begin{aligned}&\leq \mathbb{P}\left[\sup_{z'\in\mathcal{W}_r\cap\mathcal{O}} \left|\bar{f}_N(z')\right| \geq \frac{C}{4}\left(\|z'\| - r^*\right)^2\right] \\ &+ \mathbb{P}\left[\sup_{z'\in\mathcal{W}_r\cap A}\ \sup_{z\in B_r(z')} \left|\bar{f}_N(z) - \bar{f}_N(z')\right| \geq \frac{C}{4}\left(\|z'\| - r^*\right)^2\right],\end{aligned} \tag{2.14}$$

where $\mathcal{W}_r$ is the grid with spacing r in $\mathbb{R}^2$, and $z' \in \mathcal{W}_r$ is chosen such that $0 \leq \|z\| - \|z'\| < \sqrt{2}\,r$.

The argument of the second term can be uniformly bounded. Using e.g. Lemma 6.10 of [BG1], we get that

$$|f_N(z) - f_N(z')| \leq \|z - z'\|_2 \|A\|^{1/2}, \tag{2.15}$$

where A is the matrix $(1/N)\xi^T\xi$. Similarly,

$$|\mathbb{E}f_N(z) - \mathbb{E}f_N(z')| \leq \|z - z'\|_2 (\mathbb{E}\|A\|)^{1/2}. \tag{2.16}$$

Now, a trivial computation shows that

$$\mathbb{E}\|A\| \leq 1 + C/\sqrt{N} \tag{2.17}$$

and using (for instance) the same argument as in Section 4 of [BG1], but replacing Talagrand's concentration estimate for bounded r.v.'s by the standard Gaussian concentration inequality (see e.g. [LT], Ch. 1), one shows easily that

$$\mathbb{P}\left[|\|A\| - 1| \geq x\right] \leq Ce^{-Nx^2/C}. \tag{2.18}$$

Therefore,

$$\begin{aligned}
\mathbb{P}&\left[\sup_{z' \in \mathcal{W}_r \cap A} \sup_{z \in B_r(z')} |\bar{f}_N(z) - \bar{f}_N(z')| \geq \frac{C}{4}(\|z'\| - r^*)^2]\right] \\
&\leq \mathbb{P}\left[r(\|A\|^{1/2} + (\mathbb{E}\|A\|)^{1/2}) \geq \frac{C}{4}(\|z'\| - r^*)^2\right] \\
&\leq \mathbb{P}\left[(\|A\|^{1/2} + 2) \geq \frac{C\delta^2}{4r}\right],
\end{aligned} \tag{2.19}$$

Choosing the grid parameter r such that $r \leq C\delta^2/16$ the right-hand side of (2.19) is bounded by $\mathbb{P}\left[\|A\| > 4\right] \leq Ce^{-9N/C}$ This takes care of the second term in (2.14). Let us now treat the first term. The probability that the supremum over all lattice points of some function exceeds some given value is transformed into a summable series of probabilities that at each lattice point the function is greater than this value. More precisely, we have

$$\begin{aligned}
\mathbb{P}\left[\sup_{z' \in \mathcal{W}_r \cap \mathcal{O}} \left|\bar{f}_N(z')\right| \geq \frac{C}{4}\left(\|z'\| - r^*\right)^2\right] &\leq \sum_{z' \in \mathcal{W}_r \cap \mathcal{O}} \mathbb{P}\left[\left|\bar{f}_N(z')\right| \geq \frac{C}{4}\left(\|z'\| - r^*\right)^2\right] \\
&\leq \sum_{z' \in \mathcal{W}_r \cap \mathcal{O}} e^{-KC^2\left(\|z'\| - r^*\right)^4 N},
\end{aligned} \tag{2.20}$$

by Chebyshev's inequality. Then

$$\begin{aligned}
\sum_{z' \in \mathcal{W}_r \cap \mathcal{O}} e^{-KC^2\left(\|z'\| - r^*\right)^4 N} &= r^{-2} \sum_{z' \in \mathcal{W}_r \cap \mathcal{O}} r^2 e^{-KC^2\left(\|z'\| - r^*\right)^4 N} \\
&\leq r^{-2} \int_{\mathbb{R}^2 \setminus B_0(r^* + \delta - \sqrt{2}r)} e^{-KC^2\left(\|z'\| - r^*\right)^4 N} \, dz \\
&\leq r^{-2} e^{-K\frac{C^2}{16}\delta^4 N} \int_{\mathbb{R}^2 \setminus B_0(r^* + \delta/2)} e^{-K\frac{C^2}{16}\left(\|z'\| - r^*\right)^4 N} \, dz \\
&\leq r^{-2} 2\pi e^{-K\frac{C^2}{16}\delta^4 N} N^{-\frac{1}{2}} \int_{\delta/2}^{\infty} z e^{-\widetilde{K}z^4} \, dz \\
&\leq K' r^{-2} e^{-K\frac{C^2}{2}\delta^4 N} N^{-\frac{1}{2}},
\end{aligned} \tag{2.21}$$

where K' stands for an upper bound for the integral, which is independent of N (assuming $\delta > 2\sqrt{2}r$). Combining this and (2.19), and choosing δ small enough such that $C\delta^2/16 \leq \delta/(2\sqrt{2})$ concludes the proof of Lemma 2.4. $\diamondsuit$

Therefore, on a set of measure at least $1 - C_1 e^{-C_2 N\delta^4}$, the integral (2.11) can be bounded by

$$\begin{aligned} \int_{\mathcal{O}} e^{-\beta N \mathbb{E}\,\Phi_N(z)} e^{-\beta N(\Phi_N(z) - \mathbb{E}\,\Phi_N(z))}\, dz &\leq \int_{\mathcal{O}} e^{-\beta N \frac{C}{2}(\|z\| - r^*)^2}\, dz \\ &\leq 2\pi \int_{r^*+\delta}^{\infty} r e^{-\beta N C(r-r^*)^2}\, dz \\ &\leq 2\pi e^{-N\frac{C}{4}\delta^2} \int_0^{\infty} r e^{-\beta N \frac{C}{4} r^2}\, dr \\ &= 2\pi \frac{2}{\beta N C} e^{-\beta N \frac{C}{4}\delta^2}. \end{aligned} \tag{2.22}$$

We now turn to the integral on the "inner" part $\mathcal{I}$. Again, we have to control the term

$$\Phi_N(z) - \mathbb{E}\,\Phi_N(z). \tag{2.23}$$

Since $\mathcal{I}$ is compact, we can do this uniformly by using the following lemma.

Lemma 2.5: *Let $f_N(z) = 1/(\beta N)\sum_{i=1}^N \ln\cosh\beta(\xi_i, z)$ and $A \subset \mathbb{R}^2$ a bounded set. Then there exist strictly positive constants K_1, K_2, C_1, C_2 such that*

$$\mathbb{P}\left[\sup_{z\in A} |f_N(z) - \mathbb{E} f_N(z)| > \varepsilon\right] \leq K_1 e^{-K_2 N} + C_1 \varepsilon^{-2} e^{-C_2 \varepsilon^2 N}. \tag{2.24}$$

The proof is similar (if not simpler) to the proof of Lemma 2.4 and is left to the reader. $\diamondsuit$

Lemma 2.5 implies that

$$\begin{aligned} \int_{\mathcal{I}} e^{-\beta N \Phi_N(z)}\, dz &\leq e^{\varepsilon N} e^{-\beta N \mathbb{E}\,\Phi(r^*)} \int_{\mathcal{I}} e^{-\beta N \mathbb{E}\,\Phi_N(z)}\, dz \\ &\leq e^{\varepsilon\ bN} e^{-\delta^2 C \beta N} \pi r^{*2}, \end{aligned} \tag{2.25}$$

using the fact that $\mathbb{E}\Phi_N(\|z\|) - \mathbb{E}\Phi(r^*)$ can be bounded uniformly on $\mathcal{I}$ by its value for $\|z\| = r^* - \delta$.

Finally, the denominator in (2.2) can be bounded from below, using the second order Taylor expansion with remainder of $\mathbb{E}\,\Phi_N(\|z\|)$

$$\begin{aligned} &\int_{|\,\|z\| - r^*| < \delta} e^{-\beta N \Phi_N(z)}\, dz \\ &\geq e^{-\beta N \mathbb{E}\,\Phi(r^*)} \int_{|\,\|z\| - r^*| < \delta} e^{-NC(\|z\| - r^*)^2 - NC'(\|\tilde{z}\| - r^*)^3 - N\varepsilon}\, dz \\ &\geq 2\pi \frac{1}{\beta N C} e^{-\varepsilon\beta N} e^{-\beta N C'\delta^3} e^{-\beta N \mathbb{E}\,\Phi(r^*)} \left(1 - \delta e^{-\beta N C \delta^2}\right), \end{aligned} \tag{2.26}$$

on a set of measure at least $1-Ke^{-KN}-C\varepsilon^{-2}e^{-CN\varepsilon^2}$ (this error term can be estimated by Lemma 2.5). Collecting (2.22), (2.25) and (2.26), we get that on a set of measure exponentially close to one,

$$\begin{aligned}\frac{\int_{|\|z\|-r^*|\geq\delta} e^{-\beta N\Phi_N(z)}\,dz}{\int_{|\|z\|-r^*|<\delta} e^{-\beta N\Phi_N(z)}\,dz} &\leq Me^{\varepsilon\beta N}e^{\beta NC'\delta^3}(2\pi)^{-1}\beta NC\left(1-\delta e^{-\beta NC\delta^2}\right)^{-1}\\ &\quad\times\left\{e^{\varepsilon\beta N}e^{-\beta NC\delta^2}\pi r^{*2}+2\pi e^{-\beta N\frac{C}{4}\delta^2}\frac{2}{\beta NC}\right\}\\ &= MKe^{-\beta N(C\delta^2-2\varepsilon-C'\delta^3)}\beta N\left(1-\delta e^{-NC\delta^2}\right)^{-1}\\ &\quad+MK'e^{-\beta N(\frac{C}{4}\delta^2-\varepsilon-C'\delta^3)}N\left(1-\delta e^{-\beta NC\delta^2}\right)^{-1}.\end{aligned} \tag{2.27}$$

Now let us choose $\delta_N=N^{-\frac{1}{10}}$, $\varepsilon_N=N^{-\frac{1}{4}}$; then (2.27) gives

$$\begin{aligned}\frac{\int_{|\,\|z\|-r^*|\geq\delta_N} e^{-\beta N\Phi_N(z)}\,dz}{\int_{|\,\|z\|-r^*|<\delta_N} e^{-\beta N\Phi_N(z)}\,dz} &\leq M\widetilde{K}Ne^{-\beta N^{\frac{4}{5}}(C-2N^{-\frac{1}{20}}-C'N^{-\frac{1}{10}})}\\ &\quad+M\widetilde{K}Ne^{-N^{\frac{4}{5}}(\frac{C}{4}-N^{-\frac{1}{20}}-C'N^{-\frac{1}{10}})},\end{aligned} \tag{2.28}$$

on a set which is exponentially close (in N) to 1. This concludes the proof of Lemma 2.1. $\diamondsuit$

We now turn to the proof of Lemma 2.2 which is a little more delicate than the previous one.

Proof of 2.2: Let us write $I(B)$ for the integral $\int_B e^{-\beta N\Phi_N(z)}\,dz$. We will prove the concentration behaviour by a strategy similar to the one used in Lemma 2.1. Namely we replace the function Φ_N by its expectation $\mathbb{E}\Phi_N$ and control the error.

Write the fluctuation term $\Phi_N-\mathbb{E}\Phi_N$ as

$$\begin{aligned}\Phi_N(z)-\mathbb{E}\Phi_N(z) &= \frac{1}{\beta N}\sum_{i=1}^N\{\ln\cosh\beta(\xi_i,z)-\mathbb{E}\ln\cosh\beta(\xi_i,z)\}\\ &= \frac{1}{\beta N}\sum_{i=1}^N\{\ln\cosh\beta(\xi_i,z)-\ln\cosh\beta(\xi_i,z')\\ &\qquad-\mathbb{E}\ln\cosh\beta(\xi_i,z)+\mathbb{E}\ln\cosh\beta(\xi_i,z')\}\\ &\quad+\frac{1}{\beta N}\sum_{i=1}^N\{\ln\cosh\beta(\xi_i,z')-\mathbb{E}\ln\cosh\beta(\xi_i,z')\}.\end{aligned} \tag{2.29}$$

Now choose z' such that $z'=z'(z)=\lambda z$, $\lambda>0$, and $\|z'\|=r^*$ (i.e. z' is the projection of z

onto $S^1(r^*)$). Define the two functions

$$h_N(z) \equiv \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \{\ln \cosh \beta(\xi_i, z) - \ln \cosh \beta(\xi_i, z') \\ - \mathbb{E} \ln \cosh \beta(\xi_i, z) + \mathbb{E} \ln \cosh \beta(\xi_i, z')\}, \tag{2.30}$$

with z' defined as above, and

$$g_N(z) \equiv \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \{\ln \cosh \beta(\xi_i, z) - \mathbb{E} \ln \cosh \beta(\xi_i, z)\}. \tag{2.31}$$

Then the fluctuation term takes the form

$$N(\Phi_N(z) - \mathbb{E}\,\Phi_N(z)) = \frac{\sqrt{N}}{\beta}(h_N(z) - g_N(z')). \tag{2.32}$$

It is the term g_N that determines the concentration behaviour of the measure. To see this we first bound the term h_N uniformly on the "annulus of concentration" $A_N \cup A'_N$. We have the following result.

Lemma 2.6: *Let $\{\xi_i\}_{i\in\mathbb{N}}$ be i.i.d. Gaussian variables with mean zero and variance one. Let h_N be as in (2.30), and A_N, A'_N as in (2.6). Then for any $\varepsilon > 0$,*

$$\mathbb{P}\left[\sup_{z\in A_N\cup A'_N} |h_N(z)| \geq \varepsilon\right] \leq KN^2 e^{-N^{1/10}(\varepsilon - KN^{-1/10})}. \tag{2.33}$$

Proof: Let us write

$$f_i(z) \equiv \ln \cosh \beta(\xi_i, z), \tag{2.34}$$

and

$$\bar{f}_i \equiv \ln \cosh \beta(\xi_i, z) - \mathbb{E} \ln \cosh \beta(\xi_i, z). \tag{2.35}$$

We also keep the notation $z' = z'(z)$ defined above. Introduce a polar grid $\mathcal{W}_N$ in $\mathbb{R}^2$, i.e. a discrete set of points $x_{i,j}$ whose polar coordinates are given by $(\rho_i, \alpha_j) \in \mathbb{R}^+ \times [0, 2\pi)$, such that $\Delta_N\alpha \equiv |\alpha_i - \alpha_j| = KN^{-1/2}$ and $\Delta_N\rho \equiv |\rho_i - \rho_j| = KN^{-1/2}$, for some appropriate constant K. Note that for any point z in a bounded domain $A \subset \mathbb{R}^2$, the distance to the closest grid point is less than $K'N^{-1/2}$.

For any $z \in \mathbb{R}^2$, define $x = x(z) \in \mathcal{W}_N$ to be the grid point closest to z, and $y = y(z) \in \mathcal{W}_N$ the grid point closest to $z' = z'(z)$. One can easily convince oneself, that $x' = y'$, i.e. the two points x and y lie on the same ray starting at the origin. Then we can decompose the function $h_N(z)$ as

$$\begin{aligned} h_N(z) &= \frac{1}{\sqrt{N}} \sum_{i=1}^N \{\bar{f}_i(z) - \bar{f}_i(z')\} \\ &= \frac{1}{\sqrt{N}} \sum_{i=1}^N \{\bar{f}_i(z) - \bar{f}_i(x)\} + \frac{1}{\sqrt{N}} \sum_{i=1}^N \{\bar{f}_i(x) - \bar{f}_i(y)\} \\ &\quad + \frac{1}{\sqrt{N}} \sum_{i=1}^N \{\bar{f}_i(y) - \bar{f}_i(z')\}. \end{aligned} \tag{2.36}$$

Denote by $I_1(z,x)$, $I_2(x,y)$, $I_3(y,z')$ respectively the first, second and third sum on the right-hand side of (2.36). We can then write (let $\mathcal{A}_N = A_N \cup A'_N$, the "annulus of concentration")

$$\begin{aligned} \mathbb{P}\left[\sup_{z\in\mathcal{A}_N} |h_N(z)| \geq \varepsilon\right] &= \mathbb{P}\left[\sup_{z\in\mathcal{A}_N} |I_1(z,x) + I_2(x,y) + I_3(y,z')| \geq \varepsilon\right] \\ &\leq \mathbb{P}\left[\sup_{x\in\mathcal{W}_N\cap\mathcal{A}_N} \sup_{z\in B_{KN^{-1/2}}(x)} |I_1(z,x)| \geq \frac{\varepsilon}{3}\right] \\ &\quad + \mathbb{P}\left[\sup_{x\in\mathcal{W}_N\cap\mathcal{A}_N} \sup_{\substack{y\in\mathcal{W}_N\cap\mathcal{A}_N \\ y'=x'}} |I_2(x,y)| \geq \frac{\varepsilon}{3}\right] \\ &\quad + \mathbb{P}\left[\sup_{y\in\mathcal{W}_N\cap\mathcal{A}_N} \sup_{z'\in B_{KN^{-1/2}}(y)} |I_3(y,z')| \geq \frac{\varepsilon}{3}\right]. \end{aligned} \tag{2.37}$$

The first and the third term (they are equal) can be uniformly bounded by an estimate analogous to the proof of Lemma 2.2. In fact, for any u, v, we have

$$\left| \frac{1}{\sqrt{N}} \sum_{i=1}^N \{\bar{f}_i(u) - \bar{f}_i(v)\} \right| \leq \sqrt{N}\beta(\|A\|^{1/2} + (\mathbb{E}\|A\|)^{1/2})\|u-v\|_2. \tag{2.38}$$

Now, if $\|u-v\|_2 \leq 4\varepsilon' N^{-1/2}/\beta$, we have the following exponential bound.

$$\begin{aligned} \mathbb{P}\left[\left|\frac{1}{\sqrt{N}} \sum_{i=1}^N \{\bar{f}_i(u) - \bar{f}_i(v)\}\right| \geq \varepsilon' \right] &\leq \mathbb{P}\left[\|A\|^{1/2} + (\mathbb{E}\|A\|)^{1/2} \geq \frac{\varepsilon' N^{-1/2}}{\beta\|u-v\|_2}\right] \\ &\leq \mathbb{P}\left[\|A\| \geq 4\right] \leq Ke^{-KN}. \end{aligned} \tag{2.39}$$

Thus we get for the first term in (2.37),

$$\begin{aligned} &\mathbb{P}\left[\sup_{x\in\mathcal{W}_N\cap\mathcal{A}_N} \sup_{z\in B_{KN^{-1/2}}(x)} |I_1(z,x)| \geq \frac{\varepsilon}{3}\right] \\ &\leq \sum_{x\in\mathcal{W}_N\cap\mathcal{A}_N} \mathbb{P}\left[\sup_{z\in B_{KN^{-1/2}}(x)} |I_1(z,x)| \geq \frac{\varepsilon}{3}\right] \\ &\leq \sum_{x\in\mathcal{W}_N\cap\mathcal{A}_N} \mathbb{P}\left[\|A\| \geq 4\right] \leq KN^{1/10}N^{-1}e^{-KN}, \end{aligned} \tag{2.40}$$

since we know that $\|x-z\| = K'N^{-1/2}$, by the remark preceding (2.36), and the number of grid points in $\mathcal{A}_N$ is bounded by $N^1\delta_N^{-1}$ times some constant. The same estimate is valid for the term containing I_3 (since they are equal).

Let us now consider the term containing I_2. We know that $\|x-y\| \le 2\delta_N$, since those two points are supposed to lie on the same "ray". Again, we can turn the supremum into a sum,

$$\mathbb{P}\left[\sup_{x\in\mathcal{W}_N\cap\mathcal{A}_N}\sup_{\substack{y\in\mathcal{W}_N\cap\mathcal{A}_N\\ y'=x'}}|I_3(x,y)| \ge \frac{\varepsilon}{3}\right] \le \sum_{x,y}\mathbb{P}\left[|I_3(y,z')| \ge \frac{\varepsilon}{3}\right], \tag{2.41}$$

where x, y on the right-hand side satisfy the same conditions as on the left-hand side. By Chebyshev's inequality, we get that for any u, v

$$\begin{aligned}\mathbb{P}\left[\sum_{i=1}^{N}\{\bar{f}_i(u)-\bar{f}_i(v)\} \ge \sqrt{N}\varepsilon'\right] &\le \inf_{s>0} e^{-s\varepsilon'\sqrt{N}}\mathbb{E}\left[e^{s\sum_{i=1}^{N}\{\bar{f}_i(u)-\bar{f}_i(v)\}}\right]\\ &= \inf_{s>0} e^{-s\varepsilon'\sqrt{N}}\prod_{i=1}^{N}\mathbb{E}e^{s\{\bar{f}_i(u)-\bar{f}_i(v)\}}.\end{aligned} \tag{2.42}$$

Now we use the series expansion of the exponential function, the fact that the exponent in the right-hand side of (2.42) is a centered random variable, and some obvious inequalities for each term of the expansion, to get

$$\mathbb{E}e^{s\{\bar{f}_i(u)-\bar{f}_i(v)\}} \le \left\{1+\frac{s^2}{2}\mathbb{E}\left[(\bar{f}_i(u)-\bar{f}_i(v))^2 e^{s|\bar{f}_i(u)-\bar{f}_i(v)|}\right]\right\}. \tag{2.43}$$

To evaluate the expectation term, we use the inequality

$$|f_i(u)-f_i(v)| \le \beta|(\xi_i, u-v)|. \tag{2.44}$$

Then the expectation term in (2.42) is bounded by

$$\begin{aligned}\mathbb{E}\left[(\bar{f}_i(u)-\bar{f}_i(v))^2 e^{s|\bar{f}_i(u)-\bar{f}_i(v)|}\right] &\le \left(\mathbb{E}\left[(\bar{f}_i(u)-\bar{f}_i(v))^4\right]\right)^{\frac{1}{2}}\left(\mathbb{E}e^{2s|\bar{f}_i(u)-\bar{f}_i(v)|}\right)^{\frac{1}{2}}\\ &\le 4\left(\mathbb{E}\left[(f_i(u)-f_i(v))^4\right]\right)^{\frac{1}{2}}\left(\mathbb{E}e^{2s|f_i(u)-f_i(v)|}\right)^{\frac{1}{2}}\\ &\quad\times e^{s\mathbb{E}|f_i(u)-f_i(v)|},\end{aligned} \tag{2.45}$$

where the first inequality follows by Cauchy-Schwarz, and the second one is a consequence of the inequality $(a+b)^2 \le 2(a^2+b^2)$ (applied twice to the first factor), respectively the trivial fact that $|a-b| \le |a|+|b|$. All quantities in (2.45) can be bounded easily using (2.44). One gets (by calculating explicit Gaussian integrals)

$$\mathbb{E}\left[(f_i(u)-f_i(v))^4\right] = 3\|u-v\|_2^4, \tag{2.46}$$

$$\mathbb{E}e^{2s|f_i(u)-f_i(v)|} \le 2e^{2s^2\|u-v\|_2^2}, \tag{2.47}$$

$$e^{s\mathbb{E}\,|f_i(u)-f_i(v)|} \le e^{s\sqrt{2/\pi}\|u-v\|_2}. \tag{2.48}$$

Inserting (2.46)–(2.48) into (2.45), gives

$$\frac{s^2}{2}\mathbb{E}\left[(\bar{f}_i(u)-\bar{f}_i(v))^2 e^{s|\bar{f}_i(u)-\bar{f}_i(v)|}\right] \le 2\sqrt{6}s^2\|u-v\|_2^2 e^{2s^2\|u-v\|_2^2+s\sqrt{2/\pi}\|u-v\|_2}. \tag{2.49}$$

We use the above bound (2.49) in (2.42), together with the inequality $1+x\le e^x$, and the fact that $\|x-y\|_2 \le \delta_N = KN^{-1/10}$. We thus get the following estimate

$$\mathbb{P}\left[\sum_{i=1}^N \{\bar{f}_i(u)-\bar{f}_i(v)\} \ge \sqrt{N}\varepsilon'\right] \le \inf_{s>0} e^{-s\varepsilon'\sqrt{N}+Ks^2N^{4/5}e^{2s^2N^{-1/5}+\sqrt{2/\pi}N^{-1/10}}}. \tag{2.50}$$

Choosing $s=N^{-2/5}$, this gives

$$\mathbb{P}\left[\sum_{i=1}^N \{\bar{f}_i(u)-\bar{f}_i(v)\} \ge \sqrt{N}\varepsilon'\right] \le \widetilde{K}e^{-N^{1/10}(\varepsilon'-KN^{-1/10})}. \tag{2.51}$$

The same bound applies to

$$\mathbb{P}\left[\sum_{i=1}^N \{\bar{f}_i(u)-\bar{f}_i(v)\} \le -\sqrt{N}\varepsilon'\right]. \tag{2.52}$$

Inserting (2.51) and (2.52) into the left-hand side of (2.41) gives

$$\mathbb{P}\left[\sup_{x\in\mathcal{W}_N\cap\mathcal{A}_N}\ \sup_{\substack{y\in\mathcal{W}_N\cap\mathcal{A}_N\\ y'=x'}} |I_2(x,y)| \ge \varepsilon'\right] \le KN^{1/2}N^{1/10}e^{-N^{1/10}(\varepsilon'-K'N^{-1/10})}, \tag{2.53}$$

since the number of terms in the sum does not exceed a constant times $N^{1/2}$ (the number of allowed x) times $N^{1/10}$ (the number of allowed y). Using (2.40) and (2.53), (2.37) gives

$$\mathbb{P}\left[\sup_{z\in\mathcal{A}_N}|h_N(z)| \ge \varepsilon\right] \le KN^2e^{-K'N^{1/10}\varepsilon}. \tag{2.54}$$

This concludes the proof of Lemma 2.6. $\diamondsuit$

Note that we can choose ε as a function of N, and still get an exponential bound. For example, choose $\varepsilon=\varepsilon_N\equiv(\ln N)^2N^{-1/20}$. Lemma 2.6 then reads

Lemma 2.7: *Let $\{\xi_i\}_{i\in\mathbb{N}}$ be i.i.d. Gaussian variables with mean zero and variance one. Let h_N be as in (2.30), and A_N, A'_N as in (2.6). Then,*

$$\mathbb{P}\left[\sup_{z\in A_N\cup A'_N}|h_N(z)| \ge N^{-1/20}(\ln N)^2\right] \le KN^2e^{-N^{1/20}((\ln N)^2-K'N^{-1/20})}. \tag{2.55}$$

Furthermore,

$$\mathbb{P}\left[\sup_{z\in A_N\cup A'_N} |h_N(z)| \geq N^{-1/20}(\ln N)^2, i.o.\ in\ N\right] = 0. \tag{2.56}$$

Proof: The first statement (equation (2.55)) is a straightforward consequence of Lemma 2.6. Equation (2.56) then follows by the first Borel-Cantelli Lemma. $\diamondsuit$

Let us now estimate the integral $I({A'}_N)$. We get explicitly, using the bound on h_N from Lemma 2.6,

$$\begin{aligned}\int_{{A'}_N} e^{-\beta N\Phi_N(z)}\, dz &= \int_{{A'}_N} e^{-\beta N\mathbb{E}\,\Phi_N(z)} e^{-\sqrt{N}h_N(z)} e^{-\sqrt{N}g_N(z'(z))}\, dz\\ &\leq \int_{|r-r^*|<\delta_N} re^{-\beta N\mathbb{E}\,\Phi_N(r^*)} e^{\sqrt{N}\varepsilon}\, dr\\ &\quad\times \int_{g_N(\vartheta)-\min g_N>a_N} e^{-\sqrt{N}g_N(\vartheta)}\, d\vartheta\\ &= 2e^{-\beta N\mathbb{E}\,\Phi_N(r^*)} e^{\sqrt{N}\varepsilon}\int_{|r-r^*|<\delta_N} r\, dr\\ &\quad\times \int_{g_N(\vartheta)-\min g_N>a_N} e^{-\sqrt{N}g_N(\vartheta)}\, d\vartheta\\ &\leq 4e^{-\beta N\mathbb{E}\,\Phi_N(r^*)} e^{\sqrt{N}\varepsilon} r^*\delta_N\\ &\quad\times 2\pi e^{-\sqrt{N}a_N} e^{-\sqrt{N}\min g_N}.\end{aligned} \tag{2.57}$$

Thus,

$$\int_{{A'}_N} e^{-\beta N\Phi_N(z)}\, dz \leq Ke^{-\beta N\mathbb{E}\,\Phi_N(r^*)} e^{\sqrt{N}\varepsilon}\delta_N r^* e^{-\sqrt{N}a_N}. \tag{2.58}$$

We now turn to the integral $I(A_N)$. Using standard estimates for Gaussian integrals, a quadratic upper bound of g_N about its minima, and the fact that $\mathbb{E}\Phi(\|z\|)$ can be bounded from above by a quadratic function in some neighbourhood containing r^*, we get

$$\begin{aligned}\int_{A_N} e^{-\beta N\Phi_N(z)}\, dz &\geq e^{-\beta N\mathbb{E}\,\Phi_N(r^*)} e^{-\sqrt{N}\varepsilon}\int_{|r-r^*|<\delta_N} re^{-\beta NC'(r-r^*)^2}\, dr\\ &\quad\times\int_{g_N(\vartheta)-\min g_N\leq a_N} e^{-\sqrt{N}g_N(\vartheta)}\, d\vartheta\\ &\geq Ke^{-\beta N\mathbb{E}\,\Phi_N(r^*)} e^{-\sqrt{N}\varepsilon}(r^*-\delta_N)\left(\frac{\pi}{NC'}\right)^{1/2}\\ &\quad (1-e^{-NC'\delta_N})\left(\frac{\pi}{K\sqrt{N}}\right)^{1/2}(1-e^{-\sqrt{N}K'a_N}).\end{aligned} \tag{2.59}$$

We get finally for the ratio $I({A'}_N)/I(A_N)$

$$\frac{I(A'_N)}{I(A_N)} \leq K\frac{r^*}{r^*-\delta_N} N^{3/4} e^{-\sqrt{N}(a_N-2\varepsilon)}. \tag{2.60}$$

Lemma 2.7 allows us to choose $\varepsilon = \varepsilon(N) = N^{-1/20}(\ln N)^2$. Inserting this choice, together with $a_N = N^{-1/25}$, into (2.60), gives

$$\frac{I(A'_N)}{I(A_N)} \leq KN^{3/4}e^{-N^{23/50}(1-K'(\ln N)^2 N^{-1/100})}. \tag{2.61}$$

This statement is true for all $\omega \in \Omega$, for which Lemma 2.6 respectively 2.7 holds, that is on a set of $\mathbb{P}$-measure at least $KN^2 e^{-N^{1/20}((\ln N)^2 - K' N^{-1/20})}$. This proves Lemma 2.2. $\diamondsuit$

Let us now turn to the proof of Theorem 1. We again state first a result about the concentration of the induced measure $\widetilde{\mathcal{Q}}^h_{N,\beta}$.

Proposition 2.8: *Let $\{\xi_i^\mu\}_{i\in\mathbb{N},\mu=1,2}$ be i.i.d. standard Gaussian variables, and define*

$$\Phi^h_{N,\beta}(z) \equiv \frac{1}{2}\|z\|_2^2 - \frac{1}{\beta N}\sum_{i=1}^N \ln\,\cosh\beta(\xi_i, z+h). \tag{2.62}$$

Let furthermore $\delta_N = N^{-1/5}$. Then there exist strictly positive constants K, K', m such that

$$\mathbb{P}\left\{\frac{\int_{\|z-\tilde{r}^h\|\geq\delta_N} e^{-\beta N\Phi^h_{N,\beta}(z)}\,dz}{\int_{\|z-\tilde{r}^h\|<\delta_N} e^{-\beta N\Phi^h_{N,\beta}(z)}\,dz} \geq Ke^{-K'N^m},\ i.o.\ in\ N\right\} = 0, \tag{2.63}$$

where $\tilde{r}^h$ is the unique minimum of the function

$$\mathbb{E}\Phi^h_{N,\beta}(z) = \frac{1}{2}\|z\|_2^2 - \frac{1}{\beta}\mathbb{E}\ln\,\cosh\beta(\xi_1, z+h). \tag{2.64}$$

Proof: Let us decompose $\Phi^h_{N,\beta}$ in the usual way

$$\Phi^h_{N,\beta}(z) = \mathbb{E}\Phi^h_{N,\beta}(z) + \Phi^h_{N,\beta}(z) - \mathbb{E}\Phi^h_{N,\beta}(z). \tag{2.65}$$

We first treat the denominator appearing in (2.63). $\mathbb{E}\Phi^h_{N,\beta}$ can be bounded from below by some quadratic function $C\|z-\tilde{r}^h\|_2^2$ on the set $\|z-\tilde{r}^h\| \geq \delta_N > 0$. The fluctuation term can be controlled by the following analogue of Lemma 2.4.

Lemma 2.9: *Let $f_N = \frac{1}{\beta N}\sum_{i=1}^N \ln\,\cosh\beta(\xi_i, z+h)$. Then for δ small enough, sucht that $C\delta^2/80 < \delta/2$, there exist strictly positive constants C_1, C_2, K_1, K_2 such that*

$$\begin{aligned} p_N \equiv \mathbb{P}\Big[\sup_{z:\|z-\tilde{r}^h\|_2\geq\delta} |f_N(z) - \mathbb{E}f_N(z)| \geq \frac{C}{2}\|z-\tilde{r}^h\|_2^2\Big] \\ \leq K_1 e^{-K_2 N} + C_1 N^{1/2}\delta^{-2}e^{-C_2 N}. \end{aligned} \tag{2.66}$$

Proof: The proof is completely analogous to the proof of Lemma 2.4, and is left to the reader. $\diamondsuit$

Therefore, with probability greater than $1 - p_N$, $\sup(\Phi^h_{N,\beta} - \mathbb{E}\Phi^h_{N,\beta}(z)$ does not exceed one half of the lower bound of the deterministic part, which implies that

$$\begin{aligned}\int_{\|z-\tilde{r}^h\|\geq\delta_N} e^{-\beta N\Phi^h_{N,\beta}(z)}\,dz &\leq e^{-\beta N\mathbb{E}\Phi^h_{N,\beta}(\tilde{r}^h)}\int_{\|z-\tilde{r}^h\|\geq\delta_N} e^{-\beta N\frac{C}{2}\|z-\tilde{r}^h\|_2^2}\,dz\\ &\leq e^{-\beta N\mathbb{E}\Phi^h_{N,\beta}(\tilde{r}^h)}e^{-\beta N\frac{C}{4}\delta_N^2 K}.\end{aligned}\tag{2.67}$$

We now turn to the denominator in (2.63). The probability that the fluctuation term exceeds an $\varepsilon > 0$ is bounded by Lemma 2.5:

$$q_N \equiv \mathbb{P}\Big[\sup_{\|z-\tilde{r}^h\|<\delta_N} |f_N(z) - \mathbb{E}f_N(z)| \geq \varepsilon\Big] \leq K_1 e^{-K_2 N} + C_1\varepsilon^{-2}e^{-C_2\varepsilon^2 N}.\tag{2.68}$$

Using the Taylor expansion of $\mathbb{E}\Phi^h_{N,\beta}(z)$ about $\tilde{r}^h$ up to order 2, with an error term of order 3, we get that with probability higher than $1 - q_N$,

$$\begin{aligned}\int_{\|z-\tilde{r}^h\|<\delta_N} e^{-\beta N\Phi^h_{N,\beta}(z)}\,dz &\geq e^{-\beta N(\Phi^h_{N,\beta}(\tilde{r}^h+C''\delta_N^3+\varepsilon))}\int_{\|z-\tilde{r}^h\|<\delta_N} e^{-\beta NC'\|z-\tilde{r}^h\|_2^2}\,dz\\ &\geq e^{-\beta N(\Phi^h_{N,\beta}(\tilde{r}^h+C''\delta_N^3+\varepsilon))}KN^{-1/2}(1 - e^{-\beta N\frac{C'}{2}\delta_N^2}).\end{aligned}\tag{2.69}$$

Combining (2.67) and (2.69) gives

$$\frac{\int_{\|z-\tilde{r}^h\|\geq\delta_N} e^{-\beta N\Phi^h_{N,\beta}(z)}\,dz}{\int_{\|z-\tilde{r}^h\|<\delta_N} e^{-\beta N\Phi^h_{N,\beta}(z)}\,dz} \leq \widetilde{K}e^{-\beta N(\frac{C}{2}\delta_N^2-\varepsilon-C''\delta_N^3)}\tag{2.70}$$

with probability greater than $1 - (q_N + p_N)$. Choosing $\delta_N = N^{-1/5}$, $\varepsilon = N^{-1/5}$, implies that $\sum_N(p_N + q_N) < \infty$. Applying the Borel-Cantelli Lemma then gives the statement of Proposition 2.8. $\diamondsuit$

Theorem 1 is now obvious:

Proof of Theorem 1: Let f be a bounded continuous function. Then

$$\begin{aligned}\mathcal{Q}^h_{N,\beta}(f) = f(\tilde{r}^h\mathcal{Q}^h_{N,\beta}(\mathbb{1}_{\{\|z-\tilde{r}^h\|\leq\delta_N\}}) &+ \mathcal{Q}^h_{N,\beta}((f(\tilde{r}^h - f)\mathbb{1}_{\{\|z-\tilde{r}^h\|\leq\delta_N\}})\\ &+ \mathcal{Q}^h_{N,\beta}(f\mathbb{1}_{\{\|z-\tilde{r}^h\|>\delta_N\}}).\end{aligned}\tag{2.71}$$

Taking the limit $N \uparrow \infty$, we can replace $\mathcal{Q}^h_{N,\beta}$ by $\widetilde{\mathcal{Q}}^h_{N,\beta}$ and use Proposition 2.8. Since f is bounded, the third term on the right-hand side of (2.71) converges to zero, and since it is continuous, the second term also vanishes too. These statements are true $\mathbb{P}$-a.s. Finally we let $b = \|h\|_2 \to 0$. Again by continuity of f, $f(\tilde{r}^h) \to f(r^*(\cos\vartheta, \sin\vartheta))$. This proves the Theorem.$\diamondsuit\diamondsuit$

3. Uniqueness of extrema of certain gaussian processes.

In the previous chapter we have seen that the measures $\widetilde{\mathcal{Q}}_{N,\beta}$ concentrate on a circle of radius r^* at the places where the random function $g_N(\vartheta)$ takes its minimum. In this section we will show that these sets degenerate to a single point, a.s. in the limit $N \uparrow \infty$. To do so we first prove a uniqueness theorem for the absolute minimum of a certain class of strongly correlated Gaussian processes. Then we show convergence in distribution of $g_N(\vartheta)$ to such a process and finally we show that this implies also the desired convergence in distribution of our measures. We begin with the following general result.

Proposition 3.1: *Suppose $\chi(t)$ is a real stationary Gaussian process which is periodic with period T. Suppose furthermore that its covariance function $r(s,t) = r(s-t)$ is even, $\in C^\infty[0,T]$, and $r(\tau)$ is less than $r(0)$ for all $\tau \in (0,T)$. Then there exists an equivalent process $\eta(t)$ having almost surely infinitely differentiable sample paths. Moreover, the probability that there exist two or more maxima with equal height in $[0,T)$ is zero.*

Proof: Without restricting the generality, we can assume that $\mathbb{E}[\chi(t)] = 0$ and $\kappa = \mathbb{E}[\chi(t)^2] = 1$.

By its continuity properties, $r(\tau)$ can be expanded about the origin as

$$r(\tau) = 1 - \frac{\lambda_2}{2!}\tau^2 + O(\tau^4). \tag{3.1}$$

The first assertion then follows from the following result due to Cramér and Leadbetter (see [CL]), chapter 9.2).

Lemma 3.2: *Suppose that for some $a > 3$,*

$$r(\tau) = 1 - \frac{\lambda_2}{2}\tau^2 + O\left(\frac{\tau^2}{|\ln|\tau||^a}\right), \tag{3.2}$$

where λ_2 is a constant. Then there exists a process $\eta(t)$ equivalent to $\chi(t)$ and possessing, with probability one, a continuous derivative $\eta'(t)$.

Proof: See Cramér/Leadbetter [CL].

It is easily checked that by (3.1), $r(\tau)$ satisfies the condition (3.2) in Theorem 3.2, which proves the statements about continuity and existence of a continuous derivative.

Consider now the process $\chi'(t)$. Its covariance function $\tilde{r}(\tau)$ is given by $\tilde{r}(\tau) = -r''(\tau)$ (see for example Leadbetter et al. [LLR], p. 161, chapter 7.6). Then it can be expanded about the origin as

$$\tilde{r}(\tau) = \lambda_2 - \frac{\lambda_4}{2}\tau^2 + O(\tau^4). \tag{3.3}$$

Then $\tilde{r}(\tau)$ also verifies condition (3.2) in Theorem 3.2. Repeating this argument implies, together with the Borel-Cantelli Lemma, that there exists an equivalent process $\eta(t)$ having, with probability one, infinitely differentiable sample paths.

From now on, we assume that $\chi(t)$ itself has the above continuity properties. We want to find the probability that there are not two maxima with equal height in $[0, T)$, i.e.

$$\mathbb{P}\left[\exists s, t \in T \times T : |s - t| \neq kT, |\chi(t) - \chi(s)| = 0, |\chi'(t)| = |\chi'(s)| = 0\right] = 0. \tag{3.4}$$

We first show that for any $\vartheta > 0$,

$$\mathbb{P}\left[\exists s, t \in T \times T : \Big|kT - |s - t|\Big| \geq \vartheta, |\chi(t) - \chi(s)| = 0, |\chi'(t)| = |\chi'(s)| = 0\right] = 0 \tag{3.5}$$

Let us choose a collection of grid points $t_i \in T$, separated by some distance $\varepsilon > 0$. By the continuity properties, χ and χ' are Lipschitz-continuous with a.s.-finite constants C_0, C_1. Consider the set $\tilde{\Omega}_C \subset \Omega$ such that C_0 and C_1 are bounded by some number $C > 0$. Then, by Lipschitz-continuity, $\chi'(t) = 0$, $t \in [t_i, t_{i+1})$ implies that (for some $x \in [t_i, t]$)

$$|\chi'(t_i)| \leq C\varepsilon. \tag{3.6}$$

Similarly, $|\chi(t) - \chi(s)| = 0$ implies

$$|\chi(t_i) - \chi(t_j)| \leq 2C\varepsilon \tag{3.7}$$

where $t - t_i < \varepsilon$, $s - t_j < \varepsilon$. Then we can estimate the probability of the event in (3.5) (on $\tilde{\Omega}$) by

$$\begin{aligned} &\mathbb{P}\left[\exists s, t \in T \times T : \Big|kT - |s - t|\Big| \geq \vartheta, |\chi(t) - \chi(s)| = 0, |\chi'(t)| = |\chi'(s)| = 0\right] \\ &\leq \mathbb{P}\Big[\exists t_i, t_j : \big|kT - |s - t|\big| \geq \vartheta, |\chi(t_i) - \chi(t_j)| \leq 2C\varepsilon, |\chi'(t_i)| \leq C\varepsilon, \\ &\quad |\chi'(t_j)| \leq C\varepsilon\Big]. \end{aligned} \tag{3.8}$$

Let us denote the event appearing on the left-hand side of (3.8) by $\mathcal{A}_\vartheta$, and the event appearing on the right-hand side by $\mathcal{B}_{\vartheta,\varepsilon}$. The probability $\mathbb{P}[\mathcal{B}_{\vartheta,\varepsilon}]$ can be estimated by the standard bound

$$\mathbb{P}\left[\mathcal{B}_{\vartheta,\varepsilon}\right] \leq \sum_{|kT - |t_i - t_j|| \geq \vartheta} \mathbb{P}\left[|\chi(t_i) - \chi(t_j)| \leq 2C\varepsilon, |\chi'(t_i)| \leq C\varepsilon, |\chi'(t_j)| \leq C\varepsilon\right]. \tag{3.9}$$

Now, for any fixed i, j,

$$(\chi(t_i) - \chi(t_j), \chi'(t_i), \chi'(t_j)) \tag{3.10}$$

is a Gaussian vector, and due to the condition on $|t_i - t_j|$ and the assumption concerning $r(\tau)$, its distribution is non-degenerate. Therefore, each term in the sum on the right-hand side of (3.9) can be bounded by

$$\mathbb{P}\left[|\chi(t_i) - \chi(t_j)| \leq 2C\varepsilon, |\chi'(t_i)| \leq C\varepsilon, |\chi'(t_j)| \leq C\varepsilon\right] \leq K\varepsilon^3 C^3 (2\pi\sigma_{i,j})^{-1}, \tag{3.11}$$

where $\sigma_{i,j}$ is the determinant of the non-degenerate covariance matrix of the random vector (3.10). Since the t_i, t_j are chosen in a compact set, this quantity can be bounded uniformly in i, j. We thus get

$$\mathbb{P}\left[|\chi(t_i) - \chi(t_j)| \leq 2C\varepsilon, |\chi'(t_i)| \leq C\varepsilon, |\chi'(t_j)| \leq C\varepsilon\right] \leq K(\vartheta)\varepsilon^3 C^3. \tag{3.12}$$

Finally, the number of allowed pairs (i, j) in the sum in equation (3.9) does not exceed $T^2\varepsilon^{-2}$, which implies that

$$\begin{aligned} \mathbb{P}\left[\mathcal{A}_\vartheta\right] \leq& \mathbb{P}\left[\mathcal{B}_{\vartheta,\varepsilon}\right] + \mathbb{P}\left[\tilde{\Omega}^c_C\right] \\ \leq& K(\vartheta) T^2 \varepsilon^{-2} \varepsilon^3 + \mathbb{P}\left[\tilde{\Omega}^c_C\right], \end{aligned} \tag{3.13}$$

keeping track of the set $\tilde{\Omega}^c_C$ on which the above estimates are not valid. Now choose $C = C(\varepsilon) = o(\varepsilon^{-1/3})$, and observe that due to the continuity properties

$$\begin{aligned} \lim_{\varepsilon\to 0} \mathbb{P}\left[\tilde{\Omega}^c_{C(\varepsilon)}\right] &= \mathbb{P}\left[\bigcap_{n\in\mathbb{N}} \{C \geq n\}\right] \\ &= 0. \end{aligned} \tag{3.14}$$

Finally, letting ε tend to zero in (3.13) gives that the probability (3.6) is zero.

This shows that local maxima are separated with probablity one. In particular, constant pieces and no accumulation points of maxima. This concludes its proof. $\diamondsuit$

Corollary 3.3: *Suppose $\chi(t)$ satisfies the conditions in Proposition 3.1. Then $\chi(t)$ has a.s. only one global maximum in any interval $[s, s+t]$, $t < T$.*

To see that Proposition 3.1 is relevant for our problem, we will next show that the process $g_N(\vartheta)$ converges to a process of the type covered by this proposition. In fact we have

Proposition 3.4: *Let $g : \mathbb{R} \to \mathbb{R}^+$, $g \in C^\infty$ be an aperiodic even function. Suppose also that $\chi_i(\vartheta)$, $\vartheta \in [0, 2\pi]$ is the stochastic process given by*

$$\chi_i(\vartheta) = g\left(r\zeta_i \cos(\vartheta - \phi_i)\right), \tag{3.15}$$

where r is a positive constant, $\{\zeta_i\}_{i\in\mathbb{N}}$, $\{\phi_i\}_{i\in\mathbb{N}}$ are two mutually independent families of i.i.d. random variables, distributed as cxe^{-x^2} (ζ_i), and uniformly (ϕ_i). Then the process η_N given by

$$\eta_N(\vartheta) \equiv \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\{\chi_i(\vartheta) - \mathbb{E}\chi_i(\vartheta)\} \tag{3.16}$$

converges in distribution to a strictly stationary Gaussian process $\eta(\vartheta)$ having a.s. continuously differentiable sample paths. Furthermore, $\eta(\vartheta)$ has a.s. only one global maximum on any interval $[s, s+t]$, $t < \pi$

Remark: We will use this proposition of course with $g(\cdot) = \ln\cosh(\beta\cdot)$. Then the proposition implies that the process $g_N(\vartheta) - \mathbb{E}g_N(\vartheta)$ converges to a Gaussian process with the above properties.

Proof: As $\xi_i(\vartheta)$ are i.i.d. stationary processes on the circle which are infinitely differentiable, the convergence of the process to a stationary Gaussian process on the circle is a simple application of the central limit theorem in Banach spaces (see e.g. [LT]). A computation shows that the covariance of the limiting process is given by

$$\begin{aligned} f(s,t) &= \mathbb{E}[(\chi_1(s) - \mathbb{E}\chi_1(s))\,(\chi_1(t) - \mathbb{E}\chi_1(t))] \\ &= \mathbb{E}[g\,(r\zeta_1\cos(\varphi_1))\,g\,(r\zeta_1\cos(t-s-\varphi_1))] - (\mathbb{E}[g\,(r\zeta_1\cos(\varphi_1))])^2 \end{aligned} \tag{3.17}$$

We see that this function is even, and is in C^∞ as a function of $\tau = t - s$. Moreover, it is easily checked that the covariance function $f(\tau)$ is strictly smaller than $f(0)$, whenever $\tau \neq k\pi$. Proposition 3.1 and Corollary 3.3 then imply the assertions about continuity and non-existence of more than one global maximum. This concludes the proof of Proposition 3.4. $\diamondsuit$

We now check some intuitive properties of the position of the minimum of the Gaussian process from Proposition 3.1 (for those ω such that the minimum exists and is unique).

Proposition 3.5: *Suppose that the conditions of Proposition 3.1 are satisfied. Define $(\Omega', \mathcal{F}', \mathbb{P}')$ to be the restriction of $(\Omega, \mathcal{F}, \mathbb{P})$ to all ω such that the conclusions of Proposition 3.1 are true. Then the position of the minimum*

$$\vartheta^*[\omega] \equiv \arg\min_{\vartheta\in[0,\pi)} \chi[\omega](\vartheta) \tag{3.18}$$

of the sample path $\chi[\omega]$ is a random variable with uniform distribution on $[0,\pi)$.

Proof: To prove that $\vartheta^*[\omega]$ is a random variable, it is enough to show that for all intervals $\mathcal{U} = (a,b) \subseteq [0,\pi)$, the set $\vartheta^{*-1}(\mathcal{U})$ is in $\mathcal{F}'$. We note that by the continuity of χ on $[0,\pi)$

for all $\omega \in \Omega'$,

$$\begin{aligned} \vartheta^{*-1}(\mathcal{U}) &\equiv \{\omega \in \Omega : \chi[\omega](\cdot) \quad \text{assumes its minimum in} \quad \mathcal{U}\} \\ &= \{\omega \in \Omega' : \exists t \in \mathcal{U} \cap \mathbb{Q} \quad \text{such that} \quad \forall s \in \mathcal{U}^c \cap \mathbb{Q}, \chi(t) < \chi(s)\}. \end{aligned} \tag{3.19}$$

The second line can be written as

$$\bigcup_{t \in \mathcal{U} \cap \mathbb{Q}} \bigcap_{s \in \mathcal{U}^c \cap \mathbb{Q}} \{\omega \in \Omega' : \chi(t) < \chi(s)\}, \tag{3.20}$$

which clearly is in $\mathcal{F}'$.

Equation (3.20), together with the strict stationarity (since it is a real stationary process) of the process χ, implies the uniformity of the distribution. This proves Proposition 3.5. $\diamondsuit$

Finally, to get some information about the convergence of functions of the position of the minimum, we use the following two results.

Lemma 3.6: *Let $\mathcal{P}([0,\pi))$ be the space of T-periodic, continuous functions, having only one global minimum, together with the supremum norm. Then the position ϑ^* of the global minimum is a continuous function from $\mathcal{P}([0,\pi))$ to $[0,\pi)$.*

Proof: Suppose that there exists a sequence of functions $\{f_n\}$ converging to $f \in \mathcal{P}([0,\pi))$, such that the sequence of the global minima ϑ_n^* does not converge to ϑ^*, the global minimum of f. Then there exists an $\varepsilon > 0$ and a subsequence $\{f_{n_k}\}$, such that for all n_k, $|\vartheta^*_{n_k} - \vartheta^*| > \varepsilon$.

Now, since ϑ^* is the unique global minimum of f, $\exists \delta_\varepsilon > 0$ such that

$$f(\vartheta^*_{n_k}) > f(\vartheta^*) + \delta_\varepsilon. \tag{3.21}$$

Similarly, since $\vartheta^*_{n_k}$ is the unique minimum of f_{n_k}, $\exists \delta'_{\varepsilon,n_k} > 0$ such that

$$f_{n_k}(\vartheta^*) > f_{n_k}(\vartheta^*_{n_k}) + \delta'_{\varepsilon,n_k}. \tag{3.22}$$

Furthermore, since f_{n_k} converges in the supremum norm, $\forall \delta > 0$, $\exists K_\delta \in \mathbb{N}$ such that

$$\forall \vartheta \in [0,\pi), \forall k > K_\delta, \quad |f_{n_k}(\vartheta) - f(\vartheta)| < \delta. \tag{3.23}$$

For any $k > K_\delta$ one can therefore write

$$\begin{aligned} f_{n_k}(\vartheta^*) - f(\vartheta^*) &= f_{n_k}(\vartheta^*) - f_{n_k}(\vartheta^*_{n_k}) + f_{n_k}(\vartheta^*_{n_k}) - f(\vartheta^*_{n_k}) + f(\vartheta^*_{n_k}) - f(\vartheta^*) \\ &> \delta'_{\varepsilon,n_k} - \delta + \delta_\varepsilon \\ &> \delta_\varepsilon - \delta. \end{aligned} \tag{3.24}$$

Now choose $\delta = \frac{1}{3}\delta_\varepsilon$. Then for all $k > K_{\frac{1}{3}\delta}$,

$$f_{n_k}(\vartheta^*) - f(\vartheta^*) > \frac{2}{3}\delta_\varepsilon > \delta, \tag{3.25}$$

which contradicts the assumption of uniform convergence. $\diamondsuit$

The following result is crucial to link the weak convergence of the process $g_N(\vartheta)$ to the weak convergence of the measures $\mathcal{Q}_{N,\beta}$.

Proposition 3.7: *Define the random sets*

$$L_N[\omega] = \left\{\vartheta \in [0,\pi) : \eta_N[\omega](\vartheta) - \min_{\vartheta'} \eta_N[\omega](\vartheta') \leq \varepsilon_N\right\} \tag{3.26}$$

with ε_N some sequence converging to zero. Then

$$L_N \stackrel{\mathcal{D}}{\to} \vartheta^* \tag{3.27}$$

Proof: Using the *method of a single probability space* (see [Shi], Chapter 3, Section 8, Theorem 1) one can construct a probability space $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$ and random processes η_N^*, η^*, such that

$$\eta_N^* \to \eta^*, \quad \mathbb{P}^* - a.s., \tag{3.28}$$

and

$$\eta^* \stackrel{\mathcal{D}}{=} \eta, \qquad \eta_N^* \stackrel{\mathcal{D}}{=} \eta_N. \tag{3.29}$$

Now introduce the random level sets

$$L_N^*[\omega^*] = \left\{\vartheta \in [0,\pi) : \eta_N^*[\omega^*](\vartheta) - \min_{\vartheta'} \eta_N^*[\omega^*](\vartheta') \leq \varepsilon_N\right\},$$

Then L_N and L_N^* have the same distribution. But since $\eta_N^*[\omega]$ converges almost surely to $\eta^*[\omega] \in \mathcal{P}([0,\pi))$, one sees that due to Lemma 3.6 $L_N^*[\omega]$ converges $\mathbb{P}^*$-a.s. to the position of the unique absolute minimum of $\eta^*[\omega^*]$. But this minimum has the same distribution as that of η, which is the uniform distribution by Proposition 3.5. Therefore, L_N converges in distribution to a uniformly distributed point on $[0,\pi)$. $\diamondsuit$

We have finally all tools available to prove Theorem 3.

Proof of Theorem 3: We have to check convergence on the following type of functions $F : \mathcal{M}(\mathbb{R}^2) \to \mathbb{R}$

$$F(\mu) = \widetilde{F}(\mu(f_1), \dots, \mu(f_k)), \tag{3.30}$$

where $\widetilde{F}$ is a polynomial function, and $f_1, \dots, f_k$ are bounded continuous functions from $\mathbb{R}^2 \to \mathbb{R}$. Convergence in law then means that

$$\lim_{N\uparrow\infty} \mathbb{E}\Big[F(\mathcal{Q}_{N,\beta}[\omega])\Big] = \frac{1}{\pi}\int_0^\pi F(\frac{1}{2}\delta_{(m^*\cos\vartheta, m^*\sin\vartheta)} + \frac{1}{2}\delta_{(m^*\cos\vartheta+\pi, m^*\sin\vartheta+\pi)})\,d\vartheta. \tag{3.31}$$

The left-hand side of (3.31) is explicitly written as

$$\lim_{N\uparrow\infty} \mathbb{E}\Big[\widetilde{F}(\mathcal{Q}_{N,\beta}[\omega](f_1), \dots, \mathcal{Q}_{N,\beta}[\omega](f_k))\Big]. \tag{3.32}$$

We now treat the individual arguments of $\widetilde{F}$ in (3.32). Let $A_N[\omega]$ (the level sets in the previous lemmata) be decomposed into its $2l'$ connected components $A_{N,j_N}[\omega]$. As a consequence of Lemma 3.7, there exists $N[\omega]$ which is finite a.s. such that for all $N \geq N(\omega)$, $l = 1$, and the two corresponding connected components are symmetric with respect to the origin. Now choose arbitrary points $x_{N,j_N}[\omega] \in A_{N,j_N}[\omega]$. Then we can decompose

$$\begin{aligned}\widetilde{\mathcal{Q}}_{N,\beta}[\omega](f_i) &= \sum_{j_N} f_i(x_{N,j_N})\widetilde{\mathcal{Q}}_{N,\beta}[\omega](\mathbb{1}_{A_{N,j_N}}) + \sum_{j_N} \widetilde{\mathcal{Q}}_{N,\beta}(\mathbb{1}_{A_{N,j_N}}(f_i(x_{N,j_N}) - f_i)) \\ &\quad + \widetilde{\mathcal{Q}}_{N,\beta}(\mathbb{1}_{A_N^c} f_i).\end{aligned} \tag{3.33}$$

Expanding $\widetilde{F}$ using the decomposition (3.33), we get a sum consisting of two different types of terms: (i), summands that are products of the first sum on the right-hand side of (3.33) only, and (ii), summands where at least one of the second and third term from the right-hand side of (3.33) enter. Proposition 2.3 and Proposition 3.7, and the continuity and boundedness of the f_i's imply that the terms of type (ii) vanish $\mathbb{P}$-a.s., as $N \uparrow \infty$. In the limit, the only terms left are of type (i), which together sum up to

$$\widetilde{F}\left(\sum_{j_N} f_1(x_{N,j_N})\widetilde{\mathcal{Q}}_{N,\beta}[\omega](\mathbb{1}_{A_{N,j_N}}), \dots, \sum_{j_N} f_k(x_{N,j_N})\widetilde{\mathcal{Q}}_{N,\beta}[\omega](\mathbb{1}_{A_{N,j_N}})\right) \tag{3.34}$$

All arguments of $\widetilde{F}$ in (3.34) converge in distribution to

$$\frac{1}{2}f_i((m^*\cos\vartheta, m^*\sin\vartheta)) + \frac{1}{2}f_i((m^*\cos\vartheta+\pi, m^*\sin\vartheta+\pi)), \quad \forall i = 1, \dots, k \tag{3.35}$$

where ϑ is a uniformly distributed r.v. on $[0,\pi)$, by Proposition 3.7. But convergence in distribution means by definition that

$$\begin{aligned}&\lim_{N\uparrow\infty} \mathbb{E}\Big[\widetilde{F}(\sum_{j_N} f_1(x_{N,j_N})\widetilde{\mathcal{Q}}_{N,\beta}[\omega](A_{N,j_N}), \dots, \sum_{j_N} f_2(x_{N,j_N})\widetilde{\mathcal{Q}}_{N,\beta}[\omega](A_{N,j_N}))\Big] \\ &\quad = \frac{1}{\pi}\int_0^\pi \widetilde{F}(\frac{1}{2}f_i((m^*\cos\vartheta, m^*\sin\vartheta)) + \frac{1}{2}f_i((m^*\cos\vartheta+\pi, m^*\sin\vartheta+\pi))\,d\vartheta,\end{aligned} \tag{3.36}$$

which in turn is by definition equal to

$$\frac{1}{\pi}\int_0^{\pi} F(\frac{1}{2}\delta_{(m^*\cos\vartheta, m^*\sin\vartheta)} + \frac{1}{2}\delta_{(m^*\cos\vartheta+\pi, m^*\sin\vartheta+\pi)})\, d\vartheta. \tag{3.37}$$

This proves the convergence in law (1.13) in Theorem 3. To obtain the identification of the metastate, just note that the process $\eta_N(\vartheta)[\omega]$ actually converges to the same Gaussian process under any of the conditional laws $\mathbb{P}[\cdot|\mathcal{F}_n]$, where $\mathcal{F}_n$ is the sigma-algebra generated by the random variables $\xi_i, i \leq n$. $\diamondsuit\diamondsuit$

4. Volume dependence, empirical metastates, superstates

We conclude this paper with the discussion of some more sophisticated concepts that have been proposed by Newman and Stein [NS2] and Bovier and Gayrard [BG3] and that should capture in more detail the actual asymptotic volume dependence of the Gibbs measures. In fact, the first question one may ask is whether for a fixed realization as the volume grows the finite volume Gibbs states really explore all the possibilities in the support of the metastate. One way of stating that this is the case is the following

Theorem 4.1: *There exist (deterministic) sequences $N_k \uparrow \infty$ such that the empirical metastate*

$$\frac{1}{k}\sum_{\ell=1}^{k} \delta_{\mathcal{Q}_{N_k,\beta}}, \tag{4.1}$$

converges almost surely to the law of $\mathcal{Q}_{\infty,\beta}$.

Proof: We have seen that the measure $\mathcal{Q}_{N_k,\beta}$ is sharply concentrated on the circle of radius r^* and at the angle where the process $g_{N_k}(\vartheta)$ (defined in (2.3) takes its absolute minimum. The idea is to choose N_k in such a way that these angles will be virtually independent for different k. Now note that we can write

$$g_{N_k}(\vartheta) = \widetilde{g}_k(\vartheta) + R_k(\vartheta), \tag{4.2}$$

where

$$\widetilde{g}_k(\vartheta) = \frac{1}{N_k}\sum_{i=N_{k-1}+1}^{N_k} \ln\cosh(\beta(r^*\zeta_i\cos(\vartheta-\varphi_i))), \tag{4.3}$$

are independent for different k by construction and

$$R_k(\vartheta) = \frac{1}{N_k} \sum_{i=1}^{N_{k-1}} \ln\cosh(\beta(r^*\zeta_i \cos(\vartheta - \varphi_i))). \tag{4.4}$$

Now by standard estimates identical to those presented in Section 3, one shows easily that there is a constant $C < \infty$ such that

$$\mathbb{P}\left[sup_{\vartheta\in[0,\pi)} |R_k(\vartheta) - \mathbb{E}R_k(\vartheta)| \geq x\frac{N_{k-1}}{N_k} \right] \leq C\exp\left(-x^2/C\right). \tag{4.5}$$

Thus we can always choose N_k growing sufficiently rapidly (e.g. $N_k = k!$) such that R_k is totally negligible compared to $\widetilde{g}_k$ for large k, and the position of the absolute minimum of $g_{N_k}(\vartheta)$ is asymptotically equal to that of $\widetilde{g}_k(\vartheta)$. This allows us to approximate for large k the random measures $\delta_{\mathcal{Q}_{N_k,\beta}}$ by independent measures and from this the asserted result follows from the law of large numbers. $\diamondsuit$

Remark: Theorem 4.1 says that that the empirical metastate constructed with sparse subsequences converges to the Aizenman-Wehr metastate, a.s.. This is a special example of a general theorem due to Newman and Stein [NS2] (where however they require possibly subsequences ℓ_i in the definition (4.1)).

Rather than considering the empirical metastate with sparse subsequences one may be interested in the volume dependence as the volume grows at its natural pace. To capture this, the idea put forward in [BG3] is to construct a measure valued stochastic process

$$\mu_\beta^t \equiv \lim_{N\uparrow\infty} \mu_{\beta,[tN]}, \tag{4.6}$$

with $t \in (0,1]$ and to consider either the (conditional) probability distribution of this process (the “superstate” [BG3]) or the (conditional) empirical distribution of the process (the “empirical metastate” [NS2]). Let us see what this entails in our context. The reader who has been following the exposition of the last two chapters will easily be convinced that this problem amounts to study the quantity

$$\vartheta(t) \equiv \arg\min_{\theta\in[0,\pi)} (\chi_t(\theta)), \tag{4.7}$$

where $\chi_t(\theta)$ is the distributional limit of the process

$$\chi_N^t(\vartheta) \equiv g_{[tN]}(\vartheta) - \mathbb{E}g_{[tN]}(\vartheta). \tag{4.8}$$

where $g_N(\theta)$ is defined in (2.3). By completely standard arguments one shows that the following invariance principle holds:

Lemma 4.2: *The process $\chi_N^t(\vartheta)$ converges in distribution, as $N \uparrow \infty$ to the Gaussian process $\chi_t(\vartheta)$, $t \in (0,1], \vartheta \in [0,\pi)$ with mean zero and covariance*

$$C(\vartheta,\vartheta',t,t') \equiv \frac{t \wedge t'}{\sqrt{tt'}} f(\vartheta,\vartheta'), \tag{4.9}$$

where

$$f(\vartheta,\vartheta') = \mathbb{E}\left[\ln\cosh\left(\beta r \zeta_1 \cos(\varphi)\right) \ln\cosh\left(\beta r \zeta_1 \cos\left(\varphi - (\vartheta - \vartheta')\right)\right)\right]. \tag{4.10}$$

$\chi_t(\theta)$ is a rather curious Gaussian process: as a function of t, to fixed ϑ it is (normalized) Brownian motion, while for fixed t as a function of ϑ it is the C^∞ process discussed in the previous section. The question is then what can be said about the process ϑ_t, defined by (4.7)?

Some facts follow easily. For instance, the process is almost surely single valued for all $t \in (0,1]$ except possibly on some Cantor set of zero Lebesgue measure. On the other hand, it seems natural that such an exceptional set will exist and that a typical realization will have continuous pieces and "jumps". Also, for t going to zero, the process "circles" around rapidly since χ_t and χ_s become uncorrelated as $s \downarrow 0$. But otherwise we do not see any immediate more specific characterization of the process or its path-properties.

References

[AW] M. Aizenman, J. Wehr, *Rounding effects of quenched randomness of first-order phase transitions*, Comm. Math. Phys. **130**, 489–528, 1990.

[CL] H. Cramér, M. R. Leadbetter, *Stationary and Related Stochastic Processes*, Wiley, 1967.

[BGP] A. Bovier, V. Gayrard, P. Picco, *Gibbs States of the Hopfield Model in the Regime of Perfect Memory*, PTRF **100**, 329–363, 1994.

[BG1] A. Bovier, V. Gayrard, *Hopfield Models as Generalized Random Mean Field Models*, in *Mathematical Aspects of Spin Glasses and Neural Networks*, A. Bovier and P. Picco (eds.), Progress in Probability, Birkhäuser, 1998.

[BG2] A. Bovier, V. Gayrard, *The Retrieval Phase of the Hopfield Model: A Rigorous Analysis of the Overlap Distribution*, PTRF **107**, 61–98, 1997.

[BG3] A. Bovier, V. Gayrard, *Metastates in the Hopfield model in the replica symmetric regime*, MPAG **1**, ??-??, 1998.

[BF] A. Bovier, J. Fröhlich, *A heuristic theory of the spin-glass phase*, J. Stat. Phys. **44**, 347–391, 1986.

[vE] A. C. D. van Enter, *Stiffness exponent, number of pure states, and Almeida-Thouless line in spin-glasses* J. Stat. Phys. **60**, 275-279, 1990.

[FH1] D. S. Fisher, D. A. Huse, *Ordered phase of short-range spinglasses*, Phys. Rev. Lett. **56**, 1601–1604, 1986.

[FH2] D. S. Fisher, D. A. Huse, *Equilibrium behavior of the spin-glass ordered phase*, Phys. Rev. B **38**, 386–411, 1988.

[FH3] D. S. Fisher, D. A. Huse, *Pure states in spin glasses*, J. Phys. A **20**, L997-1003, 1987.

[FH4] D. S. Fisher, D. A. Huse, *Absence of many states in realistic spin glasses*, J. Phys. A **20**, L1005–1010, 1987.

[Ge] S. Geman, *A Limit Theorem for the Norms of Random Matrices*, Ann. Prob. **8**, 252–261, 1980.

[Ku1] C. Külske, *Metastates in disordered mean field models: Random field and Hopfield models*, J. Stat. Phys. **88**, 1257–1293, 1997.

[Ku2] C. Külske, *Limiting Behavior of Random Gibbs Measures: Metastates in Some Disordered Mean Field Models*, in *Mathematical Aspects of Spin Glasses and Neural Networks*, A. Bovier, P. Picco (eds.), Progress in Probability 41, Birkhäuser, Boston-Basel-Berlin, 1998.

[LLR] M. R. Leadbetter, G. Lindgren, H. Rootzén, *Extremes and Related Properties of Random Sequences and Processes*, Springer, Berlin-Heidelberg-New York, 1983.

[LT] M. Ledoux and M. Talagrand, “Probability in Banach spaces”, Springer, Berlin-Heidelberg-New York, 1991.

[MPR] E. Marinari, G. Parisi, J. J. Ruiz-Lorenzo, *Numerical Simulations of Spin Glass Systems*, in *Spin Glasses and Random Fields*, A. P. Young (ed.), World Scientific, 1998.

[MPV] M. Mézard, G. Parisi, M. A. Virasoro, *Spin Glass Theory and Beyond*, World Scientific, 1987.

[N] C. M. Newman, *Topics in Disordered Systems*, Birkhäuser, Boston-Basel-Berlin, 1997.

[NS1] C. M. Newman, D. L. Stein, *Non-mean-field behavior of realistic spin glasses* Phys. Rev. Lett. **76**, 515-518, 1996.

[NS2] C. M. Newman, D. L. Stein, *Thermodynamic Chaos and the Structure of Short-Range Spin Glasses*, in *Mathematical Aspects of Spin Glasses and Neural Networks*, A. Bovier and P. Picco (eds.), Progress in Probability 41, Birkhäuser, Boston-Basel-Berlin, 1998.

[NS3] C. M. Newman, D. L. Stein, *Spatial inhomogeneity and thermodynamic chaos*, Phys. Rev. Lett. **76**, 4821–4824, 1996.

[NS4] C. M. Newman, D. L. Stein, *Metastate approach to thermodynamic chaos*, Phys. Rev. E **55**, 5194–5211, 1997.

[NS5] C. M. Newman, D. L. Stein, *Multiple states and thermodynamic limits in short ranged Ising spin glass models*, Phys. Rev. B **46**, 973–982, 1992.

[NS6] C. M. Newman, D. L. Stein, *Simplicity of state and overlap structure in finite-volume realistic spin glasses*, Phys. Rev. E **57**, 1356–1369, 1998.

[Shi] A. N. Shiryaev, *Probability*, 2nd ed., GTM 95, Springer, Berlin-Heidelberg-New York, 1996.